Appendix A
Fourier transforms

The following is a brief introduction to the Fourier transform at the level required for this book. More complete descriptions can be found in textbooks, for example in Gaskill (1978) and Bracewell (2000) or online, for example in Wikipedia.

A.1 Fourier series

We start by recalling the mathematics of Fourier series. Consider a periodic function of time $f(t)$, with a period T corresponding to a frequency $\nu_0 = 1/T$. The function can be written as a superposition of sinusoidal waves with frequencies $\nu_0, 2\nu_0, 3\nu_0, \ldots$:

$$f(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\left(\frac{2\pi nt}{T}\right) + B_n \sin\left(\frac{2\pi nt}{T}\right)\right). \tag{A.1}$$

This decomposition into sine and cosine waves is known as a Fourier series representation of a function. The set of coefficients A_n and B_n define an alternative repesentation of $f(t)$, since $f(t)$ can be determined uniquely from these coefficients.

To find the value of a particular coefficient A_n or B_n, we multiply the above equation by $\cos\left(\frac{2\pi nt}{T}\right)$ or $\sin\left(\frac{2\pi nt}{T}\right)$ and integrate from $-T/2$ to $T/2$ to give

$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi nt}{T}\right)dt, \tag{A.2}$$

$$B_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi nt}{T}\right)dt. \tag{A.3}$$

These results rely on the fact that

$$\begin{aligned}
\int_{-T/2}^{T/2} \cos\left(\tfrac{2\pi mt}{T}\right)\cos\left(\tfrac{2\pi nt}{T}\right)dt &= 0 && \text{for } m \neq n \\
&= T/2 && \text{for } m = n
\end{aligned}$$

$$\int_{-T/2}^{T/2} \cos\left(\tfrac{2\pi mt}{T}\right)\sin\left(\tfrac{2\pi nt}{T}\right)dt = 0 \qquad \text{for all } m, n$$

$$\begin{aligned}
\int_{-T/2}^{T/2} \sin\left(\tfrac{2\pi mt}{T}\right)\sin\left(\tfrac{2\pi nt}{T}\right)dt &= 0 && \text{for } m \neq n \\
&= T/2 && \text{for } m = n.
\end{aligned}$$

A.1.1 Complex coefficients

Rather than write the series as a sum of sine and cosine terms, it often convenient to describe the Fourier components with complex coefficients which represent the amplitude and phase of each component:

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{i2\pi nt/T} = \sum_{n=-\infty}^{\infty} C_n e^{in\nu_0 t}. \tag{A.4}$$

The coefficients C_n are given by multiplying by $e^{-2\pi i m\nu_0 t}$ and integrating from $-T/2$ to $T/2$:

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-2\pi i n\nu_0 t}dt. \tag{A.5}$$

This results relies on the fact that

$$\begin{aligned}
\int_{-T/2}^{T/2} e^{2\pi i n\nu_0 t}e^{-2\pi i m\nu_0 t}dt &= 0 && \text{for } m \neq n \\
&= T && \text{for } m = n.
\end{aligned} \tag{A.6}$$

We are mostly interested in functions $f(t)$ which are real, and in this case we find that $C_{-m} = C_m^*$. This ensures that the imaginary parts of the terms in Equation (A.4) cancel to zero when summing over positive and negative n.

A.2 Generalisation to non-periodic functions

We now need to extend the idea of Fourier series to functions that do not repeat; in other words, aribitrary functions (although there exist functions which cannot be Fourier transformed; these are of more interest to mathematicians than to physicists). Conceptually, this can be seen as taking the limit of a Fourier series as $T \to \infty$. The fundamental frequency ν_0 tends to zero, and so the frequency components making up the series become infinitesimally close together. The discrete coefficients $\{C_m\}$ merge to become a continuous *spectrum* $g(\nu)$ and the sum becomes an integral, giving

$$f(t) = \int_{-\infty}^{\infty} g(\nu) e^{2\pi i\nu,t} df, \tag{A.7}$$

where

$$g(\nu) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i\nu t} dt. \tag{A.8}$$

These equations define the Fourier transform; $g(\nu)$ is the Fourier transform of $f(t)$ and $f(t)$ is the inverse Fourier transform of $g(\nu)$. The reader should be aware that there are alternative definitions of the Fourier transform, which use $\omega = 2\pi\nu$ instead of ν as the 'conjugate variable' to t, but the definitions above (sometimes called the 'unitary' form of the Fourier transform) are used in this book.

We denote the Fourier transform using the operator $\mathcal{F}$, which transforms the function f into the function g

$$\mathcal{F}[f(t)] = g(\nu), \tag{A.9}$$

and its inverse $\mathcal{F}^{-1}$, which derives f given g:

$$\mathcal{F}^{-1}[g(\nu)] = f(t). \tag{A.10}$$

The Fourier transform can be thought of in terms of analysis and synthesis: the forward transform splits a function into sinusoidal components at different frequencies, while the inverse transform synthesises a function from these components.

A.3 Generalisation to higher dimensions

Fourier transforms are not just applied to functions of time: they can equally be applied to functions of any variable. For example, we can Fourier transform a function of a spatial variable x to derive a function of spatial frequency s, i. e.

$$\mathcal{F}[f(x)] = g(s). \tag{A.11}$$

The spaces spanned by the variables ν and s are 'reciprocal spaces' complementary to t and x, respectively, since they have units of s^{-1} and m^{-1}, respectively.

In the case of spatial dimensions, it is natural to consider generalising the Fourier transform to functions of two- or higher-dimensional variables. This is straightforwardly done by replacing the scalar dimension x by a vector $\boldsymbol{x}$

and by integrating over multiple dimensions. In two dimensions we have the forward transform,

$$\mathcal{F}[f(\boldsymbol{x})] = g(\boldsymbol{s}) = \iint_{-\infty}^{\infty} f(\boldsymbol{x}) e^{-2\pi \mathrm{i} \boldsymbol{s}\cdot\boldsymbol{x}}\, dx\, dy, \tag{A.12}$$

where $\boldsymbol{x} = (x, y)$ is a two-dimensional spatial coordinate and $\boldsymbol{s} = (s_x, s_y)$ is a two-dimensional spatial frequency, and the inverse transform,

$$\mathcal{F}^{-1}[g(\boldsymbol{s})] = f(\boldsymbol{x}) = \iint_{-\infty}^{\infty} g(\boldsymbol{s}) e^{2\pi \mathrm{i} \boldsymbol{s}\cdot\boldsymbol{x}}\, ds_x\, ds_y. \tag{A.13}$$

A.4 Dirac delta functions

An function which is important in the theory of Fourier transforms is the *Dirac delta function*. This function is used to denote a sharp ‘spike’, which occurs over an infinitesimal time, but with finite area. An example of where this might be used is the idea of an ‘impulse’ in Newtonian mechanics, where a sharp ‘kick’ imparts finite momentum even though it occurs over an infinitesimally short time Δt. The momentum change is $F\Delta t$, so as Δt tends to zero, F must tend to infinity during the kick. We would represent the force as a function of time in this limiting case as a Dirac delta function.

Formally, the Dirac delta function $\delta(t)$ is defined by the property

$$\int_{-\infty}^{\infty} \delta(t) f(t)\, dt = f(0) \tag{A.14}$$

for any arbitrary function $f(t)$. This means that $\delta(t)$ is a unit-area spike at $t = 0$ and is zero everywhere else. The function $\delta(t - t_0)$ is the same spike offset from the origin by an amount t_0 so

$$\int_{-\infty}^{\infty} \delta(t - t_0) f(t)\, dt = f(t_0)\,. \tag{A.15}$$

In other words, multiplying a function by $\delta(t - t_0)$ and integrating returns a ‘sample’ of the value of the function at time t_0.

We can use this property to show that the Fourier transform of a delta function is a complex exponential:

$$\int_{-\infty}^{\infty} \delta(t - t_0) e^{-2\pi \mathrm{i}\nu t}\, dt = e^{-2\pi \mathrm{i}\nu t_0}\,. \tag{A.16}$$

Note that the Fourier transform of a delta function at the origin is a constant (i.e. independent of ν).

Delta functions can be defined by analogy in higher dimensions. A point source in two dimensions can be defined as $\delta(x, y)$ such that

$$\iint_{-\infty}^{\infty} \delta(x, y) f(x, y)\, dy\, dx = f(0, 0). \tag{A.17}$$

A.5 Convolution

We now introduce the idea of the *convolution* of two functions $f(y)$ and $g(y)$. The convolution is denoted by the operator '$*$' and defined by the equation

$$f(y) * g(y) = \int_{-\infty}^{\infty} f(u) g(y - u)\, du. \tag{A.18}$$

The easiest example of convolution to visualise is when one of the functions is a delta function:

$$f(y) * \delta(y - y_0) = \int_{-\infty}^{\infty} f(u) \delta(y - y_0 - u)\, du = f(y - y_0). \tag{A.19}$$

In other words, the function $f(y)$ is reproduced centred around the delta function rather than around zero. For general functions, we can think of $g(y)$ as being made up of the sum of an infinite number of delta functions with different 'heights':

$$g(y) = \int_{-\infty}^{\infty} g(u) \delta(y - u)\, du. \tag{A.20}$$

Hence, when we take the convolution of f and g, each of the delta functions making up g is replaced by a copy of f. The function g is therefore 'smeared out' by the function f (and vice versa).

Convolution has an important application in optics where imaging systems such as telescopes are concerned. These instruments are imperfect, so a point-source object (a δ-function) is smeared out in the image. The image produced from a delta-function object is described by the resolution function (or *point-spread function*) of the instrument. For a general object, the image produced is the convolution of the object with the resolution function. The operation of trying to remove the effects of the resolution function is known as *deconvolution*.

Convolutions are particularly useful in the context of Fourier transforms. It can be shown that the Fourier transform of the product of two functions is the convolution of the Fourier transforms of the individual functions, i.e. if

$$F(s) = \mathcal{F}[f(x)] \tag{A.21}$$

and

$$G(s) = \mathcal{F}[g(x)], \tag{A.22}$$

then

$$\mathcal{F}[f(x)g(x)] = F(s) * G(s). \tag{A.23}$$

Similarly, the Fourier transform of the convolution of two functions is the product of the Fourier transforms of the individual functions,

$$\mathcal{F}[f(x) * g(x)] = F(s)G(s). \tag{A.24}$$

A.6 Composing Fourier transforms

Evaluating the Fourier transform or inverse Fourier transform of a function by performing the integrals in Equations (A.7) and (A.8) can be time-consuming. An alternative in many cases is to make use of a few ‘building-block’ Fourier transforms and combine them using known mathematical properties of the Fourier transform. Some commonly used functions and their transforms are given in Table A.1 and the following results can be used to extend and combine the transforms:

Reciprocity: If we know the forward transform of a function, we can obtain the inverse transform of the same function by ‘flipping’ the result about the $t = 0$ axis: if the Fourier transform of $f(t)$ is $g(\nu)$, then the inverse Fourier transform of $f(\nu)$ is $g(-t)$. This follows from the similarity of the integrals defining forward and inverse transforms, and means that all the following results apply when you replace $\mathcal{F}$ with $\mathcal{F}^{-1}$.

Scaling law: If we ‘stretch’ a function horizontally by an amount a, the corresponding dimension in the transform is compressed by the same factor: $\mathcal{F}[f(t/a)] = |a|g(a\nu)$. This reciprocal scaling is related to Heisenberg’s uncertainty principle, since wavefunctions of quantum variables such as position and momentum form a Fourier-transform pair. Note that the vertical dimension of the transformed function is stretched by $|a|$.

Linearity: If a function is the superposition of two other functions, its Fourier transform is the superposition of the respective Fourier transforms: $\mathcal{F}[a_1 f_1(t) + a_2 f_2(t)] = a_1\mathcal{F}[f_1(t)] + a_2\mathcal{F}[f_2(t)]$, where a_1 and a_2 are arbitrary constants and f_1 and f_2 are arbitrary functions.

The convolution theorem: $\mathcal{F}[f_1(t)f_2(t)] = \mathcal{F}[f_1(t)] * \mathcal{F}[f_2(t)]$ and $\mathcal{F}[f_1(t) * f_2(t)] = \mathcal{F}[f_1(t)]\mathcal{F}[f_2(t)]$.

Table A.1 *Fourier transforms of some useful functions.*

	$f(t)$		$g(\nu)$
Delta function	$\delta(t - t_0)$	complex exponential	$e^{-2\pi i\nu t_0}$
One-dimensional top-hat	$\mathrm{rect}(t)$	sinc	$\frac{\sin(\pi\nu)}{\pi\nu}$
Circular top-hat	$\mathrm{rect}(\lvert\boldsymbol{x}\rvert)$	jinc	$\frac{J_1(\lvert s\rvert)}{\lvert s\rvert}$
Gaussian	$e^{-t^2/2}$	Gaussian	$\sqrt{\pi}e^{-\pi\nu^2}$
Comb function	$\sum_{n=-\infty}^{\infty}\delta(t-n)$	Comb function	$\sum_{n=-\infty}^{\infty}\delta(\nu-n)$

A.7 Symmetry

It is often helpful to know the symmetry properties of Fourier transforms in order to check that we have got the right results. It is straightforward to show from Equation A.8 that the Fourier transform of a function $f(t)$ that is purely real (something which is true of most of the physical variables we will be taking the Fourier transform of) has so-called *Hermitian symmetry*, i.e. that $g(-\nu) = g(\nu)^*$. This means that the properties of the function can be determined purely from the positive frequency components of the Fourier transform, so typically we only plot the positive half of the Fourier transform in these cases.

Similarly, we can show that if a function is both real and symmetric (i.e. $f(-t) = f(t)$) then its Fourier transform is both real and symmetric, while if the function is both real and antisymmetric (i.e. $f(-t) = -f(t)$) then its transform is purely imaginary and antisymmetric.