SOME DETAILS ABOUT REWIRING SYSTEMS

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30/10/14

Strongly based on a
présentation of Roel de Vrijer
and Alexis Marechal

RECALL

- Signature $\Sigma = \langle S, F \rangle$
- From Σ we can derive the set T_{Σ}, *i.e.*, the set of all terms generated by Σ.
- Give a set X of S indexed variables, we can derive the set $T_{\Sigma,X}$ of terms with variables.
- We can give a semantics to all of that by adding equations (called axioms), the whole pack Σ + X + Axioms = Adt

$\Sigma = \langle S,\ F \rangle$ $\qquad S = \{nat, bool\}$ $\qquad X_s = \{x_{nat}, b_{bool}\}$

$F = \{true_{\varepsilon,bool}, false_{\varepsilon,bool}, zero_{\varepsilon,nat}, suc_{nat,nat}, +_{nat,nat,nat}, not_{bool,bool}\}$

$(T_\Sigma)_s =$

zero

true

not(false)

+(zero, zero)

not(true)

suc(zero)

false

+(suc(zero), zero)

+(zero, +(suc(zero), suc(suc(zero)))

+(suc(zero), suc(suc(zero))

suc(suc(suc(zero)))

$(T_{\Sigma,X})_s =$

b

true

zero

x

not(false)

+(zero, zero)

not(not(b))

not(true)

suc(zero)

not(b)

false

+(suc(zero), zero)

+(suc(x), zero)

+(zero, +(suc(zero), suc(suc(zero)))

+(suc(zero), suc(suc(zero))

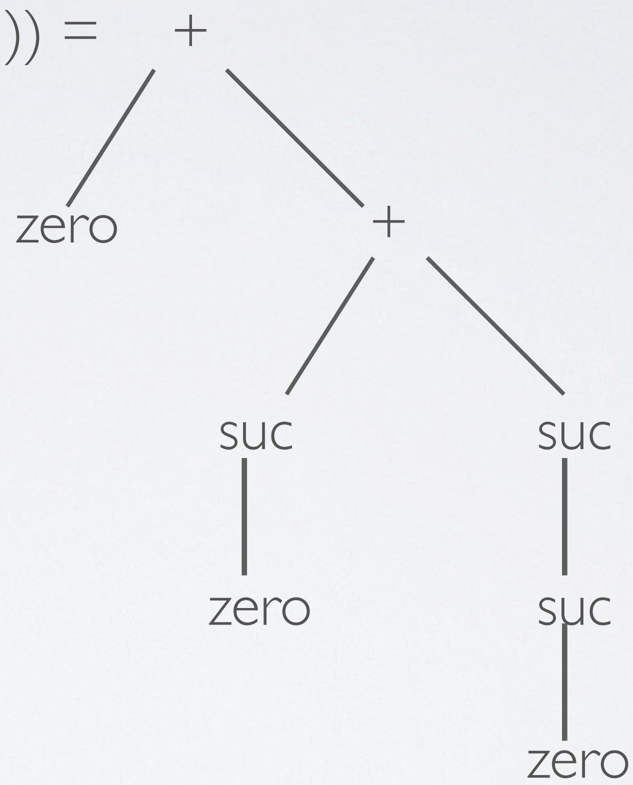
suc(suc(suc(zero)))

SYNTACTIC DERIVATION

- Two types of terms: atoms and composite terms.
  - atoms: true, false, zero; composite: suc(zero), +(zero, zero), etc.)

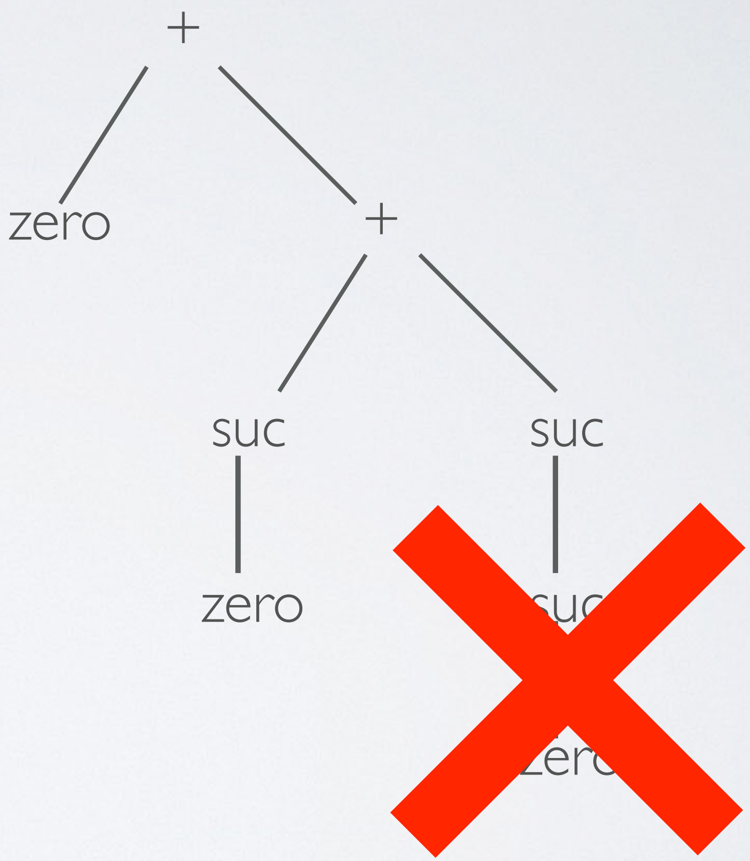
COMPOSITE TERM AS TREE

+(zero, +(suc(zero), suc(suc(zero)))) =



CONTEXT

+(zero, +(suc(zero), suc(_))) =

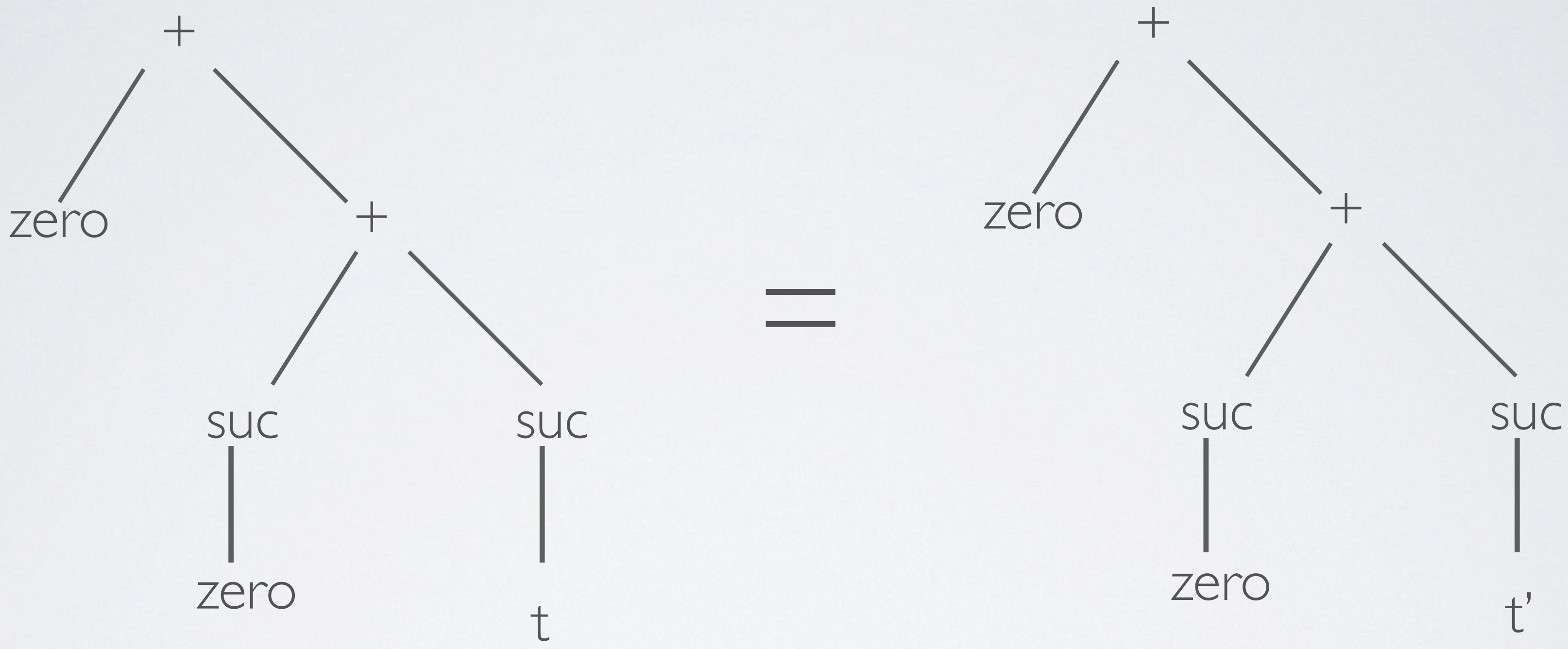


c[_] = +(zero, +(suc(zero), suc(_)))

SYNTACTIC DERIVATION PRINCIPLE

- $c[t] = c[t']$ iff $t = t'$, which is the same as
    - $c[t] = c[t']$ implies $t = t'$, and
    - $t = t'$ implies $c[t] = c[t']$
- Example: if $not(true) = false$ is the equation, and $c[\_] = not(\_)$ the context, we can **deduce by syntactic derivation that:**
    - $not(true) = false \vdash not(not(true)) = not(false)$

GRAPHICALLY



if and only if, t = t’

REPLACEMENT

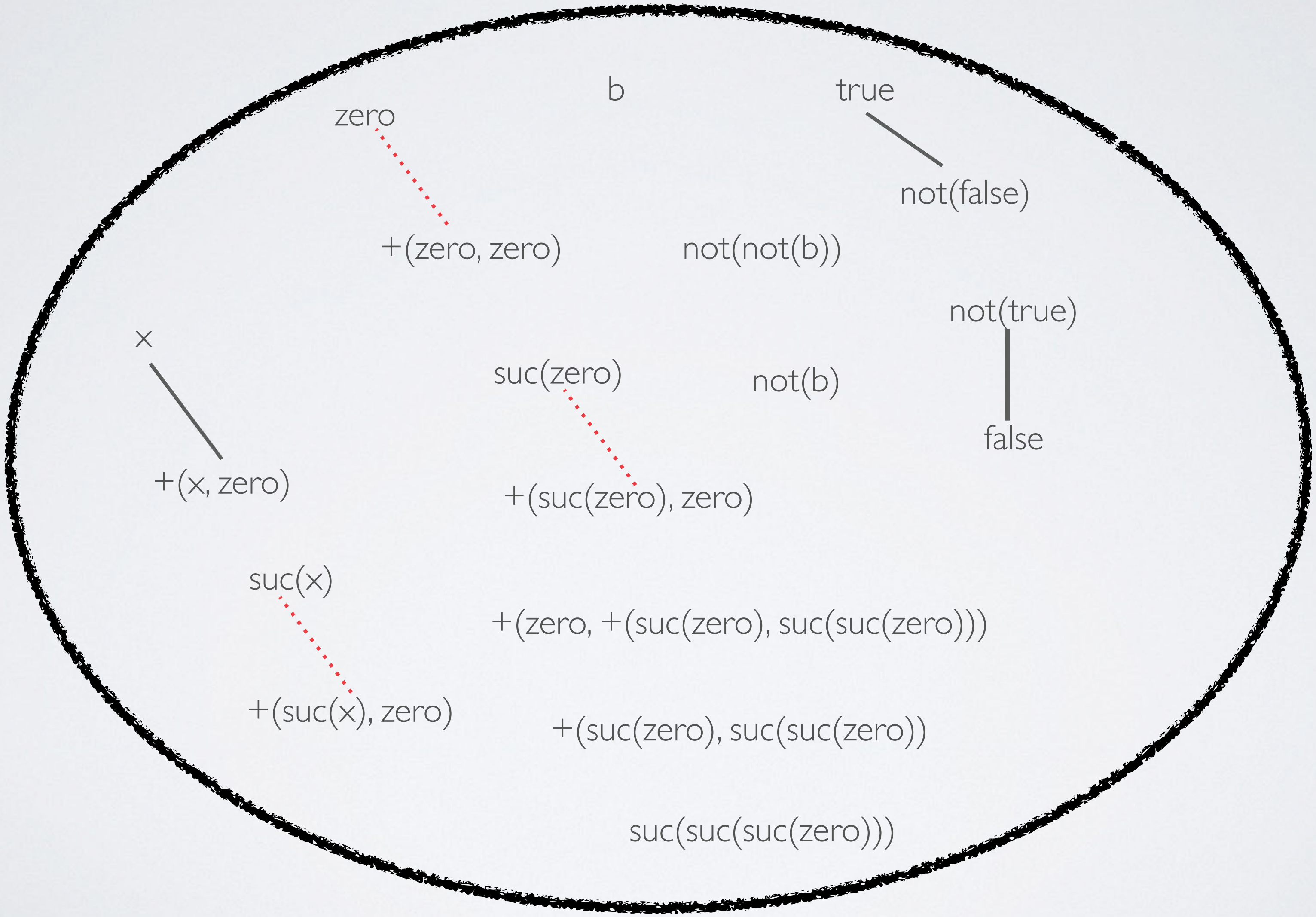
- The operation that takes a term c[t] and a new term t’ and returns a c[t’] is called a **replacement**.

$\Sigma = \langle S, F \rangle$ $\quad S = \{nat, bool\}$ $\quad X_s = \{x_{nat}, b_{bool}\}$

$F = \{true_{\varepsilon,bool}, false_{\varepsilon,bool}, zero_{\varepsilon,nat}, suc_{nat,nat}, +_{nat,nat,nat}, not_{bool,bool}\}$

$Axioms = \{not(true) = false, not(false) = true, +(x, zero) = x\}$

$(T_{\Sigma,X})_s =$



Systèmes axiomatiques:

1. f($a) = g($a)
2. g($a) = 0

Systèmes de réécriture:

1. f($a) → g($a)
2. g($a) → 0

Orientation!

Definition (Rewrite step)

Let $\Sigma = \langle S, F \rangle$ be a signature and X be a S-sorted set of variables and $l \rightsquigarrow r$,$l, r \in T_\Sigma(X)$ a rewrite rule.

- $filter(t, l) =< \sigma, c > \Leftrightarrow \exists \sigma \in S\ \exists c, c[\sigma l] = t$ [a]
- $t' = c[\sigma r]$
- $< t, t' > \in Rew_{l \rightsquigarrow r}$ a rewrite step

---

a. $c[\_]$ denotes the context of a term, i.e. a term with a place holder

Rew. rule: f($a) ⇝ $a

Rew. step: g(f(0)) ⇝ g(0)

l ⇔ f($a)	t ⇔ g(f(0))	σ ⇔ [0/a]
r ⇔ $a	t' ⇔ g(0)	c[t] ⇔ g(t)

Notations non-officielles!

Definition (Rewrite step)

Let $\Sigma = \langle S, F \rangle$ be a signature and X be a S-sorted set of variables and $l \rightsquigarrow r$,$l, r \in T_\Sigma(X)$ a rewrite rule.

- $filter(t, l) =< \sigma, c > \Leftrightarrow \exists \sigma \in S\ \exists c, c[\sigma l] = t$ [a]
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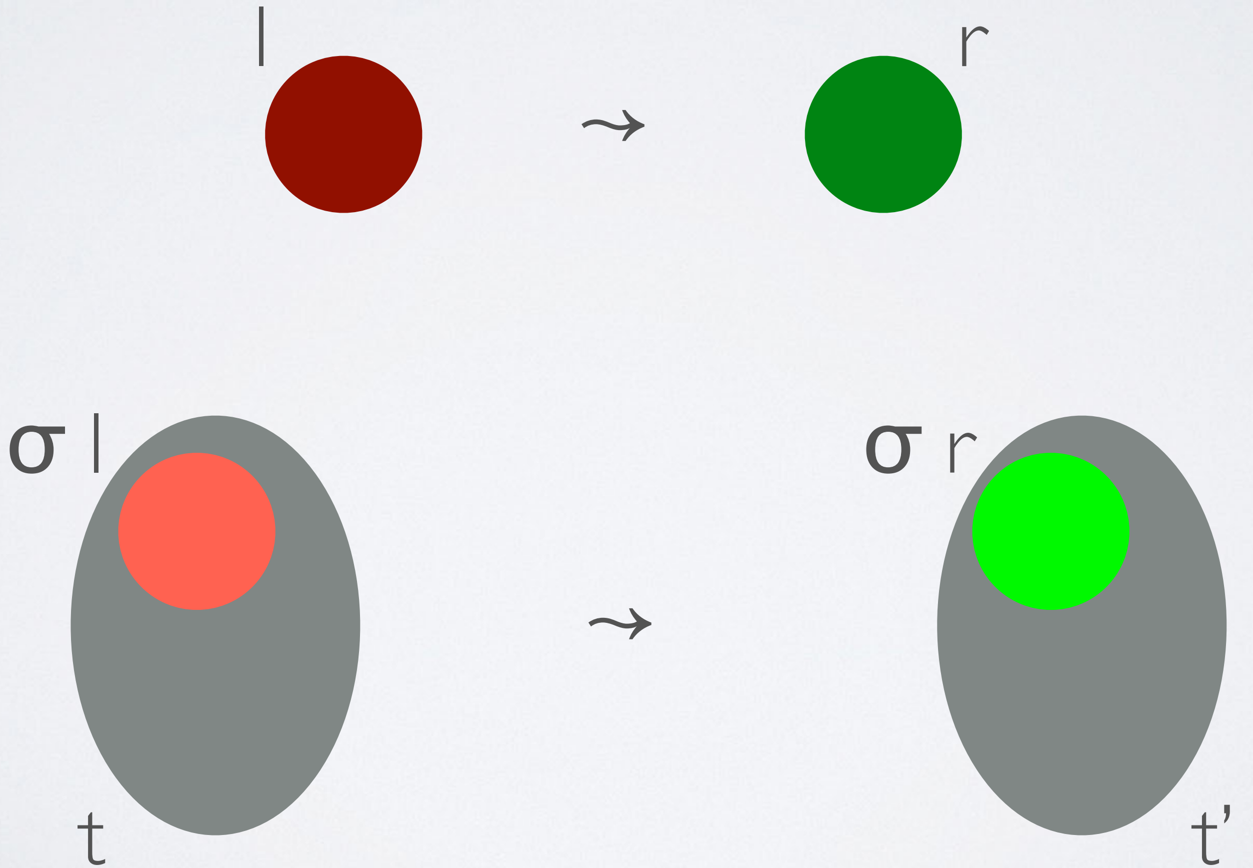
- $< t, t' > \in Rew_{l \rightsquigarrow r}$ a rewrite step

a. $c[\_]$ denotes the context of a term, i.e. a term with a place holder



$\Sigma = \langle S, F \rangle \qquad S = \{nat, bool\} \qquad X_s = \{x_{nat}, b_{bool}\}$

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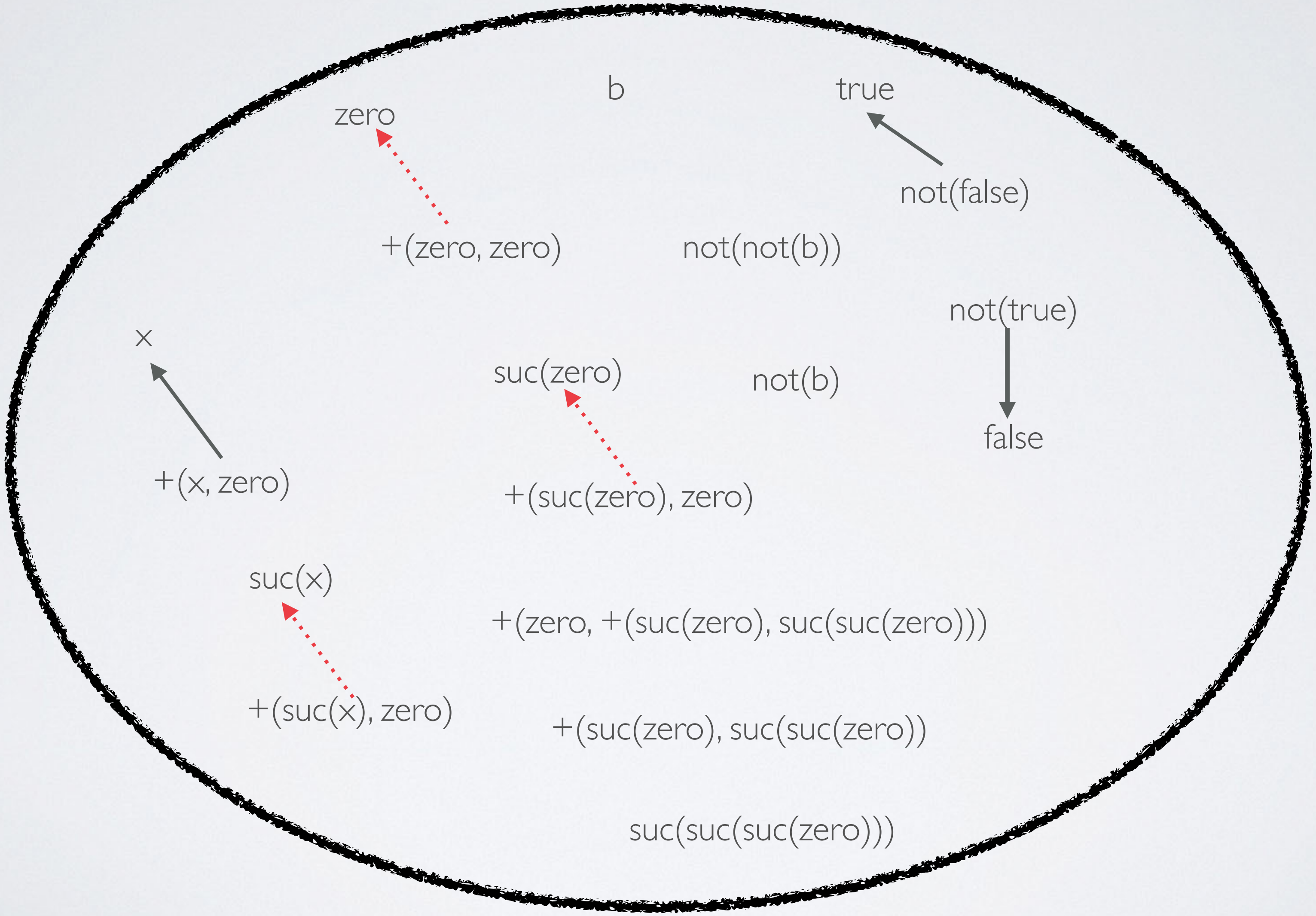
$(T_{\Sigma,X})_s =$



1 $f(0) \rightsquigarrow s(0)$

2 $g(f(\$x)) \rightsquigarrow g(\$x)$

3 $h(\$x) \rightsquigarrow h(s(\$x))$

$g(f(0)) \overset{1}{\rightsquigarrow} g(s(0))$

$g(f(0)) \overset{2}{\rightsquigarrow} g(0)$

Problème : confluence

$$
\begin{array}{cc}
1 & f(0) \rightsquigarrow s(0) \\
2 & g(f(\$x)) \rightsquigarrow g(\$x) \\
3 & h(\$x) \rightsquigarrow h(s(\$x))
\end{array}
$$

$$h(0) \overset{3}{\rightsquigarrow} h(s(0)) \overset{3}{\rightsquigarrow} h(s(s(0))) \overset{3}{\rightsquigarrow} \ldots$$

Problème: terminaison

Confluence / terminaison

$\rightsquigarrow^*$

Clôture transitive

Confluence :

Knuth-Bendix

Terminaison :

Critical Pairs - Knuth-Bendix theorem

Let $l_1 \rightsquigarrow r_1$ and $l_2 \rightsquigarrow r_2$ be two rules of a term rewriting system.
we suppose that these rules have no variables in common.
If l_1^{sub} is a subterm (and not a variable) of l_1 (or the term itself) with $l_1^{context}[l_1^{sub}] = l_1$ and there exist a most general unifier σ such that $l_1^{sub}\sigma = l_2\sigma$, then $r_1\sigma$ and $l_1^{context}[r_2\sigma]$are called a critical pair.

1 $g(f(x, y), x) \rightarrow h(x)$

2 $f(x, x) \rightarrow x$

$$\begin{tikzcd} & g(f(x, x), x) \arrow[dl, "2"'] \arrow[dr, "1"] & \\ g(x, x) \arrow[rr, "\rightarrow" description, phantom] & & h(x) \end{tikzcd}$$

$l_1^{sub} \Leftrightarrow f(x, y)$

$l_1^{context}[t] \Leftrightarrow g(t, x)$

$l_2 \Leftrightarrow f(x, x)$

$\sigma \Leftrightarrow [x/y]$

$r_1\sigma \Leftrightarrow h(x)$

$l_1^{context}[r_2\sigma] \Leftrightarrow g(x, x)$

1 $e.x \rightsquigarrow x$

2 $I(x).x \rightsquigarrow e$

3 $(x.y).z \rightsquigarrow x.(y.z)$

1. $e.x \rightsquigarrow x$
2. $I(x).x \rightsquigarrow e$
3. $(x.y).z \rightsquigarrow x.(y.z)$

Objectif:

1. $e.x \rightsquigarrow x$
2. $I(x).x \rightsquigarrow e$
3. $(x.y).z \rightsquigarrow x.(y.z)$
4. $x.e \rightsquigarrow x$
5. $x.I(x) \rightsquigarrow e$
6. $x.(y.z) \rightsquigarrow (x.y).z$
7. $I(e) \rightsquigarrow e$
8. $I(I(x)) \rightsquigarrow x$
9. $I(x.y) \rightsquigarrow I(y).I(x)$

1 $e.x \rightsquigarrow x$

2 $I(x).x \rightsquigarrow e$

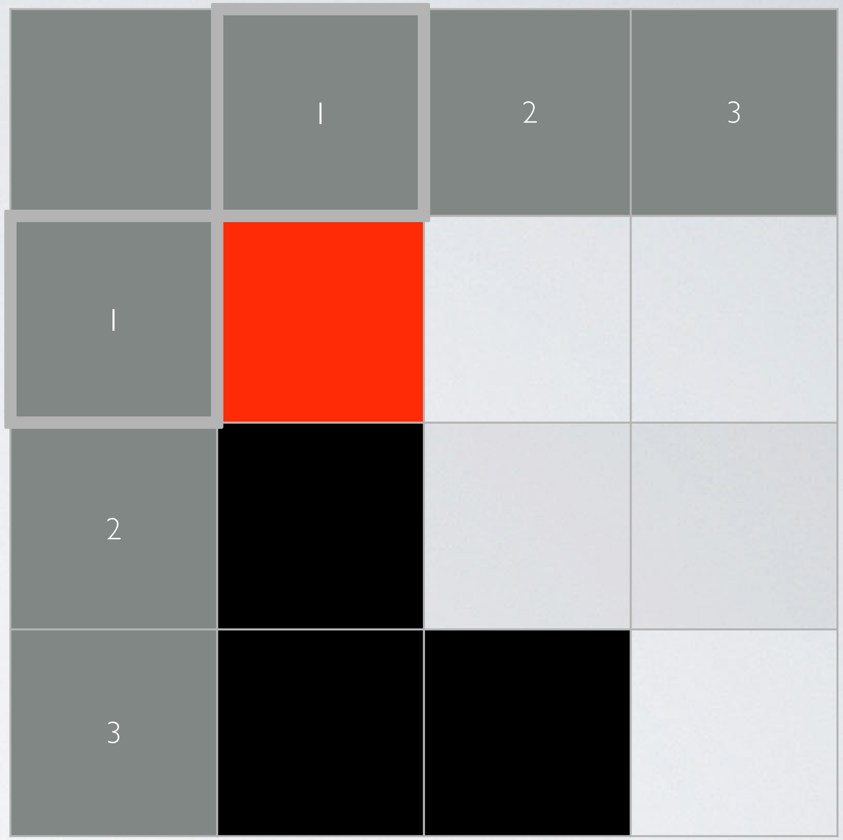
3 $(x.y).z \rightsquigarrow x.(y.z)$

	1	2	3
1			
2			
3			

1 $\quad e.x \rightsquigarrow x$

2 $\quad I(x).x \rightsquigarrow e$

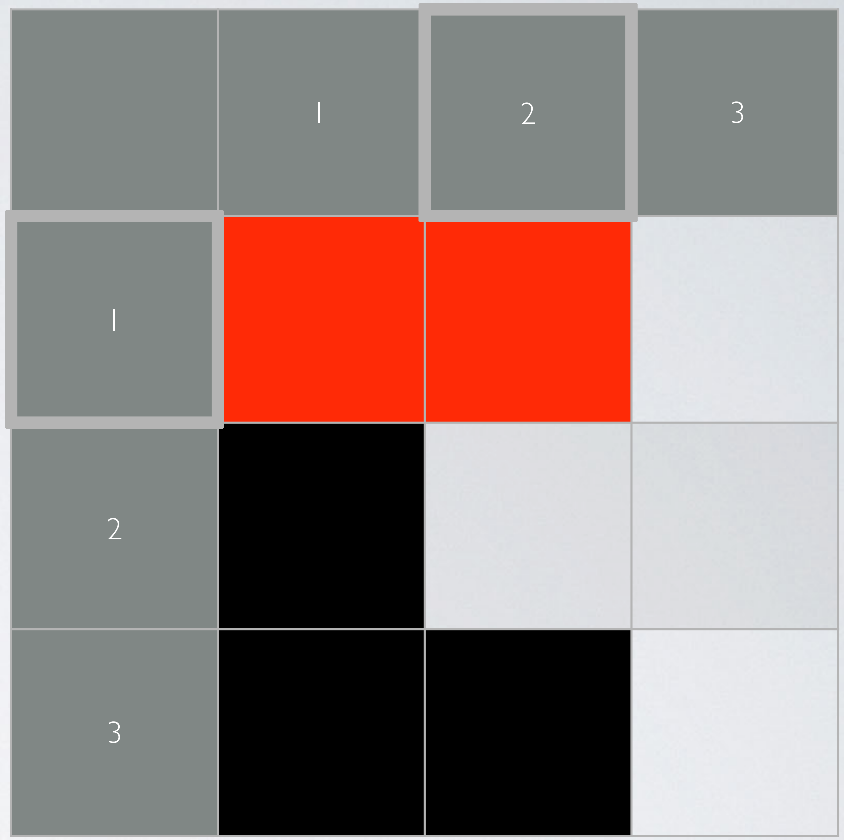
3 $\quad (x.y).z \rightsquigarrow x.(y.z)$



1 $e.x \rightsquigarrow x$

2 $I(x).x \rightsquigarrow e$

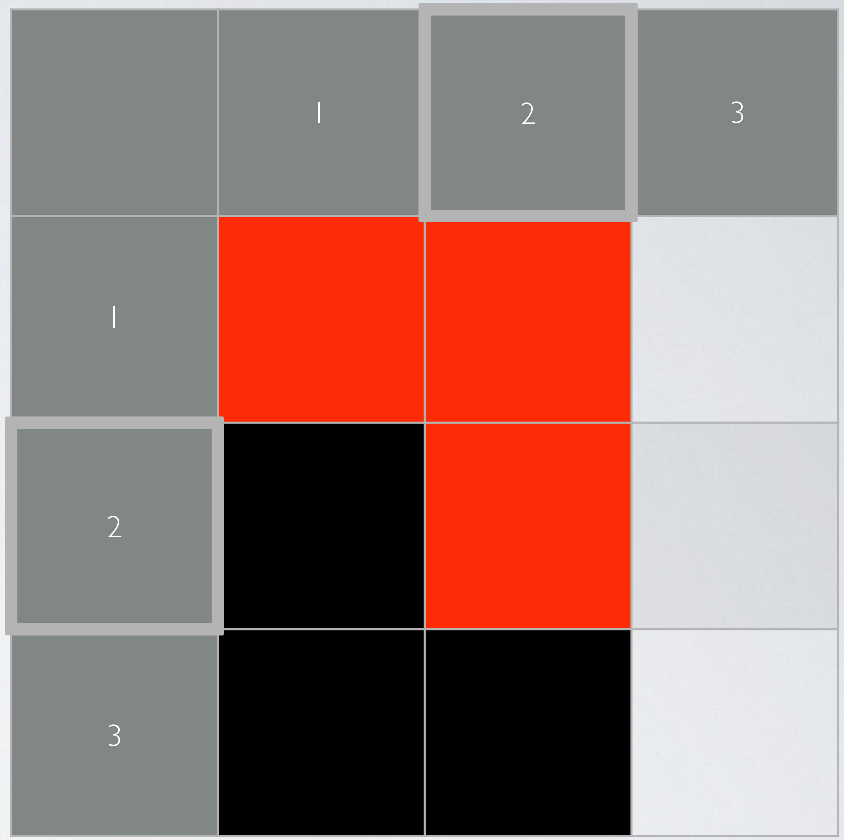
3 $(x.y).z \rightsquigarrow x.(y.z)$



1 $e.x \rightsquigarrow x$

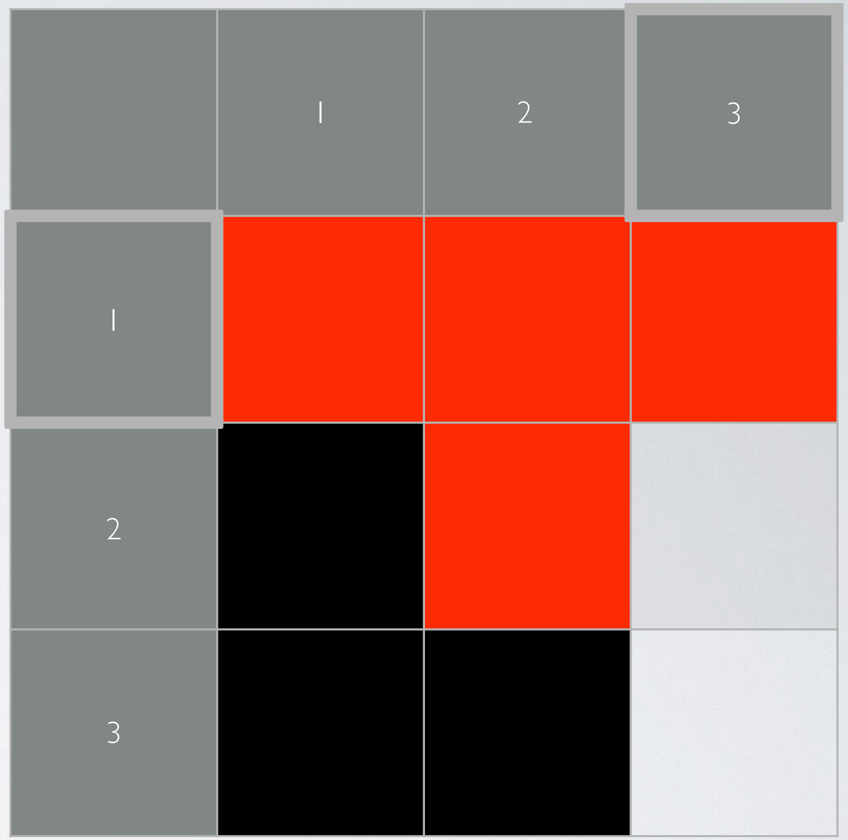
2 $I(x).x \rightsquigarrow e$

3 $(x.y).z \rightsquigarrow x.(y.z)$

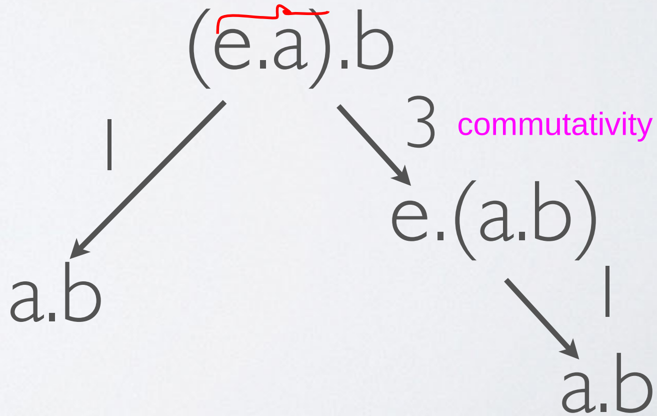


1. $e.x \rightsquigarrow x$
2. $I(x).x \rightsquigarrow e$
3. $(x.y).z \rightsquigarrow x.(y.z)$

a.



identity times a times b



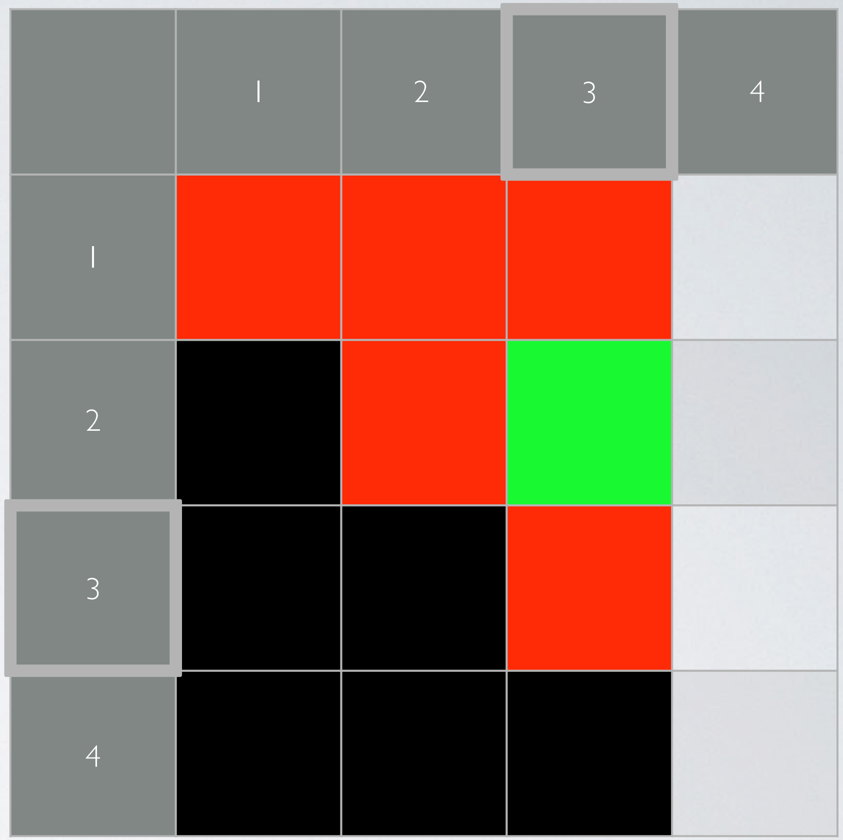
1. $e.x \rightsquigarrow x$
2. $I(x).x \rightsquigarrow e$
3. $(x.y).z \rightsquigarrow x.(y.z)$
4. $I(x).(x.z) \rightsquigarrow z$

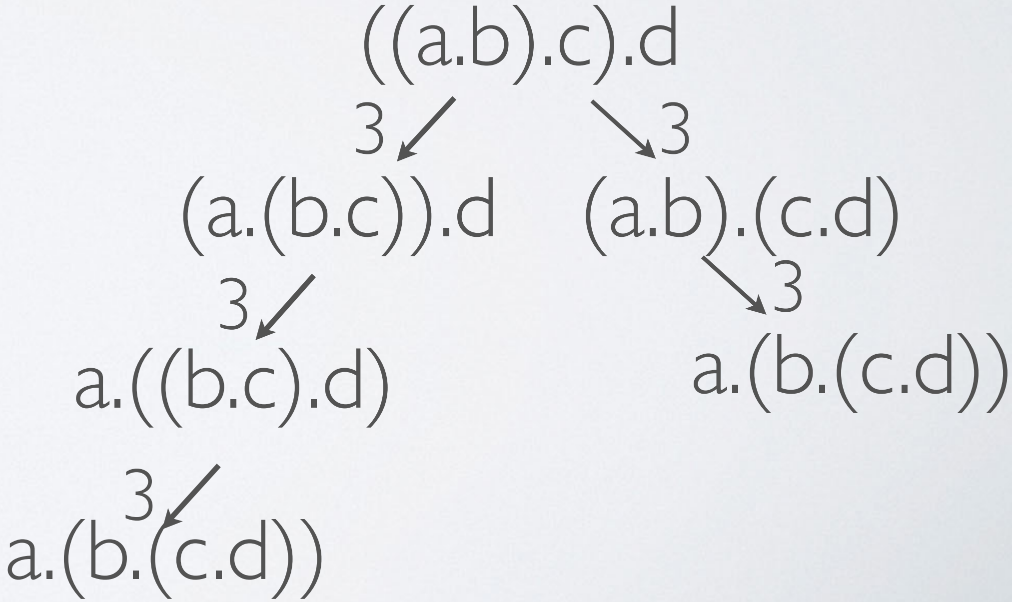
	1	2	3
1			
2			
3			

$$\begin{tikzcd}[column sep=small]
 & (I(a).a).b \arrow[dl, "2"'] \arrow[dr, "3"] & \\
e.b \arrow[d, "I"'] & & I(a).(a.b) \\
b & &
\end{tikzcd}$$

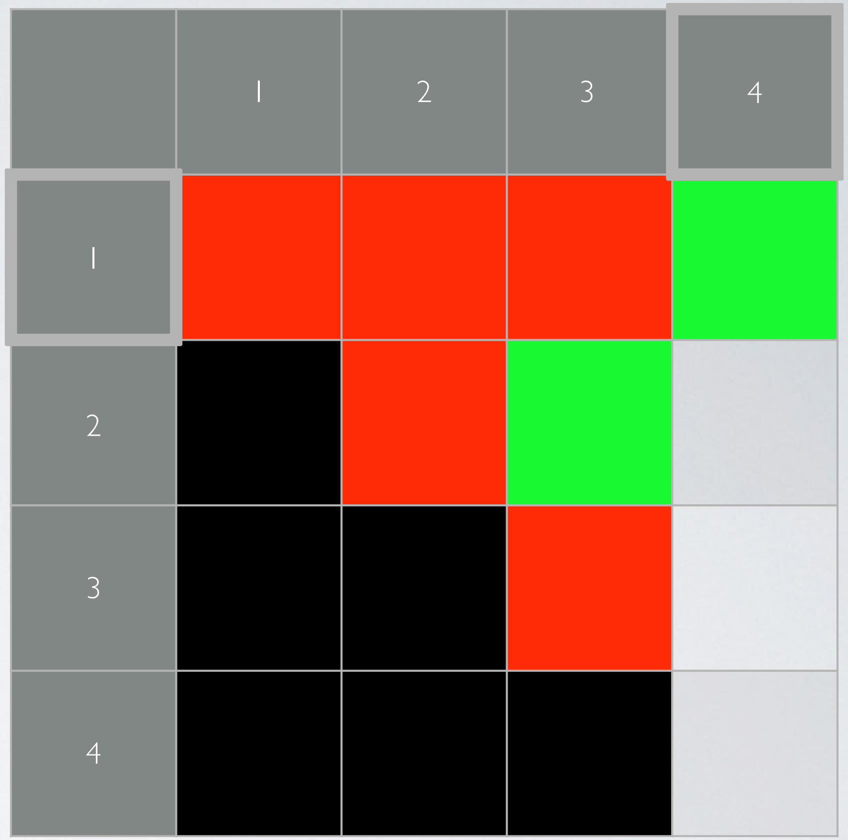
not same result => no convergence = > we need to add a rule

1. $e.x \rightsquigarrow x$
2. $I(x).x \rightsquigarrow e$
3. $(x.y).z \rightsquigarrow x.(y.z)$
4. $I(x).(x.z) \rightsquigarrow z$



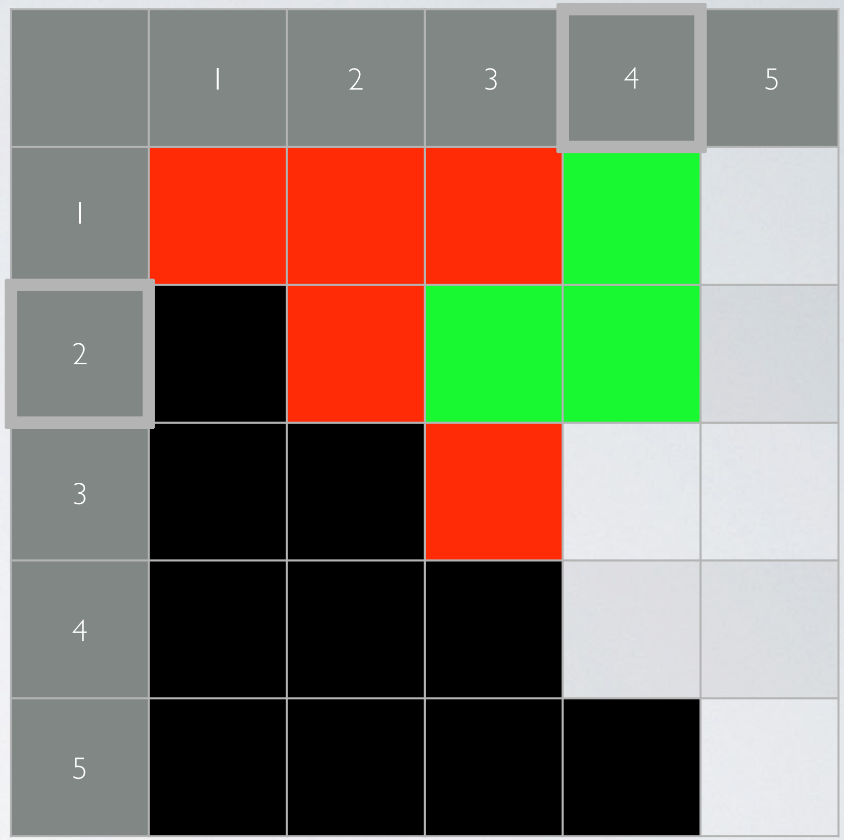


1. $e.x \rightsquigarrow x$
2. $I(x).x \rightsquigarrow e$
3. $(x.y).z \rightsquigarrow x.(y.z)$
4. $I(x).(x.z) \rightsquigarrow z$
5. $I(e).z \rightsquigarrow z$



$$\begin{tikzcd}
 & I(e).(e.a) \arrow[dl, "1"'] \arrow[dr, "4"] & \\
I(e).a & & a
\end{tikzcd}$$

1. $e.x \rightsquigarrow x$
2. $I(x).x \rightsquigarrow e$
3. $(x.y).z \rightsquigarrow x.(y.z)$
4. $I(x).(x.z) \rightsquigarrow z$
5. $I(e).z \rightsquigarrow z$
6. $I(I(x)).e \rightsquigarrow x$

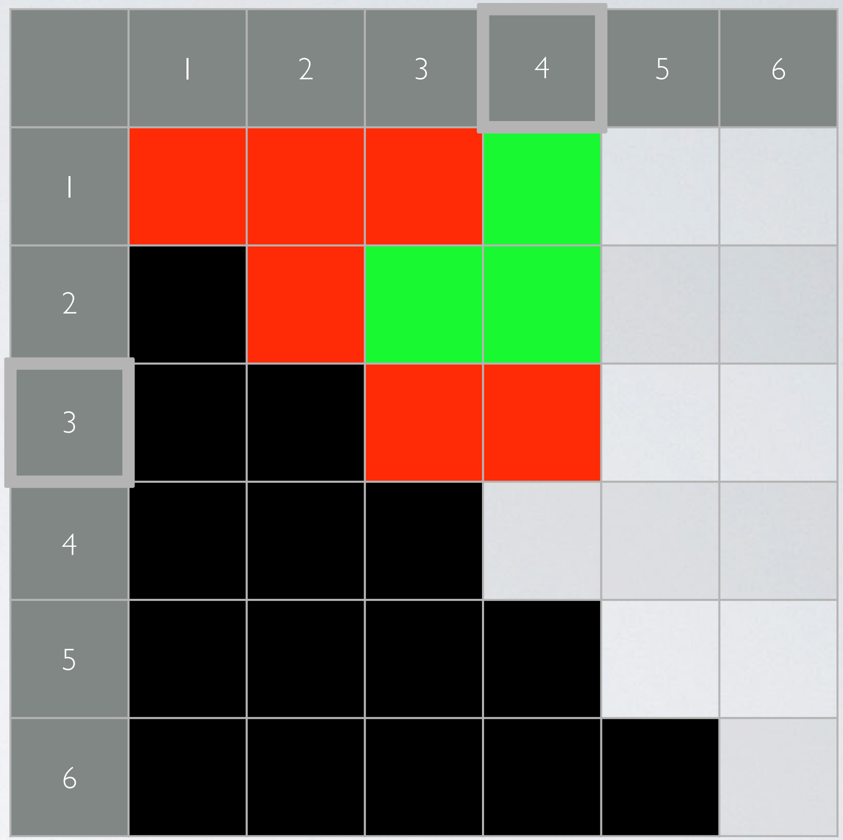


$$I(I(a)).(I(a).a)$$

$$\swarrow 2 \qquad \searrow 4$$

$$I(I(a)).e \qquad\qquad a$$

1. e.x → x
2. I(x).x → e
3. (x.y).z → x.(y.z)
4. I(x).(x.z) → z
5. I(e).z → z
6. I(I(x)).e → x



(I(a).(a.b)).c

3 ↙ ↘ 4

I(a).((a.b).c) b.c

3 ↙

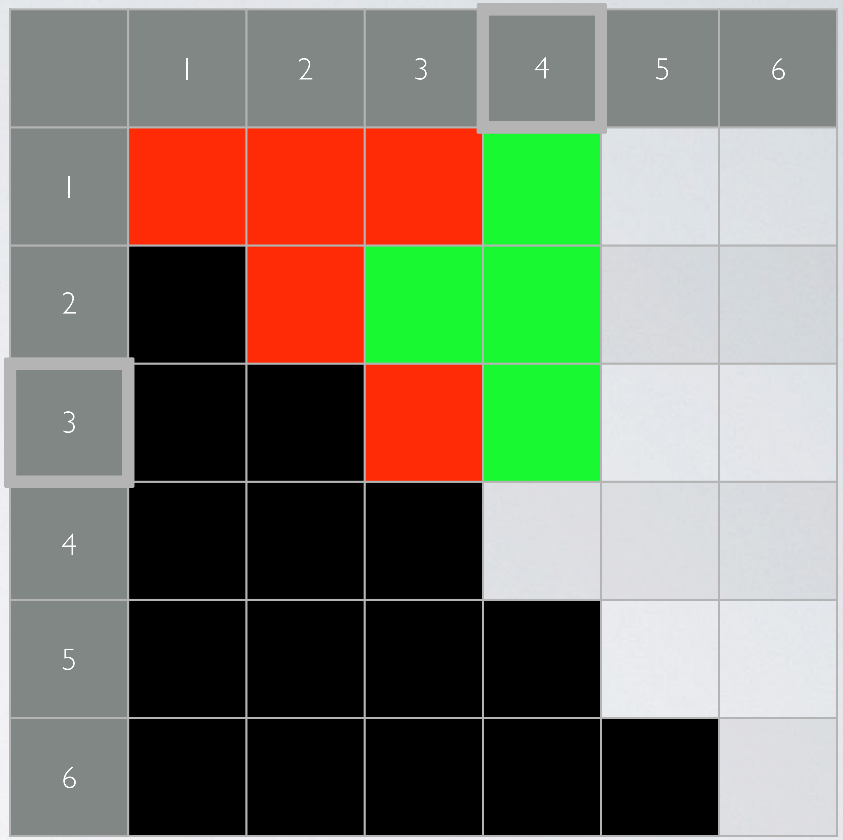
I(a).(a.(b.c))

4 ↙

b.c

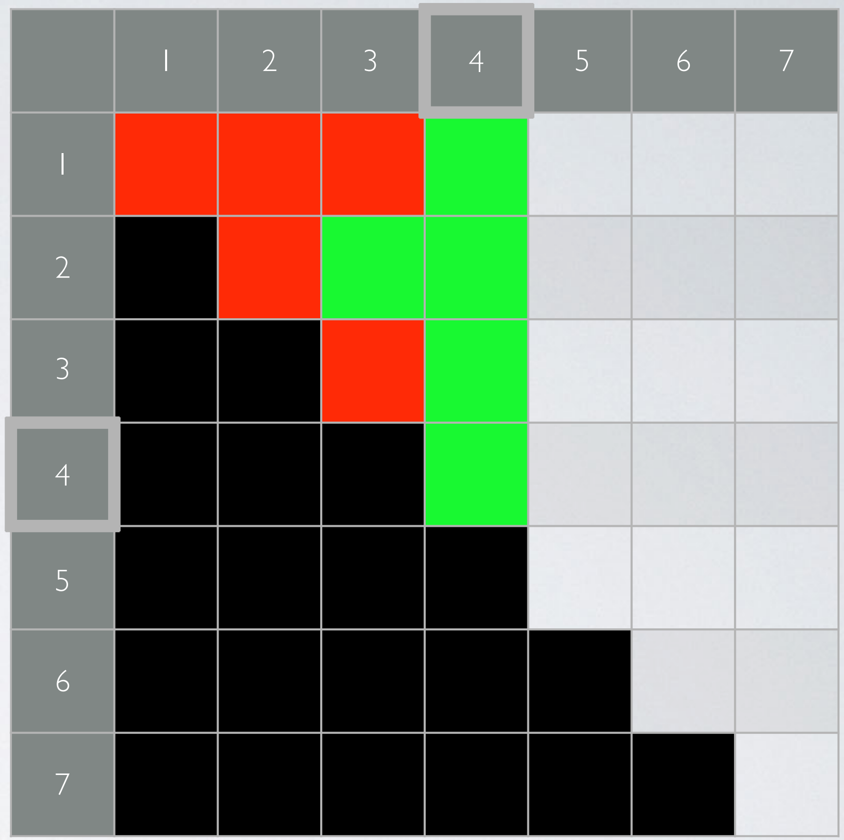
Mêmes règles, autre substitution!

1. $e.x \rightsquigarrow x$
2. $I(x).x \rightsquigarrow e$
3. $(x.y).z \rightsquigarrow x.(y.z)$
4. $I(x).(x.z) \rightsquigarrow z$
5. $I(e).z \rightsquigarrow z$
6. $I(I(x)).e \rightsquigarrow x$
7. $I(x.y).(x.(y.z)) \rightsquigarrow z$



$$
\begin{array}{ccc}
 & I(a.b).((a.b).c) & \\
3 \swarrow & & \searrow 4 \\
I(a.b).(a.(b.c)) & & c
\end{array}
$$

1. e.x ⇝ x
2. I(x).x ⇝ e
3. (x.y).z ⇝ x.(y.z)
4. I(x).(x.z) ⇝ z
5. I(e).z ⇝ z
6. I(I(x)).e ⇝ x
7. I(x.y).(x.(y.z)) ⇝ z
8. I(I(x)).y ⇝ x.y

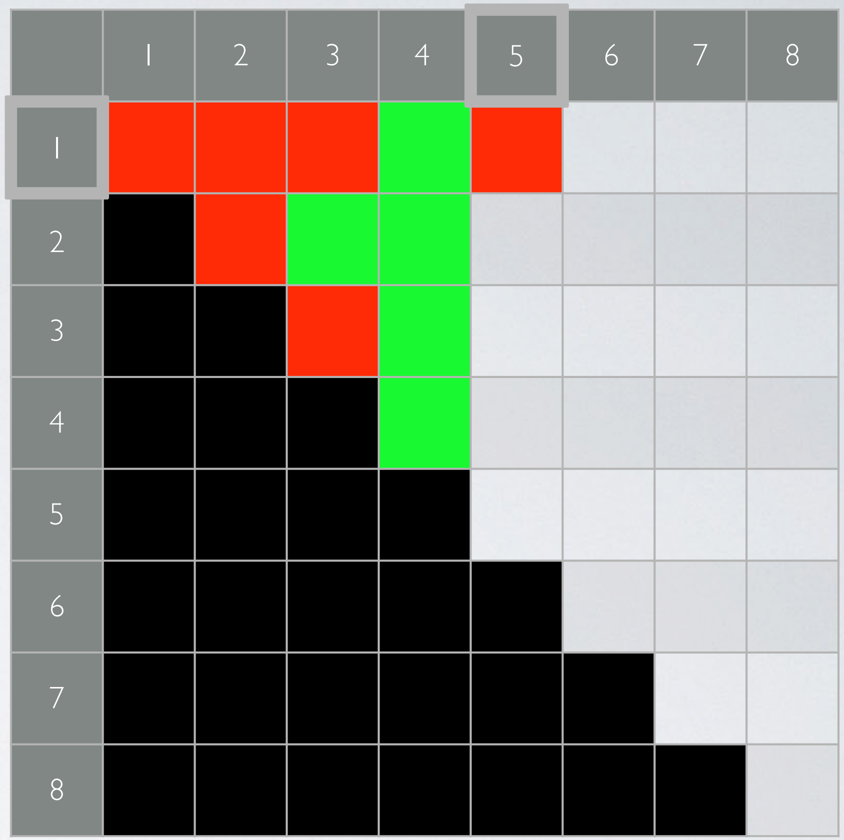


I(I(a)).(I(a).(a.b))

4 ↙ ↘ 4

I(I(a)).b (a.b)

1. $e.x \rightsquigarrow x$
2. $I(x).x \rightsquigarrow e$
3. $(x.y).z \rightsquigarrow x.(y.z)$
4. $I(x).(x.z) \rightsquigarrow z$
5. $I(e).z \rightsquigarrow z$
6. $I(I(x)).e \rightsquigarrow x$
7. $I(x.y).(x.(y.z)) \rightsquigarrow z$
8. $I(I(x)).y \rightsquigarrow x.y$



1. $e.x \rightsquigarrow x$
2. $I(x).x \rightsquigarrow e$
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4. $I(x).(x.z) \rightsquigarrow z$
5. $I(e).z \rightsquigarrow z$
6. $I(I(x)).e \rightsquigarrow x$
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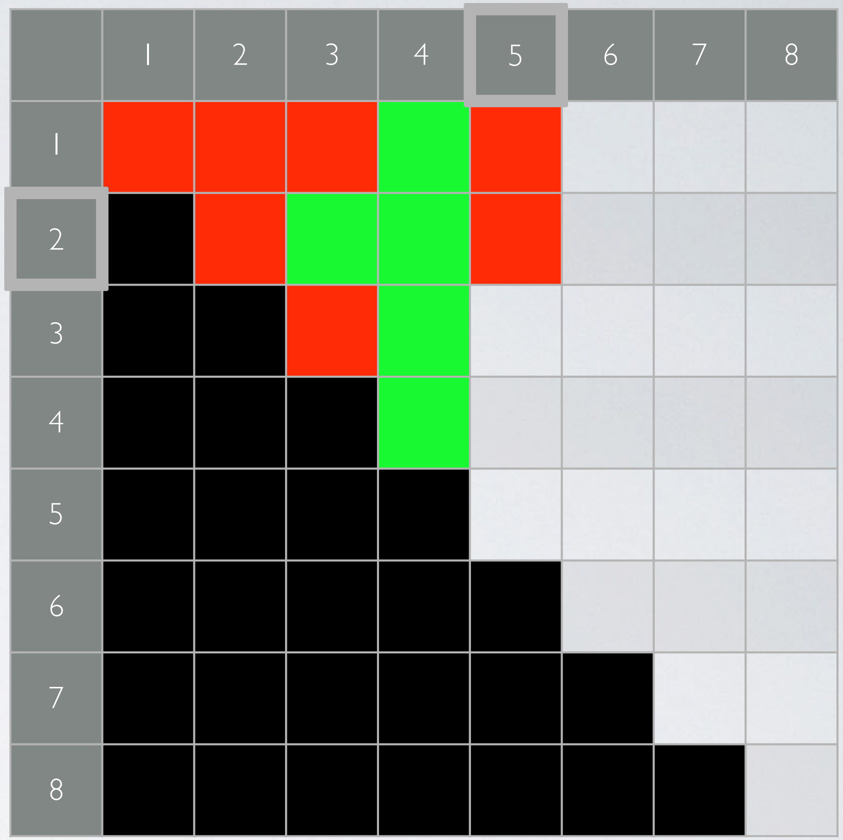

$$\begin{tikzcd} & I(e).e \arrow[dl, "2"'] \arrow[dr, "5"] & \\ e & & e \end{tikzcd}$$

1. $e.x \rightsquigarrow x$
2. $I(x).x \rightsquigarrow e$
3. $(x.y).z \rightsquigarrow x.(y.z)$
4. $I(x).(x.z) \rightsquigarrow z$
5. $I(e).z \rightsquigarrow z$
6. $I(I(x)).e \rightsquigarrow x$
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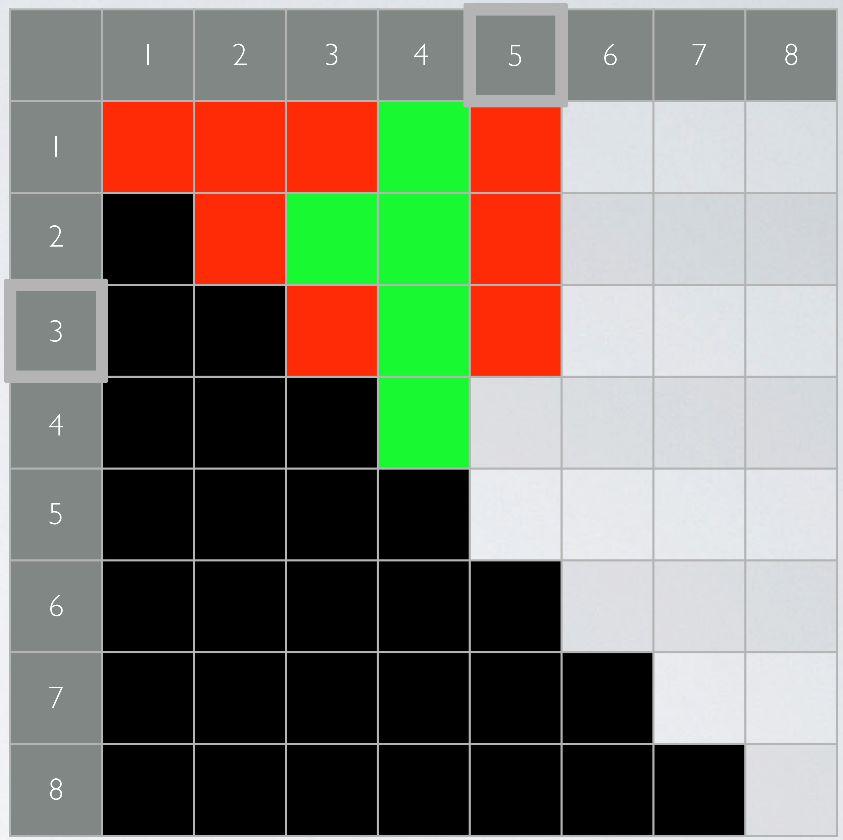


$$(I(e).a).b$$

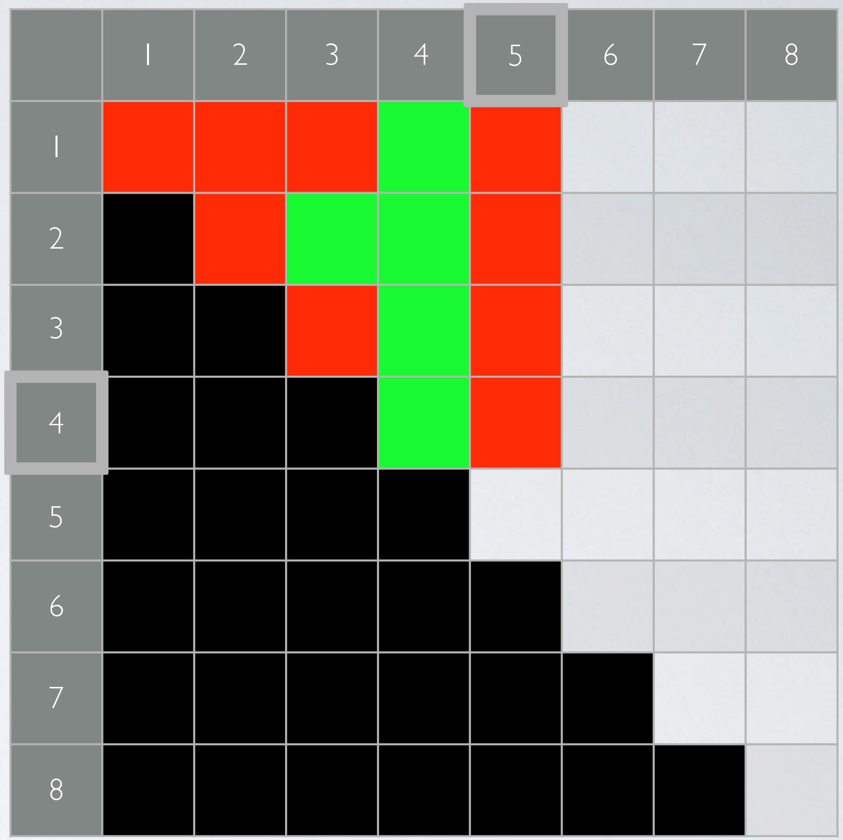
$$\swarrow^{3} \qquad \searrow^{5}$$

$$I(e).(a.b) \qquad (a.b)$$

$$\swarrow^{5}$$

$$(a.b)$$

1. $e.x \leadsto x$
2. $I(x).x \leadsto e$
3. $(x.y).z \leadsto x.(y.z)$
4. $I(x).(x.z) \leadsto z$
5. $I(e).z \leadsto z$
6. $I(I(x)).e \leadsto x$
7. $I(x.y).(x.(y.z)) \leadsto z$
8. $I(I(x)).y \leadsto x.y$



$I(I(e)).(I(e).a)$

$\swarrow 4$ $\qquad$ $\searrow 5$

a $\qquad$ $I(I(e)).a$

$\searrow 8$

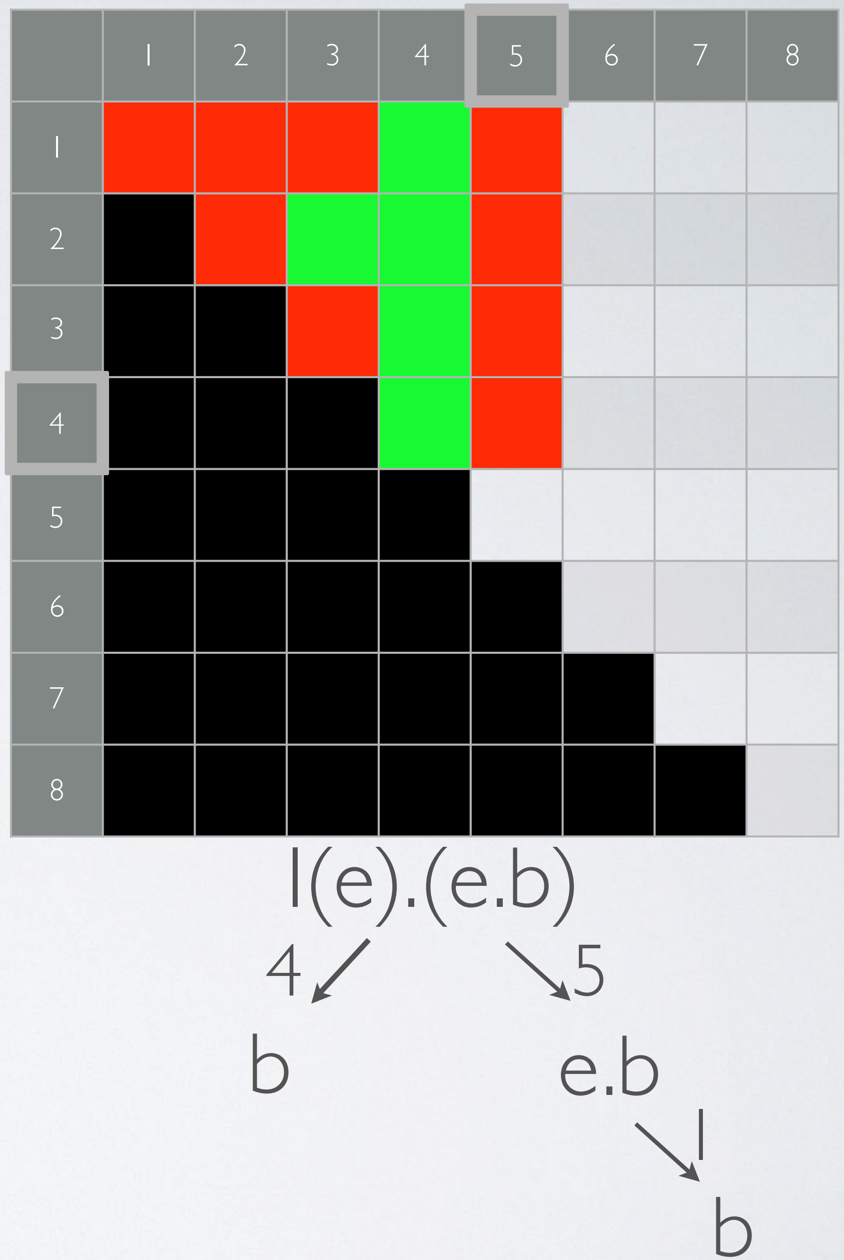
$e.a$

$\searrow 1$

a

Mêmes règles, autre substitution!

1. $e.x \rightsquigarrow x$
2. $I(x).x \rightsquigarrow e$
3. $(x.y).z \rightsquigarrow x.(y.z)$
4. $I(x).(x.z) \rightsquigarrow z$
5. $I(e).z \rightsquigarrow z$
6. $I(I(x)).e \rightsquigarrow x$
7. $I(x.y).(x.(y.z)) \rightsquigarrow z$
8. $I(I(x)).y \rightsquigarrow x.y$



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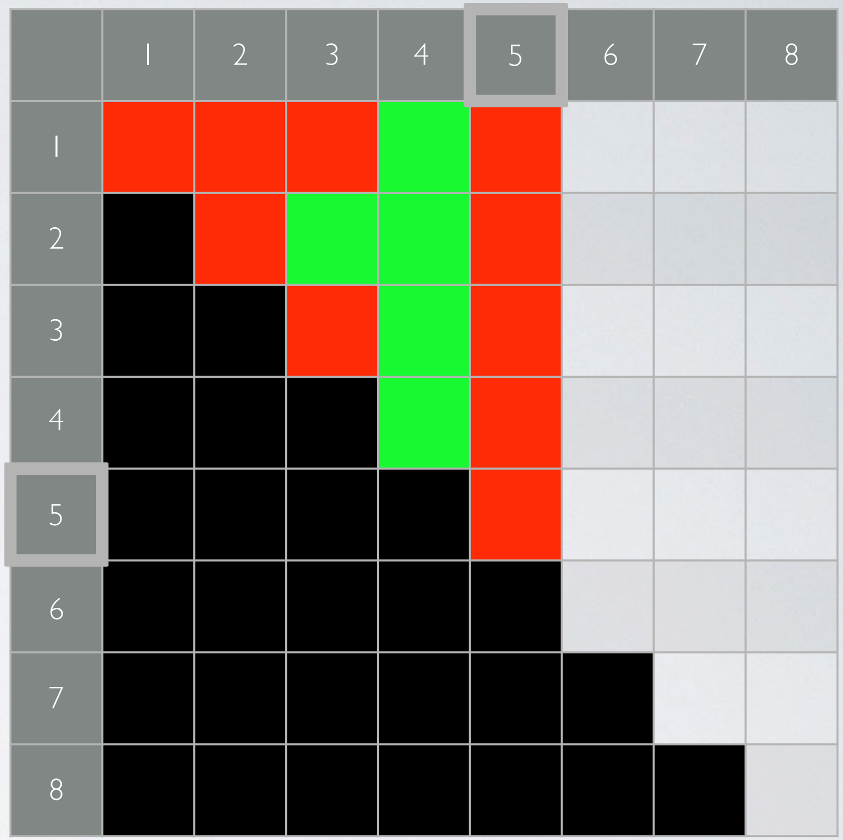
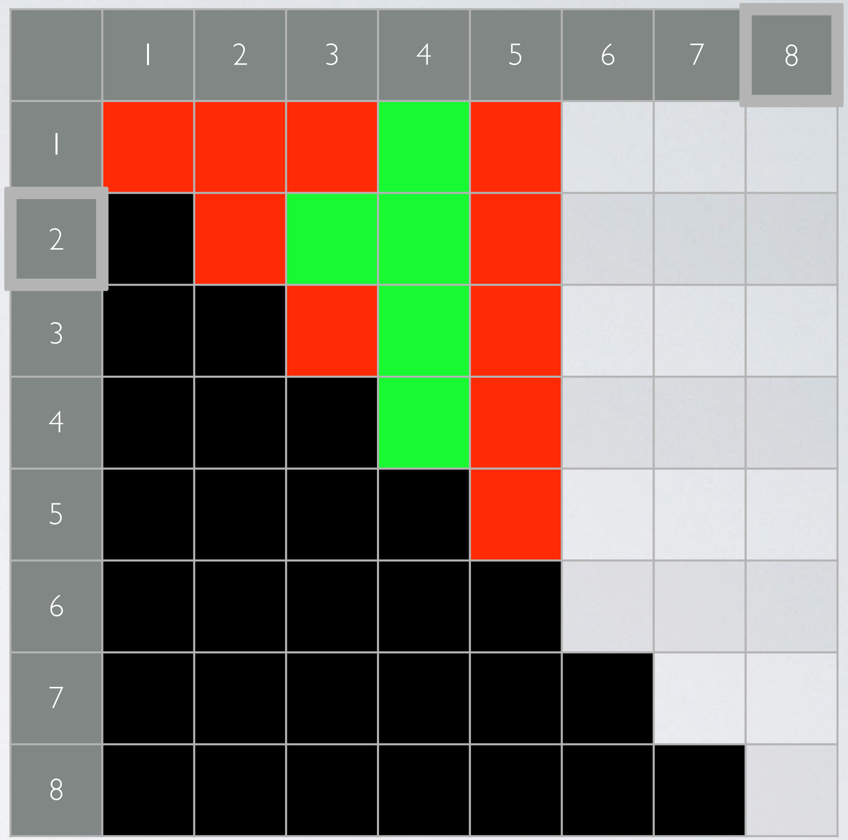
2. $I(x).x \rightsquigarrow e$
3. $(x.y).z \rightsquigarrow x.(y.z)$
4. $I(x).(x.z) \rightsquigarrow z$
5. $I(e).z \rightsquigarrow z$
6. $I(I(x)).e \rightsquigarrow x$
7. $I(x.y).(x.(y.z)) \rightsquigarrow z$
8. $I(I(x)).y \rightsquigarrow x.y$

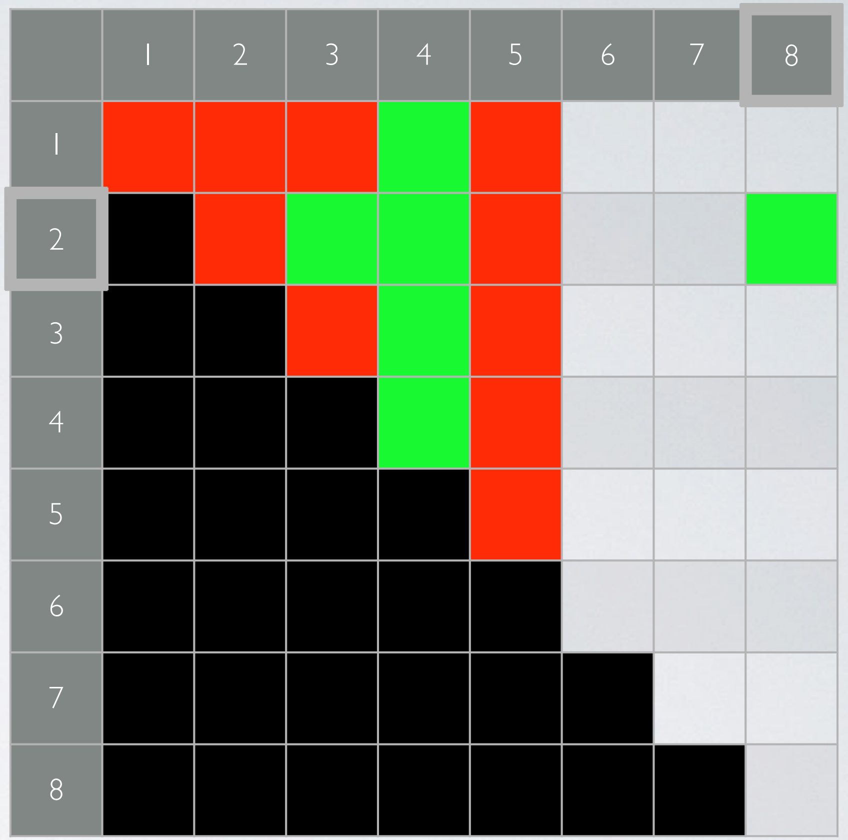


Trichons un peu pour accélérer le processus...

1. $e.x \rightsquigarrow x$
2. $I(x).x \rightsquigarrow e$
3. $(x.y).z \rightsquigarrow x.(y.z)$
4. $I(x).(x.z) \rightsquigarrow z$
5. $I(e).z \rightsquigarrow z$
6. $I(I(x)).e \rightsquigarrow x$
7. $I(x.y).(x.(y.z)) \rightsquigarrow z$
8. $I(I(x)).y \rightsquigarrow x.y$

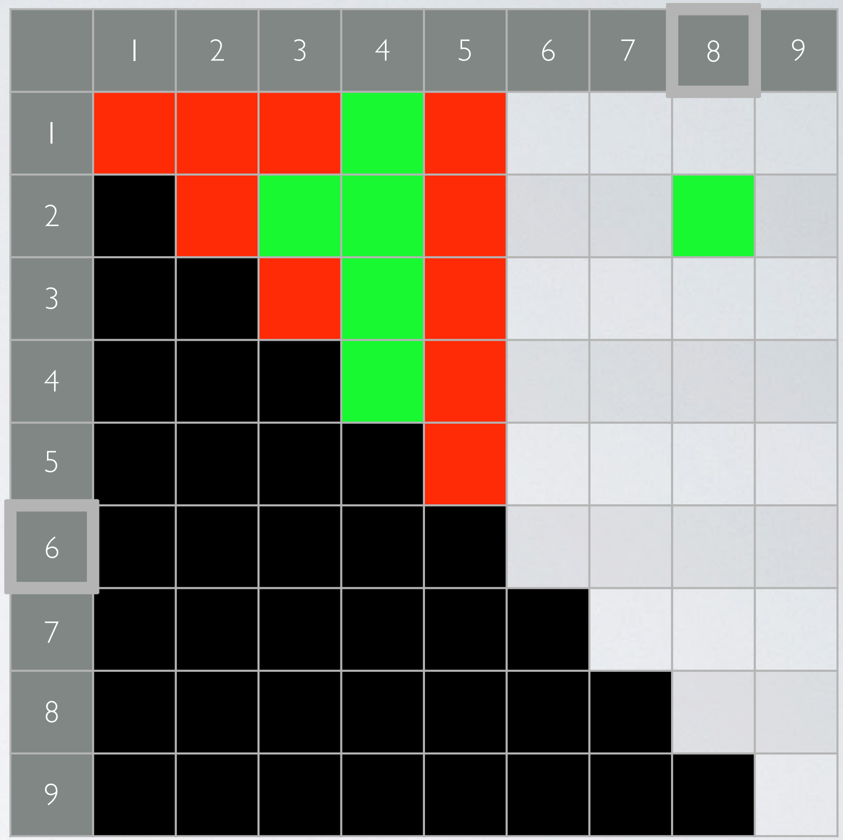


1. $e.x \rightsquigarrow x$
2. $I(x).x \rightsquigarrow e$
3. $(x.y).z \rightsquigarrow x.(y.z)$
4. $I(x).(x.z) \rightsquigarrow z$
5. $I(e).z \rightsquigarrow z$
6. $I(I(x)).e \rightsquigarrow x$
7. $I(x.y).(x.(y.z)) \rightsquigarrow z$
8. $I(I(x)).y \rightsquigarrow x.y$
9. $x.I(x) \rightsquigarrow e$

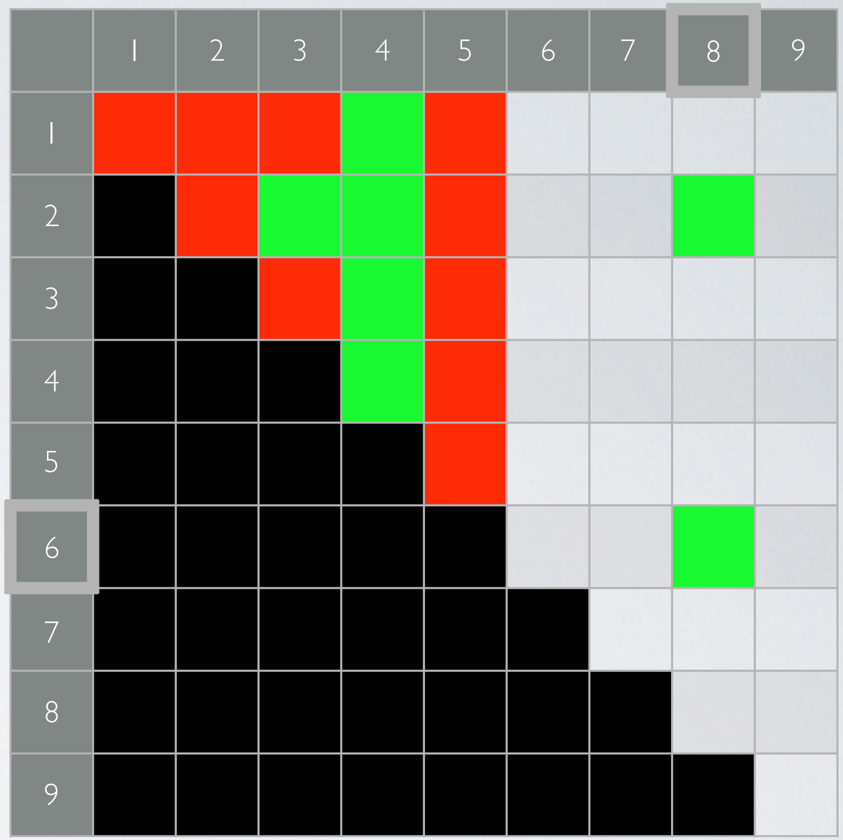


$$e \xleftarrow{2} I(I(a)).I(a) \xrightarrow{8} a.I(a)$$

1. $e.x \rightsquigarrow x$
2. $I(x).x \rightsquigarrow e$
3. $(x.y).z \rightsquigarrow x.(y.z)$
4. $I(x).(x.z) \rightsquigarrow z$
5. $I(e).z \rightsquigarrow z$
6. $I(I(x)).e \rightsquigarrow x$
7. $I(x.y).(x.(y.z)) \rightsquigarrow z$
8. $I(I(x)).y \rightsquigarrow x.y$
9. $x.I(x) \rightsquigarrow e$

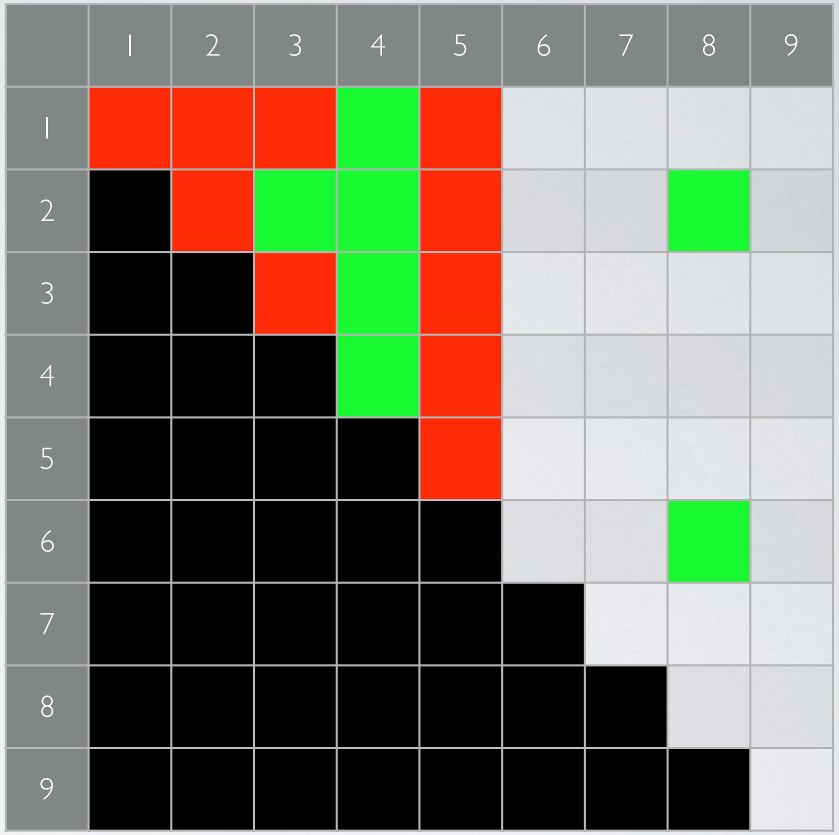


1. $e.x \rightsquigarrow x$
2. $I(x).x \rightsquigarrow e$
3. $(x.y).z \rightsquigarrow x.(y.z)$
4. $I(x).(x.z) \rightsquigarrow z$
5. $I(e).z \rightsquigarrow z$
6. $I(I(x)).e \rightsquigarrow x$
7. $I(x.y).(x.(y.z)) \rightsquigarrow z$
8. $I(I(x)).y \rightsquigarrow x.y$
9. $x.I(x) \rightsquigarrow e$
10. $x.e \rightsquigarrow x$



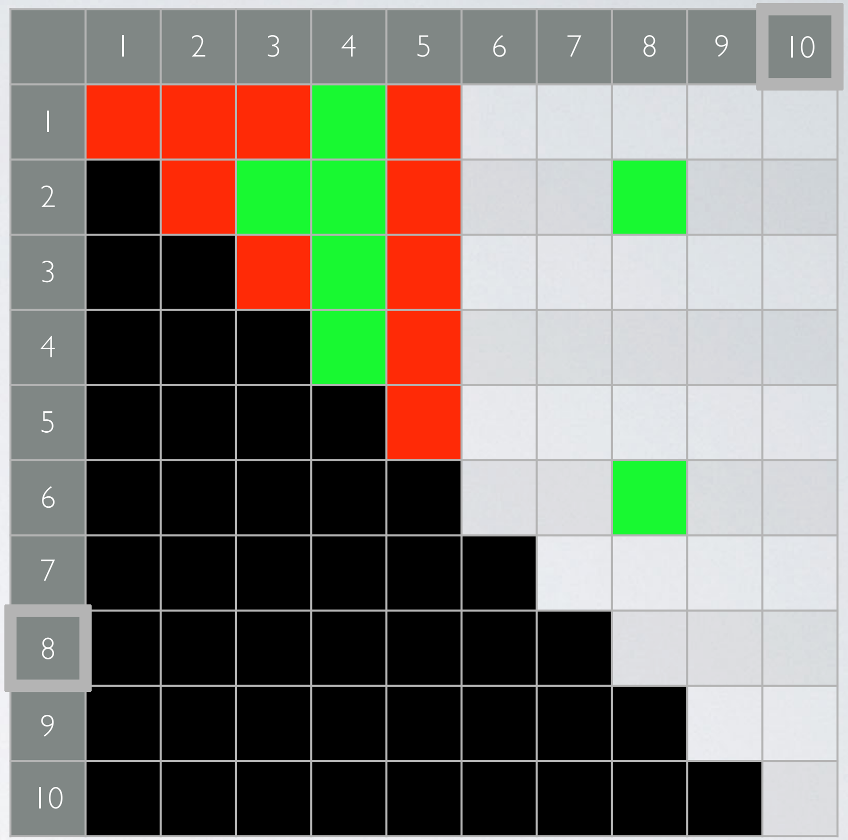
$$\begin{array}{ccc} & I(I(a)).e & \\ {}^{6}\swarrow & & \searrow^{8} \\ a & & a.e \end{array}$$

1. $e.x \rightsquigarrow x$
2. $I(x).x \rightsquigarrow e$
3. $(x.y).z \rightsquigarrow x.(y.z)$
4. $I(x).(x.z) \rightsquigarrow z$
5. $I(e).z \rightsquigarrow z$
6. ~~$I(I(x)).e \rightsquigarrow x$~~ 8 + 10
7. $I(x.y).(x.(y.z)) \rightsquigarrow z$
8. $I(I(x)).y \rightsquigarrow x.y$
9. $x.I(x) \rightsquigarrow e$
10. $x.e \rightsquigarrow x$



Useless rule!

1. $e.x \rightsquigarrow x$
2. $I(x).x \rightsquigarrow e$
3. $(x.y).z \rightsquigarrow x.(y.z)$
4. $I(x).(x.z) \rightsquigarrow z$
5. $I(e).z \rightsquigarrow z$
6. $I(x.y).(x.(y.z)) \rightsquigarrow z$
7. $I(I(x)).y \rightsquigarrow x.y$
8. $x.I(x) \rightsquigarrow e$
9. $x.e \rightsquigarrow x$



1 $e.x \to x$

2 $I(x).x \to e$

3 $(x.y).z \to x.(y.z)$

4 $I(x).(x.z) \to z$

5 $I(e).z \to z$

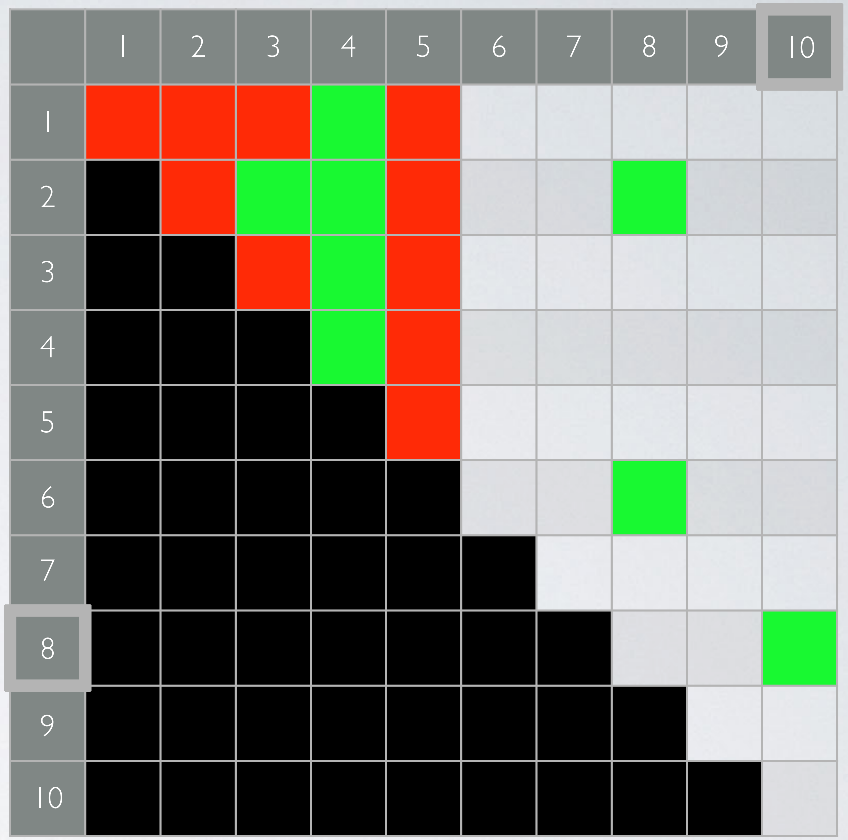
6 $I(x.y).(x.(y.z)) \to z$

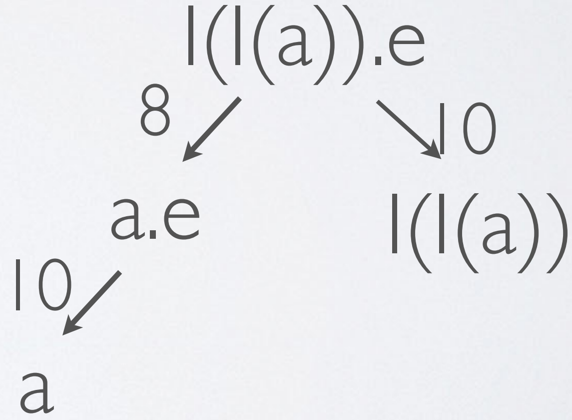
7 $I(I(x)).y \to x.y$

8 $x.I(x) \to e$

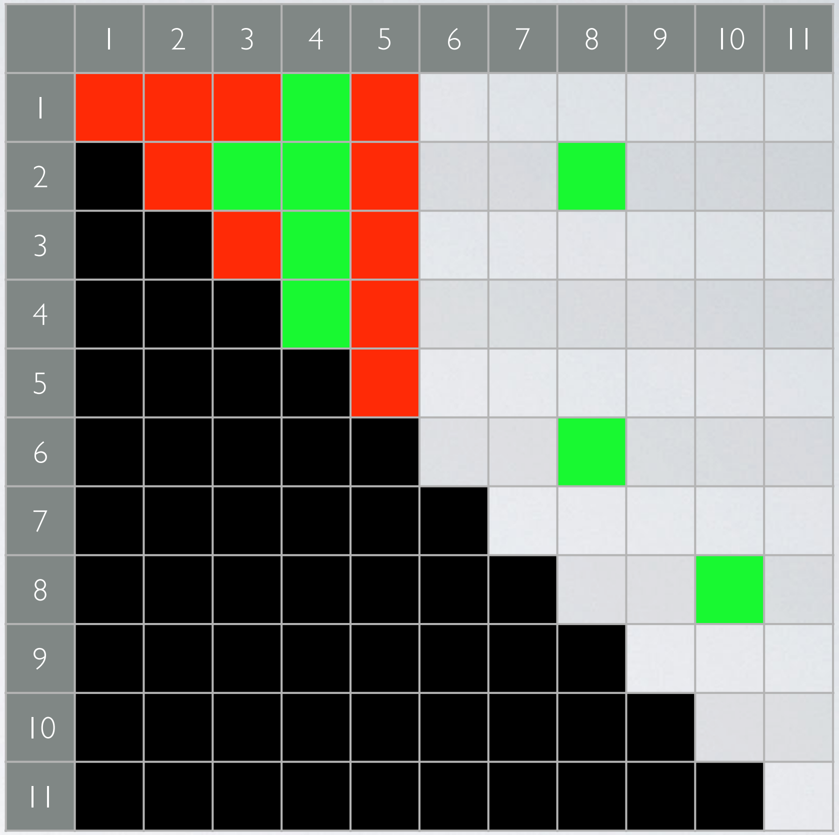
9 $x.e \to x$

10 $I(I(x)) \to x$





1. $e.x \rightsquigarrow x$
2. $I(x).x \rightsquigarrow e$
3. $(x.y).z \rightsquigarrow x.(y.z)$
4. $I(x).(x.z) \rightsquigarrow z$
5. $I(e).z \rightsquigarrow z$
6. $I(x.y).(x.(y.z)) \rightsquigarrow z$
7. $I(I(x)).y \rightsquigarrow x.y$
8. $x.I(x) \rightsquigarrow e$
9. $x.e \rightsquigarrow x$
10. $I(I(x)) \rightsquigarrow x$



1 $e.x \rightsquigarrow x$

2 $I(x).x \rightsquigarrow e$

3 $(x.y).z \rightsquigarrow x.(y.z)$

4 $I(x).(x.z) \rightsquigarrow z$

5 $I(e).z \rightsquigarrow z$

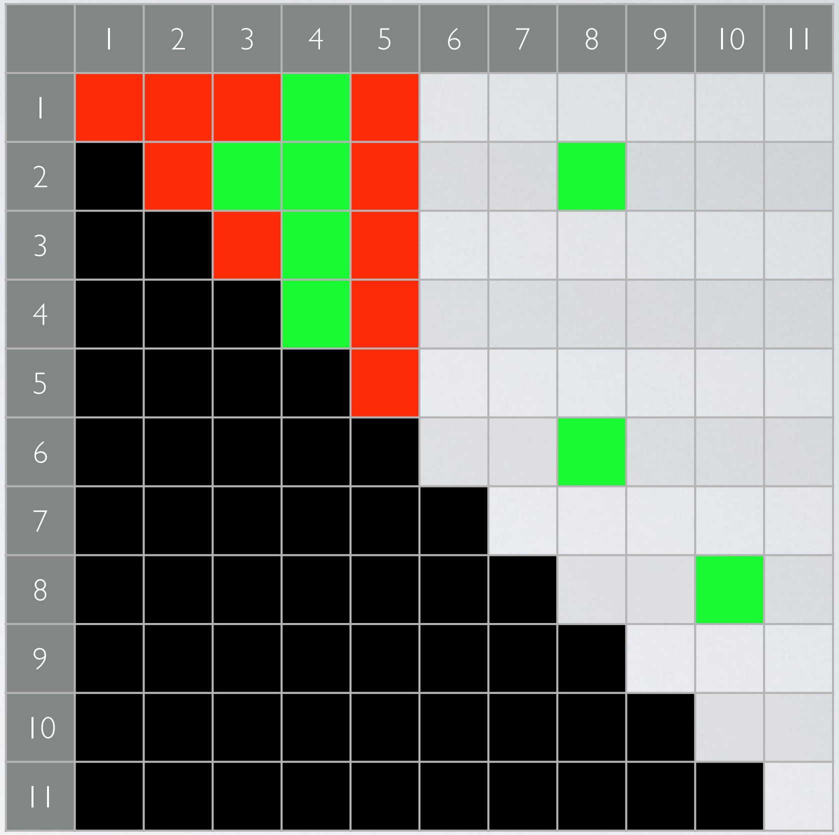
7 $I(x.y).(x.(y.z)) \rightsquigarrow z$

8 $\cancel{I(I(x)).y \rightsquigarrow x.y}$ 11

9 $x.I(x) \rightsquigarrow e$

10 $x.e \rightsquigarrow x$

11 $I(I(x)) \rightsquigarrow x$



1 $e.x \rightsquigarrow x$

2 $I(x).x \rightsquigarrow e$

3 $(x.y).z \rightsquigarrow x.(y.z)$

4 $I(x).(x.z) \rightsquigarrow z$

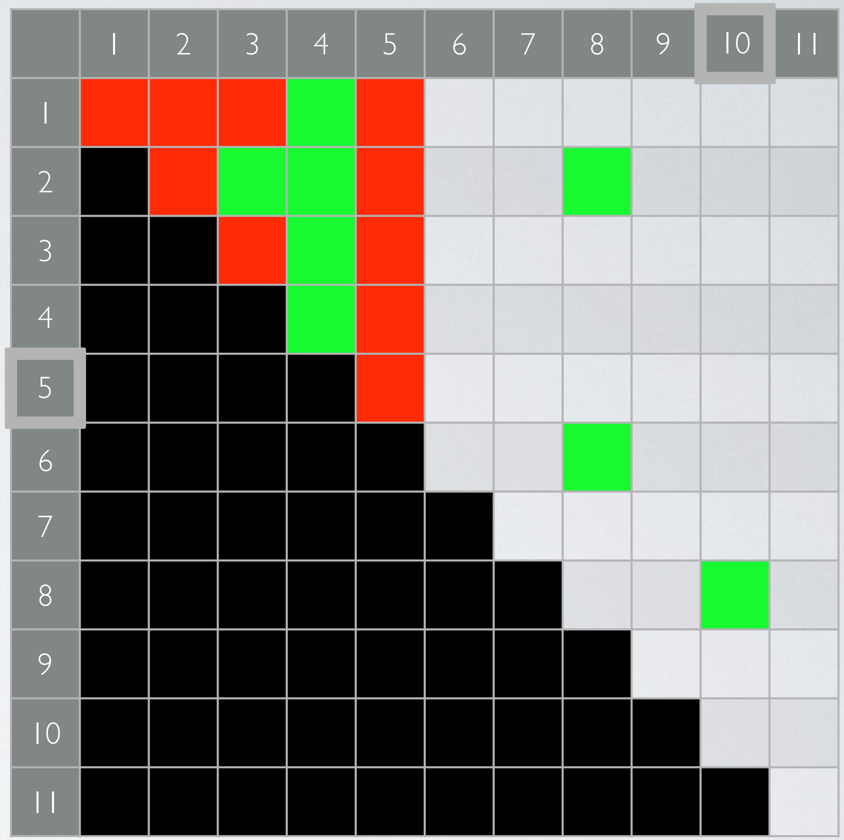
5 $I(e).z \rightsquigarrow z$

7 $I(x.y).(x.(y.z)) \rightsquigarrow z$

9 $x.I(x) \rightsquigarrow e$

10 $x.e \rightsquigarrow x$

11 $I(I(x)) \rightsquigarrow x$



1 $e.x \rightsquigarrow x$

2 $I(x).x \rightsquigarrow e$

3 $(x.y).z \rightsquigarrow x.(y.z)$

4 $I(x).(x.z) \rightsquigarrow z$

5 $I(e).z \rightsquigarrow z$

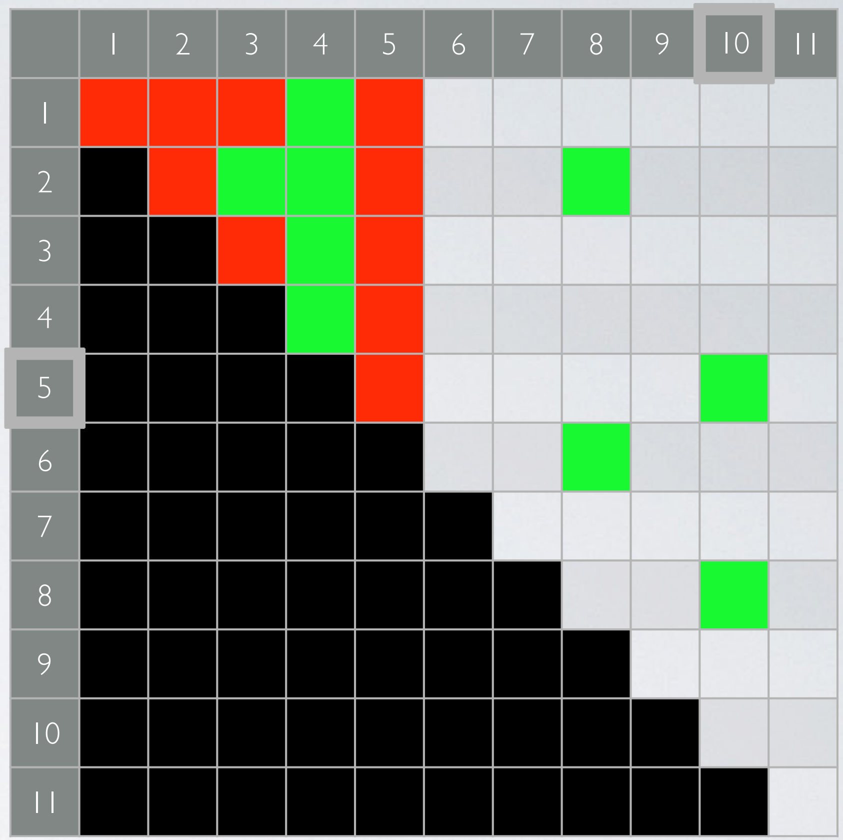
7 $I(x.y).(x.(y.z)) \rightsquigarrow z$

9 $x.I(x) \rightsquigarrow e$

10 $x.e \rightsquigarrow x$

11 $I(I(x)) \rightsquigarrow x$

12 $I(e) \rightsquigarrow e$



$$\begin{tikzcd}[column sep=small] & I(e).e \arrow[dl, "5"'] \arrow[dr, "10"] & \\ e & & I(e) \end{tikzcd}$$

1 $e.x \rightsquigarrow x$

2 $I(x).x \rightsquigarrow e$

3 $(x.y).z \rightsquigarrow x.(y.z)$

4 $I(x).(x.z) \rightsquigarrow z$

5 $\cancel{I(e).z \rightsquigarrow z}$ 12 + 1

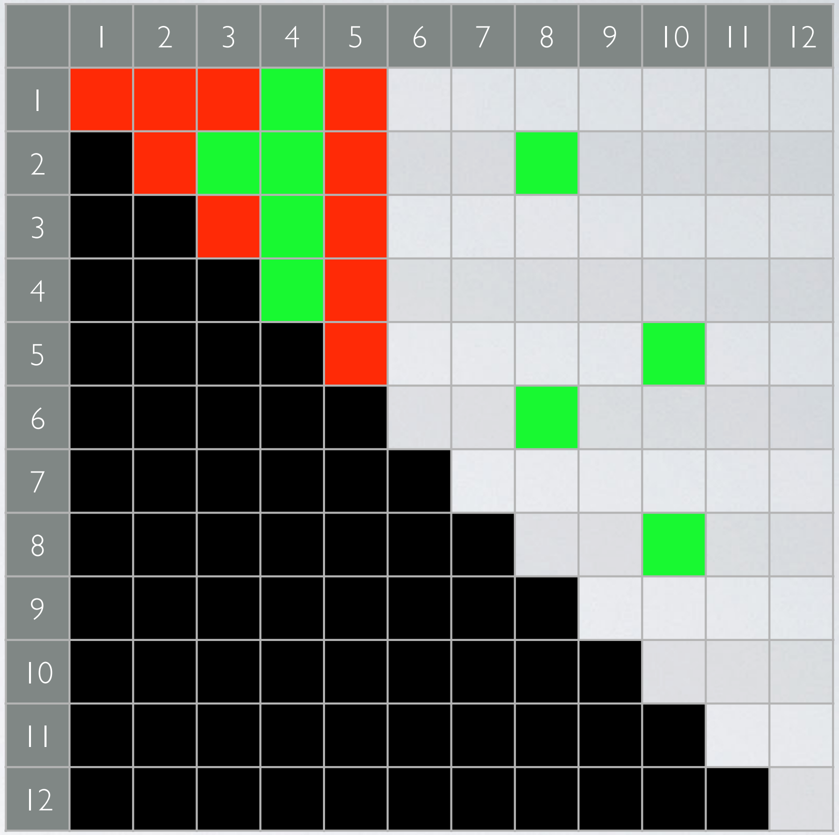
7 $I(x.y).(x.(y.z)) \rightsquigarrow z$

9 $x.I(x) \rightsquigarrow e$

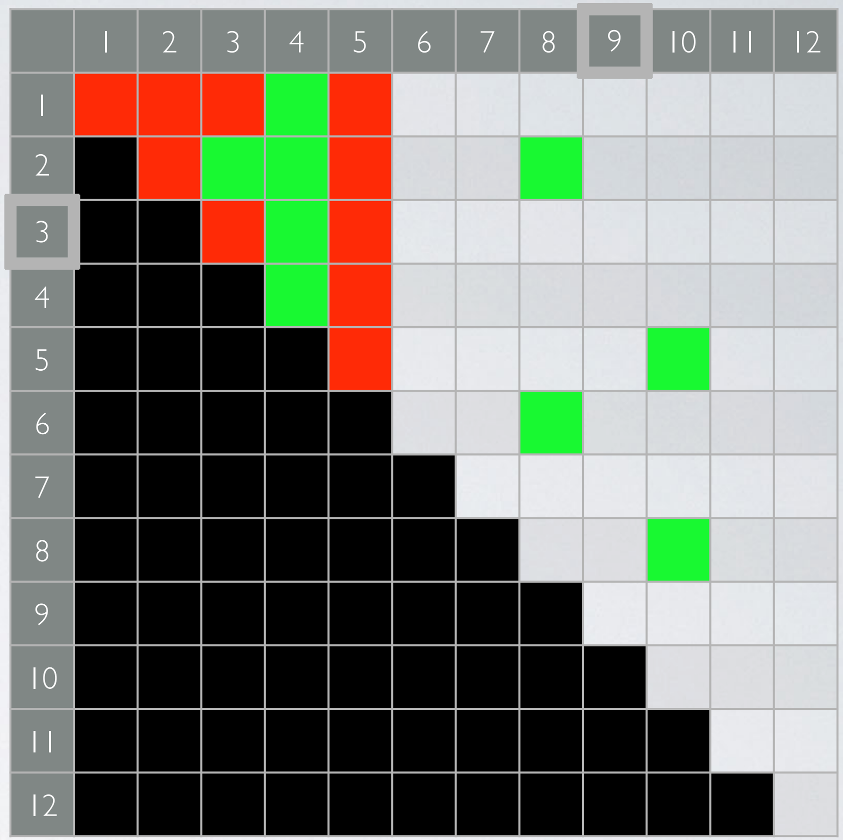
10 $x.e \rightsquigarrow x$

11 $I(I(x)) \rightsquigarrow x$

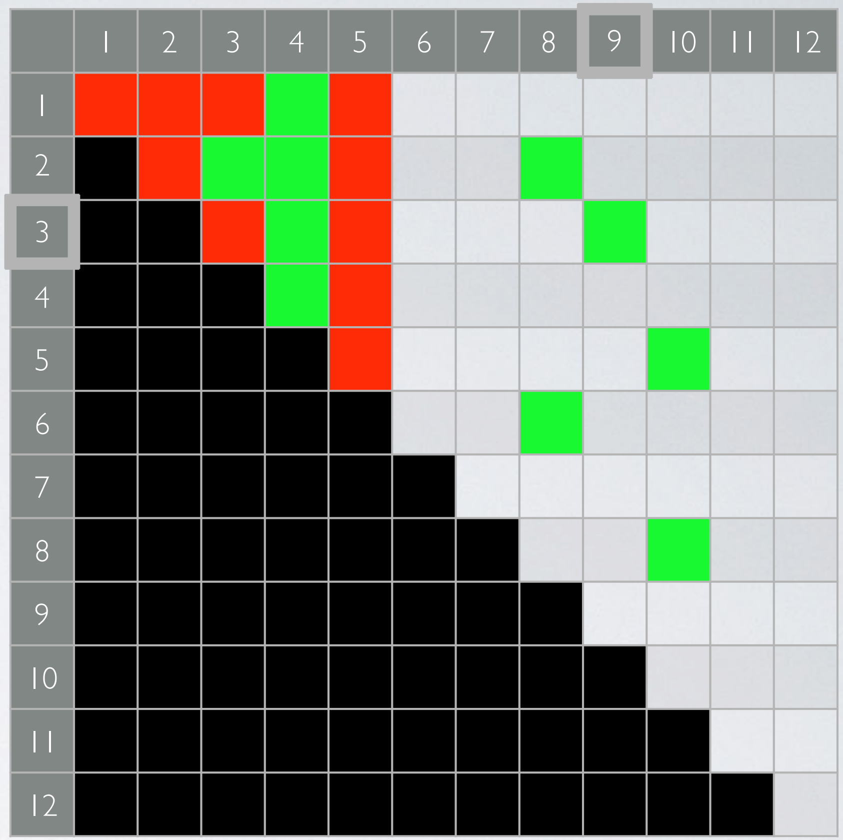
12 $I(e) \rightsquigarrow e$



1. $e.x \rightarrow x$
2. $I(x).x \rightarrow e$
3. $(x.y).z \rightarrow x.(y.z)$
4. $I(x).(x.z) \rightarrow z$
7. $I(x.y).(x.(y.z)) \rightarrow z$
9. $x.I(x) \rightarrow e$
10. $x.e \rightarrow x$
11. $I(I(x)) \rightarrow x$
12. $I(e) \rightarrow e$



1. $e.x \rightsquigarrow x$
2. $I(x).x \rightsquigarrow e$
3. $(x.y).z \rightsquigarrow x.(y.z)$
4. $I(x).(x.z) \rightsquigarrow z$
7. $I(x.y).(x.(y.z)) \rightsquigarrow z$
9. $x.I(x) \rightsquigarrow e$
10. $x.e \rightsquigarrow x$
11. $I(I(x)) \rightsquigarrow x$
12. $I(e) \rightsquigarrow e$
13. $y.(I(y).x) \rightsquigarrow x$

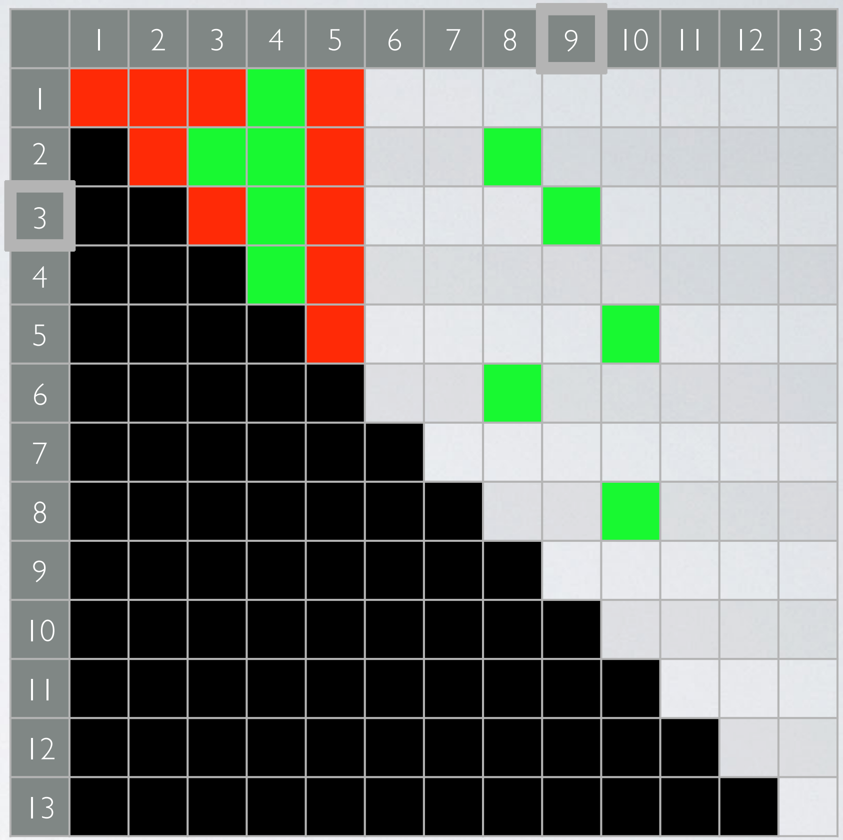


$(a.I(a)).b$

$(a.I(a)).b \xrightarrow{3} a.(I(a).b)$

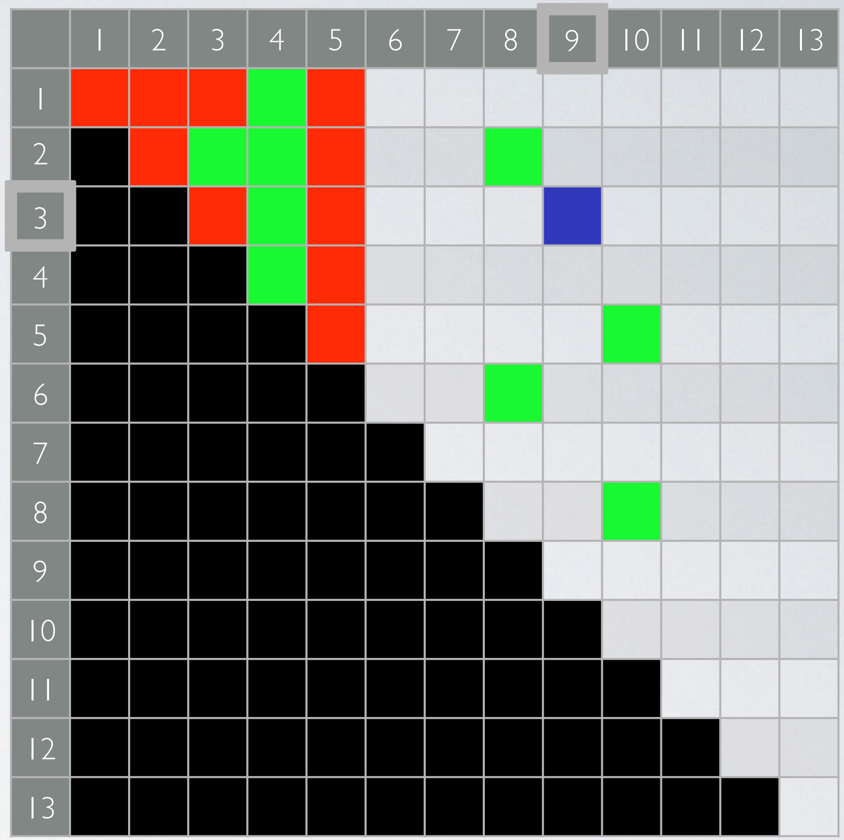
$(a.I(a)).b \xrightarrow{9} e.b \xrightarrow{1} b$

1. $e.x \rightsquigarrow x$
2. $I(x).x \rightsquigarrow e$
3. $(x.y).z \rightsquigarrow x.(y.z)$
4. $I(x).(x.z) \rightsquigarrow z$
7. $I(x.y).(x.(y.z)) \rightsquigarrow z$
9. $x.I(x) \rightsquigarrow e$
10. $x.e \rightsquigarrow x$
11. $I(I(x)) \rightsquigarrow x$
12. $I(e) \rightsquigarrow e$
13. $y.(I(y).x) \rightsquigarrow x$



Mêmes règles, autre substitution!

1. $e.x \rightarrow x$
2. $I(x).x \rightarrow e$
3. $(x.y).z \rightarrow x.(y.z)$
4. $I(x).(x.z) \rightarrow z$
7. $I(x.y).(x.(y.z)) \rightarrow z$
9. $x.I(x) \rightarrow e$
10. $x.e \rightarrow x$
11. $I(I(x)) \rightarrow x$
12. $I(e) \rightarrow e$
13. $y.(I(y).x) \rightarrow x$
14. $x.(y.I(x.y)) \rightarrow e$



$$a.(b.(I(a.b))) \xleftarrow{3} (a.b).I(a.b) \xrightarrow{9} e$$

1 $e.x \rightsquigarrow x$

2 $I(x).x \rightsquigarrow e$

3 $(x.y).z \rightsquigarrow x.(y.z)$

4 $I(x).(x.z) \rightsquigarrow z$

7 $I(x.y).(x.(y.z)) \rightsquigarrow z$

9 $x.I(x) \rightsquigarrow e$

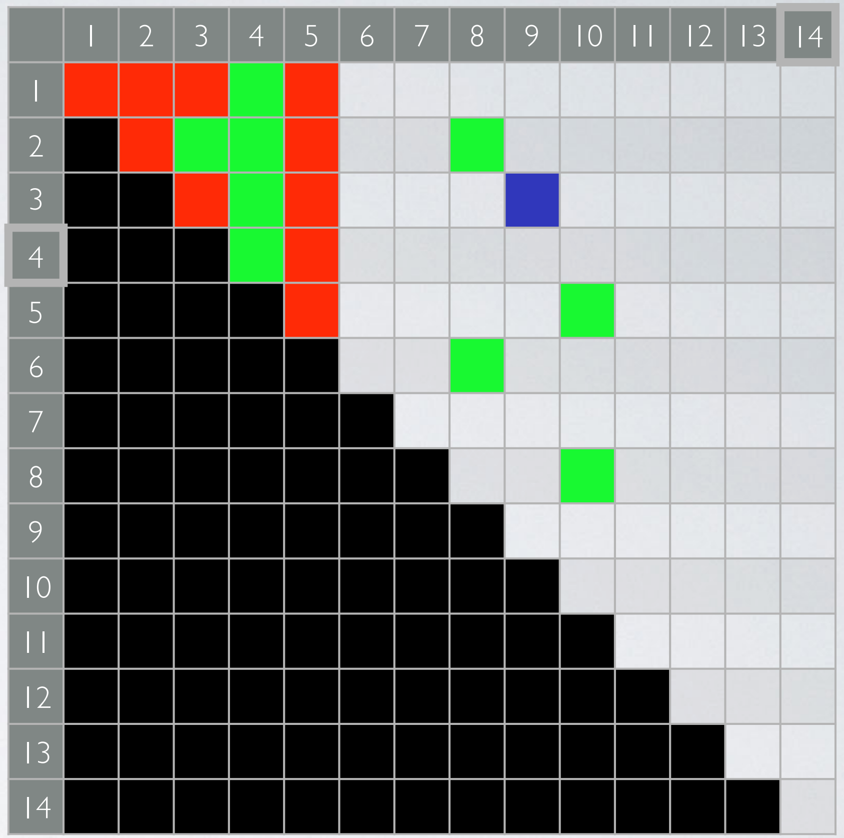
10 $x.e \rightsquigarrow x$

11 $I(I(x)) \rightsquigarrow x$

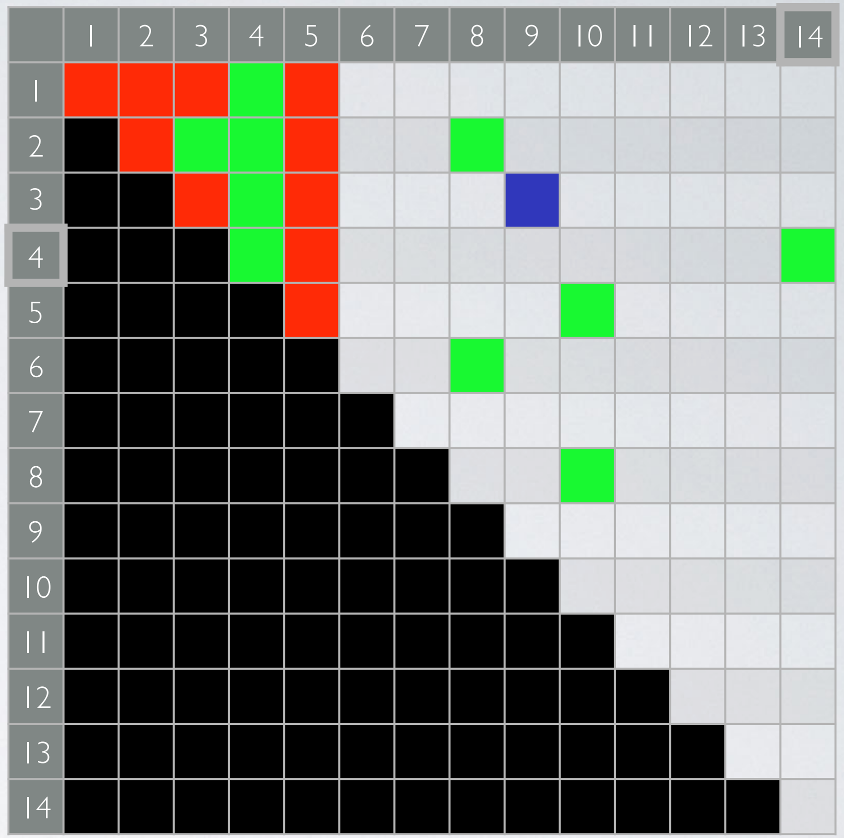
12 $I(e) \rightsquigarrow e$

13 $y.(I(y).x) \rightsquigarrow x$

14 $x.(y.I(x.y)) \rightsquigarrow e$



1. $e.x \rightsquigarrow x$
2. $I(x).x \rightsquigarrow e$
3. $(x.y).z \rightsquigarrow x.(y.z)$
4. $I(x).(x.z) \rightsquigarrow z$
7. $I(x.y).(x.(y.z)) \rightsquigarrow z$
9. $x.I(x) \rightsquigarrow e$
10. $x.e \rightsquigarrow x$
11. $I(I(x)) \rightsquigarrow x$
12. $I(e) \rightsquigarrow e$
13. $y.(I(y).x) \rightsquigarrow x$
14. $x.(y.I(x.y)) \rightsquigarrow e$
15. $x.I(y.x) \rightsquigarrow I(y)$



$$I(a).(a.(b.I(a.b)))$$

$$b.I(a.b) \xleftarrow{4} I(a).(a.(b.I(a.b))) \xrightarrow{14} I(a)$$

1 $e.x \rightarrow x$

2 $I(x).x \rightarrow e$

3 $(x.y).z \rightarrow x.(y.z)$

4 $I(x).(x.z) \rightarrow z$

7 $I(x.y).(x.(y.z)) \rightarrow z$

9 $x.I(x) \rightarrow e$

10 $x.e \rightarrow x$

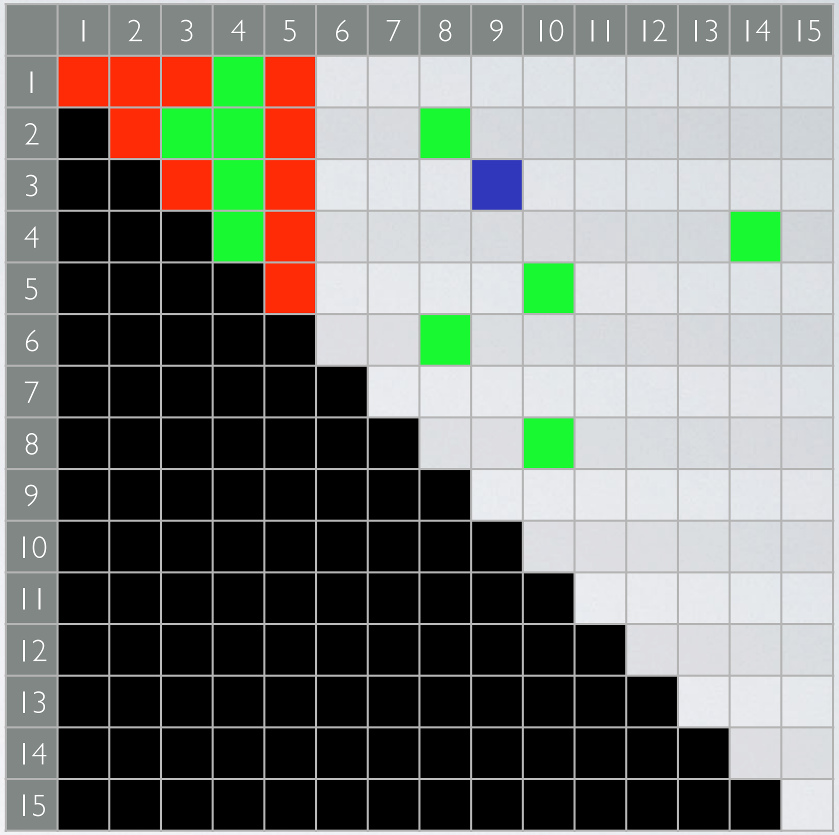
11 $I(I(x)) \rightarrow x$

12 $I(e) \rightarrow e$

13 $y.(I(y).x) \rightarrow x$

$\cancel{14 \quad x.(y.I(x.y)) \rightarrow e}$ 15 + 9

15 $x.I(y.x) \rightarrow I(y)$



1 $e.x \rightsquigarrow x$

2 $I(x).x \rightsquigarrow e$

3 $(x.y).z \rightsquigarrow x.(y.z)$

4 $I(x).(x.z) \rightsquigarrow z$

7 $I(x.y).(x.(y.z)) \rightsquigarrow z$

9 $x.I(x) \rightsquigarrow e$

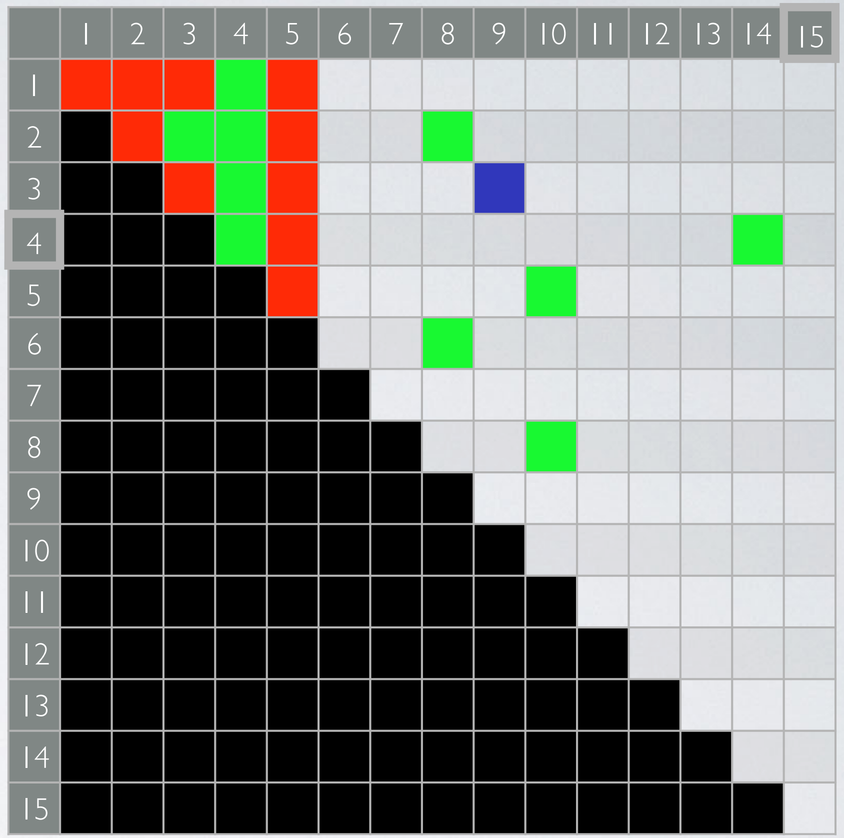
10 $x.e \rightsquigarrow x$

11 $I(I(x)) \rightsquigarrow x$

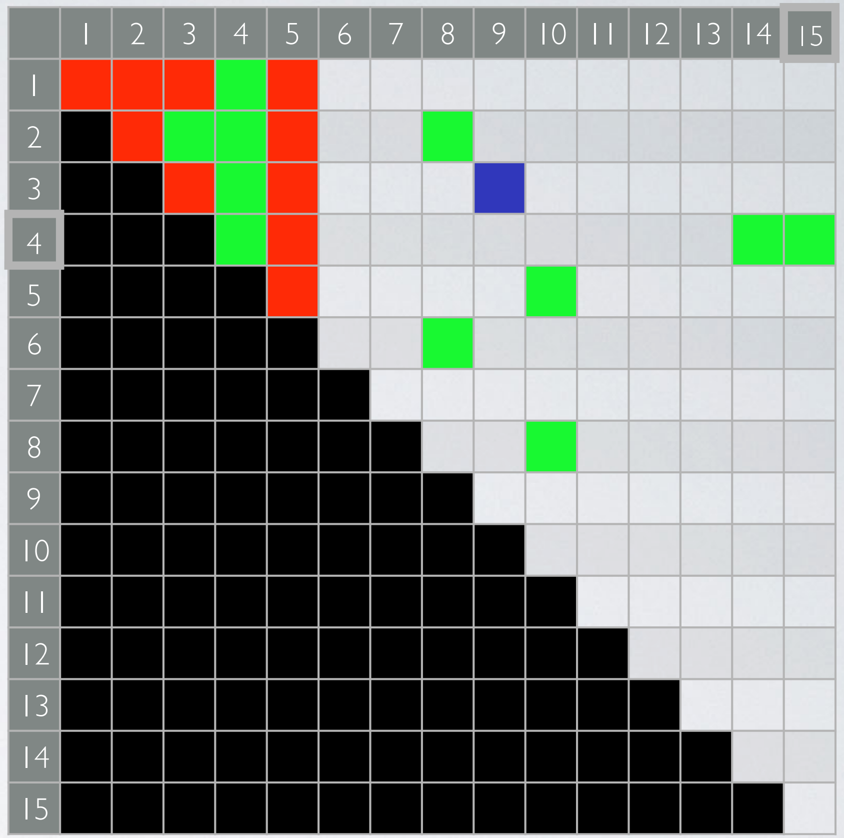
12 $I(e) \rightsquigarrow e$

13 $y.(I(y).x) \rightsquigarrow x$

15 $x.I(y.x) \rightsquigarrow I(y)$

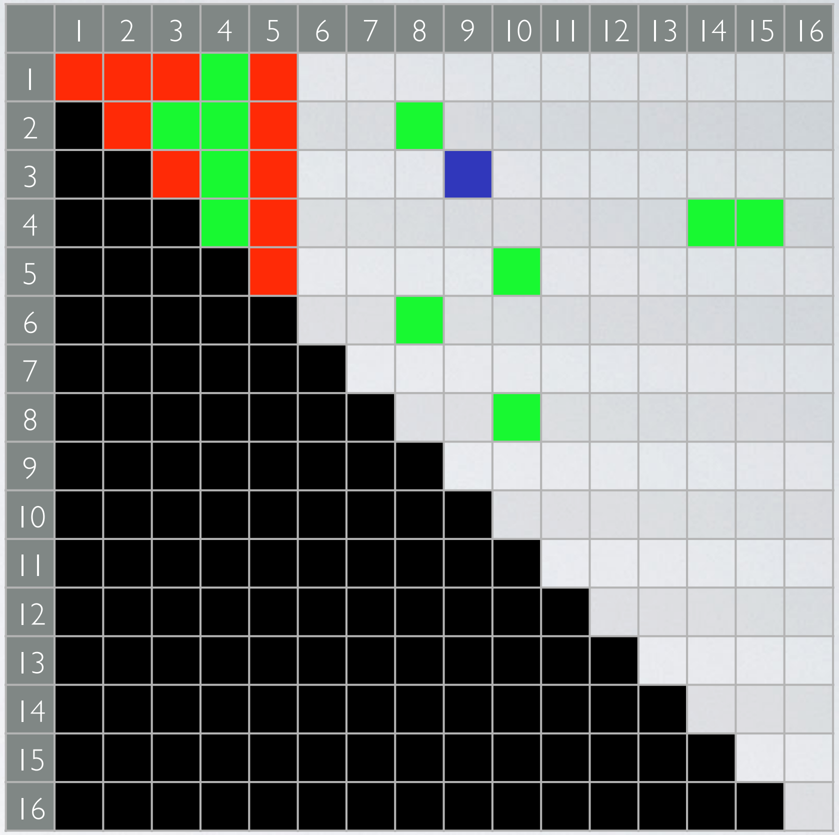


1. e.x → x
2. I(x).x → e
3. (x.y).z → x.(y.z)
4. I(x).(x.z) → z
7. I(x.y).(x.(y.z)) → z
9. x.I(x) → e
10. x.e → x
11. I(I(x)) → x
12. I(e) → e
13. y.(I(y).x) → x
15. x.I(y.x) → I(y)
16. I(x.y) → I(y).I(x)



$$
\begin{tikzcd}
 & \mathrm{I(a).(a.I(b.a))} \arrow[dl, "4"'] \arrow[dr, "15"] & \\
\mathrm{I(a.b)} & & \mathrm{I(b).I(a)}
\end{tikzcd}
$$

1. $e.x \rightsquigarrow x$
2. $I(x).x \rightsquigarrow e$
3. $(x.y).z \rightsquigarrow x.(y.z)$
4. $I(x).(x.z) \rightsquigarrow z$
7. $I(x.y).(x.(y.z)) \rightsquigarrow z$
9. $x.I(x) \rightsquigarrow e$
10. $x.e \rightsquigarrow x$
11. $I(I(x)) \rightsquigarrow x$
12. $I(e) \rightsquigarrow e$
13. $y.(I(y).x) \rightsquigarrow x$
15. ~~$x.I(y.x) \rightsquigarrow I(y)$~~ 16 + 13
16. $I(x.y) \rightsquigarrow I(y).I(x)$



1 $e.x \rightsquigarrow x$

2 $I(x).x \rightsquigarrow e$

3 $(x.y).z \rightsquigarrow x.(y.z)$

4 $I(x).(x.z) \rightsquigarrow z$

~~7~~ $\cancel{I(x.y).(x.(y.z)) \rightsquigarrow z}$ 16 + 3 + 4 + 4

9 $x.I(x) \rightsquigarrow e$

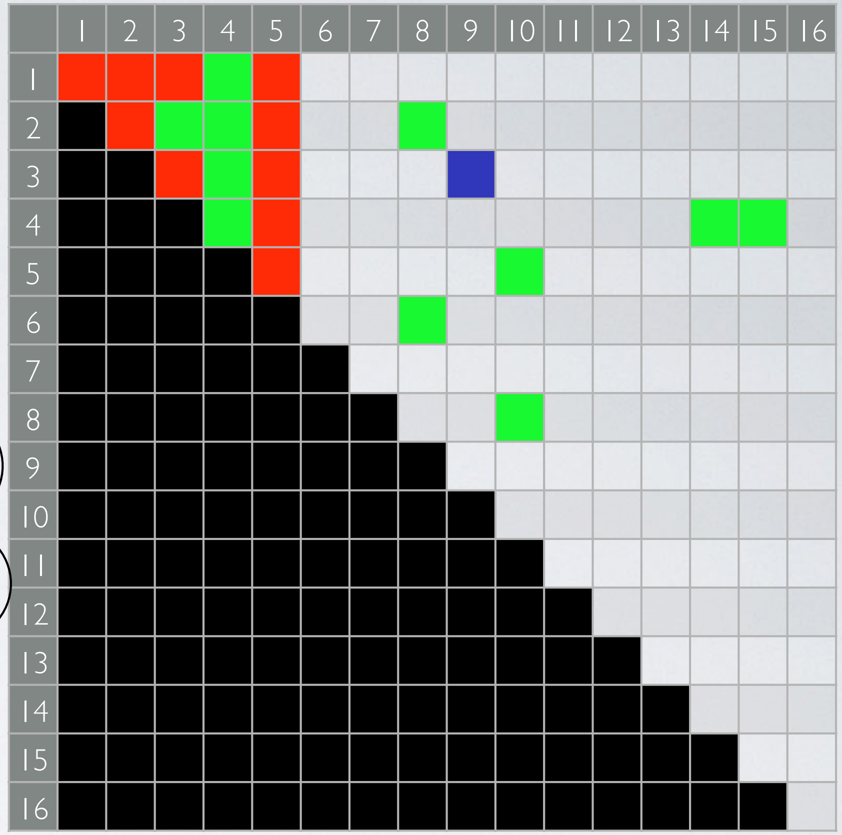
10 $x.e \rightsquigarrow x$

11 $I(I(x)) \rightsquigarrow x$

12 $I(e) \rightsquigarrow e$

13 $y.(I(y).x) \rightsquigarrow x$

16 $I(x.y) \rightsquigarrow I(y).I(x)$



1 $e.x \rightsquigarrow x$

2 $I(x).x \rightsquigarrow e$

3 $(x.y).z \rightsquigarrow x.(y.z)$

4 $I(x).(x.z) \rightsquigarrow z$

9 $x.I(x) \rightsquigarrow e$

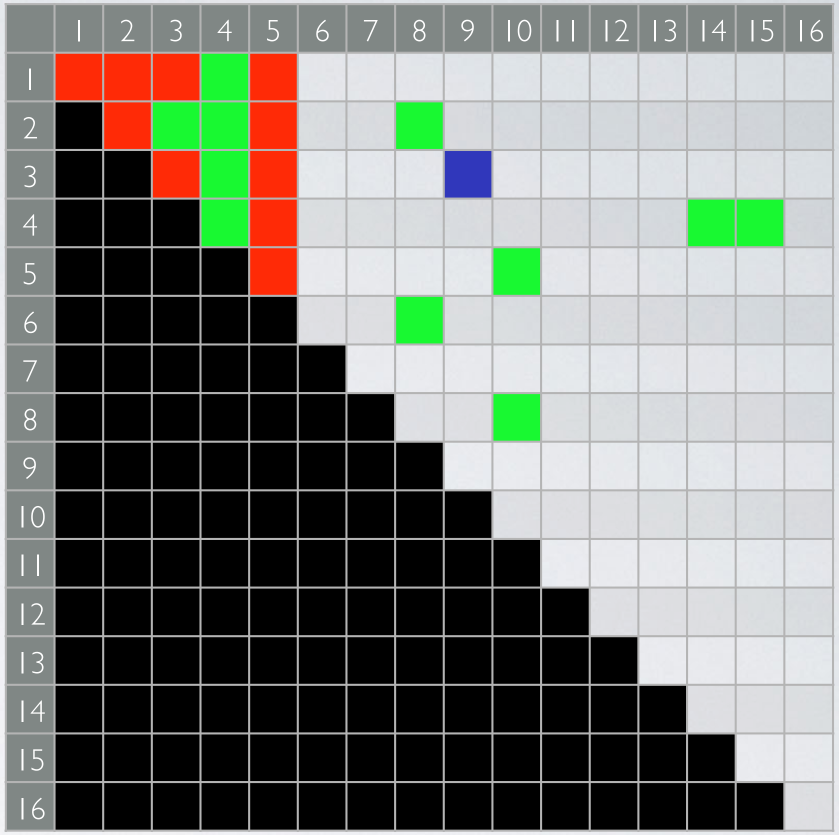
10 $x.e \rightsquigarrow x$

11 $I(I(x)) \rightsquigarrow x$

12 $I(e) \rightsquigarrow e$

13 $y.(I(y).x) \rightsquigarrow x$

16 $I(x.y) \rightsquigarrow I(y).I(x)$



$$e.x \rightsquigarrow x$$
$$I(x).x \rightsquigarrow e$$
$$(x.y).z \rightsquigarrow x.(y.z)$$
$$I(x).(x.z) \rightsquigarrow z$$
$$x.I(x) \rightsquigarrow e$$
$$x.e \rightsquigarrow x$$
$$I(I(x)) \rightsquigarrow x$$
$$I(e) \rightsquigarrow e$$
$$y.(I(y).x) \rightsquigarrow x$$
$$I(x.y) \rightsquigarrow I(y).I(x)$$

Objectif:

1. $e.x \rightsquigarrow x$
2. $I(x).x \rightsquigarrow e$
3. $(x.y).z \rightsquigarrow x.(y.z)$
4. $x.e \rightsquigarrow x$
5. $x.I(x) \rightsquigarrow e$
6. $x.(y.z) \rightsquigarrow (x.y).z$
7. $I(e) \rightsquigarrow e$
8. $I(I(x)) \rightsquigarrow x$
9. $I(x.y) \rightsquigarrow I(y).I(x)$

1 $e.x \rightsquigarrow x$

2 $I(x).x \rightsquigarrow e$

3 $(x.y).z \rightsquigarrow x.(y.z)$

$I(x).(x.z) \rightsquigarrow z$

5 $x.I(x) \rightsquigarrow e$

4 $x.e \rightsquigarrow x$

8 $I(I(x)) \rightsquigarrow x$

7 $I(e) \rightsquigarrow e$

$y.(I(y).x) \rightsquigarrow x$

9 $I(x.y) \rightsquigarrow I(y).I(x)$

Objectif:

1 $e.x \rightsquigarrow x$

2 $I(x).x \rightsquigarrow e$

3 $(x.y).z \rightsquigarrow x.(y.z)$

4 $x.e \rightsquigarrow x$

5 $x.I(x) \rightsquigarrow e$

6 $x.(y.z) \rightsquigarrow (x.y).z$

7 $I(e) \rightsquigarrow e$

8 $I(I(x)) \rightsquigarrow x$

9 $I(x.y) \rightsquigarrow I(y).I(x)$