Problem 5

Let p be a polynomial of degree $\leq n-1$ that interpolates the function $f(x) = \sinh x$ at any set of n nodes in the interval $[-1,1]$, subject only to the condition that one of the nodes is 0. Prove that the error obeys this inequality on $[-1,1]$:

$$|p(x) - f(x)| \leq \frac{2^n}{n!}|f(x)|.$$

Solution 5

By the remainder of Lagrange interpolation

$$|p(x) - f(x)| = |\frac{f^n(\xi(x))}{n!}|\prod_{i=1}^{n}(x - x_i)| \leq \frac{|f^n(\xi(x))|}{n!}|x|\prod_{i=2}^{n}|(x - x_i)|.$$

Also $x_i, x \in [-1,1]$ which implies that $|x - x_i| \leq 2$. So the error can be written as

$$|p(x) - f(x)| \leq \frac{|f^n(\xi(x))|}{n!}|x|2^{n-1}.$$

To bound $|f^n(\xi(x))|$ consider two cases:

If n is even, then $f^n(\xi(x)) = \sinh(\xi(x))$.

$$|\sinh(x)| = \max_{|x|\leq 1}|\tfrac{e^x - e^{-x}}{2}| \leq |\tfrac{e^1 - e^{-1}}{2}| \leq |\tfrac{(5)-(1)}{2}| = 2$$

If n is odd, then $f^n(\xi(x)) = \cosh(\xi(x))$.

$$|\cosh(x)| = \max_{|x|\leq 1}|\tfrac{e^x + e^{-x}}{2}| \leq |\tfrac{e^1 + e^{-1}}{2}| \leq |\tfrac{(3)+(1)}{2}| = 2$$

This shows that $|f^n(\xi(x))| \leq 2$. Now show that $|x| \leq |\sinh(x)|$ for $x \in [-1,1]$. By the mean value theorem there exists some $c(x)$ in $[-1,1]$ such that $|\sinh(x) - \sinh(0)| = |\cosh(c(x))(x-0)|$ which implies $\frac{|\sinh(x)|}{|x|} = |\cosh(c(x))|$. Since $\cosh(x) = \cosh(-x)$, $\cosh(c(x))$ is an even function and symmetric about the x- axis. Moreover, the derivative of $\cosh(x)$ is $\frac{e^x - e^{-x}}{2} \geq 0$ which shows that $\cosh(c(x))$ is an increasing function on $[0,1]$. Since $\cosh(0) = \frac{e^0 + e^0}{2} = 1$ it is a result that $|\cosh(c(x))| \geq 1$ for all $x \in [-1,1]$ and it follows that $|\sinh(x)| \geq |x|$. Therefore

$$|p(x) - f(x)| \leq \frac{|f^n(\xi(x))||x|}{n!}|\prod_{i=2}^{n}(x - x_i)| \leq \frac{2 \cdot |f(x)|}{n!}2^{n-1} = \frac{2^n|f(x)|}{n!}.$$