

Problem 5

Let p be a polynomial of degree $\leq n - 1$ that interpolates the function $f(x) = \sinh x$ at any set of n nodes in the interval $[-1, 1]$, subject only to the condition that one of the nodes is 0. Prove that the error obeys this inequality on $[-1, 1]$:

$$|p(x) - f(x)| \leq \frac{2^n}{n!} |f(x)|.$$

Solution 5

By the remainder of Lagrange interpolation

$$|p(x) - f(x)| = \left| \frac{f^n(\xi(x))}{n!} \prod_{i=1}^n (x - x_i) \right| \leq \frac{|f^n(\xi(x))|}{n!} |x| \prod_{i=2}^n |x - x_i|.$$

Also $x_i, x \in [-1, 1]$ which implies that $|x - x_i| \leq 2$. So the error can be written as

$$|p(x) - f(x)| \leq \frac{|f^n(\xi(x))|}{n!} |x| 2^{n-1}.$$

To bound $|f^n(\xi(x))|$ consider two cases:

If n is even, then $f^n(\xi(x)) = \sinh(\xi(x))$.

$$\begin{aligned} |\sinh(x)| &= \max_{|x| \leq 1} \left| \frac{e^x - e^{-x}}{2} \right| \\ &\leq \left| \frac{e^1 - e^{-1}}{2} \right| \\ &\leq \left| \frac{(5) - (1)}{2} \right| \\ &= 2 \end{aligned}$$

If n is odd, then $f^n(\xi(x)) = \cosh(\xi(x))$.

$$\begin{aligned} |\cosh(x)| &= \max_{|x| \leq 1} \left| \frac{e^x + e^{-x}}{2} \right| \\ &\leq \left| \frac{e^1 + e^{-1}}{2} \right| \\ &\leq \left| \frac{(3) + (1)}{2} \right| \\ &= 2 \end{aligned}$$

This shows that $|f^n(\xi(x))| \leq 2$. Now show that $|x| \leq |\sinh(x)|$ for $x \in [-1, 1]$. For $x = 0$, $|\sinh(0)| = 0$. For $x \geq 0$, the derivative of $\sinh(x)$

is $\cosh(x) \geq 0$ for all $x \in [-1, 1]$. Hence $\sinh(x)$ is an increasing function on $[0, 1]$ and $|x| \leq |\sinh(x)|$ as desired. Therefore

$$|p(x) - f(x)| \leq \frac{|f^n(\xi(x))||x|}{n!} \left| \prod_{i=2}^n (x - x_i) \right| \leq \frac{2 \cdot |f(x)|}{n!} 2^{n-1} = \frac{2^n |f(x)|}{n!}.$$