Strong Normalization of the Simply-Typed Lambda Calculus in Lean by Decomposition Into the Call-By-Need SK Combinators

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1 Abstract

Proofs of strong normalization of the simply-typed lambda calculus have been exhaustively enumerated in the literature. A common strategy invented by W. W. Tait known as "Tait's method," (Robert Harper, 2022) interprets types as sets of "well-behaving" terms which are known to be strongly normalizing and composed of expressions in some such set. Strong normalization of the typed call-by-need SK combinator calculus has been comparatively under-studied. Herein, I demonstrate that the typical proof of strong normalization using Tait's method holds for the typed SK combinator calculus. I also show that decomposition of the STLC into SK combinator expressions simplifies the typical proof of strong normalization.

2 A Type Discipline for the SK Combinators

I consider the usual SK combinator calculus defined as such:

$$Kxy = x \tag{1}$$
$$Sxyz = xz(yz) \tag{2}$$

A natural interpretation of the combinators as typed functions results in the dependent typing:

$$\frac{\Gamma}{\Gamma \vdash \text{Prop} : Ty_0}$$

$$\frac{\Gamma \vdash A : K \;\; \Gamma, x : A \vdash B : L}{\Gamma \vdash (\forall x : A.B) : L}$$

$$\frac{}{\Gamma \vdash T_n : T_{n+1}}$$

$$\frac{\Gamma x : \alpha, y : \beta}{\Gamma \vdash K\alpha\beta : (\forall x, y.\alpha)}$$

$$\frac{\Gamma x : (\forall x : \alpha, y : \beta.\gamma), y : (\forall x : \alpha.\alpha), z : \alpha}{\Gamma \vdash S\alpha\beta\gamma : (\forall x, y, z.\gamma}$$

$$\frac{\Gamma \alpha : T_n,\ e_1 : (\forall x : \alpha.\text{body}), e_2 : \alpha}{\Gamma \vdash e_1 e_2 : \text{body}[x = b]}$$

Where $body[x = b]$ denotes substitution within the scope of $\forall$.

I assume the existence of a one-step reduction function of the combinators: `eval_once`.

I extend the inference rules to include beta-equivalent terms:

```
inductive beta_eq : Expr → Expr → Prop
  | trivial e₁ e₂ :
    e₁ = e₂ → beta_eq e₁ e₂
  | hard e₁ e₂ :
    beta_eq e₁ (eval_once e₂) → beta_eq e₁ e₂
  | symm e₁ e₂ :
    beta_eq e₂ e₁ → beta_eq e₁ e₂
  | trans e₁ e₂ e₃ :
    beta_eq e₁ e₂ → beta_eq e₂ e₃ → beta_eq e₁ e₃
```

where Expr is a typed combinator expression.

$$\frac{\Gamma e_1 : \alpha,\ \texttt{beta\_eq}\ e_1\ e_2}{\Gamma \vdash e_2 : \alpha}$$

3 Decomposition of the Simply-Typed Lambda Calculus into Dependently Typed SK Combinators

I utilize an SK compilation scheme outlined in "The Implementation of Functional Programming Languages" (Peyton Jones, Simon L., 1987):

$$(\lambda x.e_1\ e_2)\ arg = S(\lambda x.e_1)(\lambda x.e_2)\ arg \tag{3}$$
$$(\lambda x.x) = SKK \tag{4}$$
$$(\lambda x.c) = Kc \tag{5}$$

I consider a generic simply-typed lambda calculus with base types B, a type constructor $\rightarrow$ and the type universe:

$$T = \{t \mid t \in B\} \ \cup \ \{t \mid \exists\ t_1 \in T, t_2 \in T, t = t_1 \rightarrow t_2\}$$

3.1 Type Expressivity & Equivalence

I define a mapping ($\mathrm{M_t}$) from the $\rightarrow$ type constructor to $\forall$: $(\alpha \rightarrow \beta) \mapsto \forall x : \alpha.\beta$. I also assume the existence of a mapping ($\mathrm{M_c}$) from the base types B to arbitrary objects in my dependently-typed SK combinator calculus. Type inference is trivially derived from the above inference rules: $\forall c \in B, \exists\ t, t', c : t \implies t' = M_t t \implies M_c c : t$.

It follows that every well-typed expression in our simply-typed lambda calculus has an equivalent well-typed SK expression: 1

Proof. Assume (1) that for all $c \in B, \exists!\ c' \in M_c, c' = M_c c$. Assume (2) that for all $\{t_1, t_2, t\} \subset T, t = (t_1 \rightarrow t_2), \exists!\ t' \in M_t, t' = M_t t$. Per above and induction on (1) there exists a mapping from every lambda expression to an SK combinator expression. It follows by induction on $e : t$, where e is well-typed per the inference rules that all $t \in$ the simply-typed T are in M_t. It suffices to conclude that all well-typed expressions have well-typed counterparts in the dependently-typed SK combinator calculus. □

4 Proof

In order to prove strong normalization of the STLC, it suffices to demonstrate that a) no well-typed lambda calculus expression is inexpressible in the dependently-typed SK combinator calculus; and b) all well-typed SK combinator expressions are strongly normalizing.

4.1 Comprehensiveness of the SK Mapping

Proof. Suppose (1) there exists some well-typed expression e of type $t \in T$ in the STLC which is not representible in the dependently-typed SK combinator calculus. By induction:

- If the expression is a constant, it must be contained in M_c, per the above lemma. **contradiction**

- If the expression is a well-typed expression contained in M_c which is a dependently-typed SK expression, its type is inferred per the inference rules. The expression is thus representible. **contradiction**

- If the expression is a well-typed lambda expression, its type is of the form: $\alpha \to \beta$, where $\{\alpha, \beta\} \subset T$. An image must exist in M_t per above of the form $\forall x : \alpha.\beta$.

  - Its body is also well-typed, and has a valid type. Its body is thus representible **by induction**.

  - The expression is thus representible, per the decomposition rules. **contradiction**

- If the expression is a well-typed application $e_1 e_2$, its left hand side is of type $\alpha \to \beta$, where $\{\alpha, \beta\} \subset T$. Its right hand side must be of type β. The expression is thus of type t. By induction, the expression is representible. **contradiction**

Conclusion: no expression exists which has no image in the set of well-typed dependently-typed SK combinator expressions. □

4.2 Strong Normalization of the Typed SK Combinators

I define strong-normalization inductively (where e is an SK combinator expression) as:

```
inductive strongly_normalizing : Expr → Prop
  | trivial e : eval_once e = e → strongly_normalizing e
  | hard e : strongly_normalizing (eval_once e) →
             strongly_normalizing e
```

4.2.1 Reducibility Candidates

The K combinator is trivially strongly normalizing. It invokes no function applications, although it may produce an expression which contains an application. For example:

$$K(KK)y = KK$$

Borrowing from Tait's method, I define a mapping $R(t)$ where t is a type (expression) in our dependently-typed SK combinator calculus. The image of t is a set containing every well-typed expression of type t which is composed of expressions living in $R(t')$ for their respective types t'. I constrain the set such that all one-step reduxes of K are in R.

$$\begin{gathered}
\forall \alpha : T_n,\ \beta : T_m,\ x : \alpha,\ y : \alpha,\ R(\forall x, y.\alpha) = \\
\{K \mid K : (\forall x, y.\alpha) \wedge \forall arg_1 : \alpha,\ arg_2 : \beta, \\
\texttt{eval_once}\ K\ arg_1\ arg_2 \in R(\alpha[x = arg_1])\}
\end{gathered}$$

Or, more succinctly:

$$\begin{gathered}
\forall \alpha : T_n,\ \beta : T_m,\ x : \alpha,\ y : \alpha,\ R(\forall x, y.\alpha) = \\
\{K \mid K : (\forall x,\ y.\alpha)\ \wedge\ arg_1\ \in R(\alpha[x = arg_1])\}
\end{gathered}$$

$R(t)$ can be similarly extended to include the S combinator.

$$\begin{gathered}
\forall \alpha : T_n,\ \beta : T_m,\ \gamma : T_o, \\
T_x = (\forall x : \alpha, y : \beta.\gamma),\ T_y = (\forall x : \alpha.\alpha),\ T_z = \alpha, \\
x : T_x,\ y : T_y,\ z : T_z, \\
R(\forall x, y, z.\gamma) = \{S \mid S : (\forall x, y, z.\gamma),\ \forall arg_1 : T_x,\ arg_2 : T_z,\ arg_3 : T_z, \\
arg_1 \in R(T_x[x = arg_1]) \wedge arg_2 \in R(T_y[y = arg_2]) \wedge arg_3 \in R(T_z[z = arg_3])\}
\end{gathered}$$

Expressions which are obviously reducible and inert are as follows:

$$\begin{gathered}
R(T_{n+1}) = \{T_n\} \\
\forall K : T_n,\ L : T_m,\ A : K,\ B : L,\ R(L) = \{\text{fall} \mid,\ \text{fall} = (\forall x : A.B) \wedge \text{fall} : L\}
\end{gathered}$$

4.2.2 Inductive Proof

It suffices in order to prove strong normalization of this sytem that a) all reducibility candidates in R are strongly-normalizing; and c) all well-typed expression $(e : t)$ can be expressed using expressions in $R(t)$.

1. Preservation

   In order to execute an inductive proof leveraging our definition of $R(t)$, it is useful to prove that evaluation maintains the typing of an expression.

   Lemma 4.1. *For all well-typed expressions, $e : t \implies (eval\_once\ e) : t$.*

   Proof. The proof is obvious for obviously reducible expressions of the form T_n and $(\forall x : A.B)$. The $K : t$ combinator is inert (eval_once $k = k \implies t = t'$) except when it is provided two well-typed arguments: $K(x : t_1)(y : t_2)$. Per the inference rules, $(Kt_1t_2xy) : t$ is of the type $t = t_1$. Evaluation of Kt_1t_2xy is defined to be equivalent to x. Thus, preservation is trivially achieved. The same is true of the S combinator, whose inference rules trivially prove the goal. All combinations of expressions proceed **by induction**. $\square$

2. Proof Execution

   Lemma 4.2. *All expressions e which are well-typed with type t and occupy the set $R(t)$ are strongly normalizing.*

   Proof. Inductively:

   - All obviously reducing candidates are strongly normalizing:
     - All expressions of the form T_n are strongly normalizing, as they are inert.
     - All expressions of the form $(\forall x : A.B)$ are strongly normalizing, as they are inert.
   - All $K : t$ combinators in $R(t)$ are strongly normalizing. K is insert, and invokes no function applications. By the definition of $R(t)$, evaluation of $K \in R(t)$ will produce only one-step reduxes which are in R, and which are strongly normalizing **by induction**. Thus, the expression is **strongly normalizing**.

- All $S : t$ combinators in $R(t)$ are strongly normalizing. S is not inert, and invokes $xz(yz)$. However, x, y, and z live in R, requiring that their one-step reduxes live in R and are strongly-normalizing. The expression is strongly-normalizing **by induction**.

$\square$

Lemma 4.3. *All well-typed expressions* $(e : t)$ *occupy the set* $R(t)$*.*

Proof. The proof is trivially proven for objects of the form T_n and $(\forall x : A.B)$, as above. All well-typed $K\alpha\beta : t$ combinators are of the type $t = \forall x, y.\alpha$, where x is well-typed $(x : \alpha)$ and y is well-typed $(y : \beta)$. $x \in R(\alpha) \wedge y \in R(\beta)$ **by induction**. An expression of the form $K : t$ is said to be in $R(t)$ if all its possible one-step reduxes are in $R(\alpha)$. x has been shown to occupy $R(\alpha)$ and $K\alpha\beta xy = x$. Futhermore, per the inference rules, $K\alpha\beta xy : \alpha[x = x]$. $K\alpha\beta xy : \alpha[x = x]$ is thus in $R(\alpha[x = x])$, and per the definition of R, K is in $R(t)$. The S combinator is not inert, and invokes function application. However, its arguments are in R, and only produce one-step reduxes in R. By the definition of R, the expression is in R. $\square$

All well-typed dependently-typed SK combinator expressions are well-typed, as enumerated.

4.3 Strong Normalization of the STLC

I have shown in and that every well-typed expression in our simply-typed lambda calculus has a meaningful equivalent dependently-typed SK combinator expression. I have also demonstrated that there is no well-typed expression in the STLC which cannot be described by a well-typed dependently-typed SK combinator expression. I have demonstrated above that all well-typed SK dependently-typed SK combinator expressions are strongly normalizing. It follows that all well-typed expressions in the STLC are strongly normalizing.

4.4 Encoding in Lean

I have executed this proof in Lean.

5 References

Peyton Jones, Simon L. (1987). *The Implementation of Functional Programming Languages (Prentice-Hall International Series in Computer Science)*, Prentice-Hall, Inc..

Robert Harper (2022). *How to (Re)Invent Taits Method.*