Appl. Math. J. Chinese Univ.
2010, 25(4): 429-436

Approximate merging of B-spline curves and surfaces

CHEN Jun[1,2]　　WANG Guo-jin[1]

Abstract. Applying the distance function between two B-spline curves with respect to the L_2 norm as the approximate error, we investigate the problem of approximate merging of two adjacent B-spline curves into one B-spline curve. Then this method can be easily extended to the approximate merging problem of multiple B-spline curves and of two adjacent surfaces. After minimizing the approximate error between curves or surfaces, the approximate merging problem can be transformed into equations solving. We express both the new control points and the precise error of approximation explicitly in matrix form. Based on homogeneous coordinates and quadratic programming, we also introduce a new framework for approximate merging of two adjacent NURBS curves. Finally, several numerical examples demonstrate the effectiveness and validity of the algorithm.

§1 Introduction and notation

With the availability of a fast growing variety of modeling systems, data communication between design systems becomes quite frequent [3]. The general aim when transferring geometric information between different CAD systems is to ensure a high degree of accuracy, the least possible loss of information and requiring only a small amount of geometric data for communication. Hoschek [5] introduced two approximate conversion methods: degree reduction and approximate merging. Both of them are of great importance in geometric modeling, such as data exchange, data compression and data comparison. Furthermore, because the geometric information in shape design has come to mass, these two approximate conversion methods can also be used to simplify some geometric or graphical algorithms like intersection calculation or rendering.

Degree reduction of parametric curves and surfaces has been widely studied and a wealth of literature has focused on the problem [1,6,12]. However little study has been done for approximate merging. Approximate merging of a pair of Bézier curves was discussed in [7]. The basic idea is to find conditions for precise merging of Bézier curves and perturb the control points by constrained optimization subject to satisfying the precise merging conditions. This idea was also extended to the case of B-spline curves directly [11]. At the end of [11], Tai *et al*. mentioned the future work of approximate merging of multiple B-spline curves and of surfaces, which we

Received: 2009-01-08.
MR Subject Classification: 65D18.
Keywords: Computer aided geometric design, approximate merging, B-spline curves and surfaces.
Digital Object Identifier(DOI): 10.1007/s11766-010-2169-1.
Supported by the National Natural Science Foundation of China (60873111, 60933007).

investigate in this paper. With the help of the Moore-Penrose generalized inverse matrix theory, Cheng and Wang considered the approximate merging problem of multiple adjacent Bézier curves [2].

In this paper, we use the L_2 norm instead of the Euclidean norm to measure the distance between B-spline curves or surfaces. These two norms are not similar in general [9]. The latter is easy to deal with and depends only on the control points, while the former depends not only on the control points, but also on the base functions. Although it is hard to say which norm gives better approximation effect in any case, the L_2 norm gives more information [8]. Another advantage of the L_2 norm is that, instead of error estimation for lower order curves only [11], we get the precise error of approximation for any order.

The basic idea of this paper is to obtain the distance function between two B-spline curves with respect to the L_2 norm as the error of approximation. Then by setting the partial derivatives of the error with respect to the control points to zero, we get a system of linear equations. By solving the linear system, the best least squares approximation is given. This idea can be easily extended to approximate merging problem of multiple B-spline curves and of surfaces. Based on homogeneous coordinates and quadratic programming, we also introduce a new framework for approximate merging of two adjacent NURBS curves.

§2 Approximate merging of two adjacent B-spline curves

Consider two adjacent order k B-spline curves

$$\widehat{\mathbf{P}}^1(s) = \sum_{i=0}^{n_1} N_{i,k}^{S^1}(s)\,\widehat{\mathbf{p}}_i^1, \quad s \in \left[s_{k-1}^1, s_{n_1+1}^1\right]$$

and

$$\widehat{\mathbf{P}}^2(s) = \sum_{i=0}^{n_2} N_{i,k}^{S^2}(s)\,\widehat{\mathbf{p}}_i^2, \quad s \in \left[s_{k-1}^2, s_{n_2+1}^2\right]$$

with knot vectors

$$S^1 = \left\{s_0^1, \ldots, s_{k-1}^1, \ldots, s_{n_1}^1, s_{n_1+1}^1 = s_{k-1}^2, \ldots, s_{n_1+k}^1 = s_{2k-2}^2\right\}$$

and

$$S^2 = \left\{s_0^2 = s_{n_1-k+2}^1, \ldots, s_{k-1}^2 = s_{n_1+1}^1, s_k^2, \ldots, s_{n_2+1}^2, \ldots, s_{n_2+k}^2\right\}$$

respectively. Without loss of generality, we perform a linear transformation on the knot vector S^2 and then adjust the knot vectors such that $s_{n_1+1}^1 = s_{k-1}^2$, …, $s_{n_1+k}^1 = s_{2k-2}^2$, $s_0^2 = s_{n_1-k+2}^1$, …, $s_{k-1}^2 = s_{n_1+1}^1$. More details can be found in [11].

Put the two knot vectors together to get the knot vector $S = S^1 \cup S^2 = \{s_0, s_1, \ldots, s_{n_1+n_2+2}\}$ with

$$s_i = \begin{cases} s_i^1, & i = 0, 1, \ldots, n_1+1, \\ s_{i+k-n_1-2}^2, & i = n_1+1, n_1+2, \ldots, n_1+n_2+2. \end{cases}$$

Obviously we have

$$N_{i,k}^{S^1}(s) = N_{i,k}^S(s), \quad s \in [s_{k-1}, s_{n_1+1}]; \qquad N_{i,k}^{S^2}(s) = N_{i,k}^S(s), \quad s \in [s_{n_1+1}, s_{n_1+n_2-k+3}].$$

Putting $\widehat{\mathbf{P}}^1(s)$ and $\widehat{\mathbf{P}}^2(s)$ together, we have

$$\mathbf{P}(s) = \begin{cases} \widehat{\mathbf{P}}^1(s) = \sum\limits_{i=0}^{n_1} N_{i,k}^S(s)\mathbf{p}_i^1, & s \in [s_{k-1}, s_{n_1+1}], \\ \widehat{\mathbf{P}}^2(s) = \sum\limits_{i=n_1-k+2}^{n_1+n_2-k+2} N_{i,k}^S(s)\mathbf{p}_i^2, & s \in [s_{n_1+1}, s_{n_1+n_2-k+3}], \end{cases}$$

in which $\mathbf{p}_i^1 = \widehat{\mathbf{p}}_i^1$ and $\mathbf{p}_i^2 = \widehat{\mathbf{p}}_{i-(n_1-k+2)}^2$.

We want to find an order k B-spline curve

$$\mathbf{Q}(s) = \sum_{i=0}^{n_1+n_2-k+2} N_{i,k}^S(s)\mathbf{q}_i, \qquad s \in [s_{k-1}, s_{n_1+n_2-k+3}]$$

with knot vector S such that the distance function between $\mathbf{P}(s)$ and $\mathbf{Q}(s)$ with respect to the L_2 norm is minimized. From [11], only $k-1$ control points $\mathbf{q}_i$ for $i = n_1-k+2, \ldots, n_1$ are unknown, and the rest are given by

$$\mathbf{q}_i = \begin{cases} \mathbf{p}_i^1, & i = 0, 1, \ldots, n_1-k+1, \\ \mathbf{p}_i^2, & i = n_1+1, n_1+2, \ldots, n_1+n_2-k+2. \end{cases}$$

That means the two curves $\mathbf{P}(s)$ and $\mathbf{Q}(s)$ are the same curve on the intervals $[s_{k-1}, s_{n_1-k+2}]$ and $[s_{n_1+k}, s_{n_1+n_2-k+3}]$.

The distance function ε between $\mathbf{P}(s)$ and $\mathbf{Q}(s)$ with respect to the L_2 norm is

$$\begin{aligned} \varepsilon^2 &= \mathrm{tr}\Big\{\textstyle\int_{s_{n_1-k+2}}^{s_{n_1+k}} (\mathbf{Q}(s) - \mathbf{P}(s))^2\, ds\Big\} \\ &= \mathrm{tr}\Big\{\textstyle\int_{s_{n_1-k+2}}^{s_{n_1+1}} \big(\mathbf{Q}(s) - \mathbf{P}^1(s)\big)^2 ds + \int_{s_{n_1+1}}^{s_{n_1+k}} \big(\mathbf{Q}(s) - \mathbf{P}^2(s)\big)^2 ds\Big\}. \end{aligned}$$

Let

$$\begin{aligned} \hat{\mathbf{Q}} &= [\mathbf{q}_{n_1-k+2}, \mathbf{q}_{n_1-k+3}, \ldots, \mathbf{q}_{n_1}]^T, \\ \hat{\mathbf{P}}^1 &= \big[\mathbf{p}_{n_1-k+2}^1, \mathbf{p}_{n_1-k+3}^1, \ldots, \mathbf{p}_{n_1}^1\big]^T, \\ \hat{\mathbf{P}}^2 &= \big[\mathbf{p}_{n_1-k+2}^2, \mathbf{p}_{n_1-k+3}^2, \ldots, \mathbf{p}_{n_1}^2\big]^T, \\ \hat{\mathbf{N}} &= \big[N_{n_1-k+2,k}^S(s), N_{n_1-k+3,k}^S(s), \ldots, N_{n_1,k}^S(s)\big]. \end{aligned}$$

Then ε^2 can be simplified to

$$\begin{aligned} \varepsilon^2 &= \mathrm{tr}\big\{(\hat{\mathbf{Q}} - \hat{\mathbf{P}}^1)^T(\textstyle\int_{s_{n_1-k+2}}^{s_{n_1+1}} \hat{\mathbf{N}}^T\hat{\mathbf{N}}ds)(\hat{\mathbf{Q}} - \hat{\mathbf{P}}^1) + (\hat{\mathbf{Q}} - \hat{\mathbf{P}}^2)^T(\int_{s_{n_1+1}}^{s_{n_1+k}} \hat{\mathbf{N}}^T\hat{\mathbf{N}}ds)(\hat{\mathbf{Q}} - \hat{\mathbf{P}}^2)\big\} \\ &= \mathrm{tr}\big\{\hat{\mathbf{Q}}^T(\hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2)\hat{\mathbf{Q}} - 2\big((\hat{\mathbf{P}}^1)^T\hat{\mathbf{H}}_1 + (\hat{\mathbf{P}}^2)^T\hat{\mathbf{H}}_2\big)\hat{\mathbf{Q}} + \big((\hat{\mathbf{P}}^1)^T\hat{\mathbf{H}}_1\hat{\mathbf{P}}^1 + (\hat{\mathbf{P}}^2)^T\hat{\mathbf{H}}_2\hat{\mathbf{P}}^2\big)\big\}, \end{aligned}$$

in which

$$\hat{\mathbf{H}}_1 = \int_{s_{n_1-k+2}}^{s_{n_1+1}} \hat{\mathbf{N}}^T\hat{\mathbf{N}}ds, \qquad \hat{\mathbf{H}}_2 = \int_{s_{n_1+1}}^{s_{n_1+k}} \hat{\mathbf{N}}^T\hat{\mathbf{N}}ds.$$

We have

$$\min \varepsilon^2 = \min\left(\mathrm{tr}\left\{\hat{\mathbf{Q}}^T\mathbf{A}\hat{\mathbf{Q}} - 2\mathbf{B}\hat{\mathbf{Q}} + \mathbf{C}\right\}\right),$$

in which

$$\mathbf{A} = \hat{\mathbf{H}}_1 + \hat{\mathbf{H}}_2, \qquad \mathbf{B} = (\hat{\mathbf{P}}^1)^T\hat{\mathbf{H}}_1 + (\hat{\mathbf{P}}^2)^T\hat{\mathbf{H}}_2, \qquad \mathbf{C} = (\hat{\mathbf{P}}^1)^T\hat{\mathbf{H}}_1\hat{\mathbf{P}}^1 + (\hat{\mathbf{P}}^2)^T\hat{\mathbf{H}}_2\hat{\mathbf{P}}^2.$$

When we let

$$\mathbf{D} = \hat{\mathbf{Q}} - \hat{\mathbf{P}}^1 = \hat{\mathbf{Q}} - \hat{\mathbf{P}}^2,$$

then

$$\begin{aligned} \varepsilon_{\mathbf{D}}^2 &= (\hat{\mathbf{Q}} - \hat{\mathbf{P}}^1)^T(\textstyle\int_{s_{n_1-k+2}}^{s_{n_1+1}} \hat{\mathbf{N}}^T\hat{\mathbf{N}}ds)(\hat{\mathbf{Q}} - \hat{\mathbf{P}}^1) + (\hat{\mathbf{Q}} - \hat{\mathbf{P}}^2)^T(\int_{s_{n_1+1}}^{s_{n_1+k}} \hat{\mathbf{N}}^T\hat{\mathbf{N}}ds)(\hat{\mathbf{Q}} - \hat{\mathbf{P}}^2) \\ &= \mathbf{DAD} \geq 0 \end{aligned}$$

We see that the matrix $\mathbf{A}$ is positive and $\mathbf{A}$ is a symmetric base by definition, so the inverse of $\mathbf{A}$ exists.

By setting $\varepsilon^2/\hat{\mathbf{Q}} = \mathbf{0}$ and writing the derived equations in vector form, we get a system of linear equations $\mathbf{A}\hat{\mathbf{Q}} = \mathbf{B}^T$, or equivalently $\hat{\mathbf{Q}} = \mathbf{A}^{-1}\mathbf{B}^T$, of the $k-1$ unknown control points. We have obtained the precise error of approximation in matrix form.

§3 Approximate merging of multiple adjacent B-spline curves

Given multiple adjacent B-spline curves

$$\mathbf{P}_{mult}(s)=\begin{cases}\mathbf{P}^1(s)=\sum\limits_{i=l_1-k+1}^{l_2-1}N_{i,k}^{\hat{S}}(s)\mathbf{p}_i^1, \quad s\in[s_{l_1},s_{l_2}],\\ \vdots \\ \mathbf{P}^j(s)=\sum\limits_{i=l_j-k+1}^{l_{j+1}-1}N_{i,k}^{\hat{S}}(s)\mathbf{p}_i^j, \quad s\in\left[s_{l_j},s_{l_{j+1}}\right],\\ \vdots \\ \mathbf{P}^m(s)=\sum\limits_{i=l_m-k+1}^{l_{m+1}-1}N_{i,k}^{\hat{S}}(s)\mathbf{p}_i^m, \quad s\in\left[s_{l_m},s_{l_{m+1}}\right].\end{cases}$$

with knot vector

$$\hat{S}=\left\{s_{i_1-k+1},s_{i_1-k+2},\dots,s_{i_1},s_{i_1+1},\dots,s_{i_2},\dots,s_{i_j},\dots,s_{i_{m+1}},\dots,s_{i_{m+1}+k-1}\right\},$$

we want to find an order k B-spline curve

$$\mathbf{Q}_{mult}(s)=\sum_{i=l_1-k+1}^{l_{m+1}-1}N_{i,k}^{\hat{S}}(s)\mathbf{q}_i, s\in\left[s_{l_1},s_{l_{m+1}}\right]$$

with knot vector $\hat{S}$ so that the distance function between $\mathbf{P}_{mult}(s)$ and $\mathbf{Q}_{mult}(s)$ with respect to the L_2 norm is minimized.

Let

$$\mathbf{Q}_{mul}=\left[\mathbf{q}_{l_1-k+1},\dots,\mathbf{q}_{l_2-1},\dots,\mathbf{q}_{l_m-k+1},\dots,\mathbf{q}_{l_{m+1}-1}\right]^T,$$

$$\mathbf{P}^j=\Big[\underbrace{0,0,\dots,0}_{l_j-l_1},\mathbf{p}_{l_j-k+1},\mathbf{p}_{l_j-k+2},\dots,\mathbf{p}_{l_{j+1}+1},\underbrace{0,0,\dots,0}_{l_{m+1}-l_j}\Big]^T,$$

$$\mathbf{N}=\left[N_{l_1-k+1,k}^{\hat{S}}(s),\dots,N_{l_2-1,k}^{\hat{S}}(s),\dots,N_{l_m-k+1,k}^{\hat{S}}(s),\dots,N_{l_{m+1}-1,k}^{\hat{S}}(s)\right],$$

$$\mathbf{N}^j=\Big[\underbrace{0,0,\dots,0}_{l_j-l_1},N_{l_j-k+1,k}^{S}(s),N_{l_j-k+2,k}^{S}(s),\dots,N_{l_{j+1}-1,k}^{S}(s),\underbrace{0,0,\dots,0}_{l_{m+1}-l_j}\Big],\quad j=1,\dots,m.$$

The distance function ε_{mult} between $\mathbf{P}_{mult}(s)$ and $\tilde{\mathbf{Q}}_{mult}(s)$ with respect to the L_2 norm is

$$\begin{aligned}\varepsilon_{mult}^2 &= \mathrm{tr}\left\{\int_{s_{l_1}}^{s_{m+1}}\left(\mathbf{Q}_{mult}(s)-\mathbf{P}_{mult}(s)\right)^2ds\right\}=\mathrm{tr}\Big\{\sum_{j=1}^{m}\int_{s_{l_j}}^{s_{l_{j+1}}}\left(\mathbf{Q}_{mult}(s)-\mathbf{P}_{mult}(s)\right)^2ds\Big\}\\ &=\mathrm{tr}\Big\{\sum_{j=1}^{m}\int_{s_{l_j}}^{s_{l_{j+1}}}\left(\mathbf{N}\mathbf{Q}_{mult}-\mathbf{N}^j\mathbf{P}^j\right)^2ds\Big\}\\ &=\mathrm{tr}\Big\{\mathbf{Q}_{mult}^T\big(\sum_{j=1}^{m}\int_{s_{l_j}}^{s_{l_{j+1}}}(\mathbf{N}^j)^T\mathbf{N}^jds\big)\mathbf{Q}_{mult}-2\big(\sum_{j=1}^{m}(\mathbf{P}^j)^T\int_{s_{l_j}}^{s_{l_{j+1}}}(\mathbf{N}^j)^T\mathbf{N}^jds\big)\tilde{\mathbf{Q}}_{mult}\\ &\qquad+\sum_{j=1}^{m}(\mathbf{P}^j)^T\int_{s_{l_j}}^{s_{l_{j+1}}}(\mathbf{N}^j)^T\mathbf{N}^jds\mathbf{P}^j\Big\}.\end{aligned}$$

Hence

$$\min\varepsilon_{mult}^2=\min(\mathrm{tr}\{(\mathbf{Q}_{mul})^T\mathbf{A}_{mult}\mathbf{Q}_{mul}-2\mathbf{B}_{mult}\mathbf{Q}_{mul}+\mathbf{C}_{mult}\}),$$

in which

$$\mathbf{A}_{mult}=\sum_{j=1}^{m}\int_{s_{l_j}}^{s_{l_{j+1}}}(\mathbf{N}^j)^T\mathbf{N}^jds,\qquad \mathbf{B}_{mult}=\sum_{j=1}^{m}(\mathbf{P}^j)^T\int_{s_{l_j}}^{s_{l_{j+1}}}(\mathbf{N}^j)^T\mathbf{N}^jds,$$

$$\mathbf{C}_{mult}=\sum_{j=1}^{m}(\mathbf{P}^j)^T\int_{s_{l_j}}^{s_{l_{j+1}}}(\mathbf{N}^j)^T\mathbf{N}^jds\mathbf{P}^j.$$

We get a system of linear equations $\mathbf{A}_{mul}\mathbf{Q}_{mul}=\mathbf{B}_{mul}^T$ just like we have done in Section 2.

We can also write the system as $\mathbf{Q}_{mul}=\mathbf{A}_{mul}^{-1}\mathbf{B}_{mul}^T$ and get the precise error of approximation in matrix form.

§4 Approximate merging of two adjacent B-spline surfaces

Consider two adjacent order $l\times k$ B-spline surfaces

$$\widehat{\mathbf{P}}^1(s,t)=\sum_{i=0}^{m}\sum_{j=0}^{n_1}N_{i,l}^T(t)N_{j,k}^{S^1}(s)\,\widehat{\mathbf{p}}_{i,j}^1,\qquad t\in[t_{l-1},t_{m+1}],\qquad s\in\left[s_{k-1}^1,s_{n_1+1}^1\right]$$

and

$$\widehat{\mathbf{P}}^2(s,t)=\sum_{i=0}^{m}\sum_{j=0}^{n_2}N_{i,l}^T(t)N_{j,k}^{S^2}(s)\,\widehat{\mathbf{p}}_{i,j}^2,\qquad t\in[t_{l-1},t_{m+1}],\qquad s\in\left[s_{k-1}^2,s_{n_2+1}^2\right].$$

The knot vector is $T=\{t_0,t_1,\dots,t_{m+l}\}$, and S^1,S^2 and S are defined as those in Section 2.

Put $\widehat{\mathbf{P}}^1(s,t)$ and $\widehat{\mathbf{P}}^2(s,t)$ together to get

$$\mathbf{P}(s,t)=\begin{cases}\widehat{\mathbf{P}}^1(s,t)=\sum\limits_{i=0}^{m}\sum\limits_{j=0}^{n_1}N_{i,l}^T(t)N_{j,k}^S(s)\mathbf{p}_{i,j}^1, & t\in[t_{l-1},t_{m+1}],\quad s\in[s_{k-1},s_{n_1+1}],\\ \widehat{\mathbf{P}}^2(s,t)=\sum\limits_{i=0}^{m}\sum\limits_{j=n_1-k+2}^{n_1+n_2-k+2}N_{i,l}^T(t)N_{j,k}^S(s)\mathbf{p}_{i,j}^2, & t\in[t_{l-1},t_{m+1}],\quad s\in[s_{n_1+1},s_{n_1+n_2-k+3}]\end{cases}$$

with $\mathbf{p}_{i,j}^1=\widehat{\mathbf{p}}_{i,j}^1,\mathbf{p}_{i,j}^2=\widehat{\mathbf{p}}_{i,j-(n_1-k+2)}^2$.

We have to find an order $l\times k$ B-spline surface

$$\mathbf{Q}(s,t)=\sum_{i=0}^{m}\sum_{j=0}^{n_1+n_2-k+2}N_{i,l}^T(t)N_{j,k}^S(s)\mathbf{q}_{i,j},\quad t\in[t_{l-1},t_{m+1}],\quad s\in[s_{k-1},s_{n_1+n_2-k+3}]$$

so that the distance function between $\mathbf{P}(s,t)$ and $\mathbf{Q}(s,t)$ with respect to the L_2 norm is minimized. Let

$$\begin{aligned}\bar{\mathbf{Q}}&=[\mathbf{q}_{0,n_1-k+2},\mathbf{q}_{1,n_1-k+2},\dots,\mathbf{q}_{m,n_1-k+2},\dots,\mathbf{q}_{0,n_1},\mathbf{q}_{1,n_1},\dots,\mathbf{q}_{m,n_1}]^T,\\ \bar{\mathbf{P}}^1&=\left[\mathbf{p}_{0,n_1-k+2}^1,\mathbf{p}_{1,n_1-k+2}^1,\dots,\mathbf{p}_{m,n_1-k+2}^1,\dots,\mathbf{p}_{0,n_1}^1,\mathbf{p}_{1,n_1}^1,\dots,\mathbf{p}_{m,n_1}^1\right]^T,\\ \bar{\mathbf{P}}^2&=\left[\mathbf{p}_{0,n_1-k+2}^2,\mathbf{p}_{1,n_1-k+2}^2,\dots,\mathbf{p}_{m,n_1-k+2}^2,\dots,\mathbf{p}_{0,n_1}^2,\mathbf{p}_{1,n_1}^2,\dots,\mathbf{p}_{m,n_1}^2\right]^T,\end{aligned}$$

and

$$\bar{\mathbf{N}}=\left[N_{0,l}^T(t)N_{n_1-k+2,k}^S(s),\dots,N_{m,l}^T(t)N_{n_1-k+2,k}^S(s),\dots,N_{0,l}^T(t)N_{n_1,k}^S(s),\dots,N_{m,l}^T(t)N_{n_1,k}^S(s)\right].$$

As shown in Section 2, only $(m+1)\times(k-1)$ control points in $\bar{\mathbf{Q}}$ are unknown. The distance function ε_{surf} between $\mathbf{P}(s,t)$ and $\mathbf{Q}(s,t)$ with respect to the L_2 norm is

$$\begin{aligned}\varepsilon_{surf}^2&=\operatorname{tr}\left\{\int_{t_{l-1}}^{t_{m+1}}\int_{s_{n_1-k+2}}^{s_{n_1+k}}(\mathbf{Q}(s,t)-\mathbf{P}(s,t))^2\,dsdt\right\}\\ &=\operatorname{tr}\left\{\int_{t_{l-1}}^{t_{m+1}}\int_{s_{n_1-k+2}}^{s_{n_1+1}}\left(\mathbf{Q}(s,t)-\mathbf{P}^1(s,t)\right)^2dsdt+\int_{t_{l-1}}^{t_{m+1}}\int_{s_{n_1+1}}^{s_{n_1+k}}\left(\mathbf{Q}(s,t)-\mathbf{P}^2(s,t)\right)^2dsdt\right\}\\ &=\operatorname{tr}\{(\bar{\mathbf{Q}}-\bar{\mathbf{P}}^1)^T\left(\int_{t_{l-1}}^{t_{m+1}}\int_{s_{n_1-k+2}}^{s_{n_1+1}}\bar{\mathbf{N}}^T\bar{\mathbf{N}}dsdt\right)(\bar{\mathbf{Q}}-\bar{\mathbf{P}}^1)\\ &\qquad+(\bar{\mathbf{Q}}-\bar{\mathbf{P}}^2)^T\left(\int_{t_{l-1}}^{t_{m+1}}\int_{s_{n_1+1}}^{s_{n_1+k}}\bar{\mathbf{N}}^T\bar{\mathbf{N}}dsdt\right)(\bar{\mathbf{Q}}-\bar{\mathbf{P}}^2)\}\\ &=\operatorname{tr}\{\bar{\mathbf{Q}}^T(\bar{\mathbf{H}}_1+\bar{\mathbf{H}}_2)\bar{\mathbf{Q}}-2\left((\bar{\mathbf{P}}^1)^T\bar{\mathbf{H}}_1+(\bar{\mathbf{P}}^2)^T\bar{\mathbf{H}}_2\right)\bar{\mathbf{Q}}+\left((\bar{\mathbf{P}}^1)^T\bar{\mathbf{H}}_1\bar{\mathbf{P}}^1+(\bar{\mathbf{P}}^2)^T\bar{\mathbf{H}}_2\bar{\mathbf{P}}^2\right)\}\end{aligned}$$

in which

$$\bar{\mathbf{H}}_1=\int_{t_{l-1}}^{t_{m+1}}\int_{s_{n_1-k+2}}^{s_{n_1+1}}\bar{\mathbf{N}}^T\bar{\mathbf{N}}dsdt,\qquad \bar{\mathbf{H}}_2=\int_{t_{l-1}}^{t_{m+1}}\int_{s_{n_1+1}}^{s_{n_1+k}}\bar{\mathbf{N}}^T\bar{\mathbf{N}}dsdt.$$

Hence

$$\min\varepsilon^2=\min\left(\operatorname{tr}\left\{\bar{\mathbf{Q}}^T\bar{\mathbf{A}}\bar{\mathbf{Q}}-2\bar{\mathbf{B}}\bar{\mathbf{Q}}+\bar{C}\right\}\right)$$

in which

$$\bar{\mathbf{A}} = \bar{\mathbf{H}}_1 + \bar{\mathbf{H}}_2, \qquad \bar{\mathbf{B}} = (\bar{\mathbf{P}}^1)^T\bar{\mathbf{H}}_1 + (\bar{\mathbf{P}}^2)^T\bar{\mathbf{H}}_2, \qquad \bar{\mathbf{C}} = (\bar{\mathbf{P}}^1)^T\bar{\mathbf{H}}_1\bar{\mathbf{P}}^1 + (\bar{\mathbf{P}}^2)^T\bar{\mathbf{H}}_2\bar{\mathbf{P}}^2.$$

We get a system of linear equations $\bar{\mathbf{A}}\bar{\mathbf{Q}} = \bar{\mathbf{B}}^T$ or $\bar{\mathbf{Q}} = \bar{\mathbf{A}}^{-1}\bar{\mathbf{B}}^T$ just as in Section 2 and get the precise error of approximation in matrix form.

§5 Approximate merging of two adjacent NURBS curves

Given two adjacent order k NURBS curves

$$\widehat{\mathbf{R}}^1(s) = \sum_{i=0}^{n_1} N_{i,k}^{S^1}(s)h_i^1\widehat{\mathbf{r}}_i^1 \Big/ \sum_{i=0}^{n_1} N_{i,k}^{S^1}(s)h_i^1, \qquad \widehat{\mathbf{R}}^2(s) = \sum_{i=0}^{n_2} N_{i,k}^{S^2}(s)h_i^2\widehat{\mathbf{r}}_i^2 \Big/ \sum_{i=0}^{n_2} N_{i,k}^{S^2}(s)h_i^2$$

with knot vectors S^1 and S^2 respectively, we put $\widehat{\mathbf{R}}^1(s)$ and $\widehat{\mathbf{R}}^2(s)$ together to get

$$\mathbf{R}(s) = \begin{cases} \widehat{\mathbf{R}}^1(s) = \sum\limits_{i=0}^{n_1} N_{i,k}^{S}(s)\omega_i^1\mathbf{r}_i^1 \Big/ \sum\limits_{i=0}^{n_1} N_{i,k}^{S}(s)\omega_i^1, & s \in [s_{k-1}, s_{n_1+1}], \\ \widehat{\mathbf{R}}^2(s) = \sum\limits_{i=n_1-k+2}^{n_1+n_2-k+2} N_{i,k}^{S}(s)\omega_i^2\mathbf{r}_i^2 \Big/ \sum\limits_{i=n_1-k+2}^{n_1+n_2-k+2} N_{i,k}^{S}(s)\omega_i^2, & s \in [s_{n_1+1}, s_{n_1+n_2-k+3}]. \end{cases}$$

with

$$\mathbf{r}_i^1 = \widehat{\mathbf{r}}_i^1 = (r_i^{1,x}, r_i^{1,y}, r_i^{1,z}),\ \ \omega_i^1 = h_i^1,\ \ \mathbf{r}_i^2 = \widehat{\mathbf{r}}_{i-(n_1-k+2)}^2 = (r_i^{2,x}, r_i^{2,y}, r_i^{2,z}),\ \ \omega_i^2 = h_{i-(n_1-k+2)}^2.$$

The curve $\mathbf{R}(s)$ can be presented in homogeneous coordinates as

$$\tilde{\mathbf{R}}(s) = \begin{cases} \sum\limits_{i=0}^{n_1} N_{i,k}^{S}(s)\tilde{\mathbf{r}}_i^1, & s \in [s_{k-1}, s_{n_1+1}], \\ \sum\limits_{i=n_1-k+2}^{n_1+n_2-k+2} N_{i,k}^{S}(s)\tilde{\mathbf{r}}_i^2, & s \in [s_{n_1+1}, s_{n_1+n_2-k+3}] \end{cases}$$

with

$$\tilde{\mathbf{r}}_i^1 = (\omega_i^1 r_i^{1,x}, \omega_i^1 r_i^{1,y}, \omega_i^1 r_i^{1,z}, \omega_i^1), \qquad \tilde{\mathbf{r}}_i^2 = (\omega_i^2 r_i^{2,x}, \omega_i^2 r_i^{2,y}, \omega_i^2 r_i^{2,z}, \omega_i^2).$$

We have to find an order k NURBS curve

$$\mathbf{U}(s) = \sum_{i=0}^{n_1+n_2-k+2} N_{i,k}^{S}(s)\omega_i\mathbf{u}_i \Big/ \sum_{i=0}^{n_1} N_{i,k}^{S}(s)\omega_i, \qquad s \in [s_{k-1}, s_{n_1+n_2-k+3}]$$

with knot vector $\hat{S}$ in which $\mathbf{u}_i = (u_i^x, u_i^y, u_i^z)$. The corresponding curve in homogeneous coordinates is

$$\tilde{\mathbf{U}}(s) = \sum_{i=0}^{n_1+n_2-k+2} N_{i,k}^{S}(s)\tilde{\mathbf{u}}_i, \quad s \in [s_{k-1}, s_{n_1+n_2-k+3}], \qquad \tilde{\mathbf{u}}_i = (\omega_i u_i^x, \omega_i u_i^y, \omega_i u_i^z, \omega_i).$$

The distance function between $\tilde{\mathbf{R}}(s)$ and $\tilde{\mathbf{U}}(s)$ in homogeneous coordinates with respect to the L_2 norm should be minimized.

It is well known that NURBS curve can be formally treated as polynomial B-spline curve by using homogeneous coordinates. We get the first three coordinates $(\omega_i u_i^x, \omega_i u_i^y, \omega_i u_i^z)$ of the $k-1$ unknown control points $\tilde{\mathbf{u}}_i$ for $i = n_1-k+2, \ldots, n_1$ just as in Section 2. In order to keep the weights ω_i for $i = n_1-k+2, \ldots, n_1$ positive, we use quadratic programming instead of equations solving.

Let

$$\hat{\mathbf{\Omega}} = \left[\omega_{n_1-k+2}, \omega_{n_1-k+3}, \ldots, \omega_{n_1}\right]^T,$$

$$\hat{\mathbf{\Omega}}^1 = \left[\omega_{n_1-k+2}^1, \omega_{n_1-k+3}^1, \ldots, \omega_{n_1}^1\right]^T, \qquad \hat{\mathbf{\Omega}}^2 = \left[\omega_{n_1-k+2}^2, \omega_{n_1-k+3}^2, \ldots, \omega_{n_1}^2\right]^T.$$

We have

$$\min \varepsilon_\omega^2 = \min \left(\mathrm{tr}\{\hat{\mathbf{\Omega}}^T \mathbf{A}\hat{\mathbf{\Omega}} - 2\mathbf{B}_\omega \hat{\mathbf{\Omega}} + \mathbf{C}_\omega\}\right)$$

with $\omega_l > \delta$ for $l = n-k+2, n-k+3, \ldots, n$, in which

$$\mathbf{B}_\omega = (\hat{\mathbf{\Omega}}^1)^T \hat{\mathbf{H}}_1 + (\hat{\mathbf{\Omega}}^2)^T \hat{\mathbf{H}}_2, \qquad \mathbf{C}_\omega = (\hat{\mathbf{\Omega}}^1)^T \hat{\mathbf{H}}_1 \hat{\mathbf{\Omega}}^1 + (\hat{\mathbf{\Omega}}^2)^T \hat{\mathbf{H}}_2 \hat{\mathbf{\Omega}}^2.$$

Here we set $\delta = 0.5$ instead of $\delta = 0$ in practical engineering [4]. Because $\mathbf{A}$ is positive and symmetric, we can use quadratic programming to get $\hat{\mathbf{\Omega}}$.

In order to maintain the continuity of $\tilde{\mathbf{R}}(s)$ at $s = s_{n_1+1}$, we should clamp [10] S^1 and S^2 and make $\omega^1_{n_1} = \omega^2_{n_1-k+2}$ by scaling a common factor. Such idea cannot be extended to rational cases of multiple curves or surfaces because of their discontinuity in homogeneous coordinates.

§6 Numerical examples

Example 1. In Figure 1, two adjacent order 4 B-spline curves are in dotted line. The new approximate merging curve is in solid line.

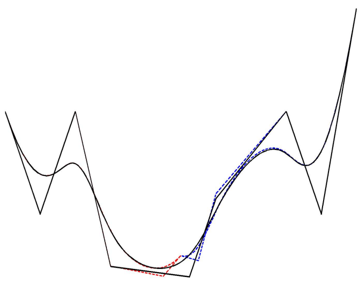

Figure 1. Two adjacent B-spline curves.

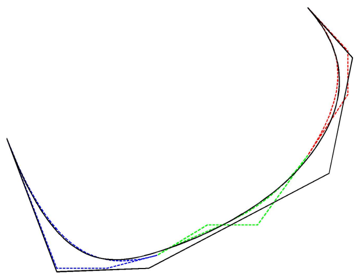

Figure 2. Three adjacent B-spline curves.

Example 2. In Figure 2, three adjacent order 4 B-spline curves are in dotted line. The new approximate merging curve is in solid line. If we approximately merge two adjacent B-spline curves twice, we have to get 6 new control points. But if we approximately merge three adjacent B-spline curves once, we only have to get 4 new control points.

Example 3. In Figure 3(a), there are two adjacent order 3×3 B-spline surfaces. The new approximate merging surface is in Figure 3(b).

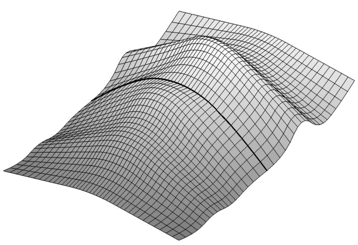

Figure 3(a). Two adjacent B-spline suefaces.

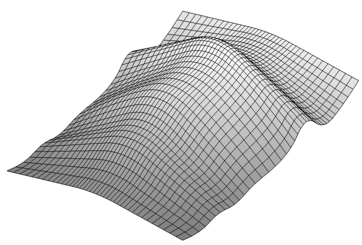

Figure 3(b). Approximate merging sueface.

Example 4. In Figure 4, two adjacent order 4 NURBS curves are in dotted line. The new approximate merging curve is in solid line.

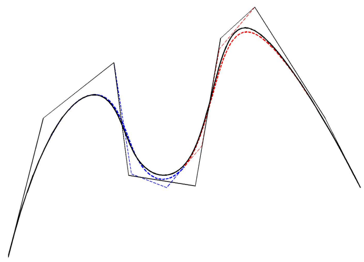

Figure 4. Approximate merging of two NURBS curves.

References

[1] G D Chen, G J Wang. *Optimal multi-degree reduction of Bézier curves with constraints of end-points continuity*, Comput Aided Geom Design, 2002, 19: 365-377.

[2] M Cheng, G J Wang. *Approximate merging of multiple Bézier segments*, Progr Natur Sci, 2008, 18: 757-762.

[3] L Danneberg, H Nowacki. *Approximate conversion of surface representations with polynomial bases*, Comput Aided Geom Design 1985, 2: 123-132.

[4] G Farin, J Hoschek, M S Kim. *Handbook of Computer Aided Geometric Design*, Elsevier, 2002.

[5] J Hoschek. *Approximate conversion of spline curves*, Comput Aided Geom Design 1987, 4: 59-66.

[6] Q Q Hu, G J Wang. *Optimal multi-degree reduction of triangular Bézier surfaces with corners continuity in the norm L_2*, J Comput Appl Math, 2008, 215: 114-126.

[7] S M Hu, R F Tong, T Ju, J G Sun. *Approximate merging of a pair of Bézier curves*, Comput Aided Design, 2001, 33: 125-136

[8] L Z Lu, G Z Wang. *Multi-degree reduction of triangular Bézier surfaces with boundary constraints*, Comput Aided Design, 2006, 18: 1215-1223.

[9] D Lutterkort, J Peters, U Reif. *Polynomial degree reduction in the L_2-norm equals best Euclidean approximation of Bézier coefficients*, Comput Aided Geom Design, 1999, 16: 607-612.

[10] L Piegl, W Tiller. *The NURBS Book*, 2nd Ed, Springer-Verlag, 1997.

[11] C L Tai, G J Wang. *Opproximate merging of B-spline curves via knot adjustment and constrained optimization*, Comput Aided Design, 2003, 35: 893-899.

[12] R J Zhang, G J Wang. *Constrained Bézier curves' best multi-degree reduction in the L_2-norm*, Progr Natur Sci, 2005, 15: 843-850.

[1] Department of Mathematics, Zhejiang University, Hangzhou 310027, China
Email: gjwang@hzcnc.com

[2] Ningbo University of Technology, Ningbo 315016, China