Chapter 1 B: Picard's Method

(a) Use Picard's method with $\psi_0(x) = 1$ to obtain the next four successive approximations of the solution to

$$y'(x) = y(x), \quad y(0) = 1$$

Show that these approximations are just the partial sums of the Maclaurin series for the actual solution e^x.

Solution Given that

$$y'(x) = y(x), \quad y(0) = 1 \tag{1}$$

Also given that

$$f(x, y) = y(x) \tag{2}$$

According to picards theorem, we have

$$\begin{aligned} \phi_{n+1}(x) &= y_0 + \int_{x_0}^{x} f(t, \phi_0(t))dt \\ &= 1 + \int_0^x 1dt \\ &= 1 + x \end{aligned} \tag{3}$$

$$\begin{aligned} \phi_2(x) &= y_0 + \int_0^x f(t, \phi_1(t))dt \\ &= 1 + \int_0^x f(t, (1+t))dt \\ &= 1 + x + \frac{x^2}{2} \end{aligned} \tag{4}$$

$$\begin{aligned} \phi_3(x) &= y_0 + \int_0^x f(t, \phi_2(t))dt \\ &= 1 + \int_0^x f\left(t, \left(1 + t + \frac{t^2}{2}\right)\right)dt \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \end{aligned} \tag{5}$$

$$\begin{aligned} \phi_4(x) &= y_0 + \int_0^x f(t, \phi_3(t))dt \\ &= 1 + \int_0^x f\left(t, \left(1 + t + \frac{t^2}{2}\right)\right)dt \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \end{aligned} \tag{6}$$

By observing the pattern as n goes, it is enough to say that

$$\phi_n(x) = 1 + x + x^2 + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!}$$

This is the partial sum of the Maclaurian series of e^x.

(b) Use Picard's method with $\psi + 0(x) = 0$ to obtain the next three successive approximations of the solution to the nonlinear problem

$$y'(x) = 3x - \left[y(x)^2\right], \qquad y(0) = 0$$

Graph these approximations for $0 \leq x \leq 1$.

Solution

$$\begin{aligned} y(x_1) &= y(x_0) + \int_{x_0}^{x} f(x,y)dx \\ &= f(x,y) = 2x - y^2 \\ y(0) &= 0 \end{aligned} \tag{1}$$

We assume that $x_0 = 0, x_1 = 0.25$ then

$$\begin{aligned} \phi_1(x) &= y(0) + \int_0^x \phi_0(x)dx = 0 + \int_0^x 2x - 0dx = x^2 \\ \phi_2(x) &= y(0) + \int_0^x \phi_1(x)dx = 0 + \int_0^x (2x - x^2)dx = x^2 - = fracx^3 3 \\ \phi_3(x) &= y(0) + \int_0^x \phi_2(x)dx = 0 + \int_0^x (2x - x^2 + \frac{x^3}{3})dx = x^2 - = fracx^3 3 + \frac{x^4}{12} \\ \phi_4(x) &= y(0) + \int_0^x \phi_3(x)dx = 0 + \int_0^x (2x - x^2 + \frac{x^3}{3} - \frac{x^4}{12})dx = x^2 - = fracx^3 3 + \frac{x^4}{12} - \frac{x^5}{60} \end{aligned} \tag{2}$$

If $x = 0.25, \phi_1(x) = 0.0625, \phi_2(x) = 0.05729, \phi_3(x) = 0.05761, \phi_4(x) = 0.0576009$. Thus, the better approximationn at $x = 0.25$ is 0.0576.

If $x = 0.5, \phi_1(x) = 0.25, \phi_2(x) = 0.2083, \phi_3(x) = 0.203125, \phi_4(x) = 0.21302$.

If $x = 0.75, \phi_1(x) = 0.5625, \phi_2(x) = 0.421875, \phi_3(x) = 0.44824, \phi_4(x) = 0.605419$.

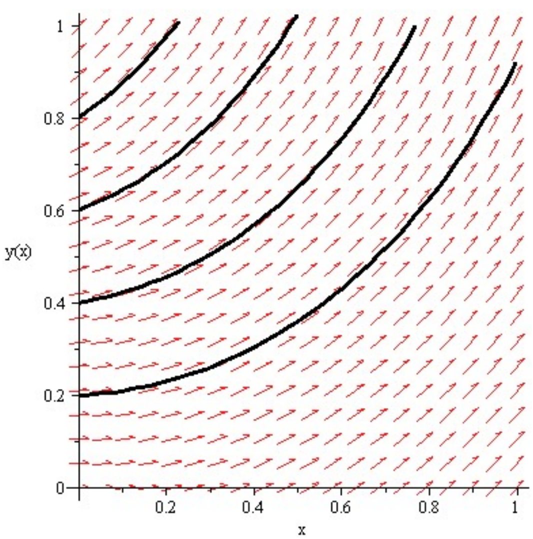

Figure 1:

(c) In Problem 29 in Exercises 1.2, we showed that the initial value problem

$$y'(x) = 3\left[y(x)\right]^{2/3}, y(2) = 0$$

does not have a unique solution. Show that Picard's method beginning with $\psi_0(x) = 0$ converges to the solution $y(x) = 0$, whereas Picard's method beginning with $\psi_0(x) = x - 2$ converges to the second solution $y(x) = (x-2)^3$.

Solution The given IVP can be written as

$$y'(t) = f(x, y(x)) \text{ where } f(x, y(x)) = 3(y(x))^{2/3} \tag{1}$$

The first iteration is given by

$$\begin{aligned} y_1(x) &= y(2) + \int_2^x f(u, y(2))du \\ &= 0 + \int_2^t f(u, 0)du \\ &= 0 + \int_0^x 0du = 0 \end{aligned} \tag{2}$$

If we repeat the procedure, then we get

$$y_2(x) = 0 \tag{3}$$

Thus, we get the trivial solution $y(x) = 0$ for the IVP.

Suppose that

$$\phi_0(x) = x - 2 \tag{4}$$

Then the first iteration is given by

$$\begin{aligned} \phi_1(x) &= \phi_0(x) + \int_2^x f\left(u, \phi_0(x)\right) du \\ &= x - 2 + \int_2^x f(u, x-2)du \\ &= x - 2 + \int_2^x 3(x-2)^{2/3}du = x - 23\frac{(x-2)^{5/3}}{5/3} \Big|_2^x \\ &= x - 2 + \frac{9}{5}(x-2)^{5/3} = (x-2)\left(1 + \frac{9}{5}(x-2)^{2/3}\right) \end{aligned} \tag{5}$$

The second iteration is given by

$$\begin{aligned} \phi_2(x) &= \phi_0(x) + \int_2^x \left(fu, \phi_1(x)\right) du \\ &= x - 2 + \int_2^x f\left(u, x-2\right) du \\ &= x - 2 + \int_2^x x - 2 + \frac{9}{5}(x-2)^{5/3}du = x - 2 + \left(\frac{(x-2)^2}{2} + \frac{9}{5}\frac{(x-2)^{8/3}}{8/3} \Big|_2^x\right) \\ &= x - 2 = \frac{(x-2)^2}{2} + \frac{27}{40}(x-2)^{8/3} \end{aligned} \tag{6}$$

The Phase Line

(a) The slopes in the direction field are all identical along horizontal lines.

Solution

(b) New solutions can be generated from old ones by time shifting [i.e., replacing $y(t)$ with $y(t - t_0)$.]

Solution

(c) Sketch the phase line for $y' = (y-1)(y-2)(y-3)$ and state the nature of its equilibria.

Solution

(d) Use the phase line for $y' = -(y-1)^{5/3}(y-2)^2(y-3)$ to predict the asymtotic behavior as $t \to \infty$ of the solution satisfying $y(0) = 2.1$.

Solution

(e) Sketch the phase line for $y' = y\text{sin}\, y$ and state the nature of its equilibria.

Solution

(f) Sketch the phase lines for $y' = y \sin y + 0.1$ and $y' = y \sin y - 0.1$. Discuss the effect of the small perturbation ± 0.1 on the equilibria.

Solution

Chapter 4 A: Nonlinear Equations Solvable by First-Order Techniques

(a) **Solution**

The given equation can ben rewritten as

$$\begin{aligned} 2x\frac{\mathrm{d}w}{\mathrm{d}x} - w + \frac{1}{w} &= 0 \\ 2x\frac{\mathrm{d}w}{\mathrm{d}x} &= w - \frac{1}{w} \\ \frac{1}{w - 1/w}dw &= \frac{1}{2x}dx \end{aligned}$$

Therefore we get

$$\frac{1}{2}\int \left(\frac{1}{w^2-1}2w\right)dw = \frac{1}{2}\int \frac{1}{b}2dx$$

That is

$$\begin{aligned} \ln w^2 - 1 &= \ln|x| + C \\ w^2 - 1 &= x + C \\ w = \frac{\mathrm{d}y}{\mathrm{d}x} &= \sqrt{x+1} \end{aligned}$$

Thus, we obtain

$$y = \int (x+1)^{\frac{1}{2}}dx = \frac{1}{2\sqrt{x+1}} + C$$

(b) **Solution**

$$\begin{aligned} (2y)(w)\frac{\mathrm{d}w}{\mathrm{d}y} &= 1 + w^2 \\ 2y\frac{\mathrm{d}w}{\mathrm{d}y} &= \frac{1}{w} + w = \frac{w^2+1}{w} \end{aligned}$$

This becomes

$$\frac{1}{2}\int 1\big(w^2+1\big)\,(2w)dw = \frac{1}{2}\int \frac{1}{y}dy$$

Then we obtain

$$w = \sqrt{y-1} + C$$

$$\int \frac{1}{\sqrt{y-1}}dy = \int dx\sqrt{y-1} = 2x + C_1$$

Thus we obtain

$$y_1 = 1 + 4x^2 + C$$

Then we need to have another equation.

$$2yw\frac{\mathrm{d}w}{\mathrm{d}y} = -yw2\frac{\mathrm{d}w}{\mathrm{d}y} = -\frac{yw}{yw} = -1$$

$$\begin{aligned}\int dw &= -\frac{1}{2}\int dy \\ -2\int \frac{1}{y}dy &= \int dx\end{aligned}$$

This becomes

$$\begin{aligned}-2\ln|y| &= x + C \\ \ln|y| &= -\frac{1}{2}x + C\end{aligned}$$

Thus, we get

$$y_2 = Ce^{-\frac{x}{2}}$$

(c) **Suspended Cable.**

Solution The given differential equation is

$$y'' = \frac{1}{a}\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}; \quad y(0) = a, \quad y'(0) = 0, \quad \text{where } a(\neq 0) \text{ is a constant.} \tag{1}$$

Let $v = y'$ then we can say that $\frac{\mathrm{d}v}{\mathrm{d}x} = y''$

Now, we solve

$$\begin{aligned}\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{1}{a}\sqrt{1+v^2} \\ \int \frac{dv}{\sqrt{1+v^2}} &= a\int dx\end{aligned} \tag{2}$$

This becomes

$$\ln\sqrt{1+v^2} + v = ax \tag{3}$$

Thus, this is enough reason to say that

$$x'(0) = 0$$

This means $C = 0$.

Therefore

$$\begin{aligned}\ln\sqrt{1+v^2} + v &= ax \\ \sqrt{1+v^2} + v &= e^{ax} \\ \left(-\sqrt{1+v^2}\right) &= (v - e^{ax})^2 \\ v &= \frac{e^{2ax} - 1}{2e^{ax}} = \frac{1}{2}\left[e^{ax} - e^{-ax}\right]\end{aligned} \tag{4}$$

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{2}\left(e^{ax} - e^{-ax}\right) \\
\int dy &= \frac{1}{2}\int \left(e^{ax} - e^{-ax}\right) dx \\
y &= \frac{1}{2}\frac{\left(e^{ax} + e^{-ax}\right)}{a} + C_2
\end{aligned} \tag{5}$$

At $y(0) = a$,

$$a = \frac{1}{2}\frac{2}{a} + C_2$$

Therefore,

$$C_2 = \frac{\left(a^2 - 1\right)}{a}$$

Then finally we get

$$y = \frac{1}{2}\left(\frac{a^{ax} + e^{-ax}}{a}\right) + \frac{a^2 - 1}{a} \tag{6}$$

Chapter 4C: Simple Pendulum

(a) **Solution** By assuming the problem consisting of mass m suspended by a rod of length l and writing the derived nonlinear initial value problem,

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{l}\sin\theta = 0, \qquad \theta(0) = \alpha(0), \theta'(0) = 0$$

Let us determine the equation of motion for the pendulum and its period.

$$\begin{aligned}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\right) + \frac{g}{l}\sin\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right) &= 0\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 - \frac{g}{l}\cos\theta\right) &= 0\\ \frac{1}{2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 - \frac{g}{l}\cos theta &= c\\ \left(\frac{\theta}{t}\right)^2 - \frac{2g}{l}\cos\theta &= C\end{aligned} \tag{1}$$

By applying the initial conditions,

$$(\theta'(0))^2 - \frac{2g}{l}\cos theta(0) = C \tag{2}$$

Then we get

$$\begin{aligned}0 - \frac{2g}{l}\cos\alpha = C\\ C = -\frac{2g}{l}\cos\alpha\end{aligned} \tag{3}$$

By substituting C value into $\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 - \frac{2g}{l}\cos\theta = C$, then we get

$$\begin{aligned}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 - \frac{2g}{l}\cos\theta &= -\frac{2g}{l}\cos\alpha\\ \left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 &= \frac{2g}{l}(\cos\theta - \cos\alpha)\\ \frac{\mathrm{d}\theta}{\mathrm{d}t} &= -\sqrt{\frac{2g}{l}(\cos\theta - \cos\alpha)}\\ dt &= -\sqrt{\frac{1}{2g}}\frac{d\theta}{\sqrt{(\cos\theta - \cos\alpha)}}\end{aligned} \tag{4}$$

(b) **Solution** By using the formula, $\cos x = 1 - 2\sin^2 x$, we can say that

$$\begin{aligned}dt &= -\sqrt{\frac{1}{2g}}\frac{d\theta}{\sqrt{\left(\left(1 - 2\sin^2\theta\right) - \left(1 - 2\sin^2\alpha\right)\right)}}\\ dt &= -\frac{1}{2}\sqrt{\frac{l}{g}}\frac{d\theta}{\sqrt{sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}}\end{aligned} \tag{5}$$

(c) **Solution** Now, let us determine the elapsed time T, for the pendulum to fall from the angle $\theta = \alpha$ to the angle $\theta = \beta$ corresponding to $\phi = \frac{\pi}{2}$ to $\phi = \Phi$

$$\begin{aligned} T = \int_0^t dt &= -\frac{1}{2}\sqrt{\frac{l}{g}}\frac{d\theta}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}} \\ &= -\frac{1}{2}\sqrt{\frac{l}{g}}\int_{\frac{\pi}{2}}^{\phi}\frac{d\theta}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}} \\ &= -\sqrt{\frac{l}{g}}\int_{\frac{\pi}{2}}^{\phi}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}, \quad \text{where } k = \sin\frac{\alpha}{2} \end{aligned} \tag{6}$$

Thus, we get

$$-T = \sqrt{\frac{1}{g}}\int_{\frac{\pi}{2}}^{\phi}\frac{d\phi}{1-k^2\sin^2\phi} \tag{7}$$

(d) **Solution** Let us determine the period P of the pendulum. The period is defined to be the time required for it swing from one extreme to other extreme and back. Therefore, from α to $-\alpha$ and from $-\alpha$ to α. Therefore, the total period is four equal parts. Thus, the period is given by

$$P = (4)\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} \tag{8}$$

Now, the integral

$$\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-k^2\sin^2(\phi)}}$$

is called an elliptic integral of the first kind and denoted by

$$F\left(k,\frac{\pi}{2}\right) = \int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$$

(e) **Solution**

$$\begin{aligned} \frac{P}{4} + (-T) &= \sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-k^2\sin^2(\phi)}} + \sqrt{\frac{l}{g}}\int_{\frac{\pi}{2}}^{\phi}\frac{d\phi}{\sqrt{1-k^2\sin^2(\phi)}} \\ -T + \frac{P}{4} &= \sqrt{\frac{l}{g}}\int_0^{\phi}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} \\ -T + \frac{P}{4} &= \sqrt{\frac{l}{g}}F(k,\Phi) \end{aligned} \tag{9}$$

This is obtained by substituting

$$F(k,\Phi) = \int_0^{\phi}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$$