

Chapter 1 B: Picard's Method

- (a) Use Picard's method with $\psi_0(x) = 1$ to obtain the next four successive approximations of the solution to

$$y'(x) = y(x), \quad y(0) = 1$$

Show that these approximations are just the partial sums of the Maclaurin series for the actual solution e^x .

Solution Given that

$$y'(x) = y(x), \quad y(0) = 1 \tag{1}$$

Also given that

$$f(x, y) = y(x) \tag{2}$$

According to Picard's theorem, we have

$$\begin{aligned} \phi_{n+1}(x) &= y_0 + \int_{x_0}^x f(t, \phi_0(t)) dt \\ &= 1 + \int_0^x 1 dt \\ &= 1 + x \end{aligned} \tag{3}$$

$$\begin{aligned} \phi_2(x) &= y_0 + \int_0^x f(t, \phi_1(t)) dt \\ &= 1 + \int_0^x f(t, (1+t)) dt \\ &= 1 + x + \frac{x^2}{2} \end{aligned} \tag{4}$$

$$\begin{aligned} \phi_3(x) &= y_0 + \int_0^x f(t, \phi_2(t)) dt \\ &= 1 + \int_0^x f\left(t, \left(1+t+\frac{t^2}{2}\right)\right) dt \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \end{aligned} \tag{5}$$

$$\begin{aligned} \phi_4(x) &= y_0 + \int_0^x f(t, \phi_3(t)) dt \\ &= 1 + \int_0^x f\left(t, \left(1+t+\frac{t^2}{2}\right)\right) dt \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \end{aligned} \tag{6}$$

By observing the pattern as n goes, it is enough to say that

$$\phi_n(x) = 1 + x + x^2 + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!}$$

This is the partial sum of the Maclaurian series of e^x .

- (b) Use Picard's method with $\psi + 0(x) = 0$ to obtain the next three successive approximations of the solution to the nonlinear problem

$$y'(x) = 3x - [y(x)^2], \quad y(0) = 0$$

Graph these approximations for $0 \leq x \leq 1$.

Solution

$$\begin{aligned} y(x_1) &= y(x_0) + \int_{x_0}^x f(x, y) dx \\ &= f(x, y) = 2x - y^2 \\ y(0) &= 0 \end{aligned} \tag{1}$$

We assume that $x_0 = 0, x_1 = 0.25$ then

$$\begin{aligned} \phi_1(x) &= y(0) + \int_0^x \phi_0(x) dx = 0 + \int_0^x 2x - 0 dx = x^2 \\ \phi_2(x) &= y(0) + \int_0^x \phi_1(x) dx = 0 + \int_0^x (2x - x^2) dx = x^2 - \frac{x^3}{3} \\ \phi_3(x) &= y(0) + \int_0^x \phi_2(x) dx = 0 + \int_0^x (2x - x^2 + \frac{x^3}{3}) dx = x^2 - \frac{x^3}{3} + \frac{x^4}{12} \\ \phi_4(x) &= y(0) + \int_0^x \phi_3(x) dx = 0 + \int_0^x (2x - x^2 + \frac{x^3}{3} - \frac{x^4}{12}) dx = x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{60} \end{aligned} \tag{2}$$

If $x = 0.25, \phi_1(x) = 0.0625, \phi_2(x) = 0.05729, \phi_3(x) = 0.05761, \phi_4(x) = 0.0576009$. Thus, the better approximationn at $x = 0.25$ is 0.0576.

If $x = 0.5, \phi_1(x) = 0.25, \phi_2(x) = 0.2083, \phi_3(x) = 0.203125, \phi_4(x) = 0.21302$.

If $x = 0.75, \phi_1(x) = 0.5625, \phi_2(x) = 0.421875, \phi_3(x) = 0.44824, \phi_4(x) = 0.605419$.

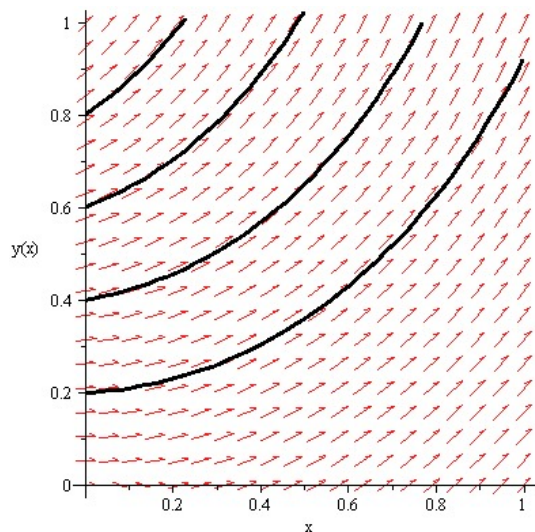


Figure 1:

(c) In Problem 29 in Exercises 1.2, we showed that the initial value problem

$$y'(x) = 3[y(x)]^{2/3}, y(2) = 0$$

does not have a unique solution. Show that Picard's method beginning with $\psi_0(x) = 0$ converges to the solution $y(x) = 0$, whereas Picard's method beginning with $\psi_0(x) = x - 2$ converges to the second solution $y(x) = (x - 2)^3$.

Solution The given IVP can be written as

$$y'(t) = f(x, y(x)) \text{ where } f(x, y(x)) = 3(y(x))^{2/3} \quad (1)$$

The first iteration is given by

$$\begin{aligned} y_1(x) &= y(2) + \int_2^x f(u, y(2)) du \\ &= 0 + \int_2^x f(u, 0) du \\ &= 0 + \int_2^x 0 du = 0 \end{aligned} \quad (2)$$

If we repeat the procedure, then we get

$$y_2(x) = 0 \quad (3)$$

Thus, we get the trivial solution $y(x) = 0$ for the IVP.

Suppose that

$$\phi_0(x) = x - 2 \quad (4)$$

Then the first iteration is given by

$$\begin{aligned} \phi_1(x) &= \phi_0(x) + \int_2^x f(u, \phi_0(x)) du \\ &= x - 2 + \int_2^x f(u, x - 2) du \\ &= x - 2 + \int_2^x 3(x - 2)^{2/3} du = x - 23 \frac{(x - 2)^{5/3}}{5/3} \Big|_2^x \\ &= x - 2 + \frac{9}{5}(x - 2)^{5/3} = (x - 2) \left(1 + \frac{9}{5}(x - 2)^{2/3} \right) \end{aligned} \quad (5)$$

The second iteration is given by

$$\begin{aligned} \phi_2(x) &= \phi_0(x) + \int_2^x (f u, \phi_1(x)) du \\ &= x - 2 + \int_2^x f(u, x - 2) du \\ &= x - 2 + \int_2^x x - 2 + \frac{9}{5}(x - 2)^{5/3} du = x - 2 + \left(\frac{(x - 2)^2}{2} + \frac{9}{5} \frac{(x - 2)^{8/3}}{8/3} \Big|_2^x \right) \\ &= x - 2 = \frac{(x - 2)^2}{2} + \frac{27}{40}(x - 2)^{8/3} \end{aligned} \quad (6)$$

The Phase Line

- (a) The slopes in the direction field are all identical along horizontal lines.

Solution

- (b) New solutions can be generated from old ones by time shifting [i.e., replacing $y(t)$ with $y(t - t_0)$.]

Solution

- (c) Sketch the phase line for $y' = (y - 1)(y - 2)(y - 3)$ and state the nature of its equilibria.

Solution

- (d) Use the phase line for $y' = -(y - 1)^{5/3}(y - 2)^2(y - 3)$ to predict the asymptotic behavior as $t \rightarrow \infty$ of the solution satisfying $y(0) = 2.1$.

Solution

- (e) Sketch the phase line for $y' = y \sin y$ and state the nature of its equilibria.

Solution

- (f) Sketch the phase lines for $y' = y \sin y + 0.1$ and $y' = y \sin y - 0.1$. Discuss the effect of the small perturbation ± 0.1 on the equilibria.

Solution

Chapter 4 A: Nonlinear Equations Solvable by First-Order Techniques

(a) Solution

The given equation can be rewritten as

$$\begin{aligned} 2x \frac{dw}{dx} - w + \frac{1}{w} &= 0 \\ 2x \frac{dw}{dx} &= w - \frac{1}{w} \\ \frac{1}{w - 1/w} dw &= \frac{1}{2x} dx \end{aligned}$$

Therefore we get

$$\frac{1}{2} \int \left(\frac{1}{w^2 - 1} 2w \right) dw = \frac{1}{2} \int \frac{1}{x} dx$$

That is

$$\begin{aligned} \ln w^2 - 1 &= \ln |x| + C \\ w^2 - 1 &= x + C \\ w &= \frac{dy}{dx} = \sqrt{x+1} \end{aligned}$$

Thus, we obtain

$$y = \int (x+1)^{\frac{1}{2}} dx = \frac{2}{3} (x+1)^{\frac{3}{2}} + C$$

(b) Solution

$$\begin{aligned} (2y)(w) \frac{dw}{dy} &= 1 + w^2 \\ 2y \frac{dw}{dy} &= \frac{1}{w} + w = \frac{w^2 + 1}{w} \end{aligned}$$

This becomes

$$\frac{1}{2} \int \frac{1}{w^2 + 1} (2w) dw = \frac{1}{2} \int \frac{1}{y} dy$$

Then we obtain

$$w = \sqrt{y-1} + C$$

$$\int \frac{1}{\sqrt{y-1}} dy = \int dx \sqrt{y-1} = 2x + C_1$$

Thus we obtain

$$y_1 = 1 + 4x^2 + C$$

Then we need to have another equation.

$$2yw \frac{dw}{dy} = -yw2 \frac{dw}{dy} = -\frac{yw}{yw} = -1$$

$$\begin{aligned} \int dw &= -\frac{1}{2} \int \frac{1}{y} dy \\ -2 \int \frac{1}{y} dy &= \int dx \end{aligned}$$

This becomes

$$\begin{aligned} -2 \ln |y| &= x + C \\ \ln |y| &= -\frac{1}{2}x + C \end{aligned}$$

Thus, we get

$$y_2 = Ce^{-\frac{x}{2}}$$

(c) **Suspended Cable.**

Solution The given differential equation is

$$y'' = \frac{1}{a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}; \quad y(0) = a, \quad y'(0) = 0, \quad \text{where } a(\neq 0) \text{ is a constant.} \quad (1)$$

Let $v = y'$ then we can say that $\frac{dv}{dx} = y''$

Now, we solve

$$\begin{aligned} \frac{dv}{dx} &= \frac{1}{a} \sqrt{1 + v^2} \\ \int \frac{dv}{\sqrt{1 + v^2}} &= a \int dx \end{aligned} \quad (2)$$

This becomes

$$\ln \sqrt{1 + v^2} + v = ax \quad (3)$$

Thus, this is enough reason to say that

$$x'(0) = 0$$

This means $C = 0$.

Therefore

$$\begin{aligned} \ln \sqrt{1 + v^2} + v &= ax \\ \sqrt{1 + v^2} + v &= e^{ax} \\ (-\sqrt{1 + v^2}) &= (v - e^{ax})^2 \\ v &= \frac{e^{2ax} - 1}{2e^{ax}} = \frac{1}{2} [e^{ax} - e^{-ax}] \end{aligned} \quad (4)$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2} (e^{ax} - e^{-ax}) \\ \int dy &= \frac{1}{2} \int (e^{ax} - e^{-ax}) dx \\ y &= \frac{1}{2} \frac{(e^{ax} + e^{-ax})}{a} + C_2\end{aligned}\tag{5}$$

At $y(0) = a$,

$$a = \frac{1}{2} \frac{2}{a} + C_2$$

Therefore,

$$C_2 = \frac{(a^2 - 1)}{a}$$

Then finally we get

$$y = \frac{1}{2} \left(\frac{e^{ax} + e^{-ax}}{a} \right) + \frac{a^2 - 1}{a}\tag{6}$$