Chapter1.1 #17 Drag Race

Solution Let us consider Alice and Kevin begin from a standing start in the race and they move foward with a constant acceleration. Let us denote A for Alice and K for Kevin then we can say that

$$a(t) = k, \text{ where } k \text{ is a constant.}$$

Then the *velocity* is

$$\begin{aligned} v(t) &= \int a(t)dt + C \\ &= \int kdt + C \\ &= kt + C \end{aligned}$$

The initial velocity is zero, Thus, we get

$$\begin{aligned} v(0) &= 0 \\ k(0) + C &= 0 \\ C &= 0 \end{aligned}$$

Therefore the velocity is

$$v(t) = kt$$

Let us denote that

$$\begin{aligned} s(t) &= \int v(t)dt + C_1 \\ &= \int ktdt + C_1 \\ &= \frac{kt^2}{2} + C_1 \end{aligned}$$

The constant value will be zero since our standing point is at the beginning of the race. Therefore,

$$s(t) = \frac{1}{2}kt^2$$

Now, the acceleration is the second derivative of the position function, or is the first derivative of the velocity function. Therefore,

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = k$$

Let D represent the total distance of the racec and T be the time then we get

$$D = s(t) = \frac{1}{2}aT^2$$

Solving this equation for T to get

$$T = \sqrt{\frac{2D}{a}}$$

For Kevin, when she has $\frac{1}{4}$ left to go she has already traveled $\frac{3}{4}$ of the entire distance D, so we can express this as follows:

$$\begin{aligned} t_{\frac{3}{4}} &= \sqrt{\frac{2\left(\frac{3}{4}\right)D}{a}} \\ &= \sqrt{\frac{3}{4}}\sqrt{\frac{2D}{a}} \\ &= \frac{\sqrt{3}}{2}T \end{aligned}$$

Thus, the total time for Kevin to complete the race is,

$$\begin{aligned} T &= t_{\frac{1}{4}} + t_{\frac{3}{4}} \\ T &= 3 + \frac{\sqrt{3}}{2}T \\ T\left(1 - \frac{\sqrt{3}}{2}\right) &= 3 \\ T\left(\frac{2-\sqrt{3}}{2}\right) &= 3 \\ T &= \frac{6}{2-\sqrt{3}} \\ T &= 6\left(2+\sqrt{3}\right) \end{aligned}$$

Thus,

$$T = 6\left(2+\sqrt{3}\right) \approx 22.3923 \text{ seconds for Kevin to finish the race.}$$

Now, we need to find the time how long it takes for Alice to finish the race in a similar manner. When Alice has $\frac{1}{3}$ to go her has already covered $\frac{2}{3}$ of the total distance of the race.

$$\begin{aligned} t_{\frac{2}{3}} &= \sqrt{\frac{2\left(\frac{2}{3}\right)D}{a}} \\ &= \sqrt{\frac{2}{3}}\sqrt{\frac{2D}{a}} \\ &= \sqrt{\frac{2}{3}}T \end{aligned}$$

Therefore, the total time for Alice to complete the race is

$$\begin{aligned} T &= t_{\frac{1}{3}} + t_{\frac{2}{3}} \\ T &= 4 + \sqrt{\frac{2}{3}}T \\ T\left(1 - \sqrt{\frac{2}{3}}\right) &= 4 \\ T\left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}}\right) &= 4 \\ T &= \frac{4\sqrt{4}}{\sqrt{3}-\sqrt{2}} \\ T &= 4\sqrt{3}\left(\sqrt{3}+\sqrt{2}\right) \\ T &= 4\left(3+\sqrt{6}\right) \approx 21.798 \end{aligned}$$

Thus, Alice wins by

$$\begin{aligned}T &= 6\left(2+\sqrt{3}\right) - 4\left(3+\sqrt{6}\right)\\ &= 6\sqrt{3} - 4\sqrt{6}\\ &\approx 0.594\end{aligned}$$

Thus, Alice wins by $\left(6\sqrt{3} - 4\sqrt{6}\right)$ seconds.

Chapter 1 B: Picard's Method

(a) Use Picard's method with $\psi_0(x) = 1$ to obtain the next four successive approximations of the solution to

$$y'(x) = y(x), \quad y(0) = 1$$

Show that these approximations are just the partial sums of the Maclaurin series for the actual solution e^x.

Solution Given that

$$y'(x) = y(x), \quad y(0) = 1 \tag{1}$$

Also given that

$$f(x, y) = y(x) \tag{2}$$

According to picards theorem, we have

$$\begin{aligned} \phi_{n+1}(x) &= y_0 + \int_{x_0}^{x} f(t, \phi_0(t))dt \\ &= 1 + \int_0^x 1dt \\ &= 1 + x \end{aligned} \tag{3}$$

$$\begin{aligned} \phi_2(x) &= y_0 + \int_0^x f(t, \phi_1(t))dt \\ &= 1 + \int_0^x f(t, (1+t))dt \\ &= 1 + x + \frac{x^2}{2} \end{aligned} \tag{4}$$

$$\begin{aligned} \phi_3(x) &= y_0 + \int_0^x f(t, \phi_2(t))dt \\ &= 1 + \int_0^x f\left(t, \left(1 + t + \frac{t^2}{2}\right)\right) dt \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \end{aligned} \tag{5}$$

$$\begin{aligned} \phi_4(x) &= y_0 + \int_0^x f(t, \phi_3(t))dt \\ &= 1 + \int_0^x f\left(t, \left(1 + t + \frac{t^2}{2}\right)\right) dt \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \end{aligned} \tag{6}$$

By observing the pattern as n goes, it is enough to say that

$$\phi_n(x) = 1 + x + x^2 + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!}$$

This is the partial sum of the Maclaurian series of e^x.

(b) Use Picard's method with $\psi + 0(x) = 0$ to obtain the next three successive approximations of the solution to the nonlinear problem

$$y'(x) = 3x - \left[y(x)^2\right], \qquad y(0) = 0$$

Graph these approximations for $0 \leq x \leq 1$.

Solution

$$\begin{aligned} y(x_1) &= y(x_0) + \int_{x_0}^{x} f(x,y)dx \\ &= f(x,y) = 2x - y^2 \\ y(0) &= 0 \end{aligned} \tag{1}$$

We assume that $x_0 = 0, x_1 = 0.25$ then

$$\begin{aligned} \phi_1(x) &= y(0) + \int_0^x \phi_0(x)dx = 0 + \int_0^x 2x - 0dx = x^2 \\ \phi_2(x) &= y(0) + \int_0^x \phi_1(x)dx = 0 + \int_0^x (2x - x^2)dx = x^2 - = fracx^3 3 \\ \phi_3(x) &= y(0) + \int_0^x \phi_2(x)dx = 0 + \int_0^x (2x - x^2 + \frac{x^3}{3})dx = x^2 - = fracx^3 3 + \frac{x^4}{12} \\ \phi_4(x) &= y(0) + \int_0^x \phi_3(x)dx = 0 + \int_0^x (2x - x^2 + \frac{x^3}{3} - \frac{x^4}{12})dx = x^2 - = fracx^3 3 + \frac{x^4}{12} - \frac{x^5}{60} \end{aligned} \tag{2}$$

If $x = 0.25, \phi_1(x) = 0.0625, \phi_2(x) = 0.05729, \phi_3(x) = 0.05761, \phi_4(x) = 0.0576009$. Thus, the better approximationn at $x = 0.25$ is 0.0576.

If $x = 0.5, \phi_1(x) = 0.25, \phi_2(x) = 0.2083, \phi_3(x) = 0.203125, \phi_4(x) = 0.21302$.

If $x = 0.75, \phi_1(x) = 0.5625, \phi_2(x) = 0.421875, \phi_3(x) = 0.44824, \phi_4(x) = 0.605419$.

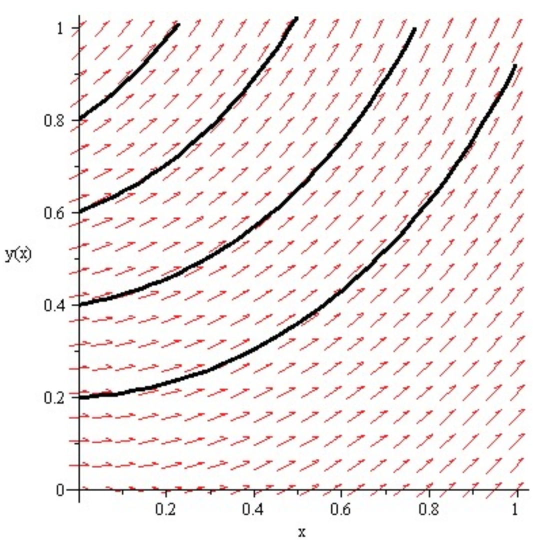

Figure 1:

(c) In Problem 29 in Exercises 1.2, we showed that the initial value problem

$$y'(x) = 3\left[y(x)\right]^{2/3}, y(2) = 0$$

does not have a unique solution. Show that Picard's method beginning with $\psi_0(x) = 0$ converges to the solution $y(x) = 0$, whereas Picard's method beginning with $\psi_0(x) = x - 2$ converges to the second solution $y(x) = (x-2)^3$.

Solution The given IVP can be written as

$$y'(t) = f(x, y(x)) \text{ where } f(x, y(x)) = 3(y(x))^{2/3} \tag{1}$$

The first iteration is given by

$$\begin{aligned} y_1(x) &= y(2) + \int_2^x f(u, y(2))du \\ &= 0 + \int_2^t f(u, 0)du \\ &= 0 + \int_0^x 0du = 0 \end{aligned} \tag{2}$$

If we repeat the procedure, then we get

$$y_2(x) = 0 \tag{3}$$

Thus, we get the trivial solution $y(x) = 0$ for the IVP.

Suppose that

$$\phi_0(x) = x - 2 \tag{4}$$

Then the first iteration is given by

$$\begin{aligned} \phi_1(x) &= \phi_0(x) + \int_2^x f\left(u, \phi_0(x)\right) du \\ &= x - 2 + \int_2^x f(u, x-2)du \\ &= x - 2 + \int_2^x 3(x-2)^{2/3}du = x - 23\frac{(x-2)^{5/3}}{5/3} \Big|_2^x \\ &= x - 2 + \frac{9}{5}(x-2)^{5/3} = (x-2)\left(1 + \frac{9}{5}(x-2)^{2/3}\right) \end{aligned} \tag{5}$$

The second iteration is given by

$$\begin{aligned} \phi_2(x) &= \phi_0(x) + \int_2^x \left(fu, \phi_1(x)\right) du \\ &= x - 2 + \int_2^x f\left(u, x-2\right) du \\ &= x - 2 + \int_2^x x - 2 + \frac{9}{5}(x-2)^{5/3}du = x - 2 + \left(\frac{(x-2)^2}{2} + \frac{9}{5}\frac{(x-2)^{8/3}}{8/3} \Big|_2^x\right) \\ &= x - 2 = \frac{(x-2)^2}{2} + \frac{27}{40}(x-2)^{8/3} \end{aligned} \tag{6}$$

The Phase Line

(a) The slopes in the direction field are all identical along horizontal lines.

Solution

(b) New solutions can be generated from old ones by time shifting [i.e., replacing $y(t)$ with $y(t-t_0)$.]

Solution

(c) Sketch the phase line for $y' = (y-1)(y-2)(y-3)$ and state the nature of its equilibria.

Solution

(d) Use the phase line for $y' = -(y-1)^{5/3}(y-2)^2(y-3)$ to predict the asymtotic behavior as $t \to \infty$ of the solution satisfying $y(0) = 2.1$.

Solution

(e) Sketch the phase line for $y' = y\sin y$ and state the nature of its equilibria.

Solution

(f) Sketch the phase lines for $y' = y\sin y + 0.1$ and $y' = y\sin y - 0.1$. Discuss the effect of the small perturbation ± 0.1 on the equilibria.

Solution

Chapter 4 A: Nonlinear Equations Solvable by First-Order Techniques

(a) **Solution**

The given equation can ben rewritten as

$$\begin{aligned} 2x\frac{\mathrm{d}w}{\mathrm{d}x} - w + \frac{1}{w} &= 0 \\ 2x\frac{\mathrm{d}w}{\mathrm{d}x} &= w - \frac{1}{w} \\ \frac{1}{w - 1/w}dw &= \frac{1}{2x}dx \end{aligned}$$

Therefore we get

$$\frac{1}{2}\int\left(\frac{1}{w^2-1}2w\right)dw = \frac{1}{2}\int\frac{1}{b}2dx$$

That is

$$\begin{aligned} \ln w^2 - 1 &= \ln|x| + C \\ w^2 - 1 &= x + C \\ w = \frac{\mathrm{d}y}{\mathrm{d}x} &= \sqrt{x+1} \end{aligned}$$

Thus, we obtain

$$y = \int (x+1)^{\frac{1}{2}}dx = \frac{1}{2\sqrt{x+1}} + C$$

(b) **Solution**

$$\begin{aligned} (2y)(w)\frac{\mathrm{d}w}{\mathrm{d}y} &= 1 + w^2 \\ 2y\frac{\mathrm{d}w}{\mathrm{d}y} &= \frac{1}{w} + w = \frac{w^2+1}{w} \end{aligned}$$

This becomes

$$\frac{1}{2}\int 1\big(w^2+1\big)\,(2w)dw = \frac{1}{2}\int\frac{1}{y}dy$$

Then we obtain

$$w = \sqrt{y-1} + C$$

$$\int\frac{1}{\sqrt{y-1}}dy = \int dx\sqrt{y-1} = 2x + C_1$$

Thus we obtain

$$y_1 = 1 + 4x^2 + C$$

Then we need to have another equation.

$$2yw\frac{\mathrm{d}w}{\mathrm{d}y} = -yw2\frac{\mathrm{d}w}{\mathrm{d}y} = -\frac{yw}{yw} = -1$$

$$\begin{aligned}\int dw &= -\frac{1}{2}\int dy\\ -2\int\frac{1}{y}dy &= \int dx\end{aligned}$$

This becomes

$$\begin{aligned}-2\ln|y| &= x + C\\ \ln|y| &= -\frac{1}{2}x + C\end{aligned}$$

Thus, we get

$$y_2 = Ce^{-\frac{x}{2}}$$

(c) **Suspended Cable.**

Solution The given differential equation is

$$y'' = \frac{1}{a}\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}; \quad y(0) = a, \quad y'(0) = 0, \quad \text{where } a(\neq 0) \text{ is a constant.} \tag{1}$$

Let $v = y'$ then we can say that $\frac{\mathrm{d}v}{\mathrm{d}x} = y''$

Now, we solve

$$\begin{aligned}\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{1}{a}\sqrt{1+v^2}\\ \int\frac{dv}{\sqrt{1+v^2}} &= a\int dx\end{aligned} \tag{2}$$

This becomes

$$\ln\sqrt{1+v^2} + v = ax \tag{3}$$

Thus, this is enough reason to say that

$$x'(0) = 0$$

This means $C = 0$.

Therefore

$$\begin{aligned}\ln\sqrt{1+v^2} + v &= ax\\ \sqrt{1+v^2} + v &= e^{ax}\\ \left(-\sqrt{1+v^2}\right) &= (v - e^{ax})^2\\ v &= \frac{e^{2ax}-1}{2e^{ax}} = \frac{1}{2}\left[e^{ax} - e^{-ax}\right]\end{aligned} \tag{4}$$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{2}\left(e^{ax} - e^{-ax}\right) \\ \int dy &= \frac{1}{2}\int \left(e^{ax} - e^{-ax}\right) dx \\ y &= \frac{1}{2}\frac{\left(e^{ax} + e^{-ax}\right)}{a} + C_2\end{aligned} \tag{5}$$

At $y(0) = a$,

$$a = \frac{1}{2}\frac{2}{a} + C_2$$

Therefore,

$$C_2 = \frac{\left(a^2 - 1\right)}{a}$$

Then finally we get

$$y = \frac{1}{2}\left(\frac{a^{ax} + e^{-ax}}{a}\right) + \frac{a^2 - 1}{a} \tag{6}$$

Chapter 4C: Simple Pendulum

(a) **Solution** By assuming the problem consisting of mass m suspended by a rod of length l and writing the derived nonlinear initial value problem,

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{l}\sin\theta = 0, \qquad \theta(0) = \alpha(0), \theta'(0) = 0$$

Let us determine the equation of motion for the pendulum and its period.

$$\begin{aligned}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\right) + \frac{g}{l}\sin\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right) &= 0\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 - \frac{g}{l}\cos\theta\right) &= 0\\ \frac{1}{2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 - \frac{g}{l}\cos theta &= c\\ \left(\frac{\theta}{t}\right)^2 - \frac{2g}{l}\cos\theta &= C\end{aligned} \tag{1}$$

By applying the initial conditions,

$$(\theta'(0))^2 - \frac{2g}{l}\cos theta(0) = C \tag{2}$$

Then we get

$$\begin{aligned}0 - \frac{2g}{l}\cos\alpha = C\\ C = -\frac{2g}{l}\cos\alpha\end{aligned} \tag{3}$$

By substituting C value into $\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 - \frac{2g}{l}\cos\theta = C$, then we get

$$\begin{aligned}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 - \frac{2g}{l}\cos\theta &= -\frac{2g}{l}\cos\alpha\\ \left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 &= \frac{2g}{l}(\cos\theta - \cos\alpha)\\ \frac{\mathrm{d}\theta}{\mathrm{d}t} &= -\sqrt{\frac{2g}{l}(\cos\theta - \cos\alpha)}\\ dt &= -\sqrt{\frac{1}{2g}}\frac{d\theta}{\sqrt{(\cos\theta - \cos\alpha)}}\end{aligned} \tag{4}$$

(b) **Solution** By using the formula, $\cos x = 1 - 2\sin^2 x$, we can say that

$$\begin{aligned}dt &= -\sqrt{\frac{1}{2g}}\frac{d\theta}{\sqrt{\left(\left(1 - 2\sin^2\theta\right) - \left(1 - 2\sin^2\alpha\right)\right)}}\\ dt &= -\frac{1}{2}\sqrt{\frac{l}{g}}\frac{d\theta}{\sqrt{sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}}\end{aligned} \tag{5}$$

(c) **Solution** Now, let us determine the elapsed time T, for the pendulum to fall from the angle $\theta = \alpha$ to the angle $\theta = \beta$ corresponding to $\phi = \frac{\pi}{2}$ to $\phi = \Phi$

$$\begin{aligned} T = \int_0^t dt &= -\frac{1}{2}\sqrt{\frac{l}{g}} \frac{d\theta}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}} \\ &= -\frac{1}{2}\sqrt{\frac{l}{g}} \int_{\frac{\pi}{2}}^{\phi} \frac{d\theta}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}} \\ &= -\sqrt{\frac{l}{g}} \int_{\frac{\pi}{2}}^{\phi} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}, \quad \text{where } k = \sin\frac{\alpha}{2} \end{aligned} \tag{6}$$

Thus, we get

$$-T = \sqrt{\frac{1}{g}} \int_{\frac{\pi}{2}}^{\phi} \frac{d\phi}{1 - k^2\sin^2\phi} \tag{7}$$

(d) **Solution** Let us determine the period P of the pendulum. The period is defined to be the time required for it swing from one extreme to other extreme and back. Therefore, from α to $-\alpha$ and from $-\alpha$ to α. Therefore, the total period is four equal parts. Thus, the period is given by

$$P = (4)\sqrt{\frac{l}{g}} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}} \tag{8}$$

Now, the integral

$$\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2\sin^2(\phi)}}$$

is called an elliptic integral of the first kind and denoted by

$$F\left(k, \frac{\pi}{2}\right) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}$$

(e) **Solution**

$$\begin{aligned} \frac{P}{4} + (-T) &= \sqrt{\frac{l}{g}} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2\sin^2(\phi)}} + \sqrt{\frac{l}{g}} \int_{\frac{\pi}{2}}^{\phi} \frac{d\phi}{\sqrt{1 - k^2\sin^2(\phi)}} \\ -T + \frac{P}{4} &= \sqrt{\frac{l}{g}} \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}} \\ -T + \frac{P}{4} &= \sqrt{\frac{l}{g}} F(k, \Phi) \end{aligned} \tag{9}$$

This is obtained by substituting

$$F(k, \Phi) = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}$$

Linearization of Nonlinear Problems

(a) **Solution**

$$\begin{aligned}
\theta'(t) &= -\sin\theta(t) \\
\theta''(0) &= -\sin(\theta(0)) \\
\theta'''(t) &= -\cos\theta(t)\theta'(t) \\
\theta^{(4)} &= \sin(\theta(t))(\theta'(t))^2 - \cos(\theta(t))\theta''(t)
\end{aligned} \tag{1}$$

$$\theta^{(4)}(0) = \sin\theta(0)(\theta'(0))^2 + \cos(\theta(0))\sin(\theta(0))$$

$$\theta^{(5)}(t) = \cos(\theta(t))(\theta'(t))^3 + 2\sin(\theta(t))\theta'(t)\theta''(t) + \frac{\mathrm{d}(\sin 2\theta(t))}{\mathrm{d}t}\left(\frac{1}{2}\right)$$

$$\begin{aligned}
\theta^{(5)}(t) &= \cos(\theta(t))(\theta'(t))^3 + 2\sin\theta(t)\theta'(t)\theta'\prime(t) + \cos 2\theta(t)\theta'(t) \\
\theta^{(5)}(0) &= \cos(\theta(0))(\theta'(0))^3 + 2\sin\theta(0)\theta'(0)\theta'\prime(0) + \cos 2\theta(0)\theta'(0)
\end{aligned}$$

(b) **Solution**

$$derivs\theta t + \theta = 0$$

Now solving for θ, then we get

$$\begin{aligned}
\theta &= C_1 e^t + C_2 e^{-t} \\
\theta(0) &= C_1 + C_2 = \frac{\pi}{2}
\end{aligned}$$

Since

$$\theta'(t) = C_1 e^t - C_2 e^{-t}$$

The constant values are

$$\begin{aligned}
C_1 - C_2 &= 0 \\
C_1 = C_2 &= \frac{\pi}{24}
\end{aligned}$$

Thus,

$$\theta(t) = \frac{\pi}{24}\left[e^t + e^{-t}\right]$$

(c) **Solution**

(d) **Solution**

(e) **Solution**

Chapter 4E: Convolution Method

(a) **Solution** The convolution of two functions g and f is the function $g \times f$ define by

$$(g \times f)(t) = \int_0^t g(t-u)f(u)du$$

And by Leibnitz's rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t h(t,u)du = \int_a^t \frac{\partial h}{\partial t}(t,u)du + h(t,t)$$

$$\begin{aligned}
(y \times f)'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[(y \times f)(t)\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t y(t-u)f(u)du \\
&= \int_0^t \frac{\partial}{\partial t}(y(t-u)f(u))du + y(t-t)f(t) \\
&= \int_0^t y'f(u)du + y(0)f(t) \\
&= (y' \times f)(t) + y(0)f(t)
\end{aligned} \tag{1}$$

Therefore we obtain

$$(y \times f)'(t) = (y' \times f)(t) + y(0)f(t)$$

Then, we can use this equation to get

$$\begin{aligned}
(y \times f)'' &= \frac{\mathrm{d}}{\mathrm{d}t}\left((y \times f)'\right)(t) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left[(y' \times f)(t) + y(0)f(t)\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}}\left((y' \times f)(t)\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(y(0)f(t)\right) \\
&= \left[(y'' \times f)(t) + y'(0)f(t)\right] + y(0)f'(t) \\
&= \left[(y'' \times f)(t) + y'(0)f(t)\right] + y(0)f'(t)
\end{aligned} \tag{2}$$

Therefore, we get

$$(y \times f)'' = \left[(y'' \times f)(t) + y'(0)f(t)\right] + y(0)f'(t)$$

(b) **Solution** Given the differential equation

$$ay'' + by' + cy = 0$$

Let $y_k(t)$ is the solution to the given differential equation. To show that $y_k \times f$ is the particular solution to $ay'' + by' + cy = f(t)$ satisfying $y(0) = y'(0) = 0$.

$$\begin{aligned}
(y_k \times f)'' &= (y_k'' \times f)(t) + y_k(0)f(t) + y_k(0)y'(t) \\
&= (y'' \times f)(t) + \frac{f(t)}{a} \\
(y_k \times f)'(t) &= (y_k \times f)'(t) + y_k(0)f(t) \\
&= (y_k' \times f)(t)
\end{aligned} \tag{3}$$

Therefore

$$
\begin{aligned}
a(y_k'' \times f)(t) + b(y_k \times f)'(t) + c(y_k \times f) &= a\left[(y_k'' \times f)(t) + \frac{f(t)}{a}\right] + b(y_k')(t) + c(y_k \times f) \\
&= f(t) + a(y_k'' \times f)(t) + b(y_k' \times f)(t) + c(y_k \times f)
\end{aligned}
$$

Then

$$
\begin{aligned}
L\left(a(y_k'' \times f)(t) + b(y_k' \times f)(t) + c(y_k \times f)\right) &= aL\left((y_k'' \times f)\right) \\
&= L(f)L\left(a_k'' + by_k' + cy_k\right) \\
&= L(f) \times 0 \\
&= 0
\end{aligned}
$$

By applying Inver Laplace Transform, we get

$$
a\left(y_k'' \times f\right)(t) + b(y_k \times f)'(t) + c(y_k \prime f) = 0
$$

Thus, $y_k \times f$ is the solution of the given differential equation. Moreover,

$$
y_k \times f\,\Big|_{t=0} = \int_0^0 f(v)y_k(0-v)dv = 0
$$

Hence showed.

(c) **Solution** Since $y_k(t)$ is a solution of the corresponding homogeneus equation and $(y_k \times f)$ is a particular solution, it is obvious to say that $(y_k \times f) + y_k(t)$ is a solution of $ay_k'' + by_k' cy_k = f(t)$. Also,

$$
\begin{aligned}
\left((y \times f) + y_k\right)(0) &= (y \times f)(0) + y_k(0) \\
&= 0 + y_0 \\
&= y_0
\end{aligned}
$$

$$
\begin{aligned}
\left((y \times f) + y_k\right)^k(0) &= (y \times f)'(0) + y_k'(0) \\
&= y_1
\end{aligned}
$$

By using **Uniqueness Theorem** of the solutions of ODE, $y_k(0)$ is the unique solution of the homogeneous equation with the initial conditions $y_k(0) = y_k,\ \ y_k'(0) = y_1$.

For the same equation $y_k(t)$ is the solution of the homogeneous equation with the initial condition $y_k(0) = 0,\ \ y_k'(0) = \frac{1}{a}$.

Thus, $(y \times f) + y_k$ must be the unique solution of $ay_k'' + by_k' + cy_k = f(t)$ with $y(0) = y_0$ and $y'(0) = y_1$.

(d) **Solution**

(a) **Finding the general solution of the homogeneous equation.**

$$
\begin{aligned}
y'' + y &= 0 \\
y &= c_1 \sin t + c_2 \cos t
\end{aligned}
$$

From part (b), $y_k = c_1 \sin t + c_2 \cos t$ satisfying $y_k(0) = 0, y_k'(0) = \frac{1}{a} = 1$.

That is

$$y_k(0) = 0 = 0 + c_2$$
$$y_k'(0) = 1 = c_1 - 0$$

Therefore $y_k = \sin t$ Now, by using part (c), we find y_k as $y(0) = 0$, $y'(0) = -1$.

$$y_k(0) = 0 = 0 + c_2$$
$$y_k'(0) = -1 = c_1 - 0$$

Therefore $y_k = -\sin t$. From part (c), the solution of the given equation is

$$\begin{aligned}
(y_k \times f)(t) + y_k(t) &= (\sin \times \tan)(t) - \sin t \\
&= \int_0^t \sin t - u u du - \sin t \\
&= \int_0^t \sin t \cos u - \sin u \cos t \tan u du - \sin t \\
&= \sin t \int_0^t \sin u du - \cos t \int_0^t \sec u - \cos u du - \sin t \\
&= \sin t \left[-\cos u\right]_0^t - \cos t \left[\ln \sec u + \tan u - \sin u_0^t\right] - \sin t \\
&= -\cos t \ln\left[\sec t + \tan t\right]
\end{aligned}$$

(b) The general solution to the homogeneous equation is

$$2y'' + y' - y = 0$$
$$y = c_1 e^{-t} + c_2 e^{\frac{t}{2}}$$

Then, $y_k = 0c_1 e^{-t} + c_2 e^{\frac{t}{2}}$ satisfying $y_k(0) = 0, y_k'(0) = \frac{1}{a} = \frac{1}{2}$.

$$y_k(0) = 0 = c_1 + c_2$$
$$y_k'(0) = \frac{1}{2} = c_1 + \frac{1}{2}c_2$$

So $c_1 = -\frac{1}{3}$, $c_2 = \frac{1}{3}$

$$y_k = \frac{1}{3}e^{\frac{t}{2}} - \frac{1}{3}e^{-t}$$

By using the part (c), $y_k = c_1 e^{-t} + c_2 e^{\frac{t}{w}}$ satisfying $y(0) = 1, y' = -1$.

$$y_k(0) = 1 = c_1 + c_2$$
$$y_1'(0) = -1 = c_1 + \frac{1}{2}c_2$$

$$y_k = \frac{4}{3}e^{\frac{t}{2}} - \frac{1}{3}e^{-t}$$

Therefore from the part (c) given equation is,

$$\begin{aligned}
(y_k \times f)(t) &= \int_0^t \left(\frac{1}{3}e^{\frac{(t-u)}{2}} - \frac{1}{3}e^{-t}\sin(u)\right) du + \frac{4}{3}e^{\frac{t}{2}} - \frac{1}{3}e^{-t} \\
&= -\frac{6}{39}e^{-t}\sin t + \frac{3}{13}e^{-t}\cos t - \frac{2}{3}e^{-t} + \frac{56}{39}e^{\frac{t}{2}}
\end{aligned}$$

(c) Now, the general solution to homogeneous equation is $2y'' - 2y' + y = 0$ is

$$y = c_1 e^t + c_2 t e^t$$

Therefore, from the part (b), $y_k = c_1 e^t + c_2 t e^t$ satisfying $y_k(0) = 0, y_0' = \frac{1}{a} = 1$.

$$\begin{aligned} y_k(0) = 0 &= c_1 \\ y_k'(0) = 1 &= c_1 + c_2 \end{aligned}$$

Thus,

$$c_1 = 0, c_2 = 1$$

Therefore $y_k = te^t$. Now by using the part (c), $c_1 e^t + c_2 t e^t$ satisfying $y(0) = 2, y'(0) = 0$ (given).

$$\begin{aligned} y_k(0) = 2 &= c_1 \\ y_k'(0) &= c_1 + c_2 \end{aligned}$$

So we get $c_1 = 2, c_2 = -2$.

Thus, from the part (c), the solution for the given equation is

$$\begin{aligned} (y_k \times f)(t) + y_k(t) &= \int_0^t (t-u)e^{t-u}\sqrt{ue^u}du + 2e^t - 2e^t \\ &= \frac{4}{15} t^{\frac{5}{2}} e^t - 2te^t + 2e^t \end{aligned}$$