Chapter 1 B: Picard's Method

(a) Use Picard's method with $\psi_0(x) = 1$ to obtain the next four successive approximations of the solution to

$$y'(x) = y(x), \quad y(0) = 1$$

Show that these approximations are just the partial sums of the Maclaurin series for the actual solution e^x.

Solution Given that

$$y'(x) = y(x), \quad y(0) = 1 \tag{1}$$

Also given that

$$f(x, y) = y(x) \tag{2}$$

According to picards theorem, we have

$$\begin{aligned} \phi_{n+1}(x) &= y_0 + \int_{x_0}^{x} f(t, \phi_0(t))dt \\ &= 1 + \int_0^x 1dt \\ &= 1 + x \end{aligned} \tag{3}$$

$$\begin{aligned} \phi_2(x) &= y_0 + \int_0^x f(t, \phi_1(t))dt \\ &= 1 + \int_0^x f(t, (1+t))dt \\ &= 1 + x + \frac{x^2}{2} \end{aligned} \tag{4}$$

$$\begin{aligned} \phi_3(x) &= y_0 + \int_0^x f(t, \phi_2(t))dt \\ &= 1 + \int_0^x f\left(t, \left(1 + t + \frac{t^2}{2}\right)\right) dt \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \end{aligned} \tag{5}$$

$$\begin{aligned} \phi_4(x) &= y_0 + \int_0^x f(t, \phi_3(t))dt \\ &= 1 + \int_0^x f\left(t, \left(1 + t + \frac{t^2}{2}\right)\right) dt \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \end{aligned} \tag{6}$$

By observing the pattern as n goes, it is enough to say that

$$\phi_n(x) = 1 + x + x^2 + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!}$$

This is the partial sum of the Maclaurian series of e^x.

(b) Use Picard's method with $\psi + 0(x) = 0$ to obtain the next three successive approximations of the solution to the nonlinear problem

$$y'(x) = 3x - \left[y(x)^2\right], \qquad y(0) = 0$$

Graph these approximations for $0 \leq x \leq 1$.

Solution

$$\begin{aligned} y(x_1) &= y(x_0) + \int_{x_0}^{x} f(x,y)dx \\ &= f(x,y) = 2x - y^2 \\ y(0) &= 0 \end{aligned} \tag{1}$$

We assume that $x_0 = 0, x_1 = 0.25$ then

$$\begin{aligned} \phi_1(x) &= y(0) + \int_0^x \phi_0(x)dx = 0 + \int_0^x 2x - 0dx = x^2 \\ \phi_2(x) &= y(0) + \int_0^x \phi_1(x)dx = 0 + \int_0^x (2x - x^2)dx = x^2 - = fracx^3 3 \\ \phi_3(x) &= y(0) + \int_0^x \phi_2(x)dx = 0 + \int_0^x (2x - x^2 + \frac{x^3}{3})dx = x^2 - = fracx^3 3 + \frac{x^4}{12} \\ \phi_4(x) &= y(0) + \int_0^x \phi_3(x)dx = 0 + \int_0^x (2x - x^2 + \frac{x^3}{3} - \frac{x^4}{12})dx = x^2 - = fracx^3 3 + \frac{x^4}{12} - \frac{x^5}{60} \end{aligned} \tag{2}$$

If $x = 0.25, \phi_1(x) = 0.0625, \phi_2(x) = 0.05729, \phi_3(x) = 0.05761, \phi_4(x) = 0.0576009$. Thus, the better approximationn at $x = 0.25$ is 0.0576.

If $x = 0.5, \phi_1(x) = 0.25, \phi_2(x) = 0.2083, \phi_3(x) = 0.203125, \phi_4(x) = 0.21302$.

If $x = 0.75, \phi_1(x) = 0.5625, \phi_2(x) = 0.421875, \phi_3(x) = 0.44824, \phi_4(x) = 0.605419$.

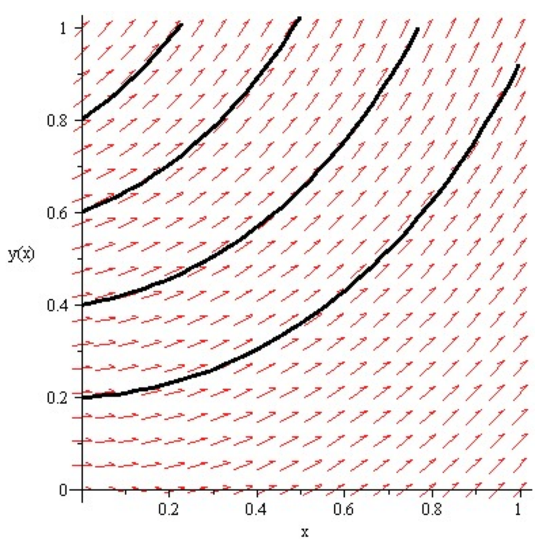

Figure 1:

(c) In Problem 29 in Exercises 1.2, we showed that the initial value problem

$$y'(x) = 3\left[y(x)\right]^{2/3}, y(2) = 0$$

does not have a unique solution. Show that Picard's method beginning with $\psi_0(x) = 0$ converges to the solution $y(x) = 0$, whereas Picard's method beginning with $\psi_0(x) = x - 2$ converges to the second solution $y(x) = (x-2)^3$.

Solution The given IVP can be written as

$$y'(t) = f(x, y(x)) \text{ where } f(x, y(x)) = 3(y(x))^{2/3} \tag{1}$$

The first iteration is given by

$$\begin{aligned} y_1(x) &= y(2) + \int_2^x f(u, y(2))du \\ &= 0 + \int_2^t f(u, 0)du \\ &= 0 + \int_0^x 0du = 0 \end{aligned} \tag{2}$$

If we repeat the procedure, then we get

$$y_2(x) = 0 \tag{3}$$

Thus, we get the trivial solution $y(x) = 0$ for the IVP.

Suppose that

$$\phi_0(x) = x - 2 \tag{4}$$

Then the first iteration is given by

$$\begin{aligned} \phi_1(x) &= \phi_0(x) + \int_2^x f\left(u, \phi_0(x)\right)du \\ &= x - 2 + \int_2^x f(u, x-2)du \\ &= x - 2 + \int_2^x 3(x-2)^{2/3}du = x - 23\frac{(x-2)^{5/3}}{5/3}\Big|_2^x \\ &= x - 2 + \frac{9}{5}(x-2)^{5/3} = (x-2)\left(1 + \frac{9}{5}(x-2)^{2/3}\right) \end{aligned} \tag{5}$$

The second iteration is given by

$$\begin{aligned} \phi_2(x) &= \phi_0(x) + \int_2^x \left(fu, \phi_1(x)\right)du \\ &= x - 2 + \int_2^x f\left(u, x-2\right)du \\ &= x - 2 + \int_2^x x - 2 + \frac{9}{5}(x-2)^{5/3}du = x - 2 + \left(\frac{(x-2)^2}{2} + \frac{9}{5}\frac{(x-2)^{8/3}}{8/3}\Big|_2^x\right) \\ &= x - 2 = \frac{(x-2)^2}{2} + \frac{27}{40}(x-2)^{8/3} \end{aligned} \tag{6}$$

The Phase Line

(a) The slopes in the direction field are all identical along horizontal lines.

(b) New solutions can be generated from old ones by time shifting [i.e., replacing $y(t)$ with $y(t-t_0)$.]

(c) Sketch the phase line for $y' = (y-1)(y-2)(y-3)$ and state the nature of its equilibria.

Solution

(d) Use the phase line for $y' = -(y-1)^{5/3}(y-2)^2(y-3)$ to predict the asymtotic behavior as $t \to \infty$ of the solution satisfying $y(0) = 2.1$.

Solution

(e) Sketch the phase line for $y' = y\sin y$ and state the nature of its equilibria.

Solution

(f) Sketch the phase lines for $y' = y\sin y + 0.1$ and $y' = y\sin y - 0.1$. Discuss the effect of the small perturbation ± 0.1 on the equilibria.

Solution