

## Chapter 1 B: Picard's Method

- (a) Use Picard's method with  $\psi_0(x) = 1$  to obtain the next four successive approximations of the solution to

$$y'(x) = y(x), \quad y(0) = 1$$

Show that these approximations are just the partial sums of the Maclaurin series for the actual solution  $e^x$ .

**Solution** Given that

$$y'(x) = y(x), \quad y(0) = 1 \tag{1}$$

Also given that

$$f(x, y) = y(x) \tag{2}$$

According to Picard's theorem, we have

$$\begin{aligned} \phi_{n+1}(x) &= y_0 + \int_{x_0}^x f(t, \phi_0(t)) dt \\ &= 1 + \int_0^x 1 dt \\ &= 1 + x \end{aligned} \tag{3}$$

$$\begin{aligned} \phi_2(x) &= y_0 + \int_0^x f(t, \phi_1(t)) dt \\ &= 1 + \int_0^x f(t, (1+t)) dt \\ &= 1 + x + \frac{x^2}{2} \end{aligned} \tag{4}$$

$$\begin{aligned} \phi_3(x) &= y_0 + \int_0^x f(t, \phi_2(t)) dt \\ &= 1 + \int_0^x f\left(t, \left(1+t+\frac{t^2}{2}\right)\right) dt \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \end{aligned} \tag{5}$$

$$\begin{aligned} \phi_4(x) &= y_0 + \int_0^x f(t, \phi_3(t)) dt \\ &= 1 + \int_0^x f\left(t, \left(1+t+\frac{t^2}{2}\right)\right) dt \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \end{aligned} \tag{6}$$

By observing the pattern as  $n$  goes, it is enough to say that

$$\phi_n(x) = 1 + x + x^2 + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!}$$

This is the partial sum of the Maclaurian series of  $e^x$ .

- (b) Use Picard's method with  $\psi + 0(x) = 0$  to obtain the next three successive approximations of the solution to the nonlinear problem

$$y'(x) = 3x - [y(x)^2], \quad y(0) = 0$$

Graph these approximations for  $0 \leq x \leq 1$ .

**Solution**

$$\begin{aligned} y(x_1) &= y(x_0) + \int_{x_0}^x f(x, y) dx \\ &= f(x, y) = 2x - y^2 \\ y(0) &= 0 \end{aligned} \tag{1}$$

We assume that  $x_0 = 0, x_1 = 0.25$  then

$$\begin{aligned} \phi_1(x) &= y(0) + \int_0^x \phi_0(x) dx = 0 + \int_0^x 2x - 0 dx = x^2 \\ \phi_2(x) &= y(0) + \int_0^x \phi_1(x) dx = 0 + \int_0^x (2x - x^2) dx = x^2 - \frac{x^3}{3} \\ \phi_3(x) &= y(0) + \int_0^x \phi_2(x) dx = 0 + \int_0^x (2x - x^2 + \frac{x^3}{3}) dx = x^2 - \frac{x^3}{3} + \frac{x^4}{12} \\ \phi_4(x) &= y(0) + \int_0^x \phi_3(x) dx = 0 + \int_0^x (2x - x^2 + \frac{x^3}{3} - \frac{x^4}{12}) dx = x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{60} \end{aligned} \tag{2}$$

If  $x = 0.25, \phi_1(x) = 0.0625, \phi_2(x) = 0.05729, \phi_3(x) = 0.05761, \phi_4(x) = 0.0576009$ . Thus, the better approximation at  $x = 0.25$  is 0.0576.

If  $x = 0.5, \phi_1(x) = 0.25, \phi_2(x) = 0.2083, \phi_3(x) = 0.203125, \phi_4(x) = 0.21302$ .

If  $x = 0.75, \phi_1(x) = 0.5625, \phi_2(x) = 0.421875, \phi_3(x) = 0.44824, \phi_4(x) = 0.605419$ .

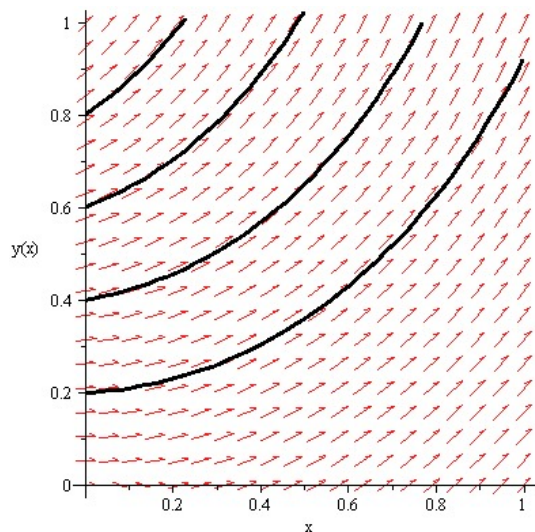


Figure 1:

(c) In Problem 29 in Exercises 1.2, we showed that the initial value problem

$$y'(x) = 3[y(x)]^{2/3}, y(2) = 0$$

does not have a unique solution. Show that Picard's method beginning with  $\psi_0(x) = 0$  converges to the solution  $y(x) = 0$ , whereas Picard's method beginning with  $\psi_0(x) = x - 2$  converges to the second solution  $y(x) = (x - 2)^3$ .

**Solution** The given IVP can be written as

$$y'(t) = f(x, y(x)) \text{ where } f(x, y(x)) = 3(y(x))^{2/3} \quad (1)$$

The first iteration is given by

$$\begin{aligned} y_1(x) &= y(2) + \int_2^x f(u, y(2)) du \\ &= 0 + \int_2^x f(u, 0) du \\ &= 0 + \int_2^x 0 du = 0 \end{aligned} \quad (2)$$

If we repeat the procedure, then we get

$$y_2(x) = 0 \quad (3)$$

Thus, we get the trivial solution  $y(x) = 0$  for the IVP.

Suppose that

$$\phi_0(x) = x - 2 \quad (4)$$

Then the first iteration is given by

$$\begin{aligned} \phi_1(x) &= \phi_0(x) + \int_2^x f(u, \phi_0(x)) du \\ &= x - 2 + \int_2^x f(u, x - 2) du \\ &= x - 2 + \int_2^x 3(x - 2)^{2/3} du = x - 2 + 3 \frac{(x - 2)^{5/3}}{5/3} \Big|_2^x \\ &= x - 2 + \frac{9}{5}(x - 2)^{5/3} = (x - 2) \left( 1 + \frac{9}{5}(x - 2)^{2/3} \right) \end{aligned} \quad (5)$$

The second iteration is given by

$$\begin{aligned} \phi_2(x) &= \phi_0(x) + \int_2^x (f(u, \phi_1(x))) du \\ &= x - 2 + \int_2^x f(u, x - 2) du \\ &= x - 2 + \int_2^x x - 2 + \frac{9}{5}(x - 2)^{5/3} du = x - 2 + \left( \frac{(x - 2)^2}{2} + \frac{9}{5} \frac{(x - 2)^{8/3}}{8/3} \Big|_2^x \right) \\ &= x - 2 = \frac{(x - 2)^2}{2} + \frac{27}{40}(x - 2)^{8/3} \end{aligned} \quad (6)$$

## The Phase Line

- (a) The slopes in the direction field are all identical along horizontal lines.

**Solution**

- (b) New solutions can be generated from old ones by time shifting [i.e., replacing  $y(t)$  with  $y(t - t_0)$ .]

**Solution**

- (c) Sketch the phase line for  $y' = (y - 1)(y - 2)(y - 3)$  and state the nature of its equilibria.

**Solution**

- (d) Use the phase line for  $y' = -(y - 1)^{5/3}(y - 2)^2(y - 3)$  to predict the asymptotic behavior as  $t \rightarrow \infty$  of the solution satisfying  $y(0) = 2.1$ .

**Solution**

- (e) Sketch the phase line for  $y' = y \sin y$  and state the nature of its equilibria.

**Solution**

- (f) Sketch the phase lines for  $y' = y \sin y + 0.1$  and  $y' = y \sin y - 0.1$ . Discuss the effect of the small perturbation  $\pm 0.1$  on the equilibria.

**Solution**

## Chapter 4 A: Nonlinear Equations Solvable by First-Order Techniques

### (a) Solution

The given equation can be rewritten as

$$\begin{aligned} 2x \frac{dw}{dx} - w + \frac{1}{w} &= 0 \\ 2x \frac{dw}{dx} &= w - \frac{1}{w} \\ \frac{1}{w - 1/w} dw &= \frac{1}{2x} dx \end{aligned}$$

Therefore we get

$$\frac{1}{2} \int \left( \frac{1}{w^2 - 1} 2w \right) dw = \frac{1}{2} \int \frac{1}{x} dx$$

That is

$$\begin{aligned} \ln w^2 - 1 &= \ln |x| + C \\ w^2 - 1 &= x + C \\ w &= \frac{dy}{dx} = \sqrt{x+1} \end{aligned}$$

Thus, we obtain

$$y = \int (x+1)^{\frac{1}{2}} dx = \frac{2}{3} (x+1)^{\frac{3}{2}} + C$$

### (b) Solution

$$\begin{aligned} (2y)(w) \frac{dw}{dy} &= 1 + w^2 \\ 2y \frac{dw}{dy} &= \frac{1}{w} + w = \frac{w^2 + 1}{w} \end{aligned}$$

This becomes

$$\frac{1}{2} \int \frac{1}{w^2 + 1} (2w) dw = \frac{1}{2} \int \frac{1}{y} dy$$

Then we obtain

$$w = \sqrt{y-1} + C$$

$$\int \frac{1}{\sqrt{y-1}} dy = \int dx \sqrt{y-1} = 2x + C_1$$

Thus we obtain

$$y_1 = 1 + 4x^2 + C$$

Then we need to have another equation.

$$2yw \frac{dw}{dy} = -yw2 \frac{dw}{dy} = -\frac{yw}{yw} = -1$$

$$\begin{aligned} \int dw &= -\frac{1}{2} \int \frac{1}{y} dy \\ -2 \int \frac{1}{y} dy &= \int dx \end{aligned}$$

This becomes

$$\begin{aligned} -2 \ln |y| &= x + C \\ \ln |y| &= -\frac{1}{2}x + C \end{aligned}$$

Thus, we get

$$y_2 = Ce^{-\frac{x}{2}}$$

(c) **Suspended Cable.**

**Solution** The given differential equation is

$$y'' = \frac{1}{a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}; \quad y(0) = a, \quad y'(0) = 0, \quad \text{where } a(\neq 0) \text{ is a constant.} \quad (1)$$

Let  $v = y'$  then we can say that  $\frac{dv}{dx} = y''$

Now, we solve

$$\begin{aligned} \frac{dv}{dx} &= \frac{1}{a} \sqrt{1 + v^2} \\ \int \frac{dv}{\sqrt{1 + v^2}} &= a \int dx \end{aligned} \quad (2)$$

This becomes

$$\ln \sqrt{1 + v^2} + v = ax \quad (3)$$

Thus, this is enough reason to say that

$$x'(0) = 0$$

This means  $C = 0$ .

Therefore

$$\begin{aligned} \ln \sqrt{1 + v^2} + v &= ax \\ \sqrt{1 + v^2} + v &= e^{ax} \\ (-\sqrt{1 + v^2}) &= (v - e^{ax})^2 \\ v &= \frac{e^{2ax} - 1}{2e^{ax}} = \frac{1}{2} [e^{ax} - e^{-ax}] \end{aligned} \quad (4)$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2} (e^{ax} - e^{-ax}) \\ \int dy &= \frac{1}{2} \int (e^{ax} - e^{-ax}) dx \\ y &= \frac{1}{2} \frac{(e^{ax} + e^{-ax})}{a} + C_2\end{aligned}\tag{5}$$

At  $y(0) = a$ ,

$$a = \frac{1}{2} \frac{2}{a} + C_2$$

Therefore,

$$C_2 = \frac{(a^2 - 1)}{a}$$

Then finally we get

$$y = \frac{1}{2} \left( \frac{e^{ax} + e^{-ax}}{a} \right) + \frac{a^2 - 1}{a}\tag{6}$$

## Chapter 4C: Simple Pendulum

- (a) **Solution** By assuming the problem consisting of mass  $m$  suspended by a rod of length  $l$  and writing the derived nonlinear initial value problem,

$$\frac{d^2\theta}{dt^2} + \frac{g}{l} \sin \theta = 0, \quad \theta(0) = \alpha(0), \theta'(0) = 0$$

Let us determine the equation of motion for the pendulum and its period.

$$\begin{aligned} \left(\frac{d\theta}{dt}\right) \left(\frac{d^2\theta}{dt^2}\right) + \frac{g}{l} \sin \theta \left(\frac{d\theta}{dt}\right) &= 0 \\ \frac{d}{dt} \left( \frac{1}{2} \left(\frac{d\theta}{dt}\right)^2 - \frac{g}{l} \cos \theta \right) &= 0 \\ \frac{1}{2} \left(\frac{d\theta}{dt}\right)^2 - \frac{g}{l} \cos \theta &= c \\ \left(\frac{\theta}{t}\right)^2 - \frac{2g}{l} \cos \theta &= C \end{aligned} \tag{1}$$

By applying the initial conditions,

$$(\theta'(0))^2 - \frac{2g}{l} \cos \theta(0) = C \tag{2}$$

Then we get

$$\begin{aligned} 0 - \frac{2g}{l} \cos \alpha &= C \\ C &= -\frac{2g}{l} \cos \alpha \end{aligned} \tag{3}$$

By substituting  $C$  value into  $\left(\frac{d\theta}{dt}\right)^2 - \frac{2g}{l} \cos \theta = C$ , then we get

$$\begin{aligned} \left(\frac{d\theta}{dt}\right)^2 - \frac{2g}{l} \cos \theta &= -\frac{2g}{l} \cos \alpha \\ \left(\frac{d\theta}{dt}\right)^2 &= \frac{2g}{l} (\cos \theta - \cos \alpha) \\ \frac{d\theta}{dt} &= -\sqrt{\frac{2g}{l} (\cos \theta - \cos \alpha)} \\ dt &= -\sqrt{\frac{1}{2g}} \frac{d\theta}{\sqrt{(\cos \theta - \cos \alpha)}} \end{aligned} \tag{4}$$

- (b) **Solution** By using the formula,  $\cos x = 1 - 2 \sin^2 x$ , we can say that

$$\begin{aligned} dt &= -\sqrt{\frac{1}{2g}} \frac{d\theta}{\sqrt{((1 - 2 \sin^2 \theta) - (1 - 2 \sin^2 \alpha))}} \\ dt &= -\frac{1}{2} \sqrt{\frac{l}{g}} \frac{d\theta}{\sqrt{\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\theta}{2}}} \end{aligned} \tag{5}$$



- (c) **Solution** Now, let us determine the elapsed time  $T$ , for the pendulum to fall from the angle  $\theta = \alpha$  to the angle  $\theta = \beta$  corresponding to  $\phi = \frac{\pi}{2}$  to  $\phi = \Phi$

$$\begin{aligned}
 T &= \int_0^t dt = -\frac{1}{2} \sqrt{\frac{l}{g}} \frac{d\theta}{\sqrt{\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\theta}{2}}} \\
 &= -\frac{1}{2} \sqrt{\frac{l}{g}} \int_{\frac{\pi}{2}}^{\phi} \frac{d\theta}{\sqrt{\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\theta}{2}}} \\
 &= -\sqrt{\frac{l}{g}} \int_{\frac{\pi}{2}}^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \quad \text{where } k = \sin \frac{\alpha}{2}
 \end{aligned} \tag{6}$$

Thus, we get

$$-T = \sqrt{\frac{l}{g}} \int_{\frac{\pi}{2}}^{\phi} \frac{d\phi}{1 - k^2 \sin^2 \phi} \tag{7}$$

- (d) **Solution** Let us determine the period  $P$  of the pendulum. The period is defined to be the time required for it swing from one extreme to other extreme and back. Therefore, from  $\alpha$  to  $-\alpha$  and from  $-\alpha$  to  $\alpha$ . Therefore, the total period is four equal parts. Thus, the period is given by

$$P = (4) \sqrt{\frac{l}{g}} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \tag{8}$$

Now, the integral

$$\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2(\phi)}}$$

is called an elliptic integral of the first kind and denoted by

$$F\left(k, \frac{\pi}{2}\right) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}$$

- (e) **Solution**

$$\begin{aligned}
 \frac{P}{4} + (-T) &= \sqrt{\frac{l}{g}} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2(\phi)}} + \sqrt{\frac{l}{g}} \int_{\frac{\pi}{2}}^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2(\phi)}} \\
 -T + \frac{P}{4} &= \sqrt{\frac{l}{g}} \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \\
 -T + \frac{P}{4} &= \sqrt{\frac{l}{g}} F(k, \Phi)
 \end{aligned} \tag{9}$$

This is obtained by substituting

$$F(k, \Phi) = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}$$

## Linearization of Nonlinear Problems

(a) **Solution**

$$\begin{aligned}
 \theta'(t) &= -\sin \theta(t) \\
 \theta''(0) &= -\sin(\theta(0)) \\
 \theta'''(t) &= -\cos \theta(t)\theta'(t) \\
 \theta^{(4)} &= \sin(\theta(t))(\theta'(t))^2 - \cos(\theta(t))\theta''(t)
 \end{aligned} \tag{1}$$

$$\theta^{(4)}(0) = \sin \theta(0)(\theta'(0))^2 + \cos(\theta(0)) \sin(\theta(0))$$

$$\theta^{(5)}(t) = \cos(\theta(t))(\theta'(t))^3 + 2 \sin(\theta(t))\theta'(t)\theta''(t) + \frac{d(\sin 2\theta(t))}{dt} \left(\frac{1}{2}\right)$$

$$\begin{aligned}
 \theta^{(5)}(t) &= \cos(\theta(t))(\theta'(t))^3 + 2 \sin \theta(t)\theta'(t)\theta''(t) + \cos 2\theta(t)\theta'(t) \\
 \theta^{(5)}(0) &= \cos(\theta(0))(\theta'(0))^3 + 2 \sin \theta(0)\theta'(0)\theta''(0) + \cos 2\theta(0)\theta'(0)
 \end{aligned}$$

(b) **Solution**

$$derivs\theta t + \theta = 0$$

Now solving for  $\theta$ , then we get

$$\begin{aligned}
 \theta &= C_1 e^t + C_2 e^{-t} \\
 \theta(0) &= C_1 + C_2 = \frac{\pi}{2}
 \end{aligned}$$

Since

$$\theta'(t) = C_1 e^t - C_2 e^{-t}$$

The constant values are

$$\begin{aligned}
 C_1 - C_2 &= 0 \\
 C_1 = C_2 &= \frac{\pi}{24}
 \end{aligned}$$

Thus,

$$\theta(t) = \frac{\pi}{24} [e^t + e^{-t}]$$

(c) **Solution**

(d) **Solution**

(e) **Solution**