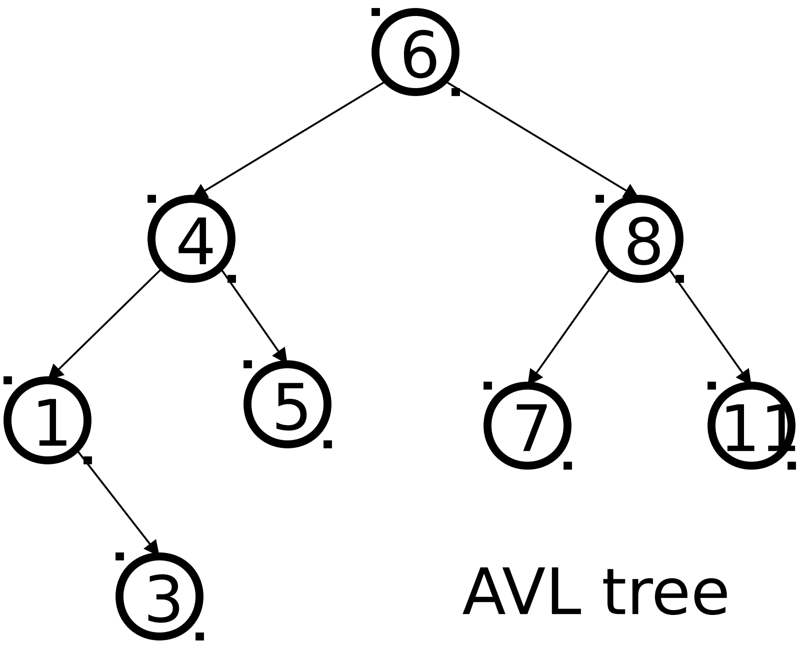
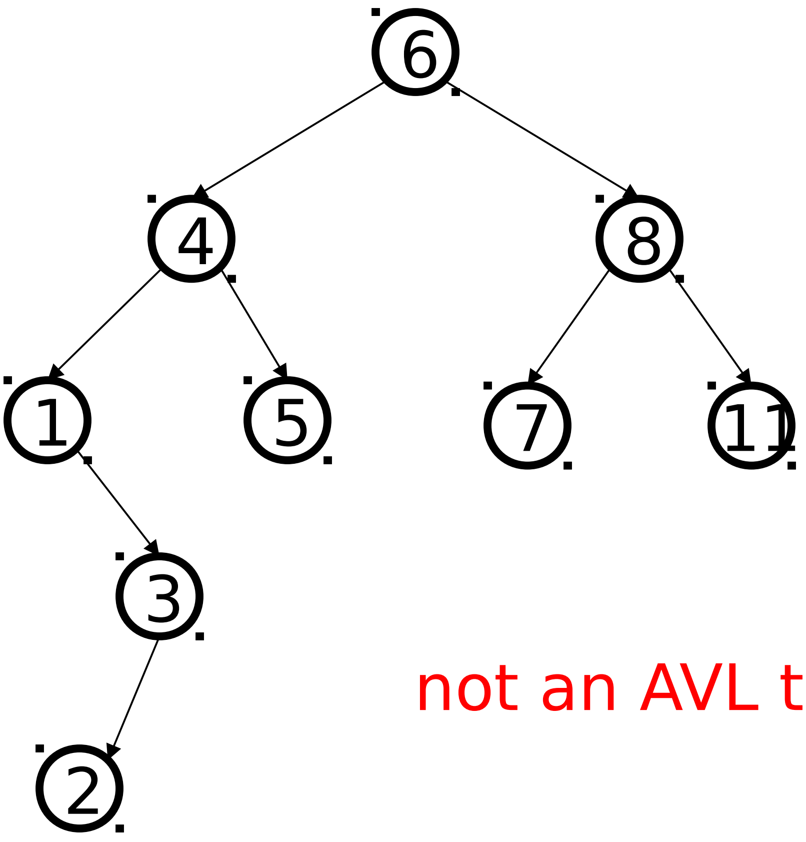
AVL Trees

- Motivation: we want to guarantee O(log n) running t
on the find/insert/remove operations.
- Idea: keep the tree balanced after each operation.

- Solution: AVL (Adelson-Velski and Landis) trees.
- AVL tree property: for every node in the tree, the height c the left and right subtrees dif by at most 1.

2/EXP/TRE/VIS/1/A





AVL trees: find, insert

- AVL tree find is the same as BST find.
- AVL insert: same as BST inse except that we might have to "fix" the AVL tree after an insert.

- These operations will take tin O(d), where d is the depth of the node being found/inserte
- What is the maximum height an n-node AVL tree?

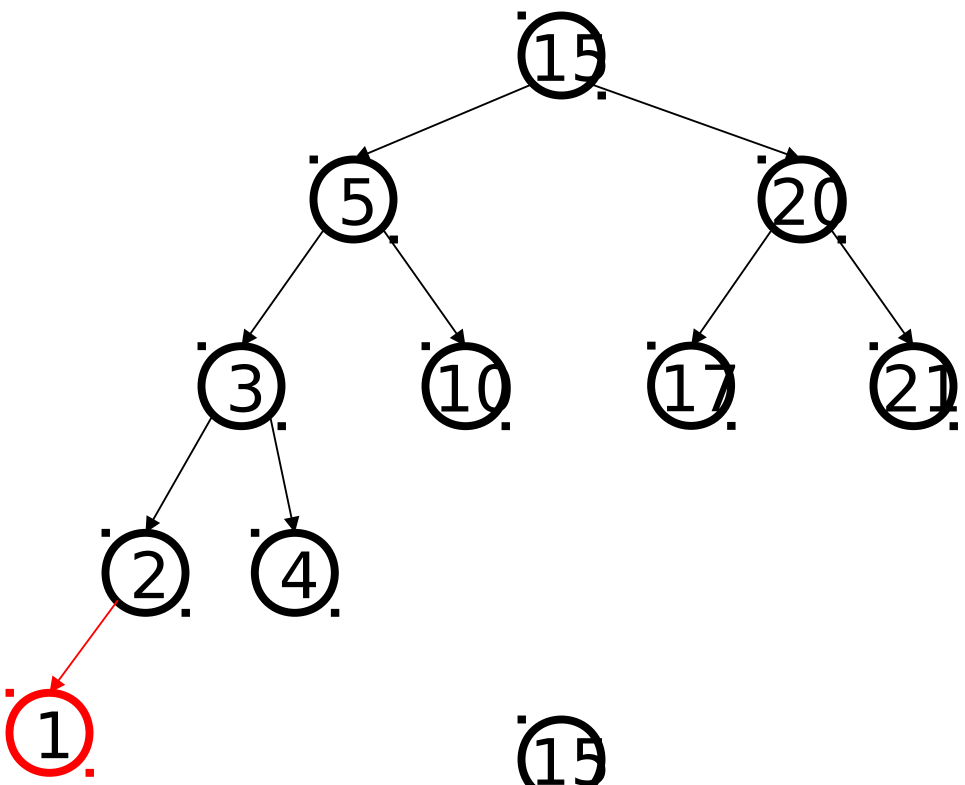
AVL tree insert

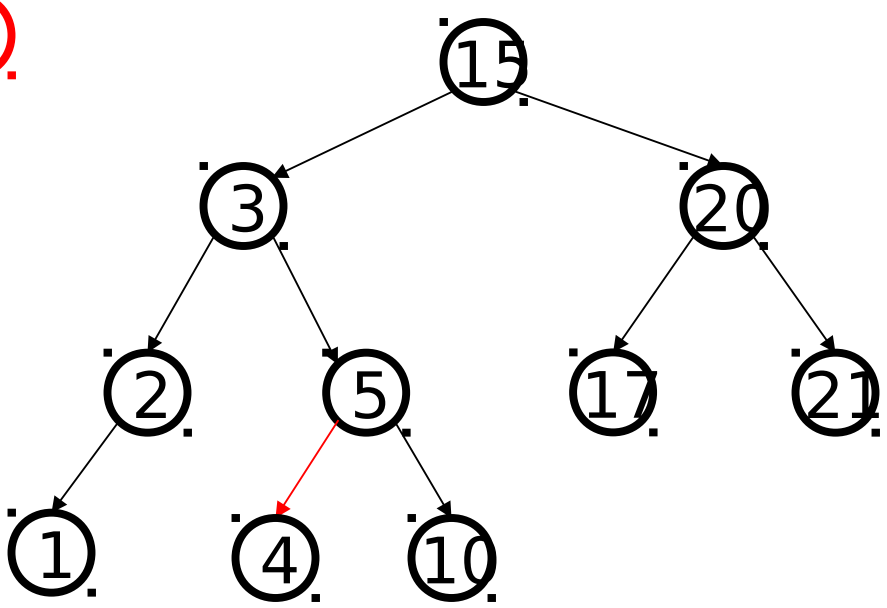
- Let x be the deepest node wh
  an imbalance occurs.
- Four cases to consider. The
  insertion is in the
  1. left subtree of the left child of x.
  2. right subtree of the left child o x.
  3. left subtree of the right child o x.
  4. right subtree of the right child x.

Idea: Cases 1 & 4 are solved by
single rotation.

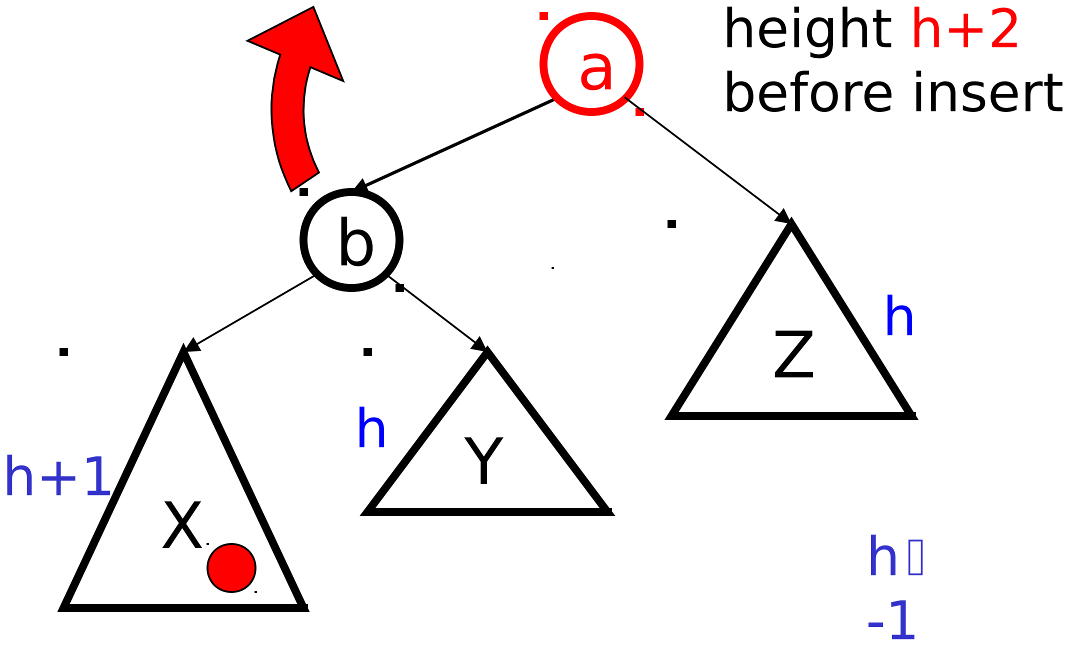
Cases 2 & 3 are solved by a dou
rotation.

Single rotation exampl

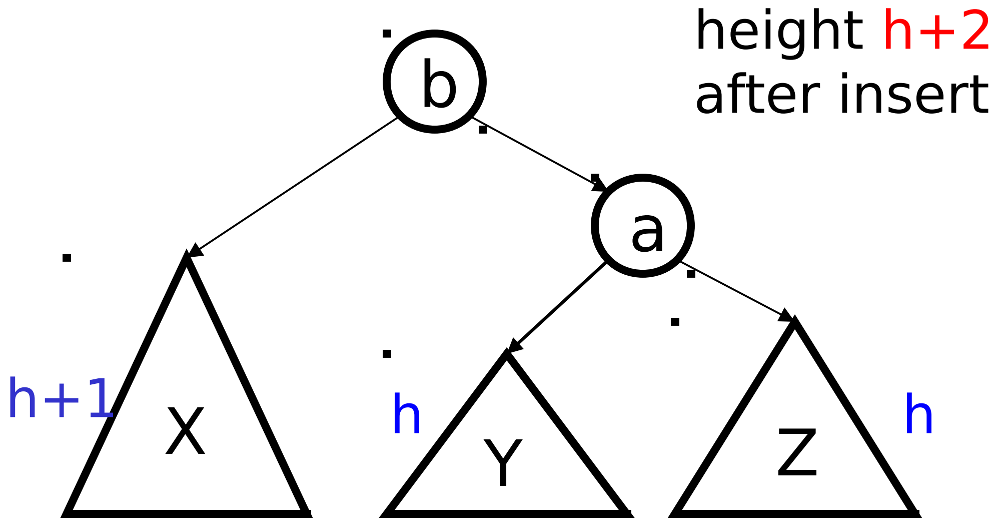




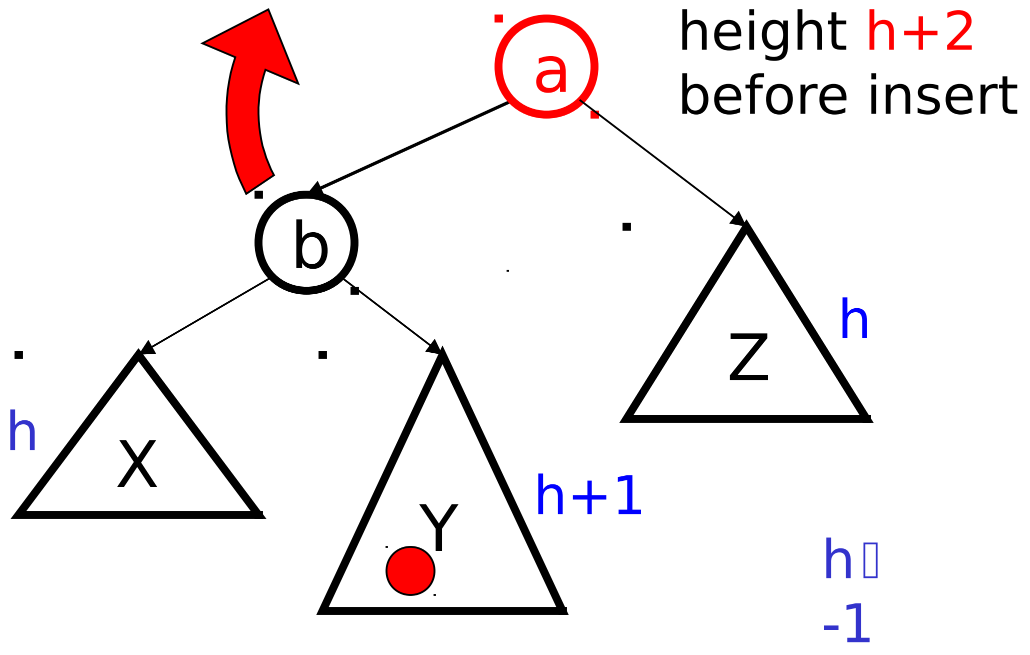
Single rotation in gene

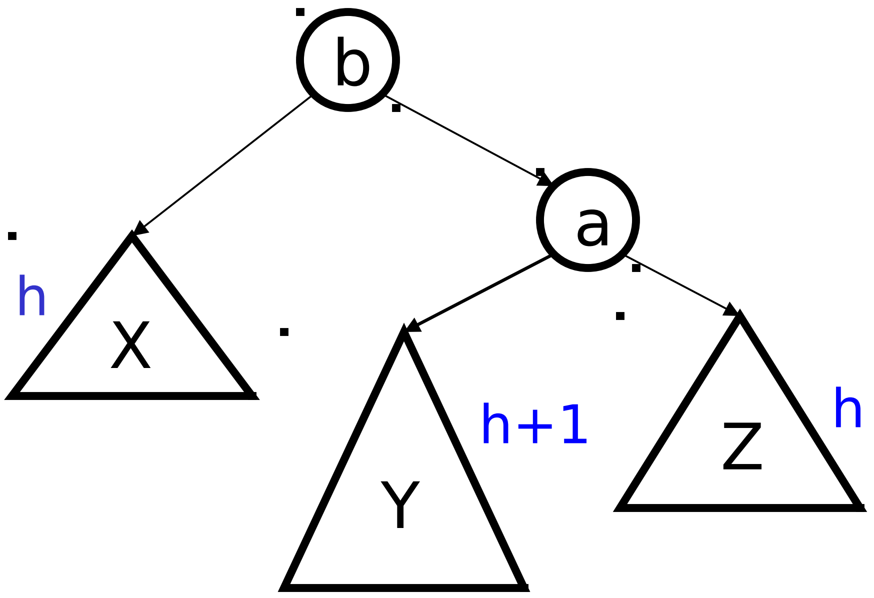


X < b < Y < a < Z



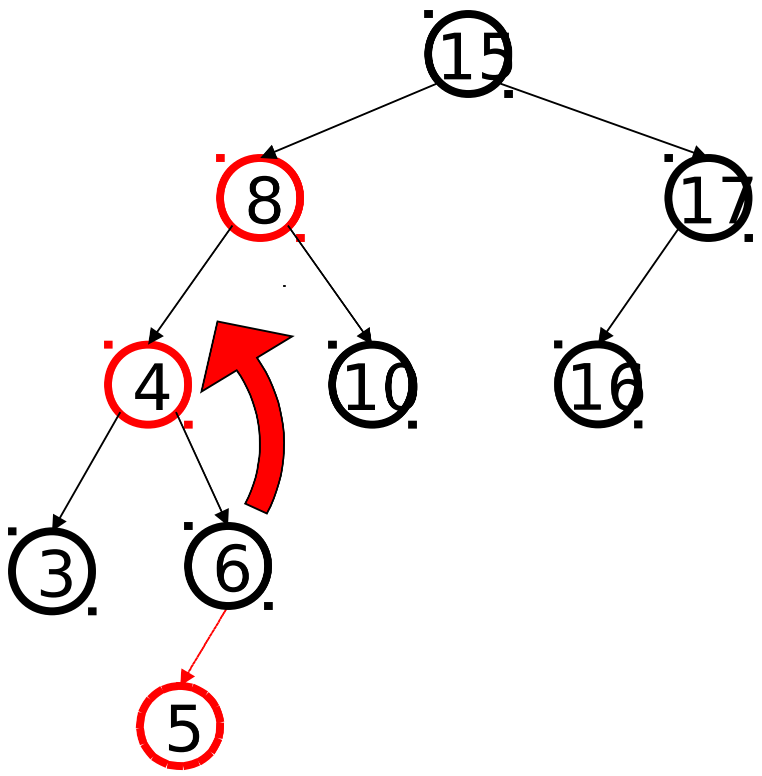
Cases 2 & 3

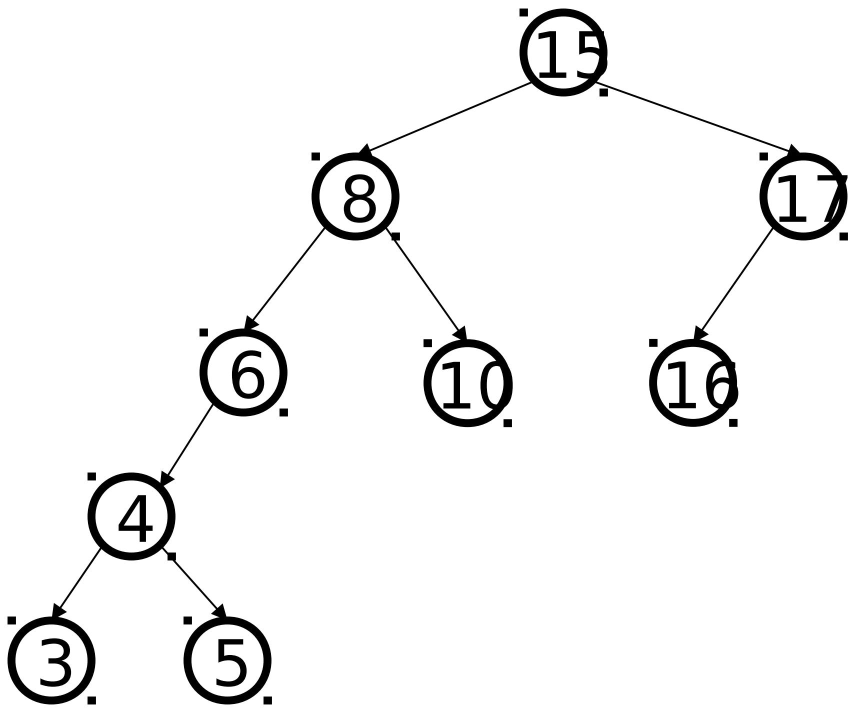




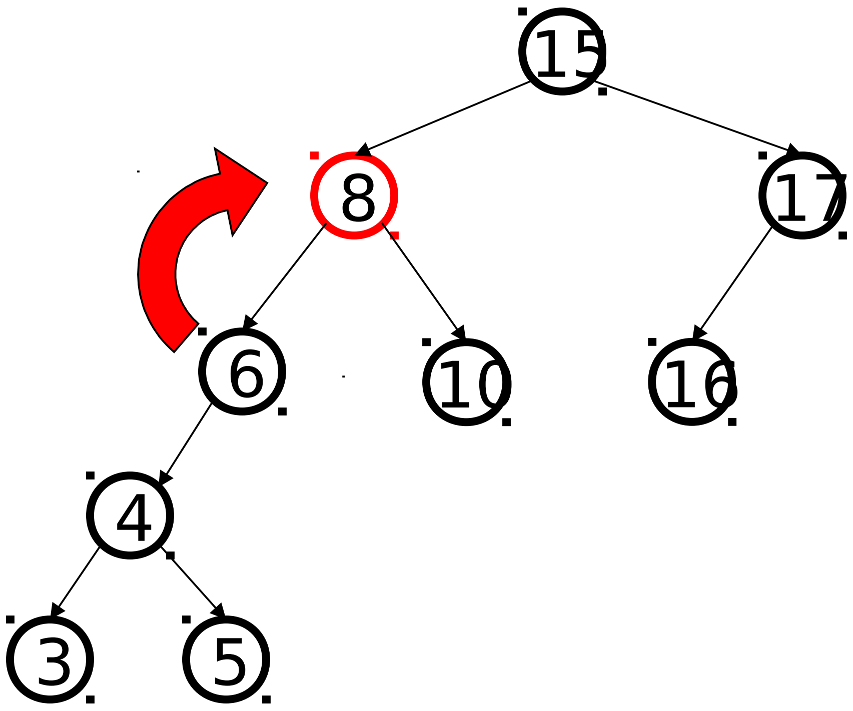
single rotation fails

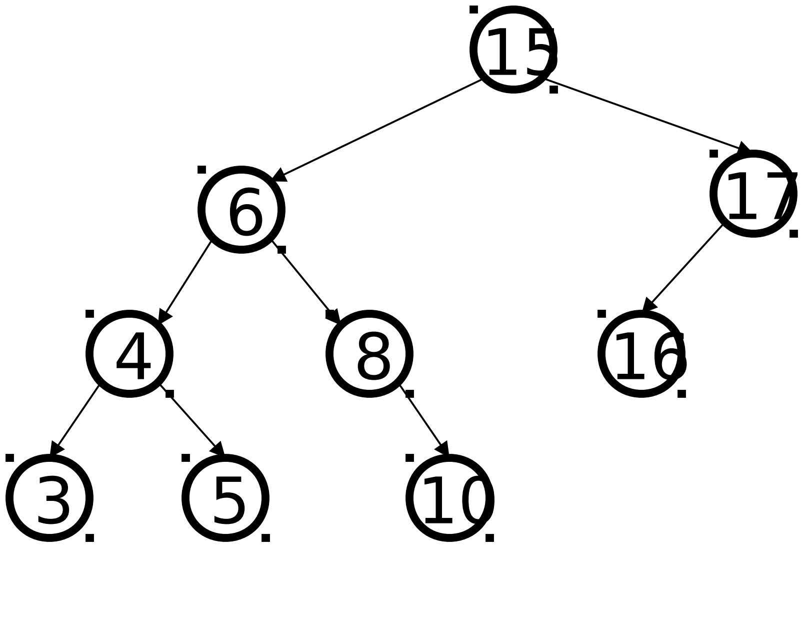
Double rotation, step 1





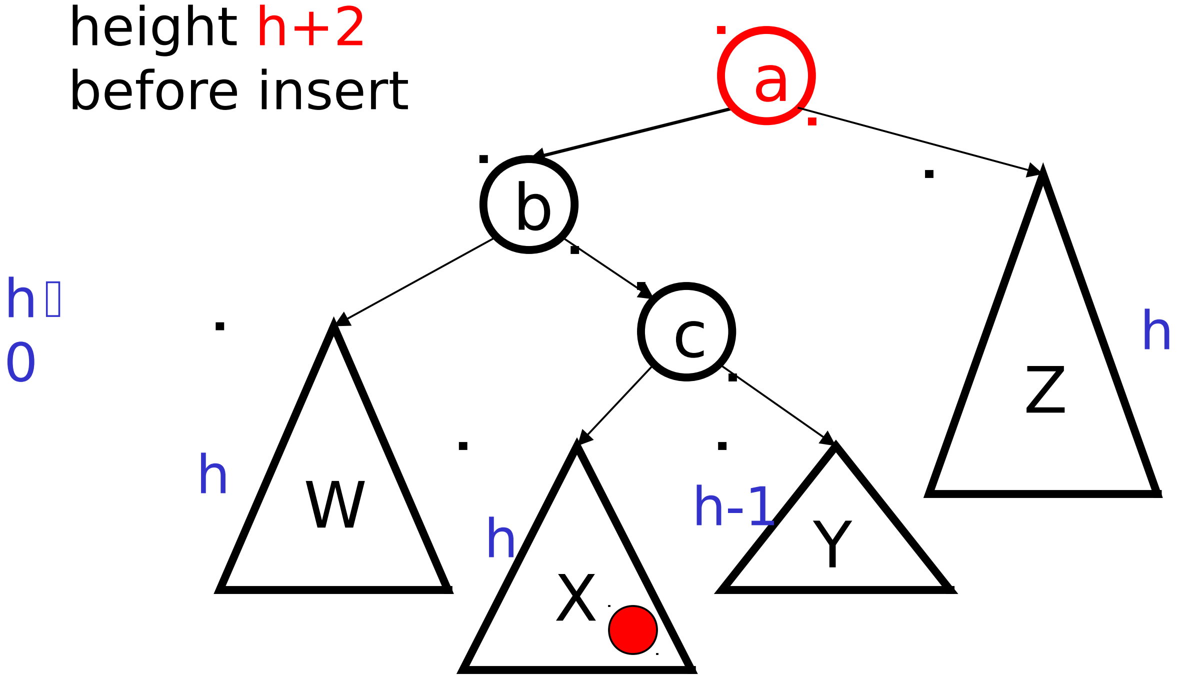
Double rotation, step 2





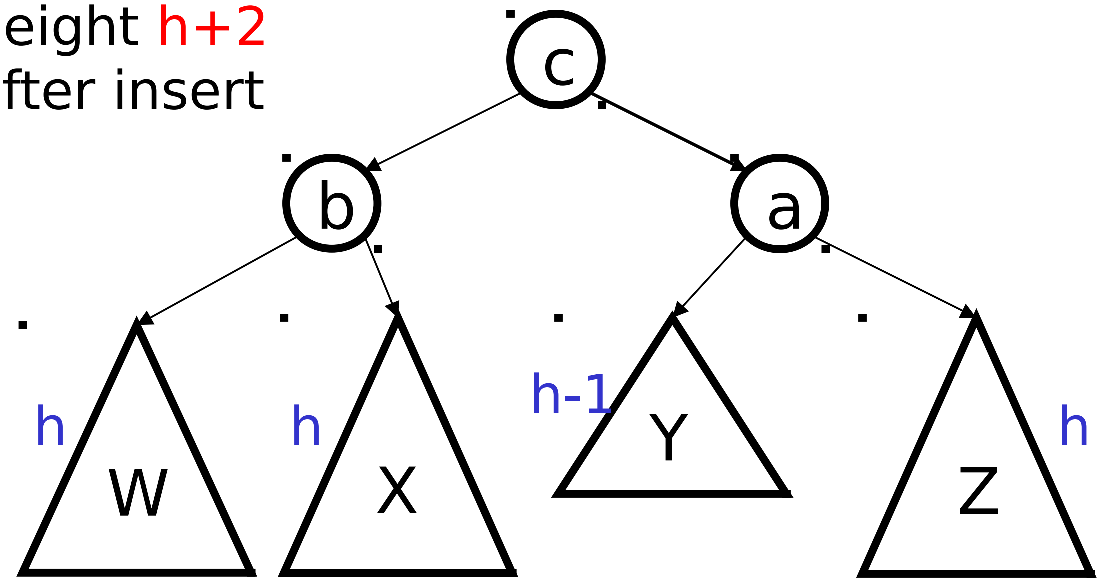
Double rotation in gene

height h+2
before insert



W < b <X < c < Y < a < Z

height h+2
after insert



Depth of an AVL tree

Theorem: Any AVL tree with n nodes has height less than 1.441 log n.

Proof: Given an n-node AVL tree, we want to find an upper bou
on the height of the tree.

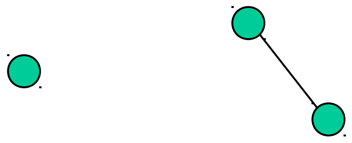
Fix h. What is the smallest n such that there is an AVL tree
height h with n nodes?

Let S_h be the set of all AVL trees of height h that have as few nodes as possible.

Let w_n be the number of nodes

any one of these trees.

$w_0 = 1$, $w_1 = 2$



Suppose $T \in S_n$, where $h \geq 2$. Let T_L and T_R be T's left and right subtrees. Since T has height h, either T_L or T_R has height $h-1$. Suppose it's T_R.

By definition, both T_L and T_R are AVL trees. In fact, $T_R \in S_{n-1}$ or else it could be replaced by a smaller AVL tree of height $h-1$ to give an AVL tree of height h that is smaller than T.

Similarly, $T_L \in S_{h-2}$.

Therefore, $w_h = 1 + w_{h-2} + w_{h-1}$.

Claim: For $h \geq 0$, $w_h \geq \phi^h$, where $\phi = (1 + \sqrt{5}) / 2 \approx 1.6$.

Proof: The proof is by induction on h.

Basis step: $h=0$. $w_0 = 1 = \phi^0$.

$h=1$. $w_1 = 2 > \phi^1$.

Induction step: Suppose the claim is true for $0 \leq m \leq h$, where $h \geq 1$.

Then

$w_{h+1} = 1 + w_{h-1} + w_h$

$\geq 1 + \varphi^{h-1} + \varphi^{h}$ (by the i.h.)

$= 1 + \varphi^{h-1}(1 + \varphi)$

$= 1 + \varphi^{h+1}$ $(1 + \varphi = \varphi^2)$

$> \varphi^{h+1}$

Thus, the claim is true.

From the claim, in an n-node AVL tree of height h,

$n \geq w_h \geq \varphi^h$ (from the Claim

$h \leq \log_{\varphi} n$

$= (\log n) / (\log \varphi)$

$< 1.441 \log n$

AVL tree: Running time

- find takes O(log n) time, because height of the tree is always O(log n).
- insert: O(log n) time because w do a find (O(log n) time), and then we may have to visit ev node on the path back to the root, performing up to 2 sing rotations (O(1) time each) to the tree.

- remove: O(log n) time. Left as an exercise.