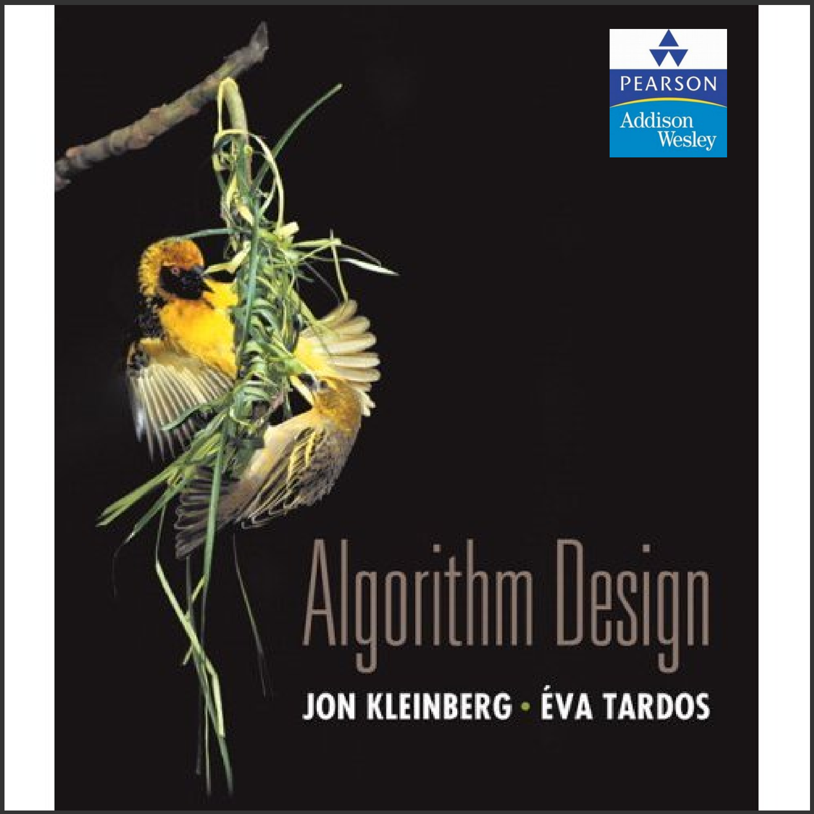


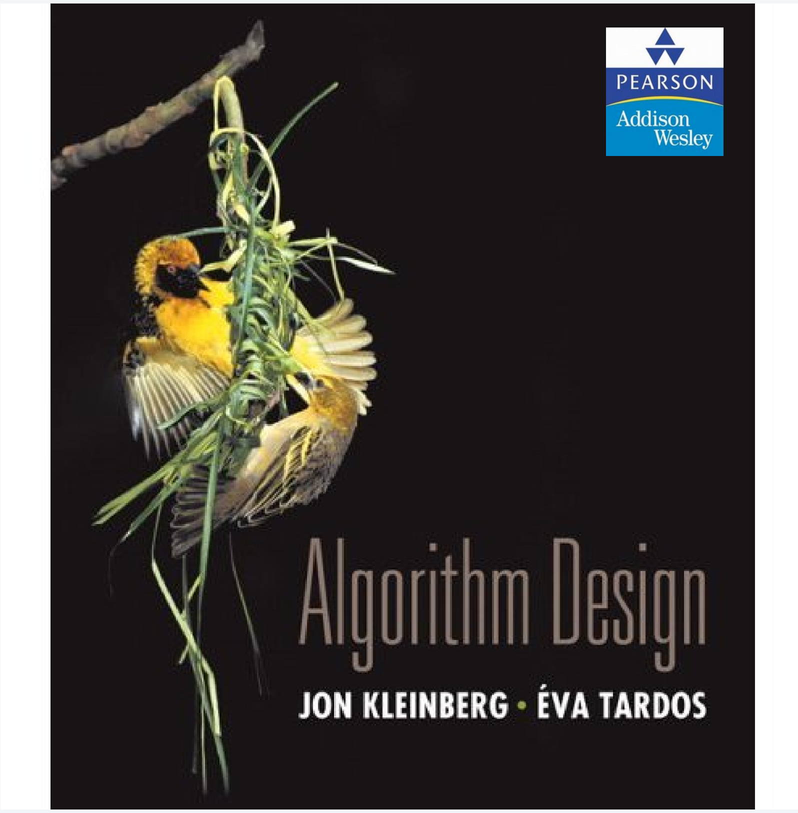
4. GREEDY ALGORITHMS II

- *Dijkstra's algorithm*
- *minimum spanning trees*
- *Prim, Kruskal, Boruvka*
- *single-link clustering*
- *min-cost arborescences*

Lecture slides by Kevin Wayne

http://www.cs.princeton.edu/~wayne/kleinberg-tardos

Last updated on Sep 8, 2013 6:30 AM



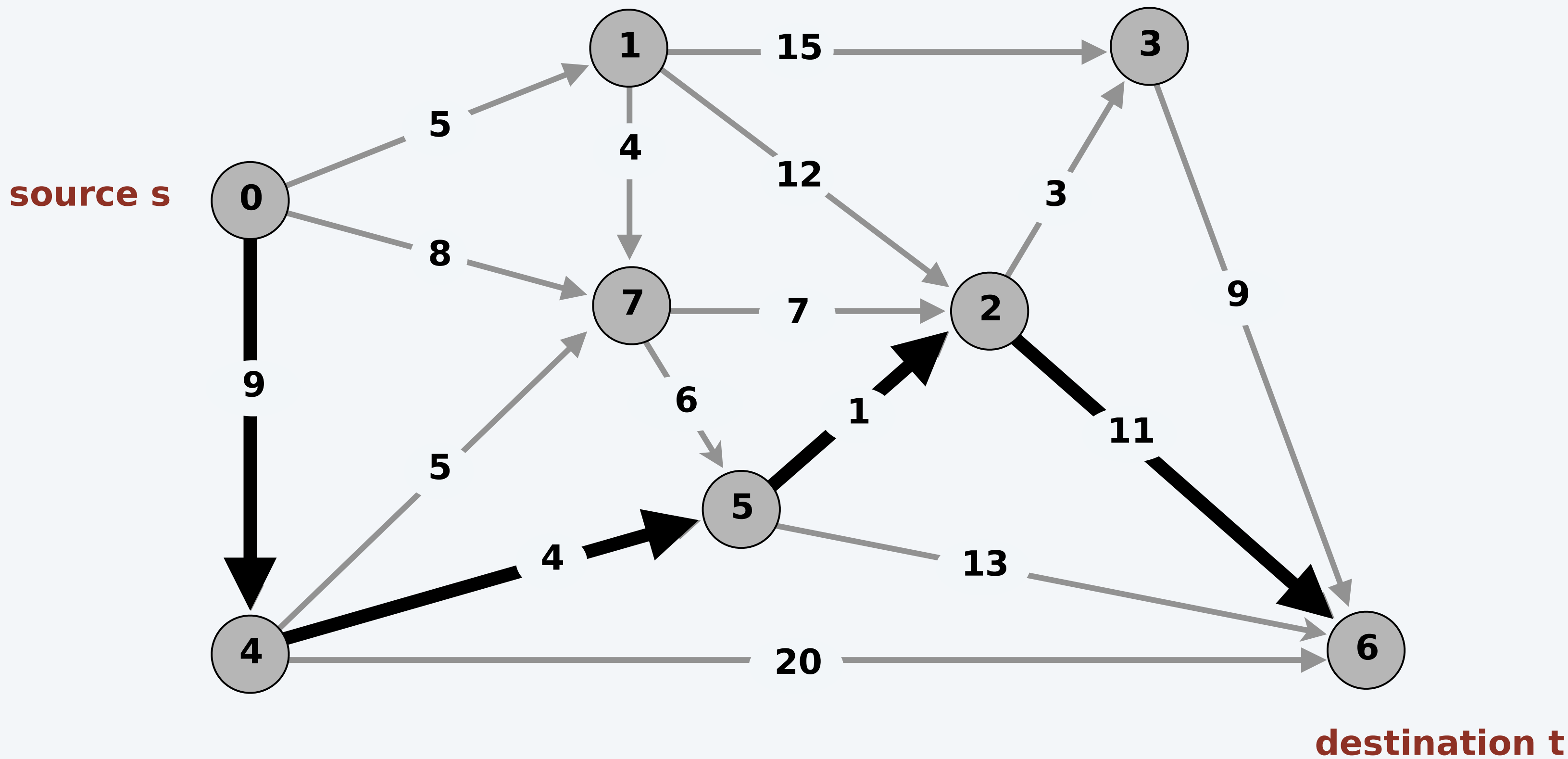
SECTION 4.4

4. GREEDY ALGORITHMS II

- ***Dijkstra's algorithm***
- *minimum spanning trees*
- *Prim, Kruskal, Boruvka*
- *single-link clustering*
- *min-cost arborescences*

Shortest-paths problem

Problem. **Given a digraph $G = (V, E)$, edge lengths $\ell_e \geq 0$, source $s \in V$, and destination $t \in V$, find the shortest directed path from s to t.**



length of path = 9 + 4 + 1 + 11 = 25

Car navigation

Shortest path applications

PERT/CPM.

Map routing.

Seam carving.

Robot navigation.

Texture mapping.

Typesetting in LaTeX.

Urban traffic planning.

Telemarketer operator scheduling.

Routing of telecommunications messages.

Network routing protocols (OSPF, BGP, RIP).

Optimal truck routing through given traffic congestion pattern.

Reference: Network Flows: Theory, Algorithms, and Applications, R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, Prentice Hall, 1993.

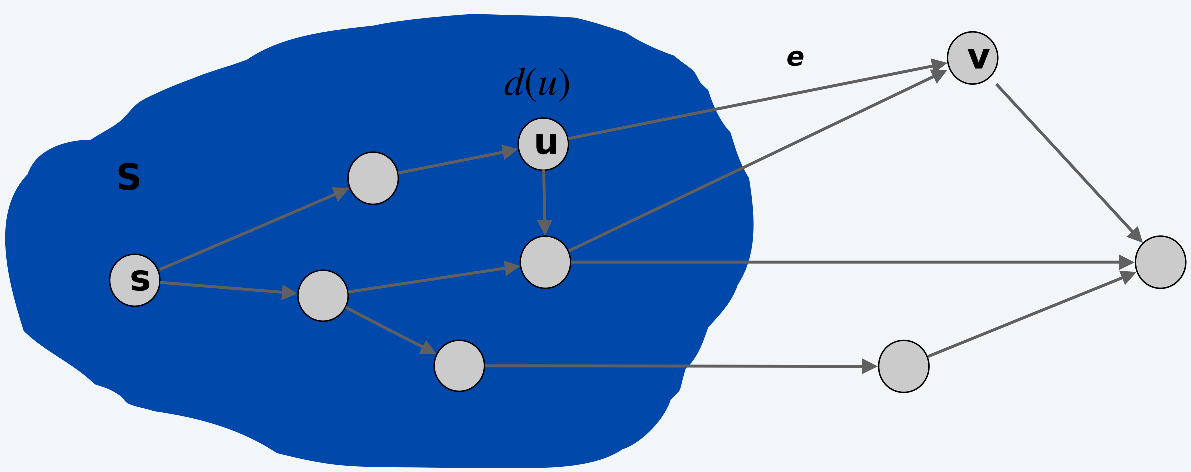
Dijkstra's algorithm

Greedy approach. **Maintain a set of explored nodes S for which algorithm has determined the shortest path distance $d(u)$ from s to u.**

- **Initialize** $S = \{ s \},\ d(s) = 0$.
- **Repeatedly choose unexplored node v which minimizes**

$$\pi(v) = \min_{e = (u,v) : u \in S} d(u) + \ell_e ,$$

shortest path to some node u in explored part, followed by a single edge (u, v)



Dijkstra's algorithm

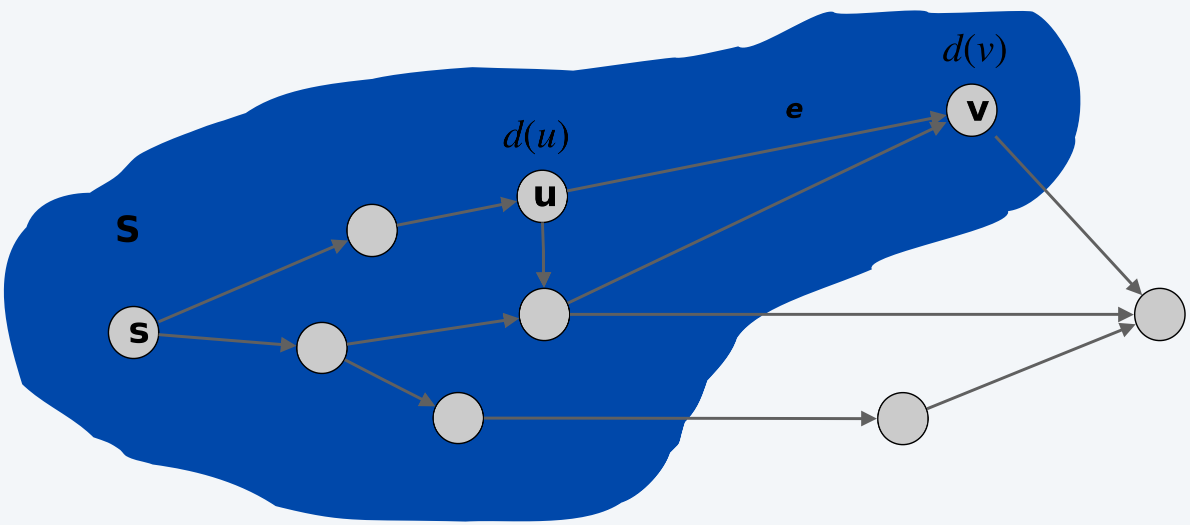
Greedy approach. **Maintain a set of explored nodes S for which algorithm has determined the shortest path distance $d(u)$ from s to u.**

- **Initialize** $S = \{ s \},\ d(s) = 0$**.**
- **Repeatedly choose unexplored node v which minimizes**

$$\pi(v) = \min_{e = (u,v)\,:\,u \in S} d(u) + \ell_e ,$$

shortest path to some node u in explored part, followed by a single edge (u, v)

add v to S, and set $d(v) = \pi(v)$.



8/STR/ALG/XPL/1/A

Dijkstra's algorithm: proof of correctness

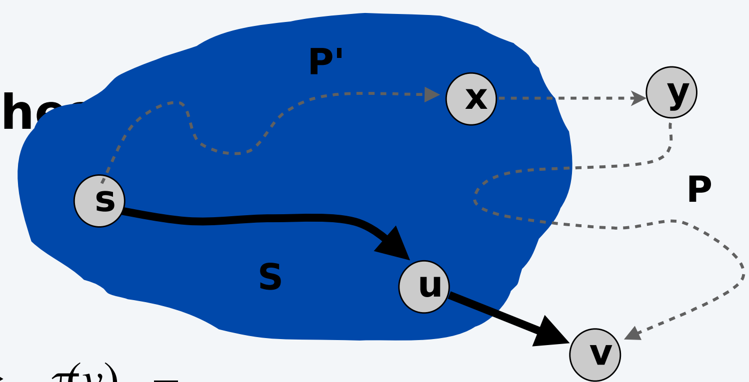
Invariant. **For each node $u \in S$, $d(u)$ is the length of the shortest $s \leadsto u$ path.**

Pf. **[by induction on $|S|$]**

Base case: **$|S| = 1$ is easy since $S = \{ s \}$ and $d(s) = 0$.**

Inductive hypothesis: **Assume true for $|S| = k \geq 1$.**

- **Let v be next node added to S, and let (u, v) be the final edge.**
- **The shortest $s \leadsto u$ path plus (u, v) is an $s \leadsto v$ path of length $\pi(v)$.**
- **Consider any $s \leadsto v$ path P. We show that it is no shorter than $\pi(v)$.**
- **Let (x, y) be the first edge in P that leaves S, and let P' be the subpath to x.**
- **P is already too long as soon as it reaches y.**



$$(P) \;\geq\; (P') + (x, y) \;\geq\; d(x) + (x, y) \;\geq\; \pi(y) \;\geq\; \pi(v) \quad \blacksquare$$

nonnegative lengths

inductive hypothesis

definition of π(y)

Dijkstra chose v instead of y

Dijkstra's algorithm: efficient implementation

Critical optimization 1. **For each unexplored node v, explicitly maintain $\pi(v)$ instead of computing directly from formula:**

$$\pi(v) = \min_{e = (u,v)\,:\,u \in S} d(u) + \ell_e .$$

- **For each $v \notin S$, $\pi(v)$ can only decrease (because S only increases).**
- **More specifically, suppose u is added to S and there is an edge (u, v) leaving u. Then, it suffices to update:**

$$\pi(v) = \min \{\, \pi(v),\ d(u) + \quad (u, v) \,\}$$

Critical optimization 2. **Use a priority queue to choose the unexplored node that minimizes $\pi(v)$.**

Dijkstra's algorithm: efficient implementation

Implementation.

- **Algorithm stores** $d(v)$ **for each explored node** v
- **Priority queue stores** $\pi(v)$ **for each unexplored node** v
- **Recall:** $d(u) = \pi(u)$ **when** u **is deleted from priority queue.**

DIJKSTRA (V, E, s)

Create an empty priority queue.

FOR EACH $v \neq s$: $d(v) \leftarrow \infty$; $d(s) \leftarrow 0$.

FOR EACH $v \in V$: *insert* v with key $d(v)$ into priority queue.

WHILE (the priority queue *is not empty*)

  $u \leftarrow$ *delete-min* from priority queue.

  FOR EACH edge $(u, v) \in E$ leaving u:

    IF $d(v) > d(u) + (u, v)$

      decrease-key of v to $d(u) + (u, v)$ in priority queue.

      $d(v) \leftarrow d(u) + (u, v)$.

Dijkstra's algorithm: which priority queue?

Performance. **Depends on PQ: n insert, n delete-min, m decrease-key.**

- **Array implementation optimal for dense graphs.**
- **Binary heap much faster for sparse graphs.**
- **4-way heap worth the trouble in performance-critical situations.**
- **Fibonacci/Brodal best in theory, but not worth implementing.**

PQ implementation	insert	delete-min	decrease-key	total
unordered array	$O(1)$	$O(n)$	$O(1)$	$O(n^2)$
binary heap	$O(\log n)$	$O(\log n)$	$O(\log n)$	$O(m \log n)$
d-way heap (Johnson 1975)	$O(d \log_d n)$	$O(d \log_d n)$	$O(\log_d n)$	$O(m \log_{m/n} n)$
Fibonacci heap (Fredman-Tarjan 1984)	$O(1)$	$O(\log n)$ †	$O(1)$ †	$O(m + n \log n)$
Brodal queue (Brodal 1996)	$O(1)$	$O(\log n)$	$O(1)$	$O(m + n \log n)$

† amortized

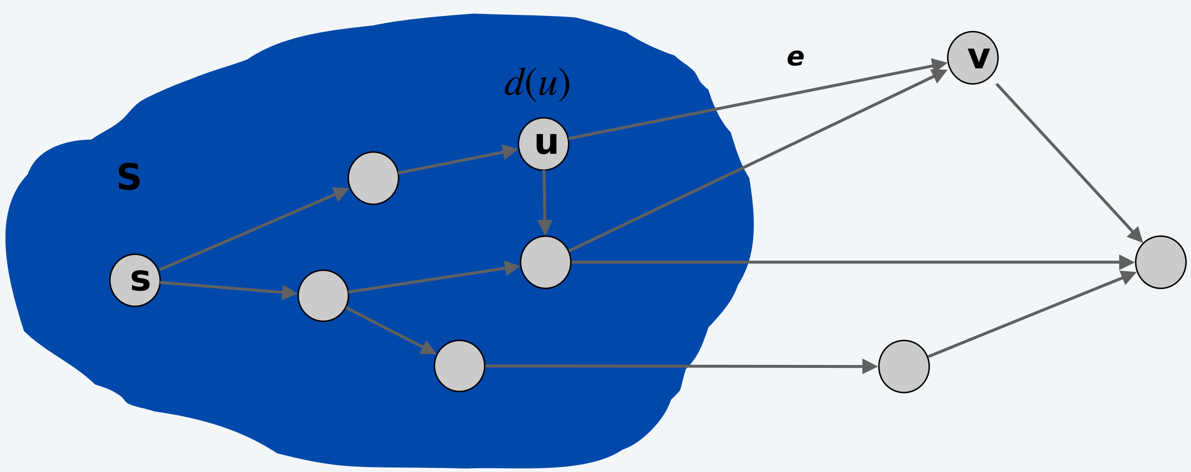
12/EXP/APP/DEF/1/A

Extensions of Dijkstra's algorithm

Dijkstra's algorithm and proof extend to several related problems:

- **Shortest paths in undirected graphs:** $d(v) \le d(u) + (u, v)$.
- **Maximum capacity paths:** $d(v) \ge \min \{ \pi(u), c(u, v) \}$.
- **Maximum reliability paths:** $d(v) \ge d(u)$ □ $\gamma(u, v)$.
- **...**

Key algebraic structure. **Closed semiring (tropical, bottleneck, Viterbi).**



4. Greedy Algorithms II

- *Dijkstra's algorithm*
- ***minimum spanning trees***
- *Prim, Kruskal, Boruvka*
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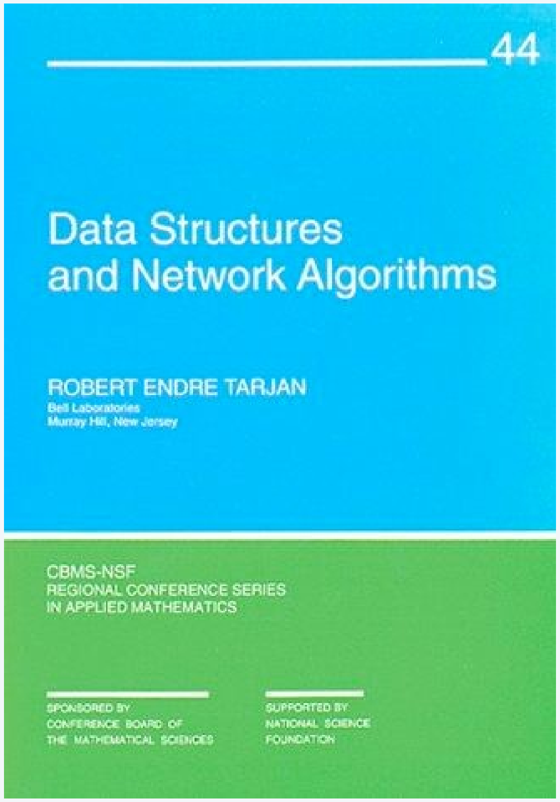
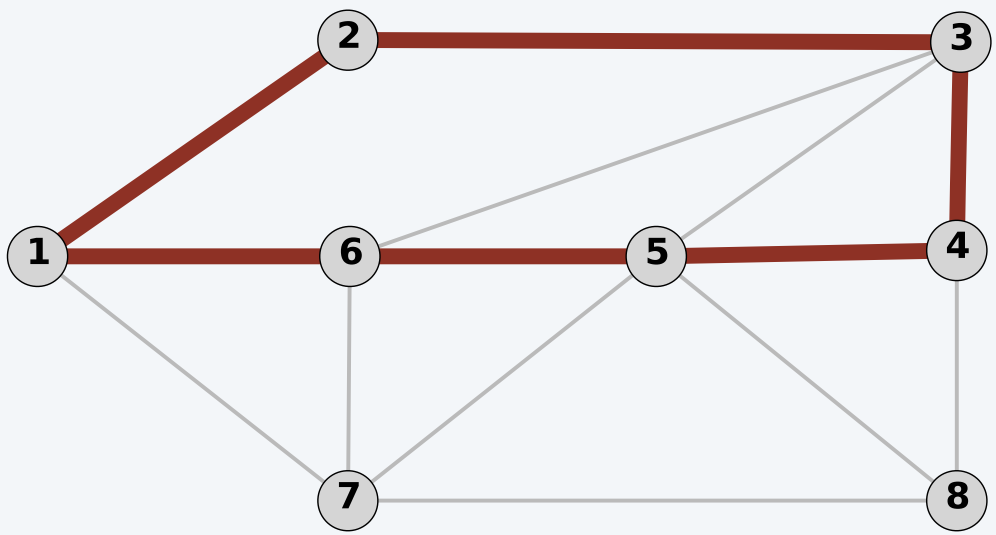


Section 6.1

Cycles and cuts

Def. **A path is a sequence of edges which connects a sequence of nodes**

Def. **A cycle is a path with no repeated nodes or edges other than the starting and ending nodes.**

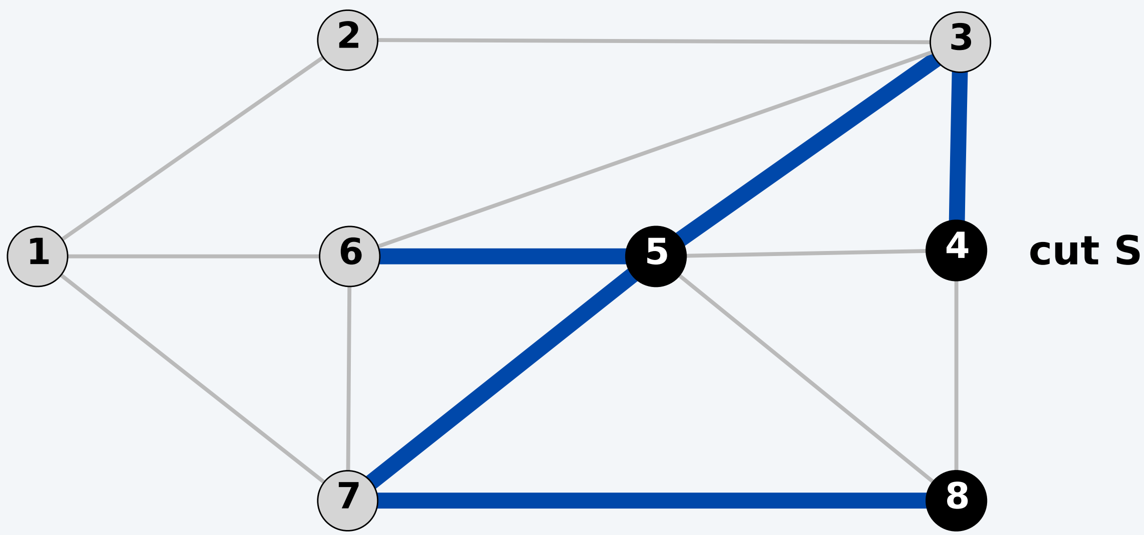


cycle C = { (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 1) }

Cycles and cuts

Def. **A cut is a partition of the nodes into two nonempty subsets S and $V - S$.**

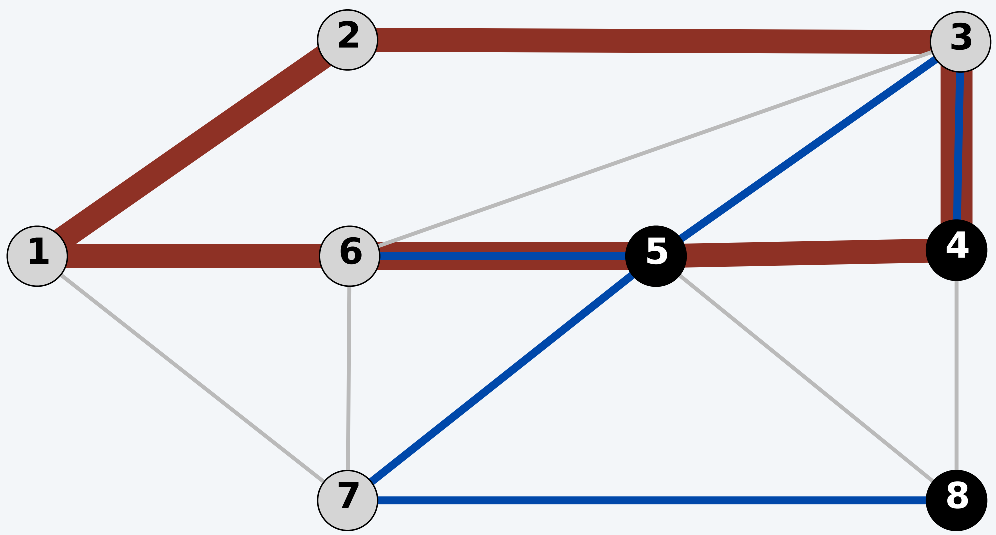
Def. **The cutset of a cut S is the set of edges with exactly one endpoint in S.**



cutset D = { (3, 4), (3, 5), (5, 6), (5, 7), (8, 7) }

Cycle-cut intersection

Proposition. **A cycle and a cutset intersect in an even number of edges.**



cutset D = { (3, 4), (3, 5), (5, 6), (5, 7), (8, 7) }

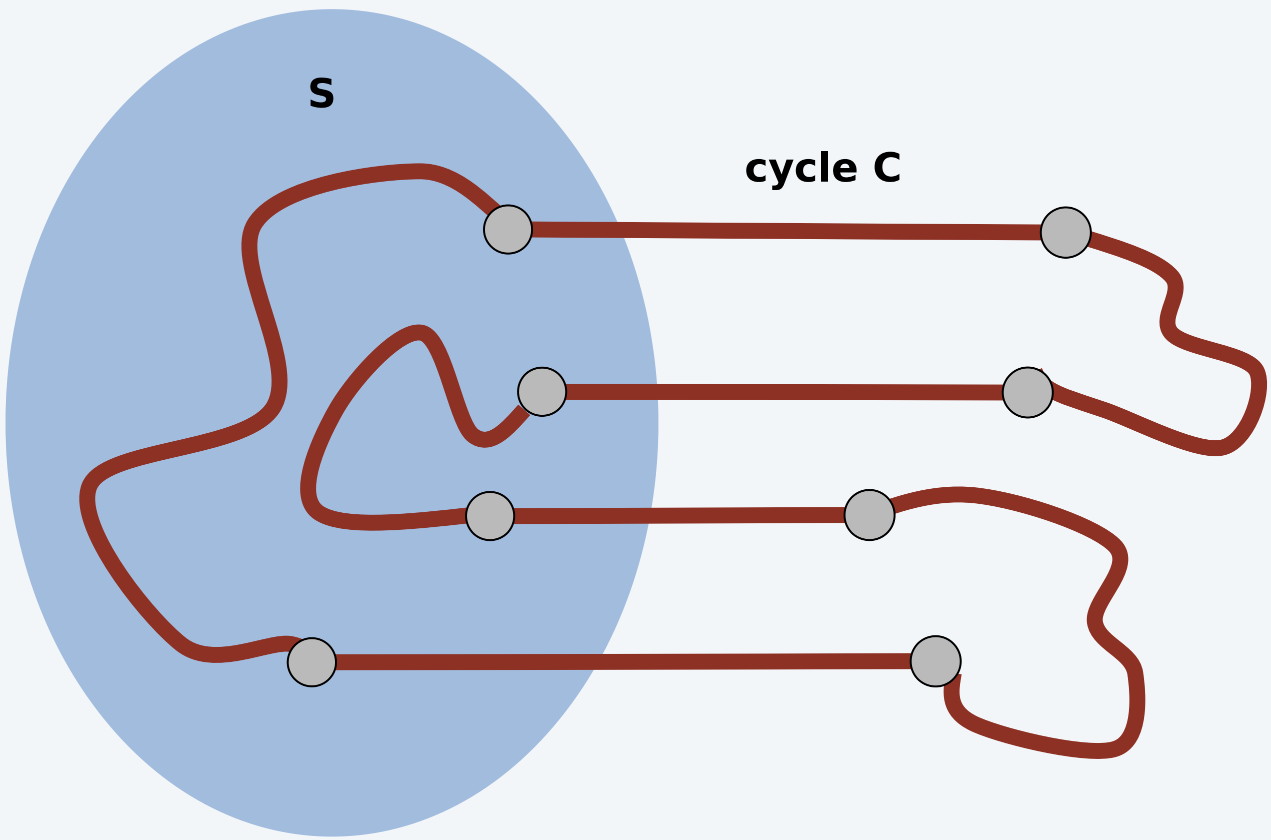
cycle C = { (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 1) }

intersection C ∩ D = { (3, 4), (5, 6) }

Cycle-cut intersection

Proposition. A cycle and a cutset intersect in an even number of edges.

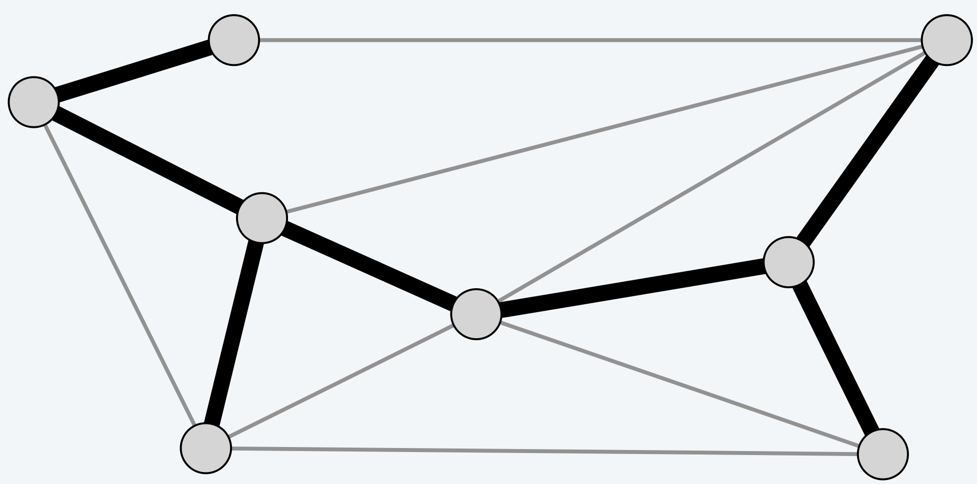
Pf. [by picture]



Spanning tree properties

Proposition. **Let $T = (V, F)$ be a subgraph of $G = (V, E)$. TFAE:**

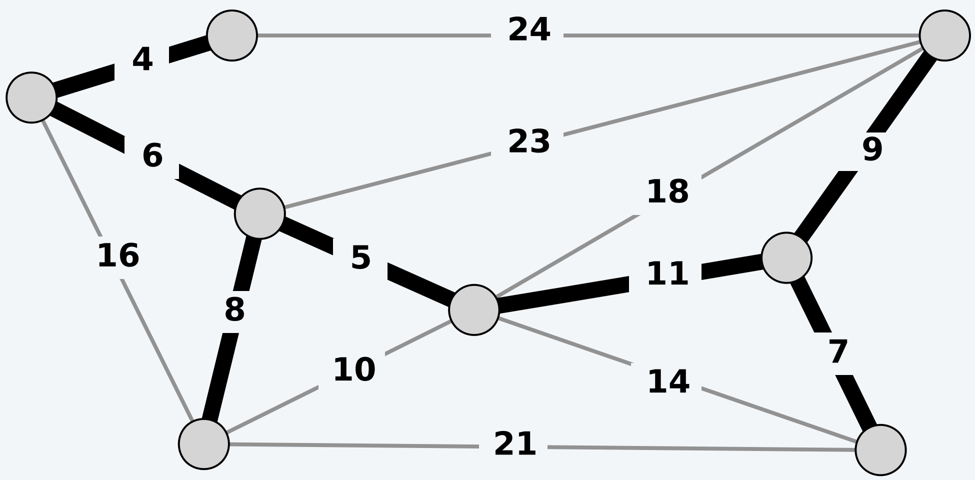
- T **is a spanning tree of G.**
- T **is acyclic and connected.**
- T **is connected and has $n - 1$ edges.**
- T **is acyclic and has $n - 1$ edges.**
- T **is minimally connected: removal of any edge disconnects it.**
- T **is maximally acyclic: addition of any edge creates a cycle.**
- T **has a unique simple path between every pair of nodes.**



T = (V, F)

Minimum spanning tree

Given a connected graph $G = (V, E)$ with edge costs c_e, an MST is a subset of the edges $T \subseteq E$ such that T is a spanning tree whose sum of edge costs is minimized.



MST cost = 50 = 4 + 6 + 8 + 5 + 11 + 9 + 7

Cayley's theorem. **There are n^{n-2} spanning trees of K_n.** ← can't solve by brute force

Applications

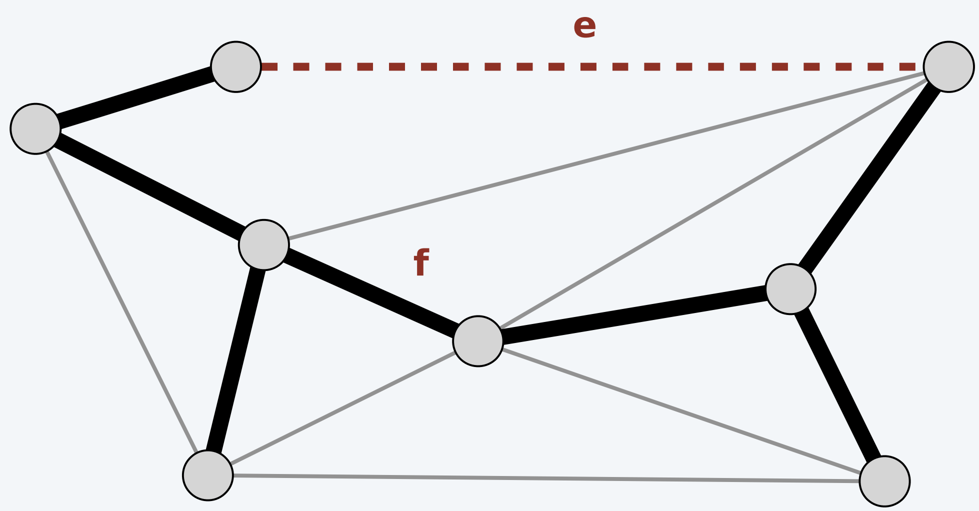
MST is fundamental problem with diverse applications.

- **Dithering.**
- **Cluster analysis.**
- **Max bottleneck paths.**
- **Real-time face verification.**
- **LDPC codes for error correction.**
- **Image registration with Renyi entropy.**
- **Find road networks in satellite and aerial imagery.**
- **Reducing data storage in sequencing amino acids in a protein.**
- **Model locality of particle interactions in turbulent fluid flows.**
- **Autoconfig protocol for Ethernet bridging to avoid cycles in a networ**
- **Approximation algorithms for NP-hard problems (e.g., TSP, Steiner tr**
- **Network design (communication, electrical, hydraulic, computer, roa**

Fundamental cycle

Fundamental cycle.

- **Adding any non-tree edge e to a spanning tree T forms unique cycle C.**
- **Deleting any edge $f \in C$ from $T \cup \{ e \}$ results in new spanning tree.**



T = (V, F)

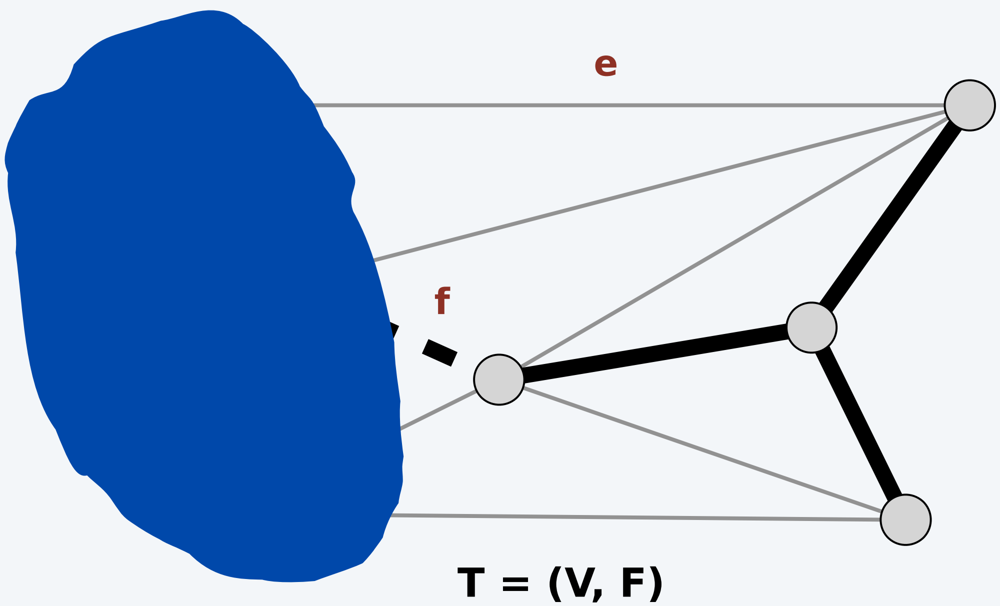
Observation. **If $c_e < c_f$, then T is not an MST.**

Fundamental cutset

Fundamental cutset.

Deleting any tree edge f from a spanning tree T divide nodes into two connected components. Let D be cutset.

Adding any edge $e \in D$ to $T - \{ f \}$ results in new spanning tree.



Observation. **If $c_e < c_f$, then T is not an MST.**

The greedy algorithm

Red rule.

Let C be a cycle with no red edges.

Select an uncolored edge of C of max weight and color it red.

Blue rule.

Let D be a cutset with no blue edges.

Select an uncolored edge in D of min weight and color it blue.

Greedy algorithm.

Apply the red and blue rules (non-deterministically!) until all edges are colored. The blue edges form an MST.

Note: can stop once $n - 1$ edges colored blue.

Greedy algorithm: proof of correctness

Color invariant. **There exists an MST T^* containing all of the blue edges and none of the red edges.**

Pf. **[by induction on number of iterations]**

Base case. **No edges colored $\Rightarrow$ every MST satisfies invariant.**

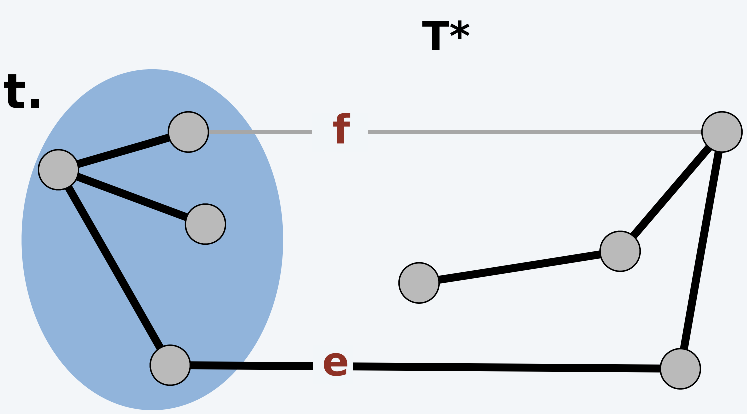
Greedy algorithm: proof of correctness

Color invariant. **There exists an MST T^* containing all of the blue edges and none of the red edges.**

Pf. **[by induction on number of iterations]**

Induction step (blue rule). **Suppose color invariant true before blue rule**

- **let D be chosen cutset, and let f be edge colored blue.**
- **if $f \in T^*$, T^* still satisfies invariant.**
- **Otherwise, consider fundamental cycle C by adding f to T^*.**
- **let $e \in C$ be another edge in D.**
- **e is uncolored and $c_e \geq c_f$ since**
  - **$e \in T^* \Rightarrow e$ not red**
  - **blue rule $\Rightarrow e$ not blue and $c_e \geq c_f$**
- **Thus, $T^* \cup \{ f \} - \{ e \}$ satisfies invariant.**



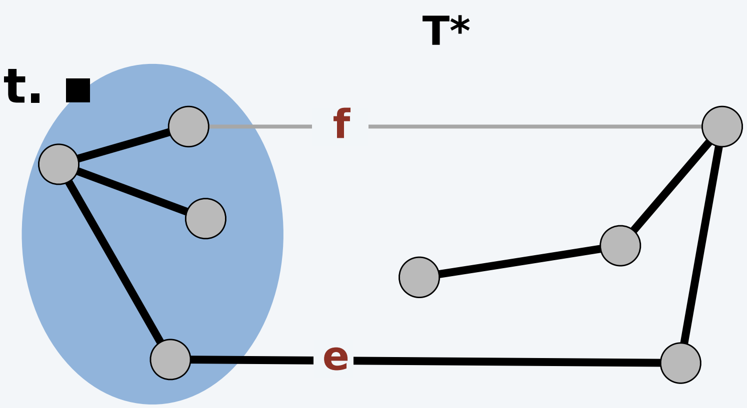
Greedy algorithm: proof of correctness

Color invariant. **There exists an MST T^* containing all of the blue edges and none of the red edges.**

Pf. **[by induction on number of iterations]**

Induction step (red rule). **Suppose color invariant true before red rule.**

- **let C be chosen cycle, and let e be edge colored red.**
- **if $e \notin T^*$, T^* still satisfies invariant.**
- **Otherwise, consider fundamental cutset D by deleting e from T^*.**
- **let $f \in D$ be another edge in C.**
- **f is uncolored and $c_e \geq c_f$ since**
  - **$f \notin T^* \Rightarrow f$ not blue**
  - **red rule $\Rightarrow f$ not red and $c_e \geq c_f$**
- **Thus, $T^* \cup \{ f \} - \{ e \}$ satisfies invariant. ■**

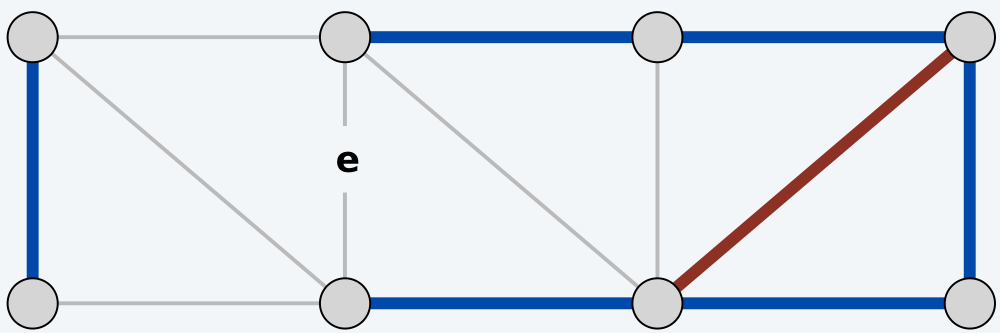


Greedy algorithm: proof of correctness

Theorem. **The greedy algorithm terminates. Blue edges form an MST.**

Pf. **We need to show that either the red or blue rule (or both) applies.**

- **Suppose edge e is left uncolored.**
- **Blue edges form a forest.**
- **Case 1: both endpoints of e are in same blue tree.**
- ⇒ **apply red rule to cycle formed by adding e to blue forest.**



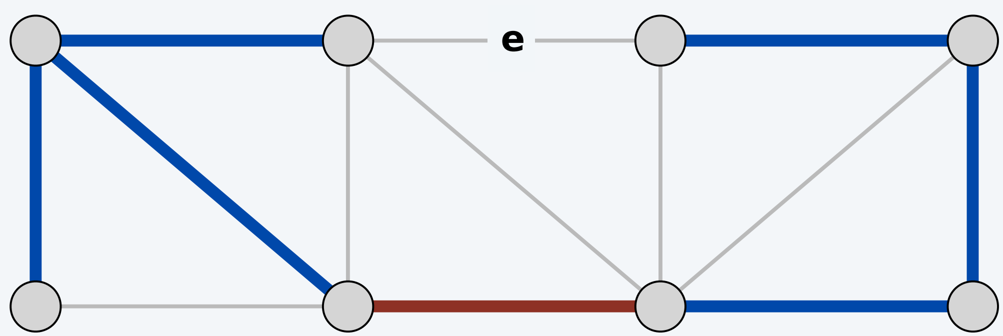
Case 1

Greedy algorithm: proof of correctness

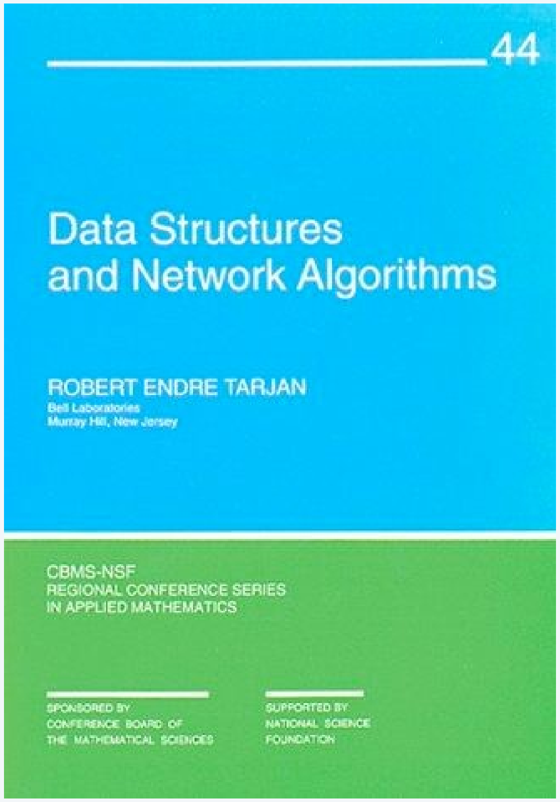
Theorem. **The greedy algorithm terminates. Blue edges form an MST.**

Pf. **We need to show that either the red or blue rule (or both) applies.**

- **Suppose edge e is left uncolored.**
- **Blue edges form a forest.**
- **Case 1: both endpoints of e are in same blue tree.**
  - ⇒ **apply red rule to cycle formed by adding e to blue forest.**
- **Case 2: both endpoints of e are in different blue trees.**
  - ⇒ **apply blue rule to cutset induced by either of two blue trees.** ▪



Case 2



SECTION 6.2

4. GREEDY ALGORITHMS II

- *Dijkstra's algorithm*
- *minimum spanning trees*
- ***Prim, Kruskal, Boruvka***
- *single-link clustering*
- *min-cost arborescences*

Prim's algorithm

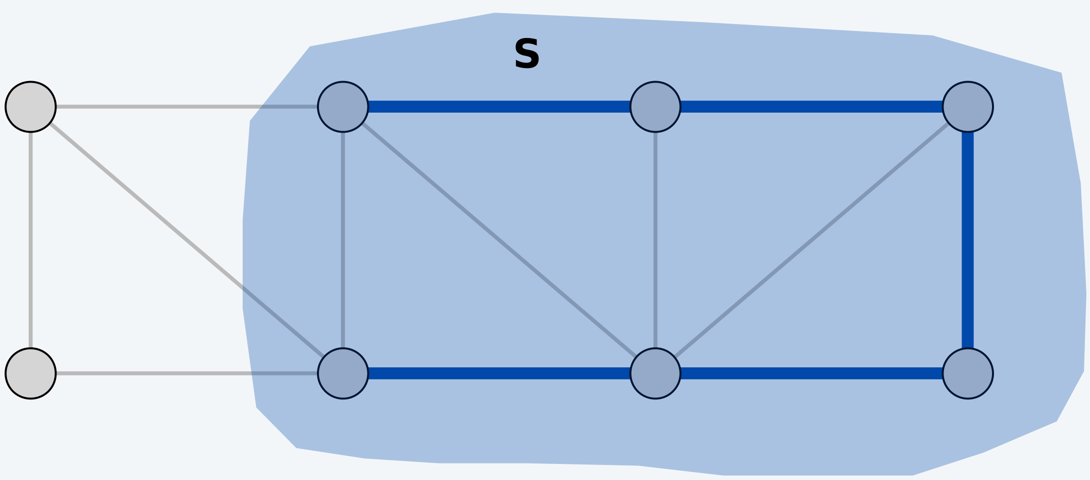
Initialize S = any node.

Repeat $n-1$ times:

- **Add to tree the min weight edge with one endpoint in S.**
- **Add new node to S.**

Theorem. **Prim's algorithm computes the MST.**

Pf. **Special case of greedy algorithm (blue rule repeatedly applied to S). ▪**



Prim's algorithm: implementation

Theorem. **Prim's algorithm can be implemented in $O(m \log n)$ time.**

Pf. **Implementation almost identical to Dijkstra's algorithm.**

[$d(v)$ = **weight of cheapest known edge between** v **and** S **]**

PRIM (V, E, c)

Create an empty priority queue.

$s \leftarrow$ any node in V.

FOR EACH $v \neq s$: $d(v) \leftarrow \infty$; $d(s) \leftarrow 0$.

FOR EACH v : *insert* v with key $d(v)$ into priority queue.

WHILE (the priority queue *is not empty*)

  $u \leftarrow$ *delete-min* from priority queue.

  FOR EACH edge $(u, v) \in E$ incident to u:

    IF $d(v) > c(u, v)$

      decrease-key of v to $c(u, v)$ in priority queue.

      $d(v) \leftarrow c(u, v)$.

Kruskal's algorithm

Consider edges in ascending order of weight:

Add to tree unless it would create a cycle.

Theorem. Kruskal's algorithm computes the MST.

Pf. Special case of greedy algorithm.

Case 1: both endpoints of e in same blue tree.

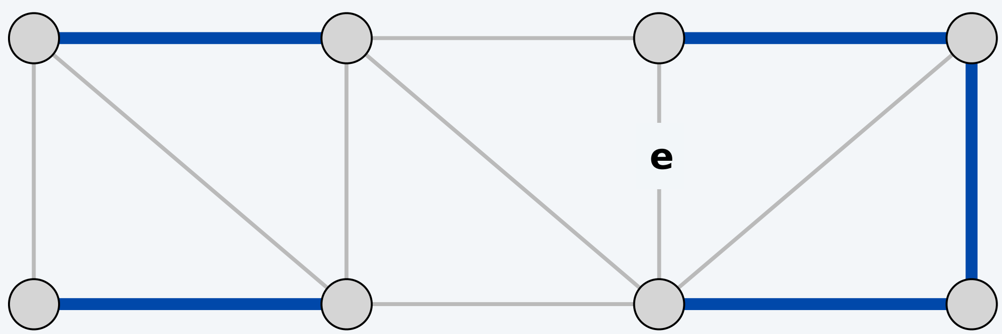
all other edges in cycle are blue

⇒ color red by applying red rule to unique cycle.

Case 2. If both endpoints of e are in different blue trees.

⇒ color blue by applying blue rule to cutset defined by either tree.

no edge in cutset has smaller weight
(since Kruskal chose it first)



Kruskal's algorithm: implementation

Theorem. **Kruskal's algorithm can be implemented in $O(m \log m)$ time.**

- **Sort edges by weight.**
- **Use union-find data structure to dynamically maintain connected components.**

KRUSKAL (V, E, c)

SORT m edges by weight so that $c(e_1) \leq c(e_2) \leq \ldots \leq c(e_m)$

$S \leftarrow \varphi$

FOREACH $v \in V$: MAKESET(v).

FOR $i = 1$ TO m

  $(u, v) \leftarrow e_i$

  IF FINDSET$(u) \neq$ FINDSET(v) ← **are u and v in same component?**

    $S \leftarrow S \cup \{ e_i \}$

    UNION(u, v). ← **make u and v in same component**

RETURN S

Reverse-delete algorithm

Consider edges in descending order of weight:

Remove edge unless it would disconnect the graph.

Theorem. **The reverse-delete algorithm computes the MST.**

Pf. **Special case of greedy algorithm.**

Case 1: removing edge e does not disconnect graph.

⇒ apply red rule to cycle C formed by adding e to existing path between its two endpoints

any edge in C with larger weight would have been deleted when considered

Case 2: removing edge e disconnects graph.

⇒ apply blue rule to cutset D induced by either component. ▪

e is the only edge in the cutset (any other edges must have been colored red / deleted)

Fact. **[Thorup 2000]** **Can be implemented in $O(m \log n (\log \log n)^3)$ time.**

Review: the greedy MST algorithm

Red rule.

Let C be a cycle with no red edges.

Select an uncolored edge of C of max weight and color it red.

Blue rule.

Let D be a cutset with no blue edges.

Select an uncolored edge in D of min weight and color it blue.

Greedy algorithm.

Apply the red and blue rules (non-deterministically!) until all edges are colored. The blue edges form an MST.

Note: can stop once $n - 1$ edges colored blue.

Theorem. **The greedy algorithm is correct.**

Special cases. **Prim, Kruskal, reverse-delete, ...**

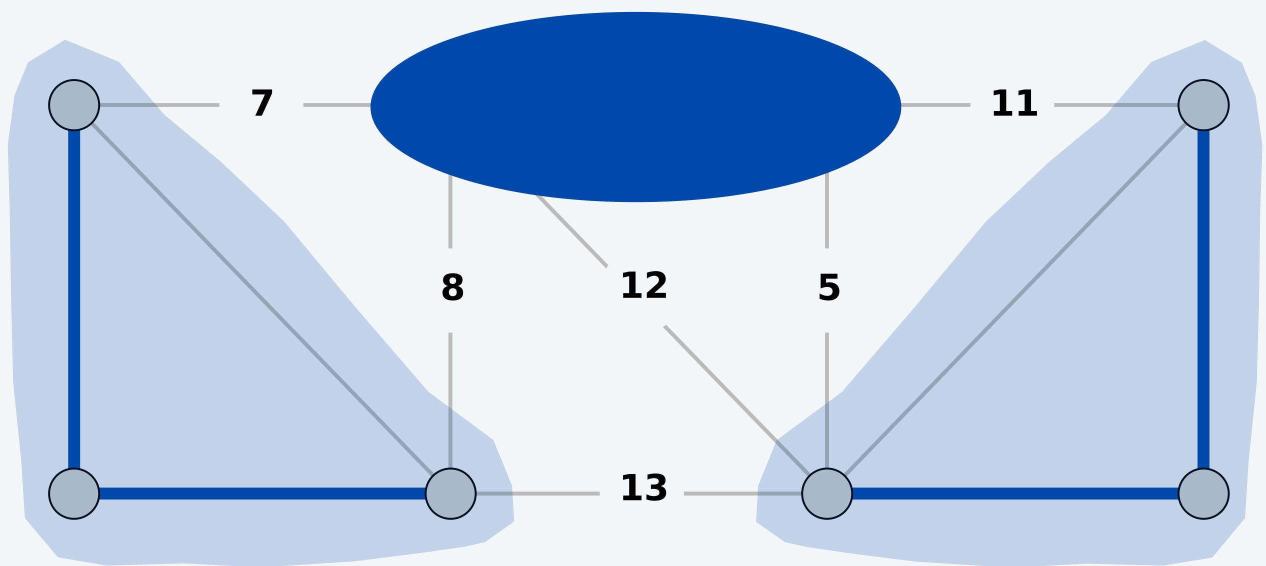
Borůvka's algorithm

Repeat until only one tree.

- **Apply blue rule to cutset corresponding to each blue tree.**
- **Color all selected edges blue.**

Theorem. **Borůvka's algorithm computes the MST.** assume edge costs are distinct

Pf. **Special case of greedy algorithm (repeatedly apply blue rule).** ▪

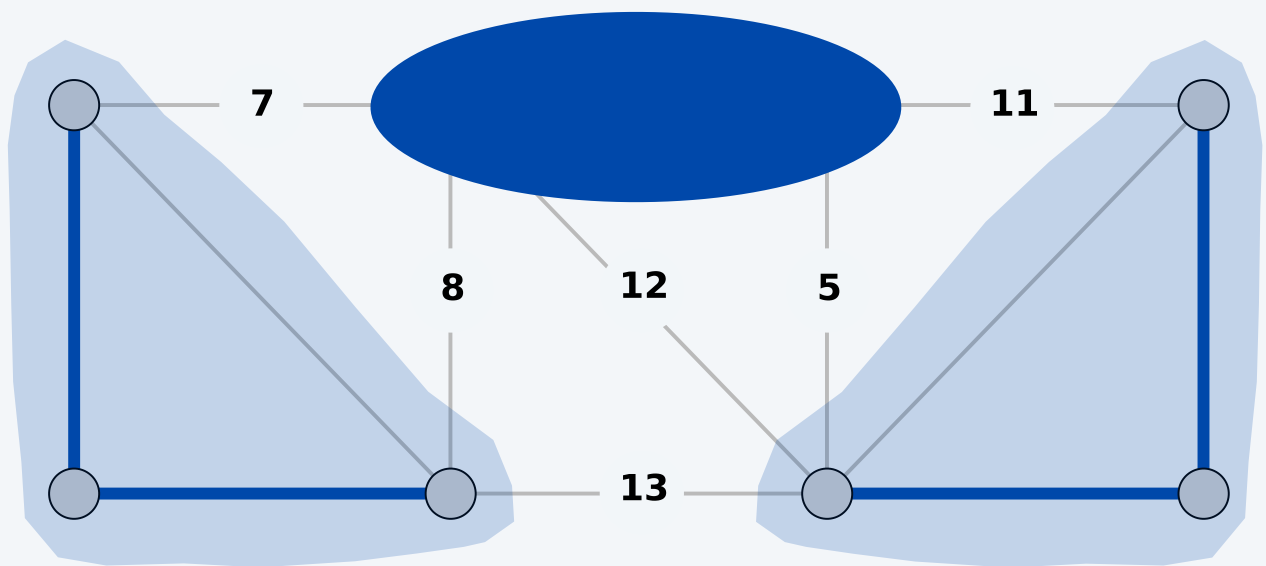


Borůvka's algorithm: implementation

Theorem. **Borůvka's algorithm can be implemented in $O(m \log n)$ time.**

Pf.

- **To implement a phase in $O(m)$ time:**
  - **compute connected components of blue edges**
  - **for each edge $(u, v) \in E$, check if u and v are in different components; if so, update each component's best edge in cutset**
- **At most $\log_2 n$ phases since each phase (at least) halves total # trees.**

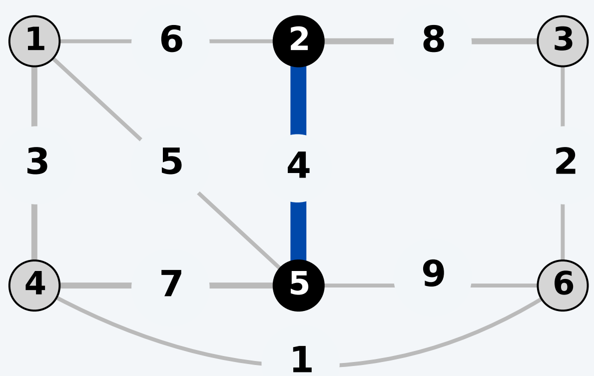


Borůvka's algorithm: implementation

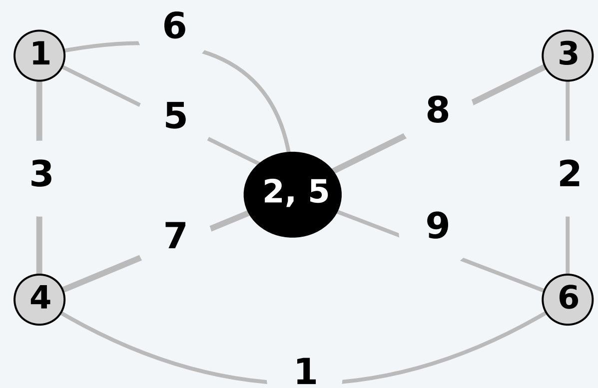
Node contraction version.

- **After each phase, contract each blue tree to a single supernode.**
- **Delete parallel edges (keeping only cheapest one) and self loops.**
- **Borůvka phase becomes: take cheapest edge incident to each node.**

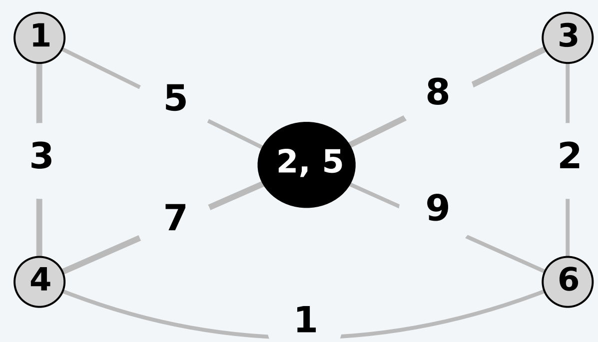
graph G



contract nodes 2 and 5



delete parallel edges and self loops

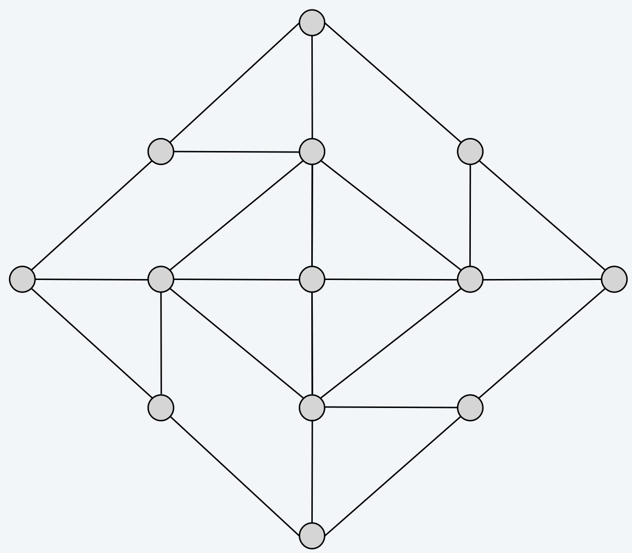


Borůvka's algorithm on planar graphs

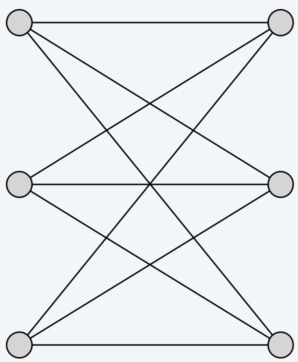
Theorem. Borůvka's algorithm runs in $O(n)$ time on planar graphs.

Pf.

- To implement a Borůvka phase in $O(n)$ time:
  - use contraction version of algorithm
  - in planar graphs, $m \leq 3n - 6$.
  - graph stays planar when we contract a blue tree
- Number of nodes (at least) halves.
- At most $\log_2 n$ phases: $cn + cn/2 + cn/4 + cn/8 + \ldots = O(n)$. ▪



planar



not planar

Borůvka-Prim algorithm

Borůvka-Prim algorithm.

- Run Borůvka (contraction version) for $\log_2 \log_2 n$ phases.
- Run Prim on resulting, contracted graph.

Theorem. The Borůvka-Prim algorithm computes an MST and can be implemented in $O(m \log \log n)$ time.

Pf.

- Correctness: special case of the greedy algorithm.
- The $\log_2 \log_2 n$ phases of Borůvka's algorithm take $O(m \log \log n)$ time; resulting graph has at most $n / \log_2 n$ nodes and m edges.
- Prim's algorithm (using Fibonacci heaps) takes $O(m + n)$ time on a graph with $n / \log_2 n$ nodes and m edges. ▪

$$O\left(m + \frac{n}{\log n} \log \left(\frac{n}{\log n}\right)\right)$$

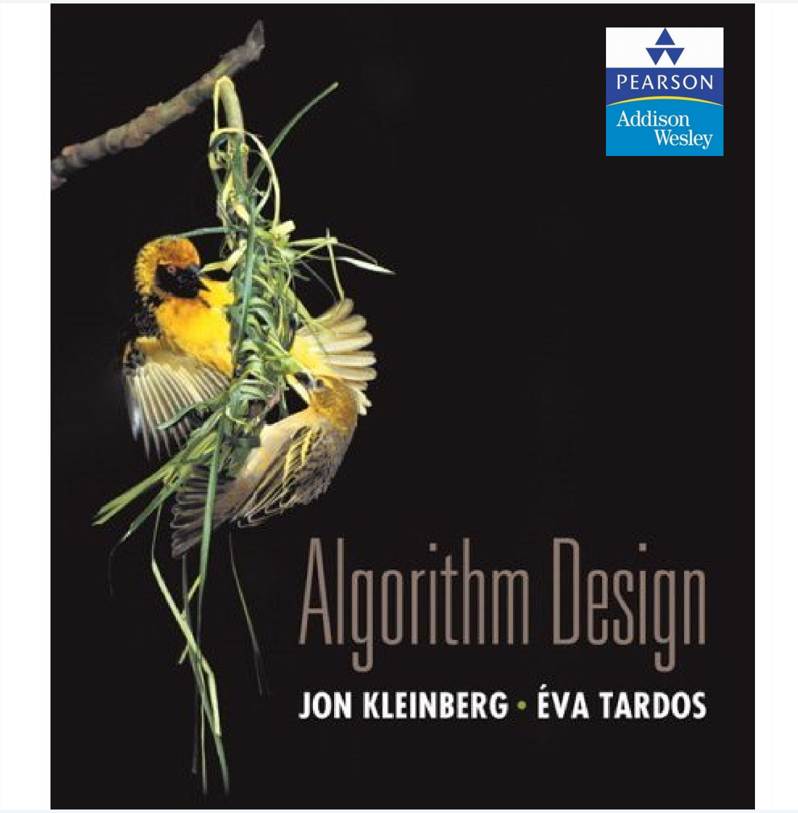
Does a linear-time MST algorithm exist?

deterministic compare-based MST algorithms

year	worst case	discovered by
1975	$O(m \log \log n)$	Yao
1976	$O(m \log \log n)$	Cheriton-Tarjan
1984	$O(m \log^* n)$ $O(m + n \log n)$	Fredman-Tarjan
1986	$O(m \log (\log^* n))$	Gabow-Galil-Spencer-Tarjan
1997	$O(m\, \alpha(n) \log \alpha(n))$	Chazelle
2000	$O(m\, \alpha(n))$	Chazelle
2002	*optimal*	Pettie-Ramachandran
20xx	$O(m)$	???

Remark 1. $O(m)$ **randomized MST algorithm.** [Karger-Klein-Tarjan 1995]

Remark 2. $O(m)$ **MST verification algorithm.** [Dixon-Rauch-Tarjan 1992]



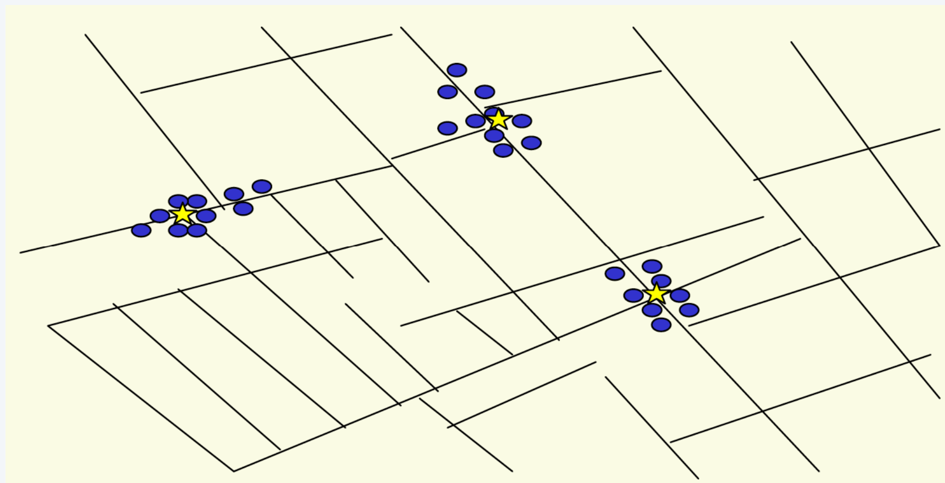
SECTION 4.7

4. GREEDY ALGORITHMS II

- *Dijkstra's algorithm*
- *minimum spanning trees*
- *Prim, Kruskal, Boruvka*
- ***single-link clustering***
- *min-cost arborescences*

Clustering

Goal. Given a set U of n objects labeled $p_1, \ldots, p_n$, partition into clusters so that objects in different clusters are far apart.



outbreak of cholera deaths in London in 1850s (Nina Mishra)

Applications.

- Routing in mobile ad hoc networks.
- Document categorization for web search.
- Similarity searching in medical image databases
- Skycat: cluster 10^9 sky objects into stars, quasars, galaxies.
- ...

Clustering of maximum spacing

k-clustering. Divide objects into k non-empty groups.

Distance function. Numeric value specifying "closeness" of two objects.

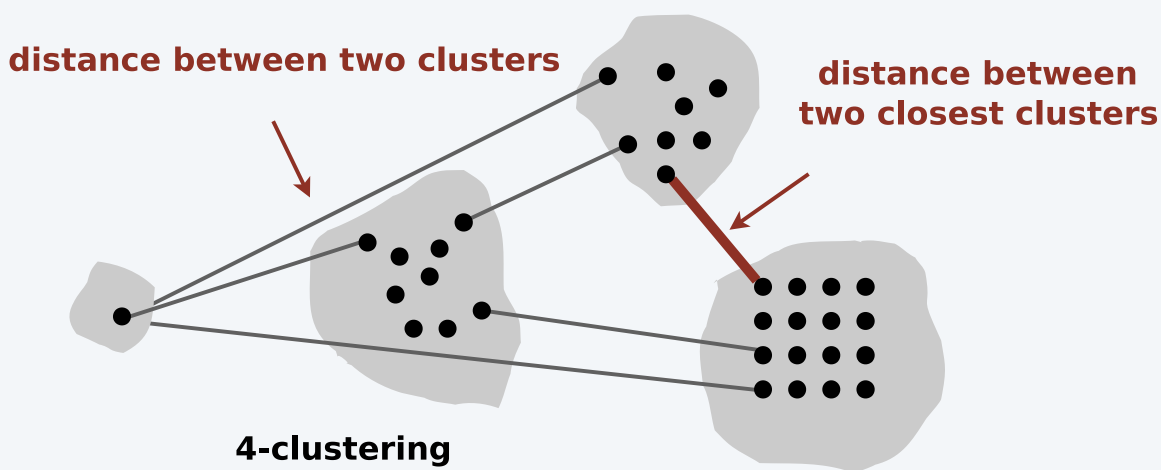
$d(p_i, p_j) = 0$ **iff** $p_i = p_j$ [identity of indiscernibles]

$d(p_i, p_j) \geq 0$ [nonnegativity]

$d(p_i, p_j) = d(p_j, p_i)$ [symmetry]

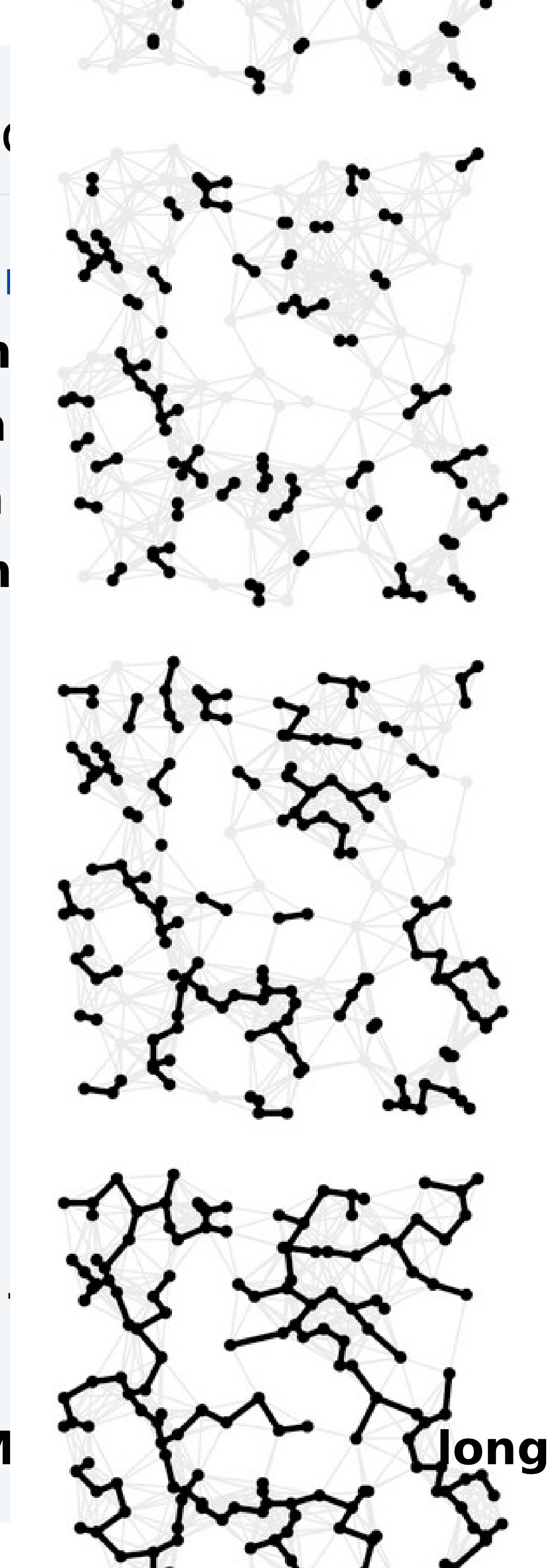
Spacing. Min distance between any pair of points in different clusters.

Goal. Given an integer k, find a k-clustering of maximum spacing.



4-clustering

Greedy clustering alg



“Well-known” algorith ... re for single-linkage k-cluster

- **Form a graph on th ... ing to n clusters.**
- **Find the closest pa ... t each object is in a different cluster, and add an ... .**
- **Repeat $n - k$ times un ... sters.**

Key observation. **This ... ly Kruskal's algorithm (except we stop when ... nponents).**

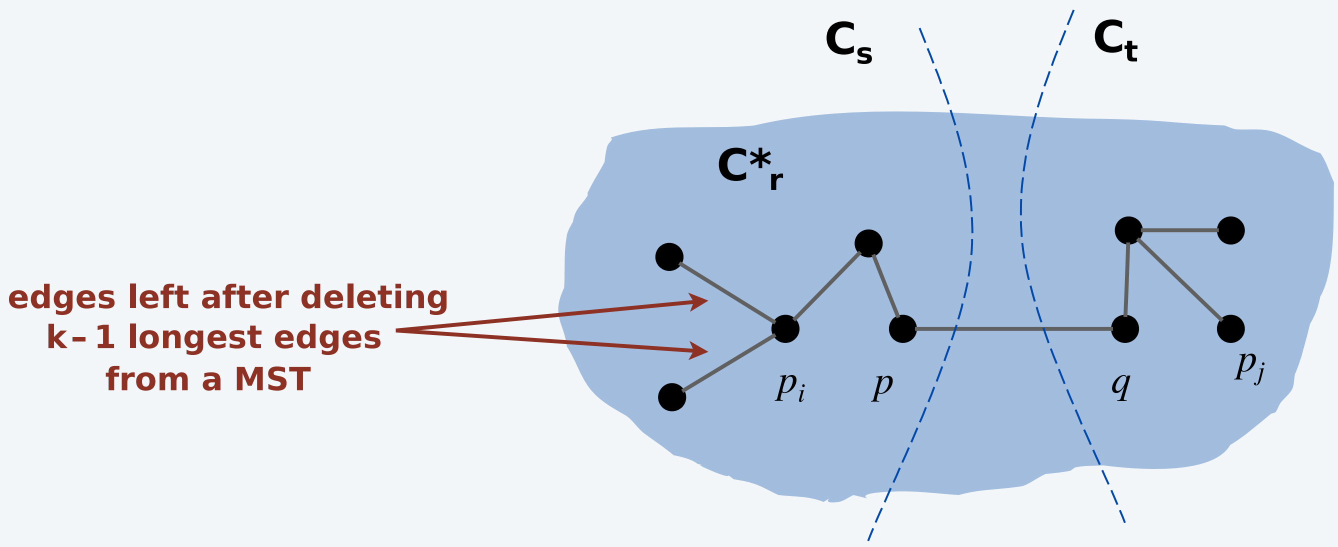
Alternative. **Find an M ... longest edges.**

Greedy clustering algorithm: analysis

Theorem. **Let C^* denote the clustering $C^*_1, \ldots, C^*_k$ formed by deleting the $k-1$ longest edges of an MST. Then, C^* is a k-clustering of max spacing.**

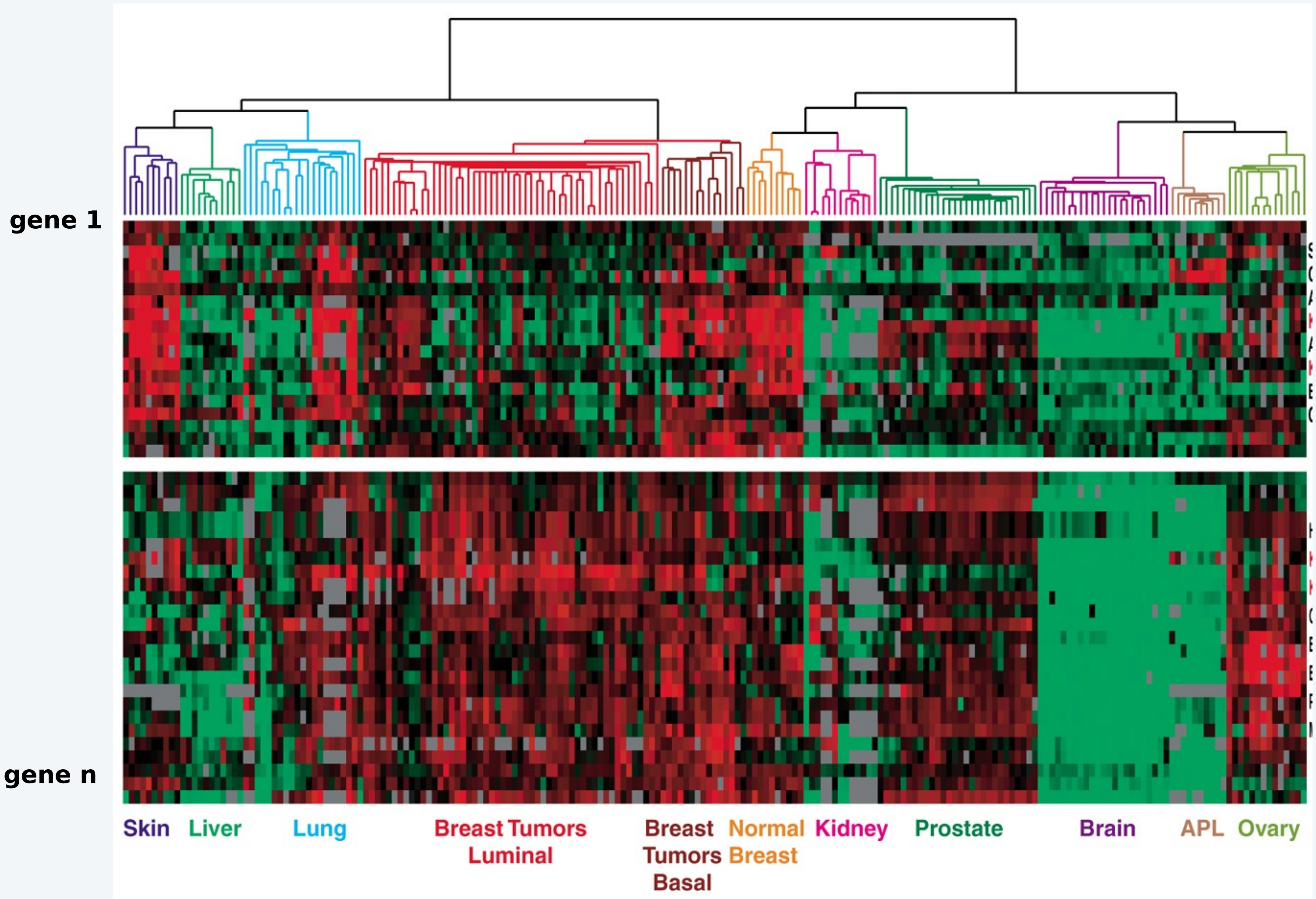
Pf. **Let C denote some other clustering $C_1, \ldots, C_k$.**

- **The spacing of C^* is the length d^* of the $(k-1)^{st}$ longest edge in MST.**
- **Let p_i and p_j be in the same cluster in C^*, say C^*_r, but different clusters in C, say C_s and C_t.**
- **Some edge (p, q) on $p_i - p_j$ path in C^*_r spans two different clusters in C.**
- **Edge (p, q) has length $\le d^*$ since it wasn't deleted.**
- **Spacing of C is $\le d^*$ since p and q are in different clusters. ▪**



Dendrogram of cancers in human

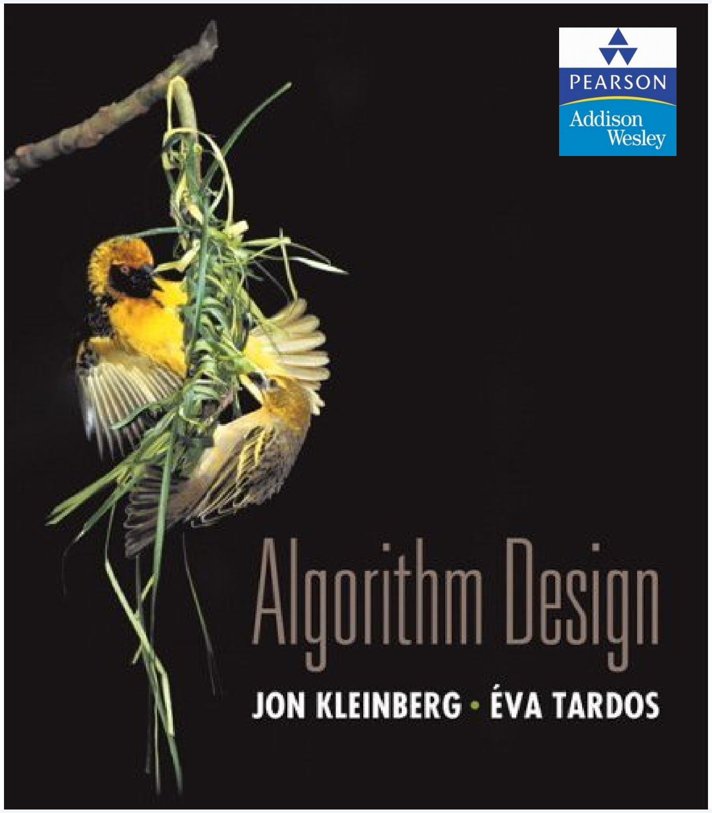
Tumors in similar tissues cluster together.



Reference: Botstein & Brown group

gene expressed

gene not expressed



SECTION 4.9

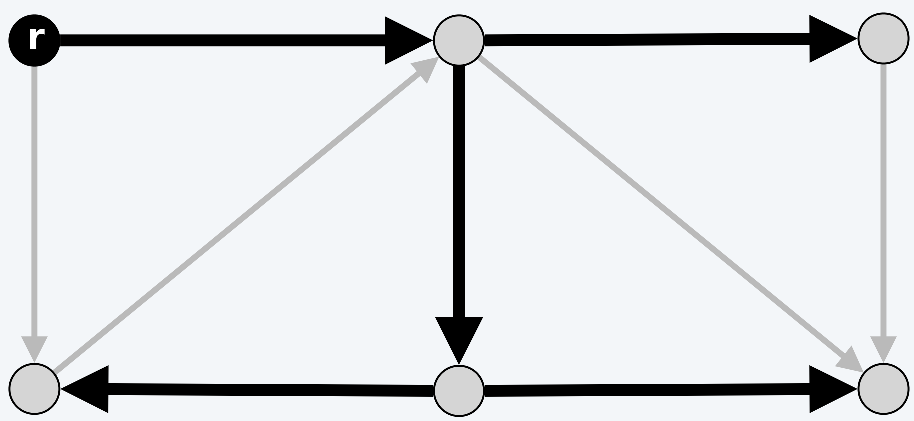
4. GREEDY ALGORITHMS II

- *Dijkstra's algorithm*
- *minimum spanning trees*
- *Prim, Kruskal, Boruvka*
- *single-link clustering*
- ***min-cost arborescences***

Arborescences

Def. Given a digraph $G = (V, E)$ and a root $r \in V$, an arborescence (rooted at r) is a subgraph $T = (V, F)$ such that

- T is a spanning tree of G if we ignore the direction of edges.
- There is a directed path in T from r to each other node $v \in V$.



Warmup. Given a digraph G, find an arborescence rooted at r (if one exists).

Algorithm. BFS or DFS from r is an arborescence (iff all nodes reachable).

Arborescences

Def. Given a digraph $G = (V, E)$ and a root $r \in V$, an arborescence (rooted at r) is a subgraph $T = (V, F)$ such that

- T is a spanning tree of G if we ignore the direction of edges.
- There is a directed path in T from r to each other node $v \in V$.

Proposition. A subgraph $T = (V, F)$ of G is an arborescence rooted at r iff T has no directed cycles and each node $v \neq r$ has exactly one entering edge.

Pf.

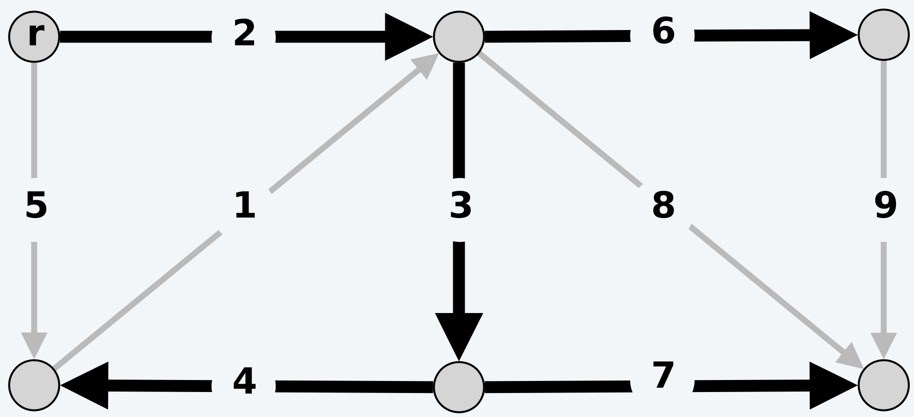
⇒ If T is an arborescence, then no (directed) cycles and every node $v \neq r$ has exactly one entering edge—the last edge on the unique $r \leadsto v$ path.

⇐ Suppose T has no cycles and each node $v \neq r$ has one entering edge.

- To construct an $r \leadsto v$ path, start at v and repeatedly follow edges in the backward direction.
- Since T has no directed cycles, the process must terminate.
- It must terminate at r since r is the only node with no entering edge. ■

Min-cost arborescence problem

Problem. **Given a digraph G with a root node r and with a nonnegative cost $c_e \geq 0$ on each edge e, compute an arborescence rooted at r of minimum cost.**



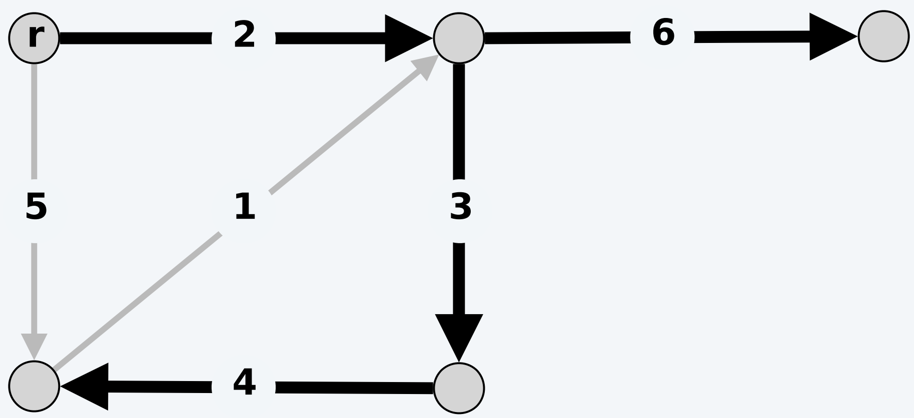
Assumption 1. **G has an arborescence rooted at r.**

Assumption 2. **No edge enters r (safe to delete since they won't help).**

Simple greedy approaches do not work

Observations. **A min-cost arborescence need not:**

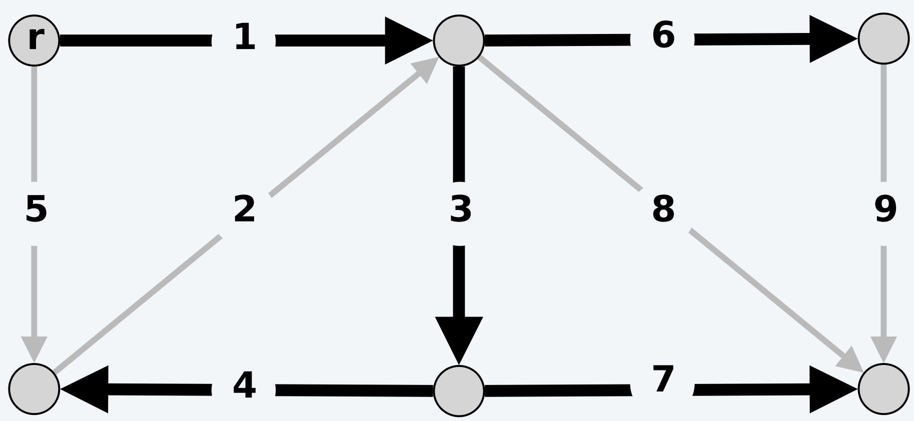
- **Be a shortest-paths tree.**
- **Include the cheapest edge (in some cut).**
- **Exclude the most expensive edge (in some cycle).**



A sufficient optimality condition

Property. **For each node $v \neq r$, choose one cheapest edge entering v and let F^* denote this set of $n - 1$ edges. If (V, F^*) is an arborescence, then it is a min-cost arborescence.**

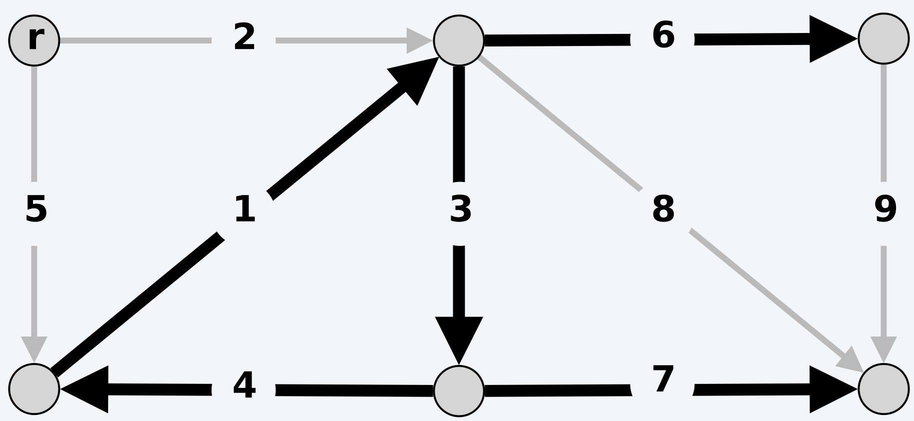
Pf. **An arborescence needs exactly one edge entering each node $v \neq r$ and (V, F^*) is the cheapest way to make these choices. ▪**



A sufficient optimality condition

Property. **For each node $v \neq r$, choose one cheapest edge entering v and let F^* denote this set of $n - 1$ edges. If (V, F^*) is an arborescence, then it is a min-cost arborescence.**

Note. F^* **may not be an arborescence (since it may have directed cycles).**

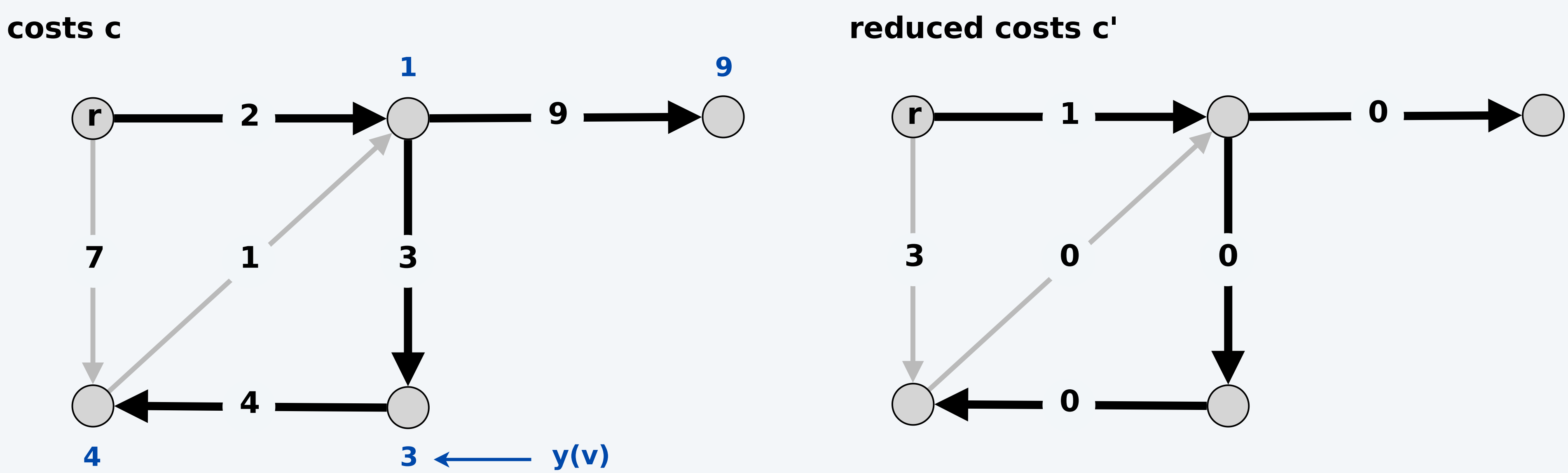


Reduced costs

Def. For each $v \neq r$, let $y(v)$ denote the min cost of any edge entering v.
The **reduced cost** of an edge (u, v) is $c'(u, v) = c(u, v) - y(v) \ \geq\ 0$.

Observation. T is a min-cost arborescence in G using costs c iff
T is a min-cost arborescence in G using reduced costs c'.

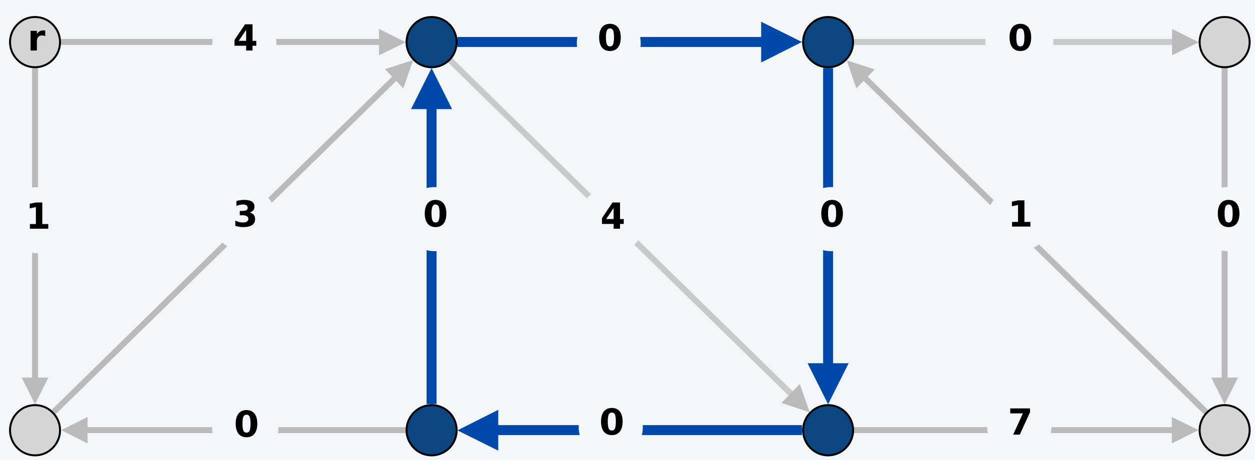
Pf. Each arborescence has exactly one edge entering v.



Edmonds branching algorithm: intuition

Intuition. **Recall F^* = set of cheapest edges entering v for each $v \neq r$.**

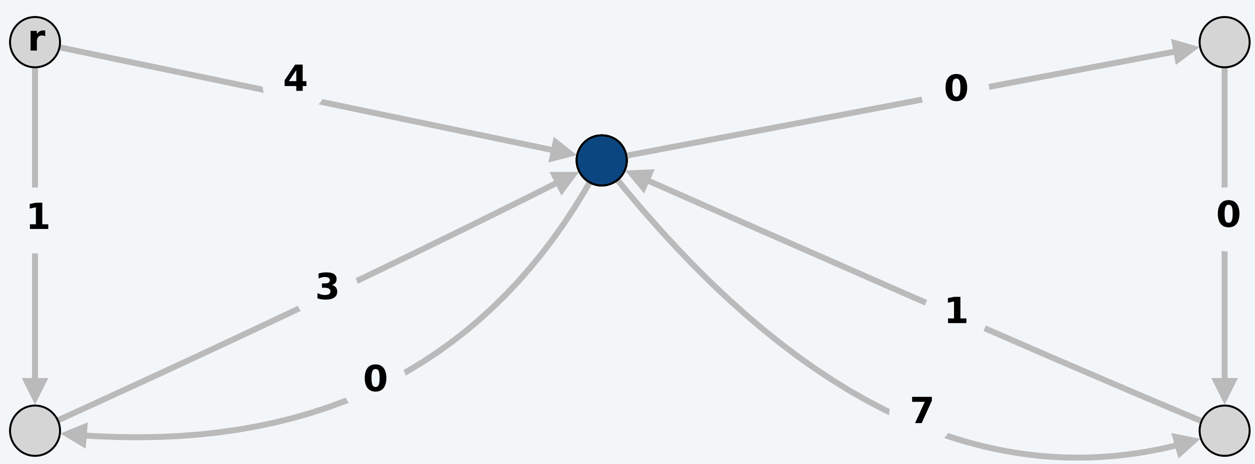
- **Now, all edges in F^* have 0 cost with respect to costs $c'(u, v)$.**
- **If F^* does not contain a cycle, then it is a min-cost arborescence.**
- **If F^* contains a cycle C, can afford to use as many edges in C as desired.**
- **Contract nodes** **in C into a supernode.**
- **Recursively solve problem in contracted network G' with costs $c'(u, v)$.**



Edmonds branching algorithm: intuition

Intuition. Recall F^* = set of cheapest edges entering v for each $v \neq r$.

- Now, all edges in F^* have 0 cost with respect to costs $c'(u, v)$.
- If F^* does not contain a cycle, then it is a min-cost arborescence.
- If F^* contains a cycle C, can afford to use as many edges in C as desired.
- Contract nodes C into a supernode (removing any self-loops).
- Recursively solve problem in contracted network G' with costs $c'(u, v)$.



Edmonds branching algorithm

EDMONDSBRANCHING(G, r, c)

FOREACH $v \neq r$

$y(v) \leftarrow$ min cost of an edge entering v.

$c'(u, v) \leftarrow c(u, v) - y(v)$ for each edge (u, v) entering v.

FOREACH $v \neq r$: choose one 0-cost edge entering v and let F^* be the resulting set of edges.

IF F^* forms an arborescence, RETURN $T = (V, F^*)$.

ELSE

$C \leftarrow$ directed cycle in F^*.

Contract C to a single supernode, yielding $G' = (V', E')$.

$T' \leftarrow$ EDMONDSBRANCHING(G', r, c')

Extend T' to an arborescence T in G by adding all but one edge of C.

RETURN T.

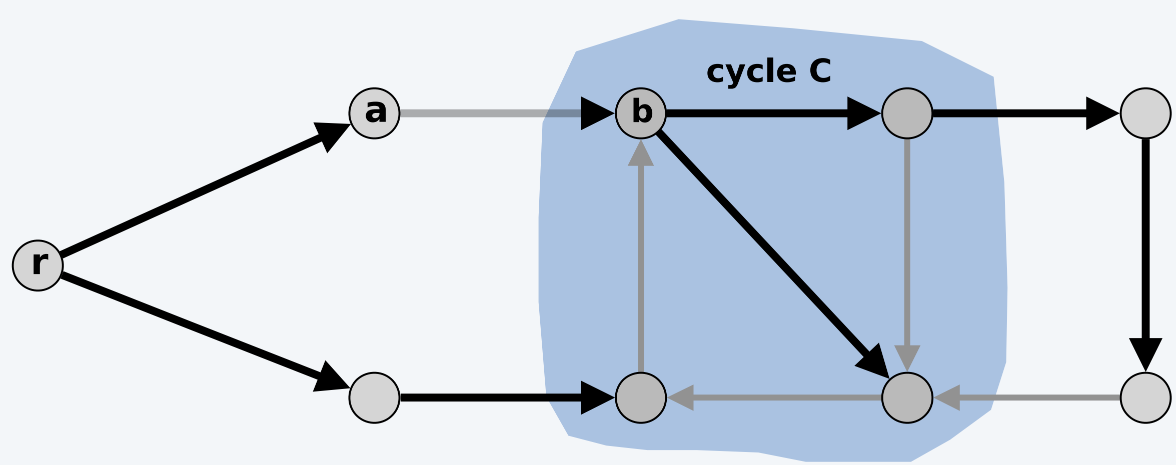
Edmonds branching algorithm

Q. **What could go wrong?**

A.

- **Min-cost arborescence in G' has exactly one edge entering a node in C (since C is contracted to a single node)**
- **But min-cost arborescence in G might have more edges entering C.**

min-cost arborescence in G



Edmonds branching algorithm: key lemma

Lemma. **Let C be a cycle in G consisting of 0-cost edges. There exists a min-cost arborescence rooted at r that has exactly one edge entering C.**

Pf. **Let T be a min-cost arborescence rooted at r.**

Case 0. **T has no edges entering C.**
Since T is an arborescence, there is an $r \leadsto v$ path for each node v $\Rightarrow$ at least one edge enters C.

Case 1. **T has exactly one edge entering C.**
T satisfies the lemma.

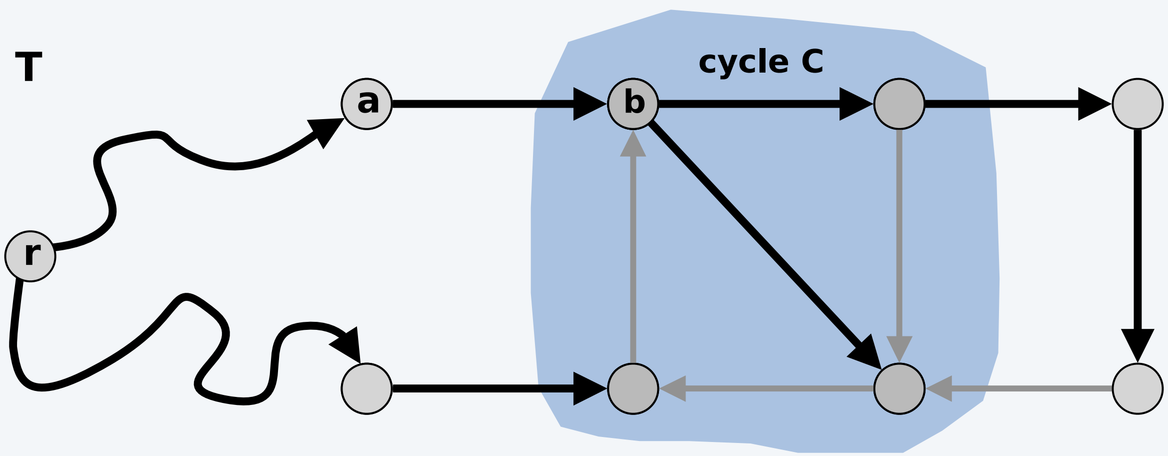
Case 2. **T has more than one edge that enters C.**
We construct another min-cost arborescence T' that has exactly one edge entering C.

Edmonds branching algorithm: key lemma

Case 2 construction of T'

- Let (a, b) be an edge in T entering C that lies on a shortest path from r.
- We delete all edges of T that enter a node in C except (a, b).
- We add in all edges of C except the one that enters b.

path from r to C uses only one node in C



Edmonds branching algorithm: key lemma

Case 2 construction of T'.

- **Let (a, b) be an edge in T entering C that lies on a shortest path from r.**
- **We delete all edges of T that enter a node in C except (a, b).**
- **We add in all edges of C except the one that enters b.**

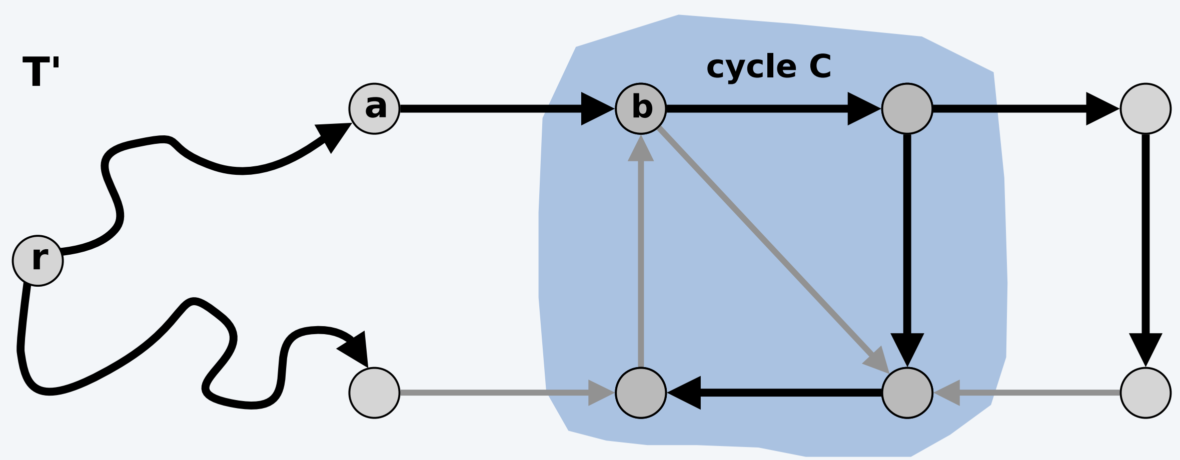
path from r to C uses only one node in C

Claim. T' is a min-cost arborescence.

- **The cost of T' is at most that of T since we add only 0-cost edges.**
- **T' has exactly one edge entering each node $v \neq r$.**
- **T' has no directed cycles.**
  (T had no cycles before; no cycles within C; now only (a, b) enters C)

T is an arborescence rooted at r

and the only path in T' to a is the path from r to a (since any path must follow unique entering edge back to r)



Edmonds branching algorithm: analysis

Theorem. [Chu-Liu 1965, Edmonds 1967] **The greedy algorithm finds a min-cost arborescence.**

Pf. [by induction on number of nodes in G]

- If the edges of F^* form an arborescence, then min-cost arborescence.
- Otherwise, we use reduced costs, which is equivalent.
- After contracting a 0-cost cycle C to obtain a smaller graph G', the algorithm finds a min-cost arborescence T' in G' (by induction).
- Key lemma: there exists a min-cost arborescence T in G that corresponds to T'. ▪

Theorem. The greedy algorithm can be implemented in $O(mn)$ time.

Pf.

- At most n contractions (since each reduces the number of nodes).
- Finding and contracting the cycle C takes $O(m)$ time.
- Transforming T' into T takes $O(m)$ time. ▪

Min-cost arborescence

Theorem. [Gabow-Galil-Spencer-Tarjan 1985] **There exists an $O(m + n \log n)$ time algorithm to compute a min-cost arborescence.**

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EFFICIENT ALGORITHMS FOR FINDING MINIMUM SPANNING TREES IN UNDIRECTED AND DIRECTED GRAPHS

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Received 23 January 1985
Revised 1 December 1985

Recently, Fredman and Tarjan invented a new, especially efficient form of heap (priority queue). Their data structure, the *Fibonacci heap* (or F-heap) supports arbitrary deletion in $O(\log n)$ amortized time and other heap operations in $O(1)$ amortized time. In this paper we use F-heaps to obtain fast algorithms for finding minimum spanning trees in undirected and directed graphs. For an undirected graph containing n vertices and m edges, our minimum spanning tree algorithm runs in $O(m \log \beta(m, n))$ time, improved from $O(m\beta(m, n))$ time, where $\beta(m, n) = \min \{i \mid \log^{(i)} n \leq m/n\}$. Our minimum spanning tree algorithm for directed graphs runs in $O(n \log n + m)$ time, improved from $O(n \log n + m \log \log \log_{(m/n+2)} n)$. Both algorithms can be extended to allow a degree constraint at one vertex.

1. Introduction

A *heap* (sometimes called a *priority queue*) is an abstract data structure consisting of a collection of *items*, each with a real-valued *key*, on which at least the following operations are possible: