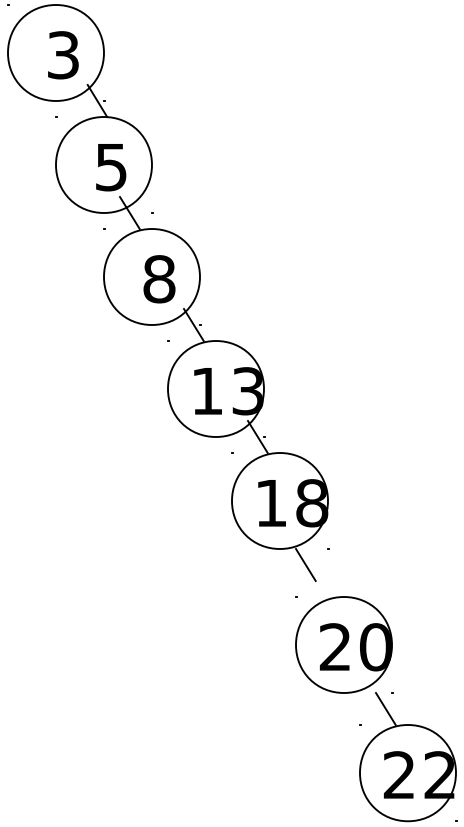
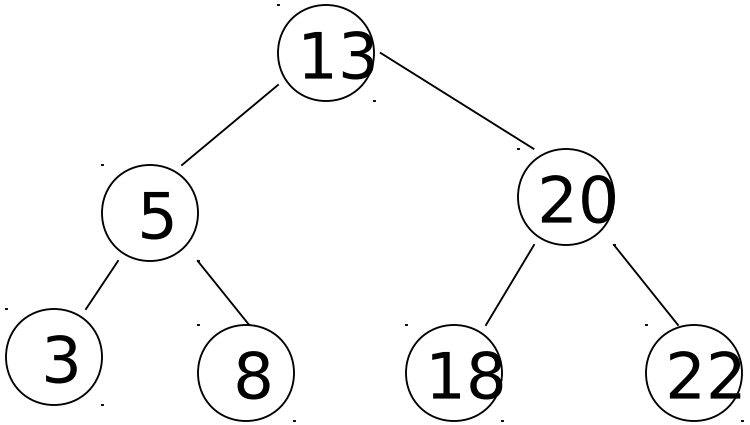
# *Trees 4: AVL Trees*

- *Section 4.4*

# *Motivation*

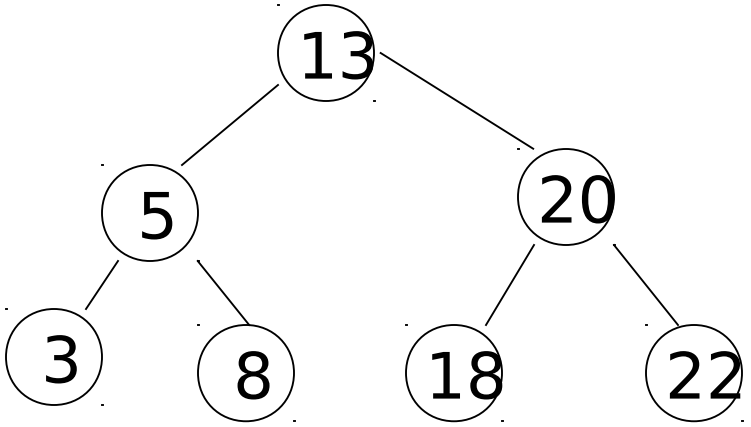
- *When building a binary search tree, what type of trees would we like? Example: 3, 5, 8, 20, 18, 13, 22*

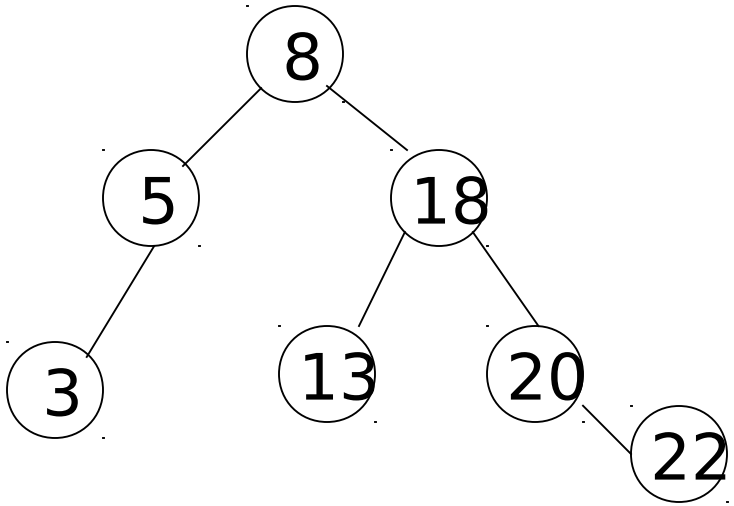


# *Motivation*

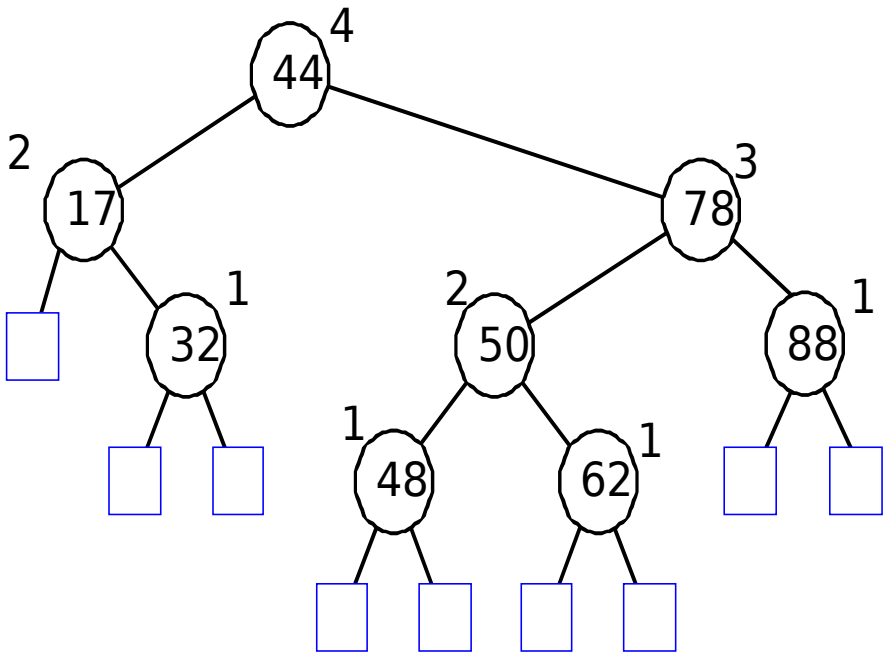
- *Complete binary tree is hard to build when we allow dynamic insert and remove.*
  - *We want a tree that has the following properties*
    - *Tree height = O(log(N))*
    - *allows dynamic insert and remove with O(log(N)) time complexity.*
  - *The AVL tree is one of this kind of trees.*



# *AVL (Adelson-Velskii and Landis) Trees*

- *An AVL Tree is a* **binary search tree** *such that for every internal node v of T, the heights of the children of v can differ by at most 1.*

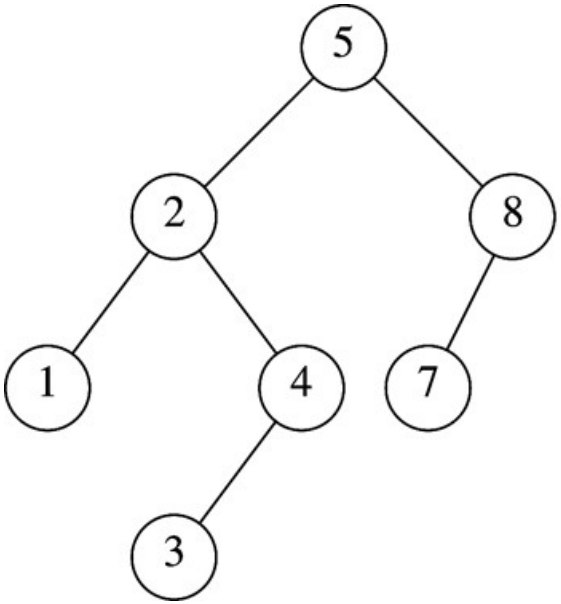


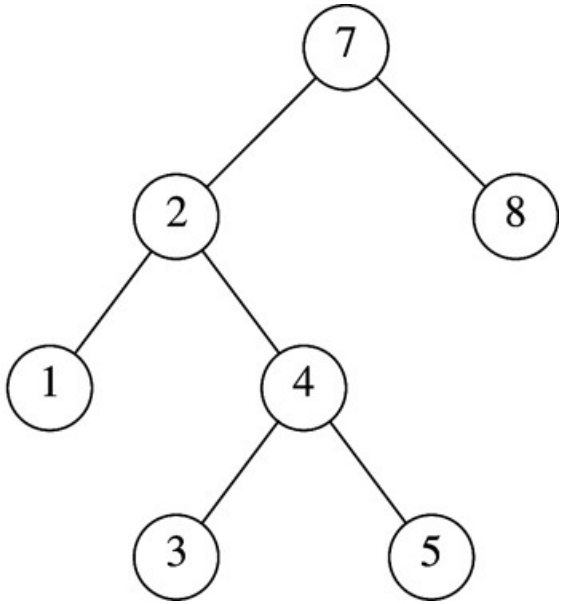

An example of an AVL tree where th
heights are shown next to the node

# AVL (Adelson-Velskii and Landis) Trees

- AVL tree is a binary search tree with balance condition
  - To ensure depth of the tree is O(log(N))
  - And consequently, search/insert/remove complexity bound O(log(N))
- Balance condition
  - For every node in the tree, height of left and right subtree can differ by at most 1

# *Which is an AVL Tree?*



# *Height of an AVL tree*

- *Theorem: The* **height** *of an AVL tree storing n keys is O(log n).*
- **Proof***:*
  - *Let us bound* **n(h),** *the minimum number of internal nodes of an AVL tree of height h.*
  - *We easily see that n(0) = 1 and n(1) = 2*
  - *For h > 2, an AVL tree of height h contains the root node, one AVL subtree of height h-1 and another of height h-2 (at worst).*
  - *That is, n(h) >= 1 + n(h-1) + n(h-2)*
  - *Knowing n(h-1) > n(h-2), we get n(h) > 2n(h-2). So*

    $n(h) > 2n(h-2)$, $n(h) > 4n(h-4)$, $n(h) > 8n(n-6)$, … *(by induction),*

    $n(h) > 2^i n(h-2i)$
  - *Solving the base case we get:* $n(h) > 2^{h/2-1}$
  - *Taking logarithms:* $h < 2\log n(h) + 2$
  - *Since n>=n(h),* $h < 2\log(n)+2$ *and the height of an AVL tree is O(log n)*

# *AVL Tree Insert and Remove*

- *Do binary search tree insert and remove*
- *The balance condition can be violated sometimes*
  - *Do something to fix it : rotations*
  - *After rotations, the balance of the whole tree is maintained*

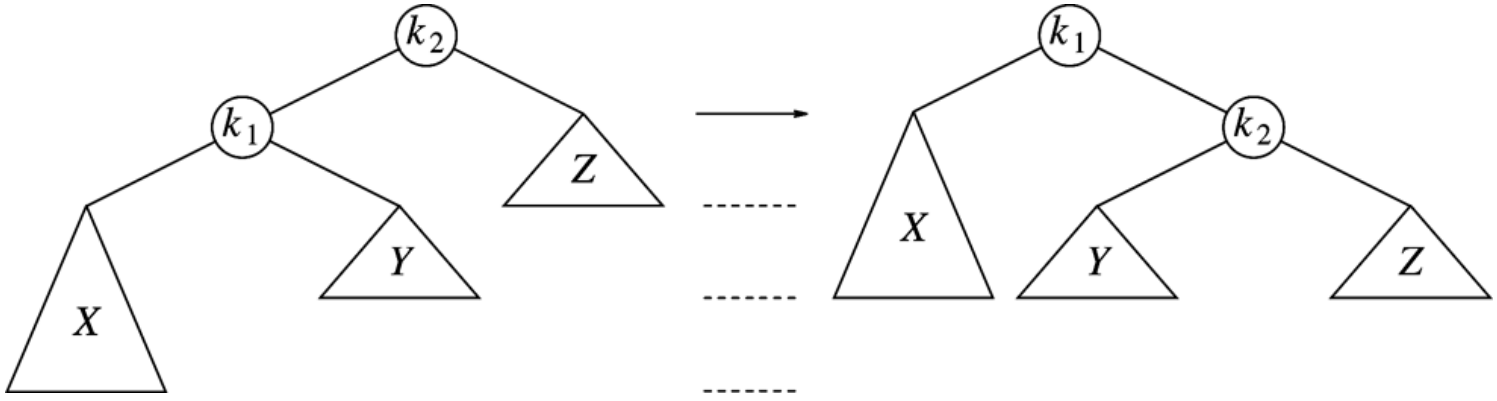
# *Balance Condition Violation*

- *If condition violated after a node insertion*
  - *Which nodes do we need to rotate?*
  - *Only nodes on path from insertion point to root may have their balance altered*
- *Rebalance the tree through rotation at the deepest node with balance violated*
  - *The entire tree will be rebalanced*
- *Violation cases at node k (deepest node)*
  1. *An insertion into left subtree of left child of k*
  2. *An insertion into right subtree of left child of k*
  3. *An insertion into left subtree of right child of k*
  4. *An insertion into right subtree of right child of k*
  - *Cases 1 and 4 equivalent*
    - *Single rotation to rebalance*
  - *Cases 2 and 3 equivalent*
    - *Double rotation to rebalance*

# *AVL Trees Complexity*

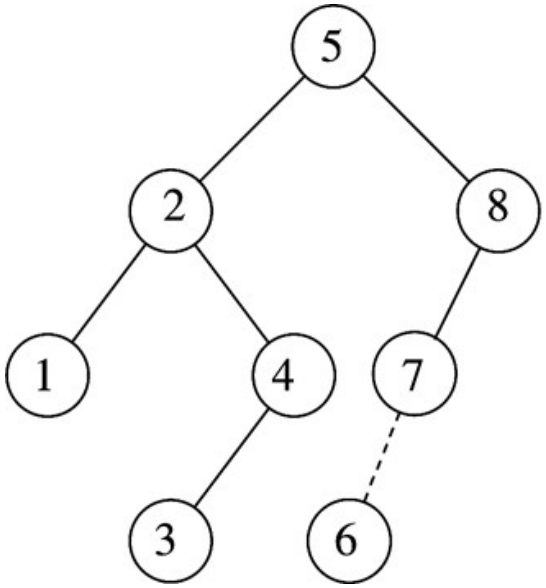
- *Overhead*
  - *Extra space for maintaining height information at each node*
  - *Insertion and deletion become more complicated, but still O(log N)*
- *Advantage*
  - *Worst case O(log(N)) for insert, delete, and search*

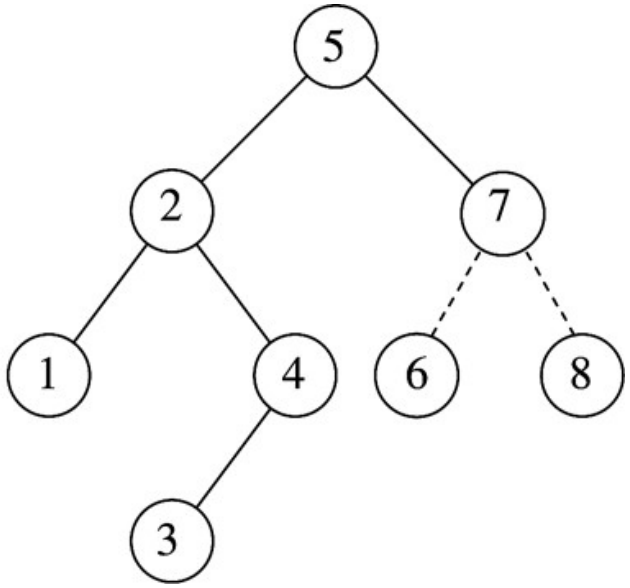
# *Single Rotation (Case 1)*




- *Replace node $k_2$ by node $k_1$*
- *Set node $k_2$ to be right child of node $k_1$*
- *Set subtree Y to be left child of node $k_2$*
- *Case 4 is similar*

# *Example*

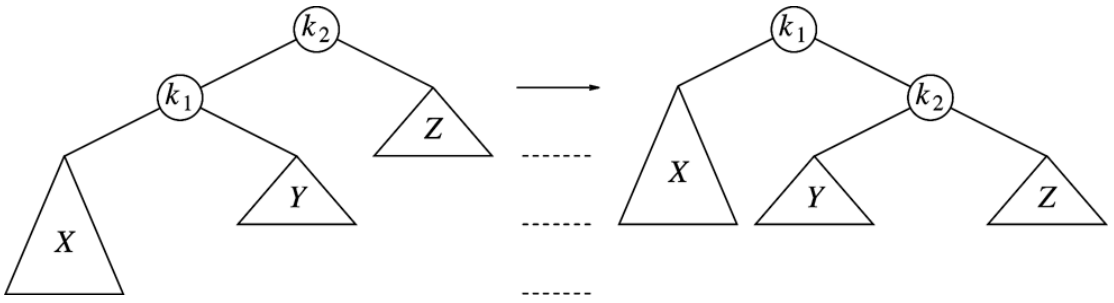



- *After inserting 6*
  - *Balance condition at node 8 is violated*

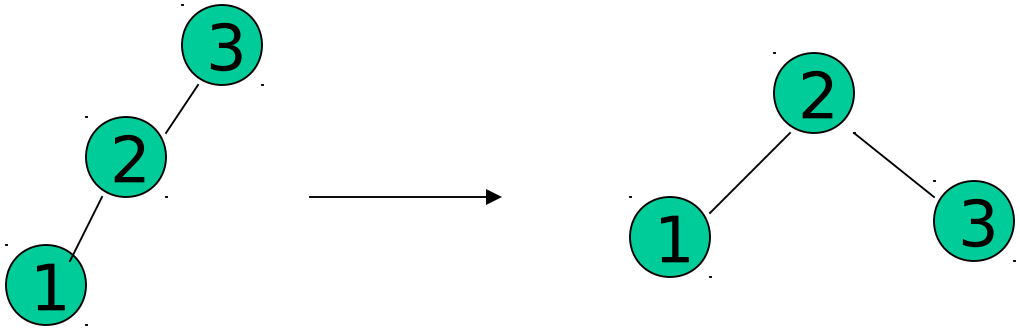
# *Single Rotation (Case 1)*

```
1       /**
2        * Rotate binary tree node with left child.
3        * For AVL trees, this is a single rotation for case 1.
4        * Update heights, then set new root.
5        */
6       void rotateWithLeftChild( AvlNode * & k2 )
7       {
8           AvlNode *k1 = k2->left;
9           k2->left = k1->right;
10          k1->right = k2;
11          k2->height = max( height( k2->left ), height( k2->right ) ) + 1;
12          k1->height = max( height( k1->left ), k2->height ) + 1;
13          k2 = k1;
14      }
```



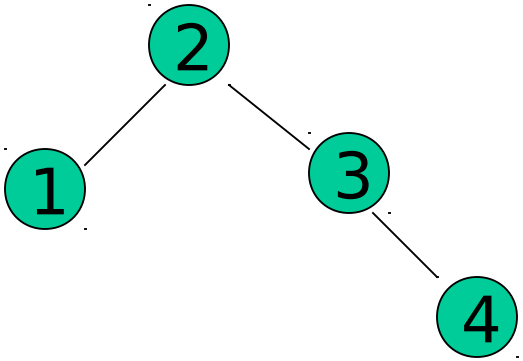
# *Example*

- *Inserting 3, 2, 1, and then 4 to 7 sequentially into empty AVL tree*

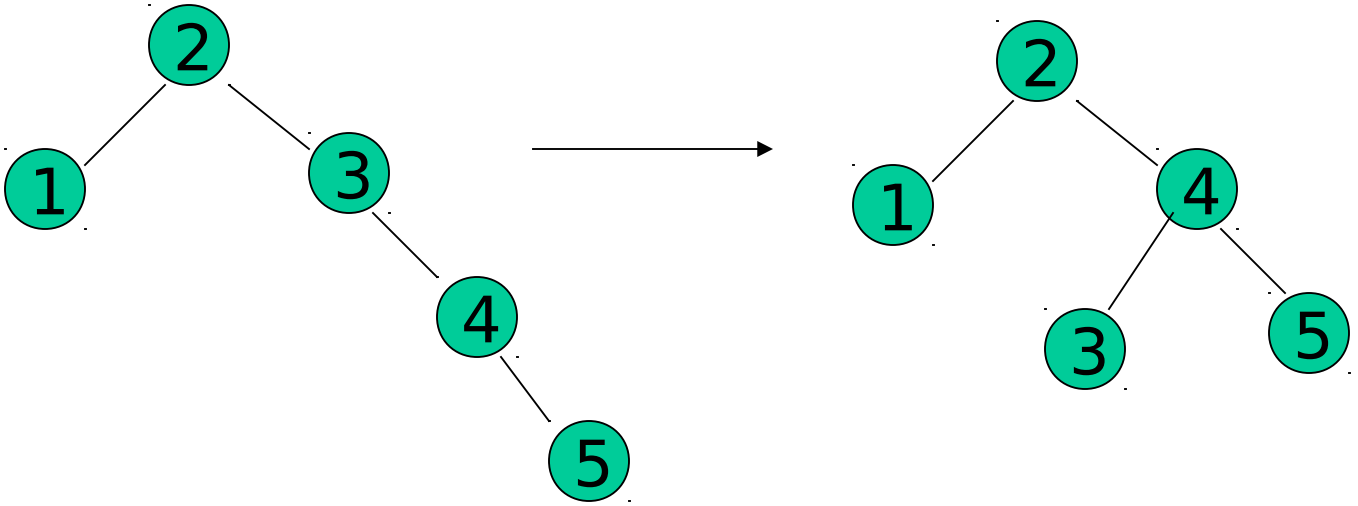


# *Example (Cont'd)*

- *Inserting 4*

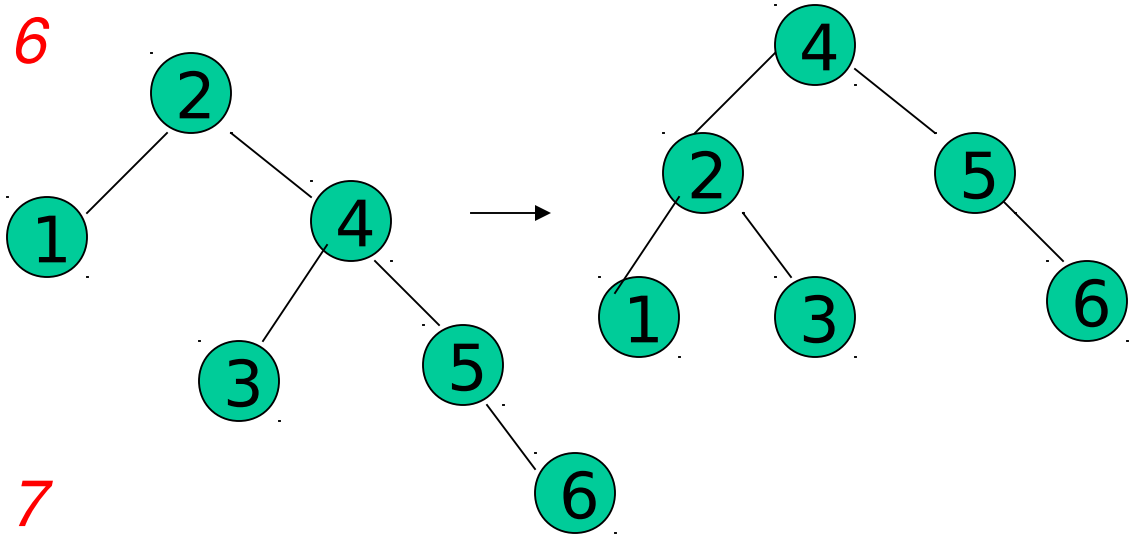



- *Inserting 5*

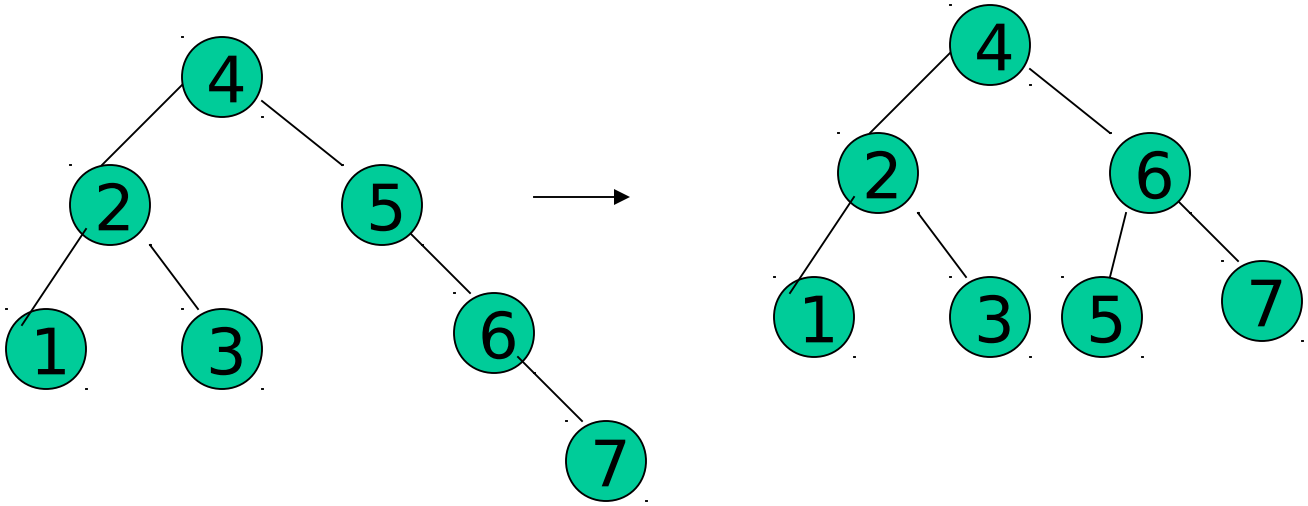


# Example (Cont'd)

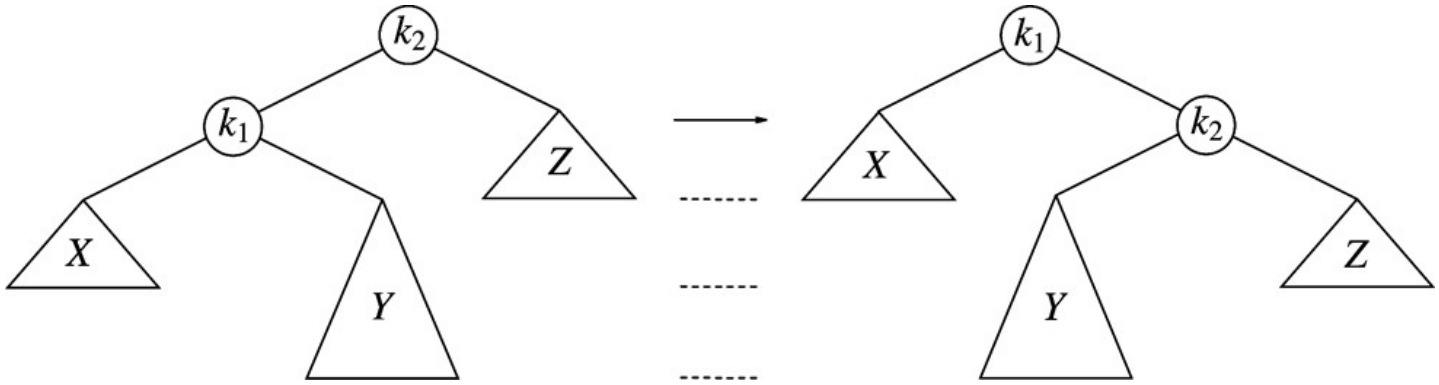
- *Inserting 6*




- *Inserting 7*

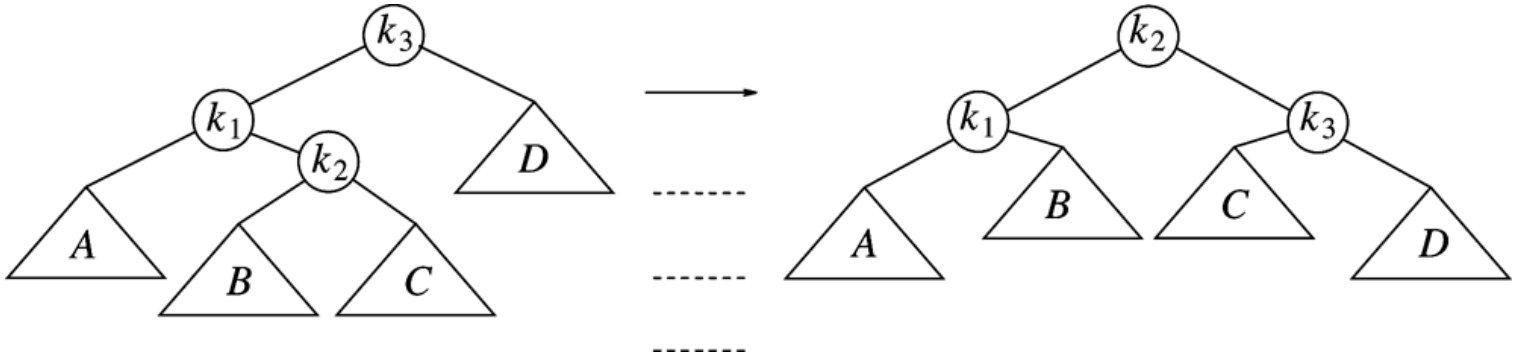


# *Single Rotation Will Not Work for the Other Case*




- *For case 2*
- *After single rotation, $k_1$ still not balanced*
- *Double rotations needed for case 2 and case 3*

# *Double Rotation (Case 2)*

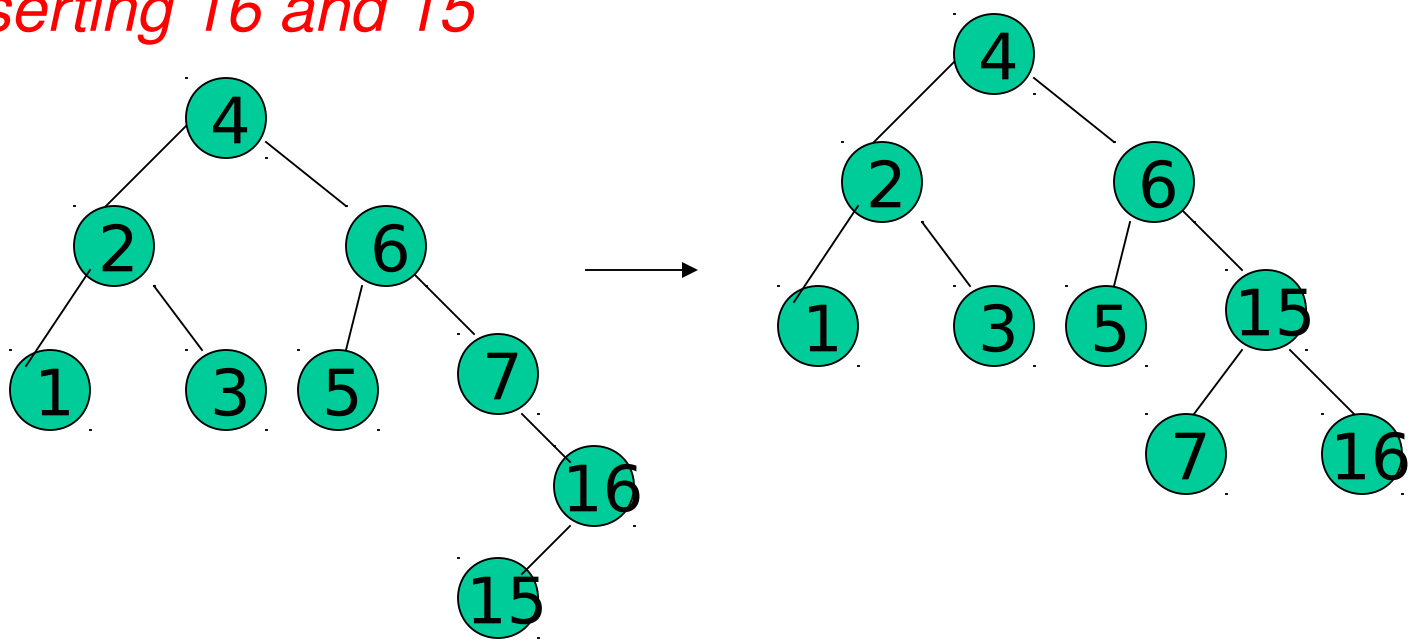



- *Left-right double rotation to fix case 2*
- *First rotate between* $\mathbf{k_1}$ *and* $\mathbf{k_2}$
- *Then rotate between* $\mathbf{k_2}$ *and* $\mathbf{k_3}$
- *Case 3 is similar*

# Example

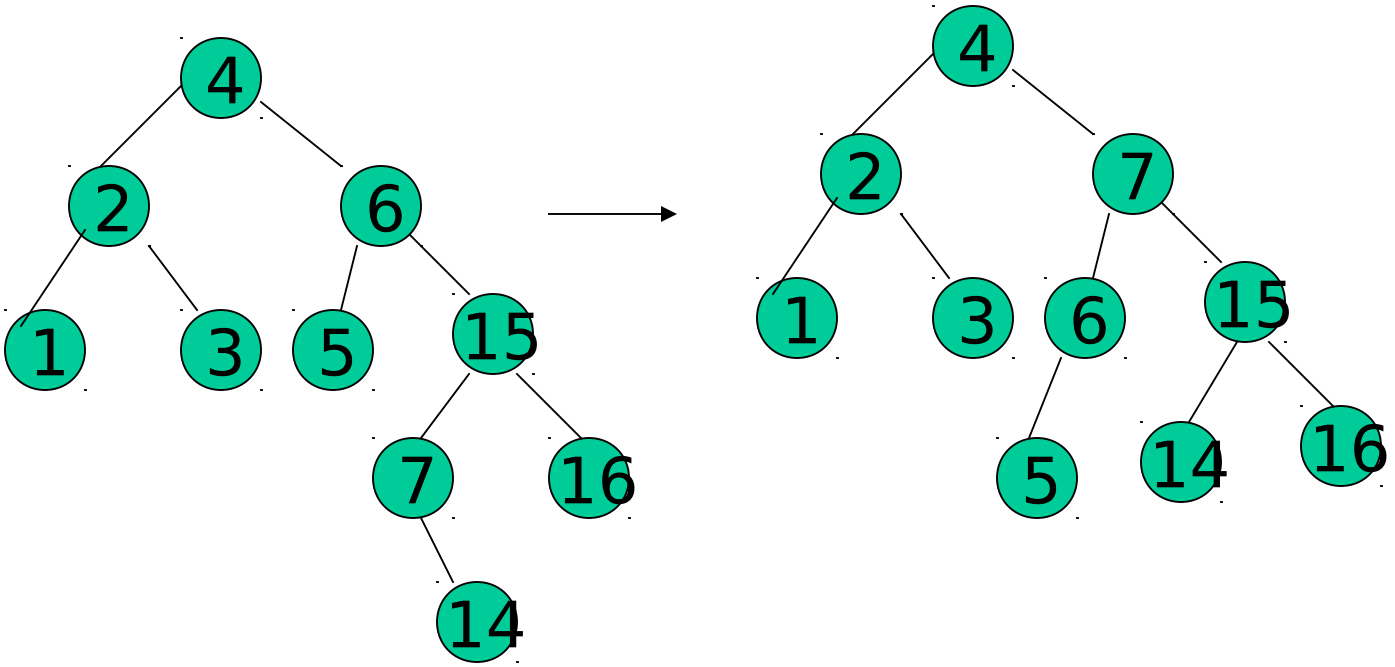
- *Continuing the previous example by inserting*
  - *16 down to 10, and then 8 and 9*

- *Inserting 16 and 15*



# *Example (Cont'd)*

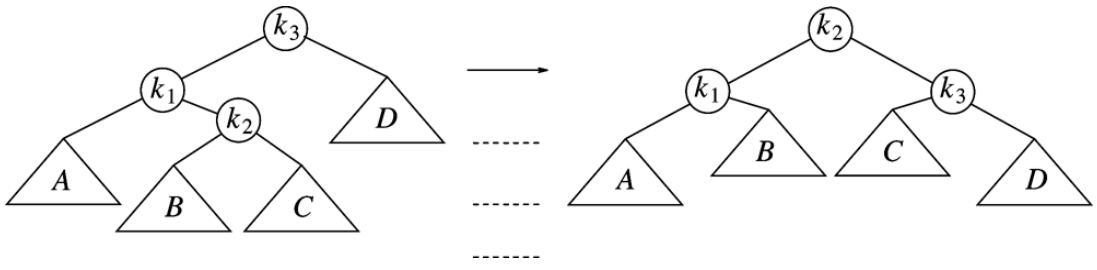
- *Inserting 14*




- *Other cases as exercises*

# *Double Rotation (Case 2)*

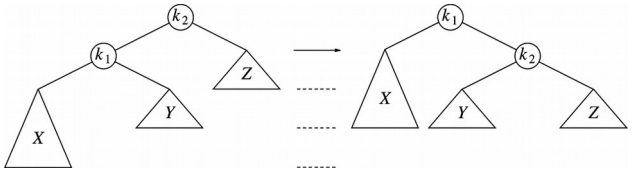
```
1       /**
2        * Double rotate binary tree node: first left child
3        * with its right child; then node k3 with new left child.
4        * For AVL trees, this is a double rotation for case 2.
5        * Update heights, then set new root.
6        */
7       void doubleWithLeftChild( AvlNode * & k3 )
8       {
9           rotateWithRightChild( k3->left );
10          rotateWithLeftChild( k3 );
11      }
```



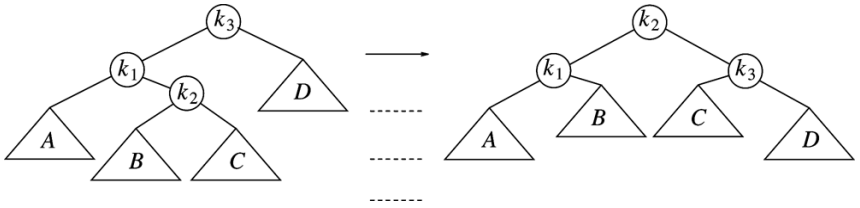
# *Summary*

Violation cases at node k (deepest node)

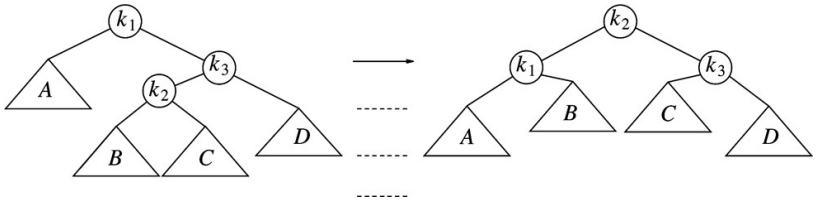
1. An insertion into left subtree of left child of k
2. An insertion into right subtree of left child of k
3. An insertion into left subtree of right child of k
4. An insertion into right subtree of right child of k




Case 1




Case 2




Case 3

Case 4?

# *Implementation of AVL Tree*

```
1       struct AvlNode
2       {
3           Comparable element;
4           AvlNode   *left;
5           AvlNode   *right;
6           int       height;
7
8           AvlNode( const Comparable & theElement, AvlNode *lt,
9                                                   AvlNode *rt, int h = 0 )
10            : element( theElement ), left( lt ), right( rt ), height( h )
11      };
```

```
1        /**
2         * Return the height of node t or -1 if NULL.
3         */
4        int height( AvlNode *t ) const
5        {
6            return t == NULL ? -1 : t->height;
7        }
```

```
1       /**
2        * Internal method to insert into a subtree.
3        * x is the item to insert.
4        * t is the node that roots the subtree.
5        * Set the new root of the subtree.
6        */
7       void insert( const Comparable & x, AvlNode * & t )
8       {
9           if( t == NULL )
10              t = new AvlNode( x, NULL, NULL );
11          else if( x < t->element )
12          {
13              insert( x, t->left );
14              if( height( t->left ) - height( t->right ) == 2 )
15                  if( x < t->left->element )
16                      rotateWithLeftChild( t );
17                  else
18                      doubleWithLeftChild( t );
19          }
20          else if( t->element < x )
21          {
22              insert( x, t->right );
23              if( height( t->right ) - height( t->left ) == 2 )
24                  if( t->right->element < x )
25                      rotateWithRightChild( t );
26                  else
27                      doubleWithRightChild( t );
28          }
29          else
30              ;  // Duplicate; do nothing
31          t->height = max( height( t->left ), height( t->right ) ) + 1;
32      }
```

Case 1

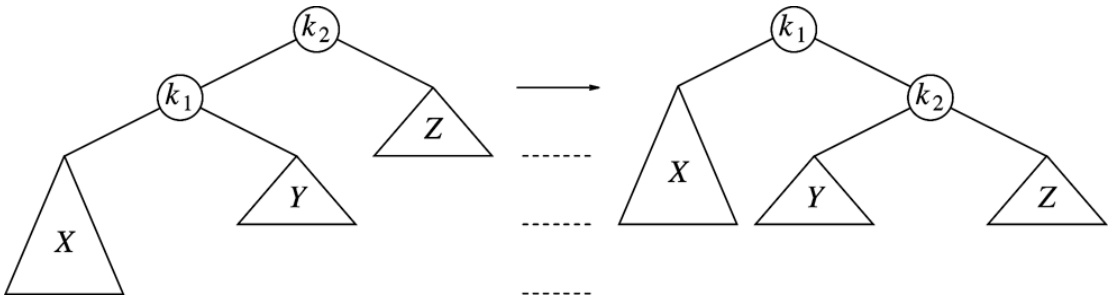
Case 2

Case 4

Case 3

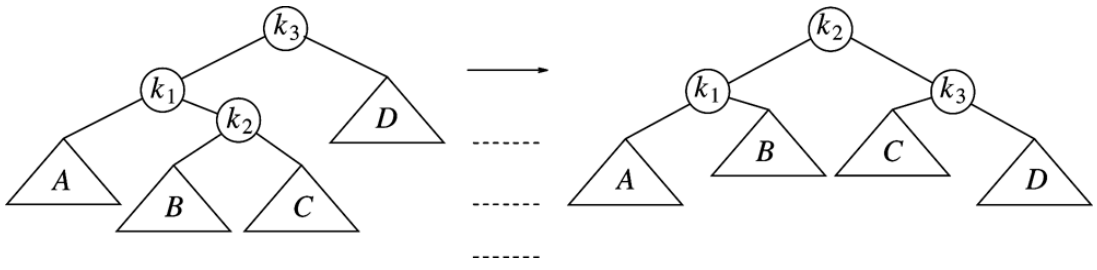
# Single Rotation (Case 1)

```
1   /**
2    * Rotate binary tree node with left child.
3    * For AVL trees, this is a single rotation for case 1.
4    * Update heights, then set new root.
5    */
6   void rotateWithLeftChild( AvlNode * & k2 )
7   {
8       AvlNode *k1 = k2->left;
9       k2->left = k1->right;
10      k1->right = k2;
11      k2->height = max( height( k2->left ), height( k2->right ) ) + 1;
12      k1->height = max( height( k1->left ), k2->height ) + 1;
13      k2 = k1;
14  }
```

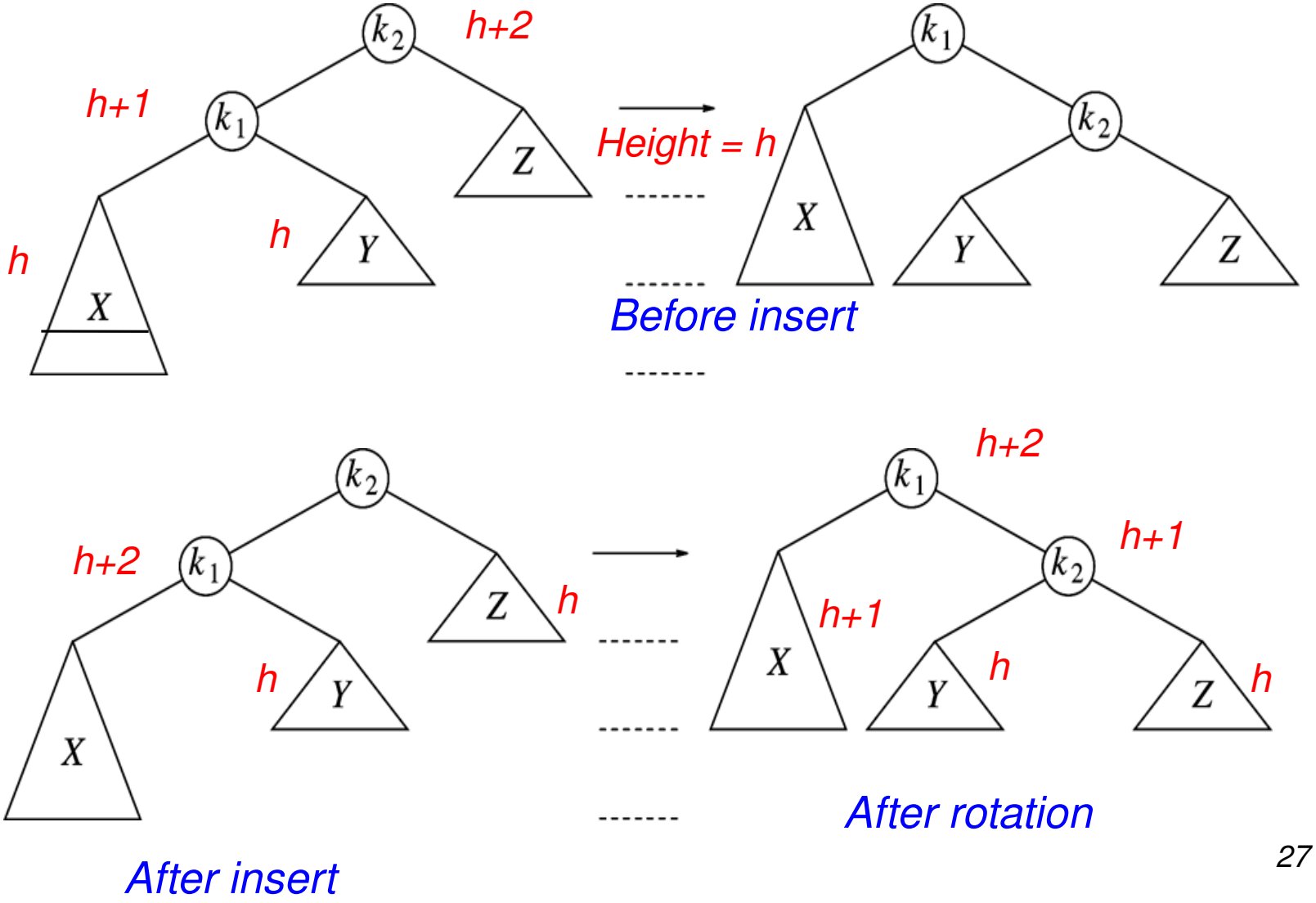


# Double Rotation (Case 2)

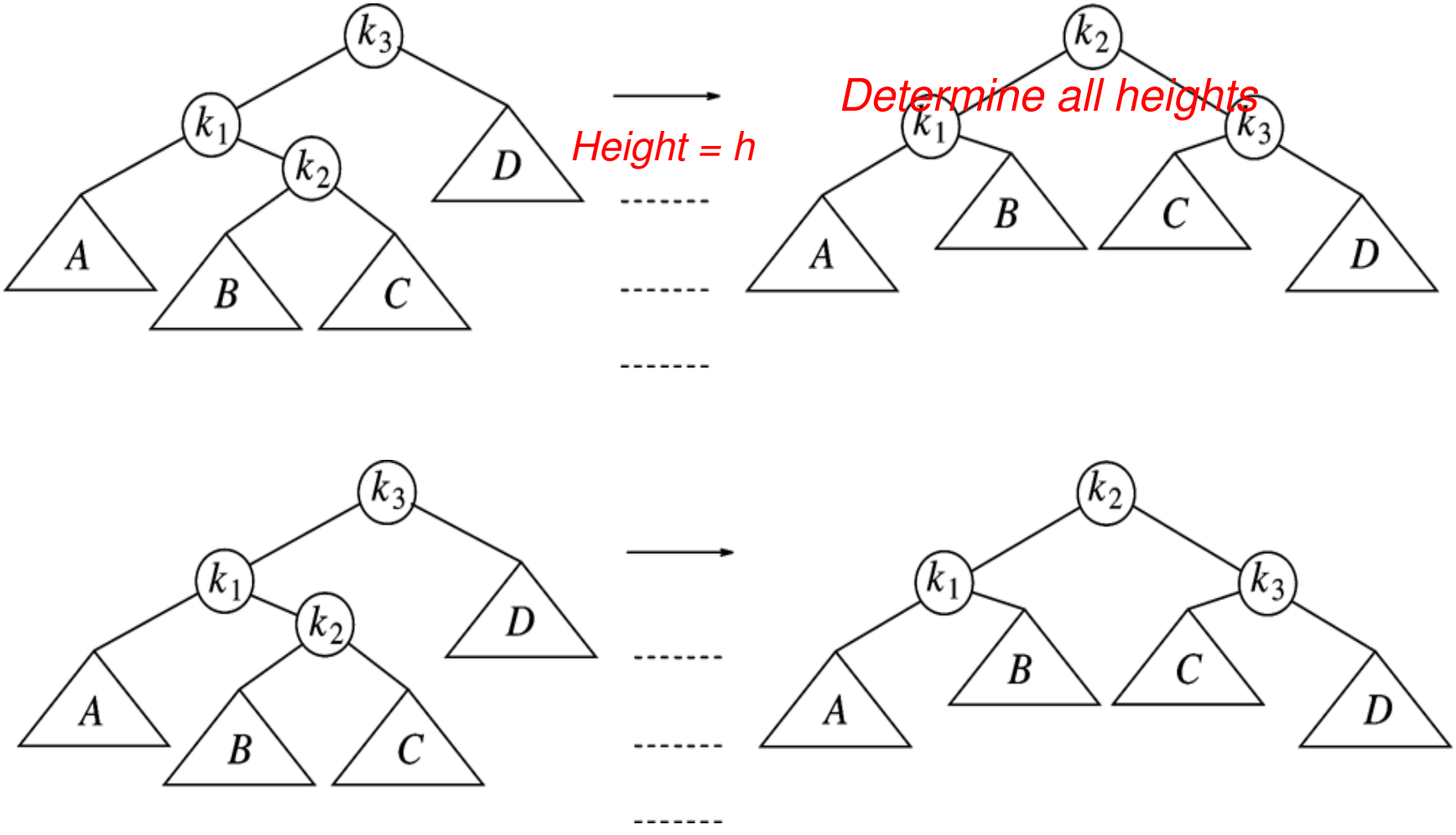
```
1     /**
2      * Double rotate binary tree node: first left child
3      * with its right child; then node k3 with new left child.
4      * For AVL trees, this is a double rotation for case 2.
5      * Update heights, then set new root.
6      */
7     void doubleWithLeftChild( AvlNode * & k3 )
8     {
9         rotateWithRightChild( k3->left );
10        rotateWithLeftChild( k3 );
11    }
```



# Review Insertion -- Case 1

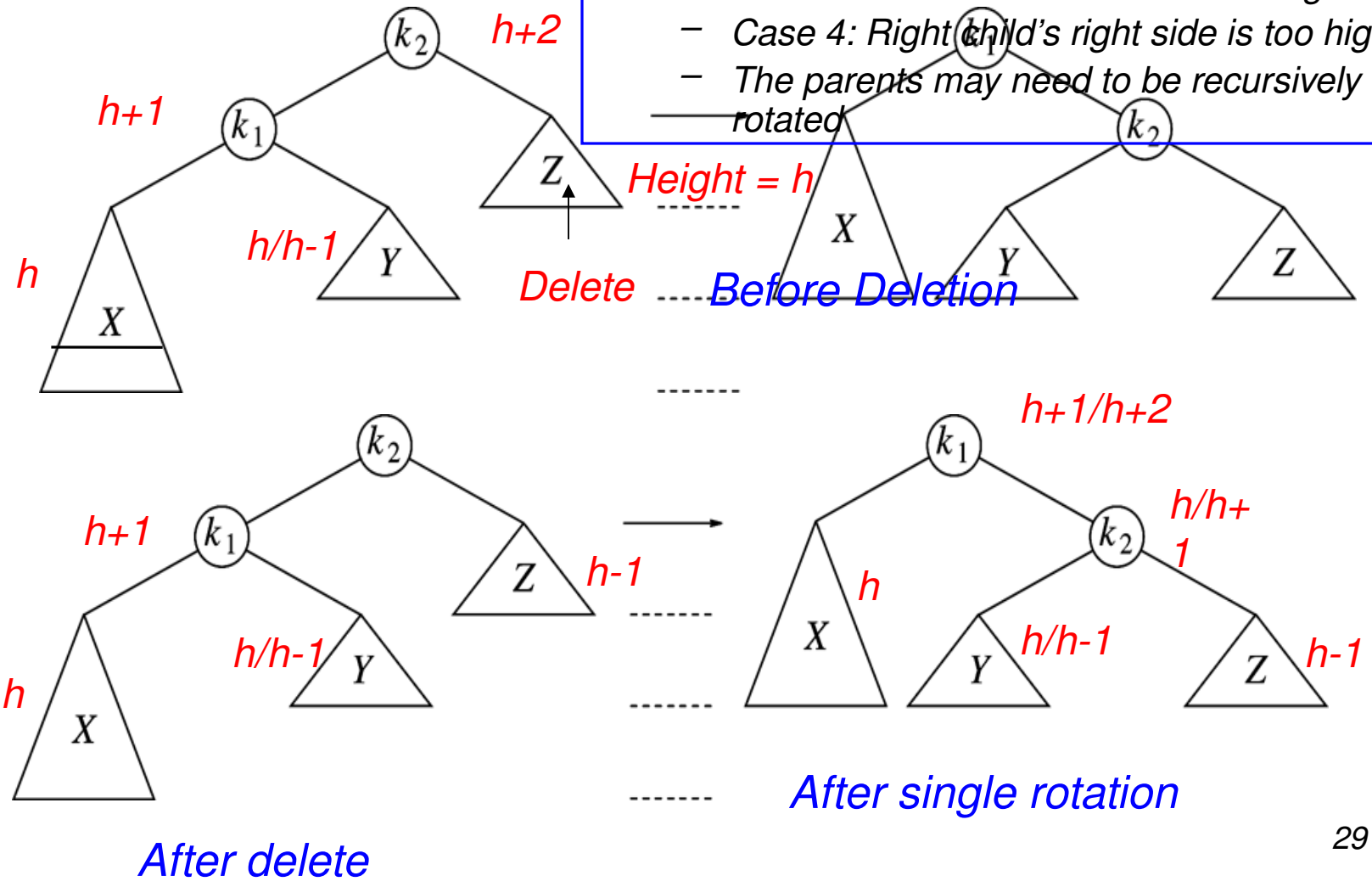


# Review Insertion -- Case 2



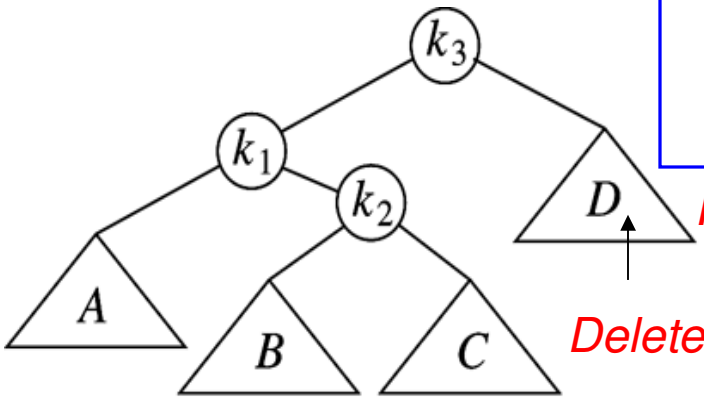
# Delete -- Case 1

- *Consider deepest unbalanced node*
  - *Case 1: Left child's left side is too high*
  - *Case 4: Right child's right side is too high*
  - *The parents may need to be recursively rotated*



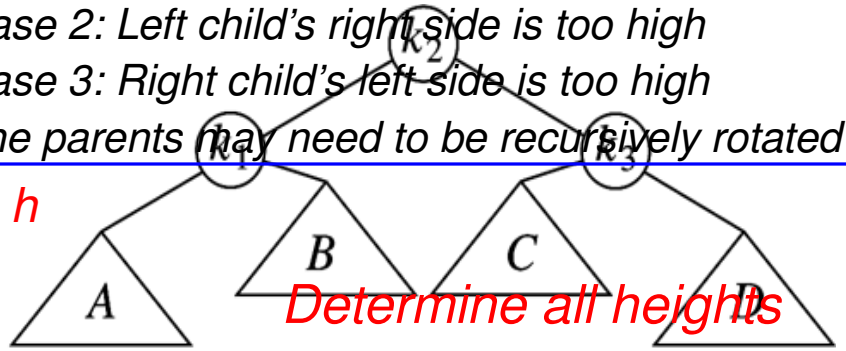
# Delete -- Case 2

- Consider deepest unbalanced node
  - Case 2: Left child's right side is too high
  - Case 3: Right child's left side is too high
  - The parents may need to be recursively rotated

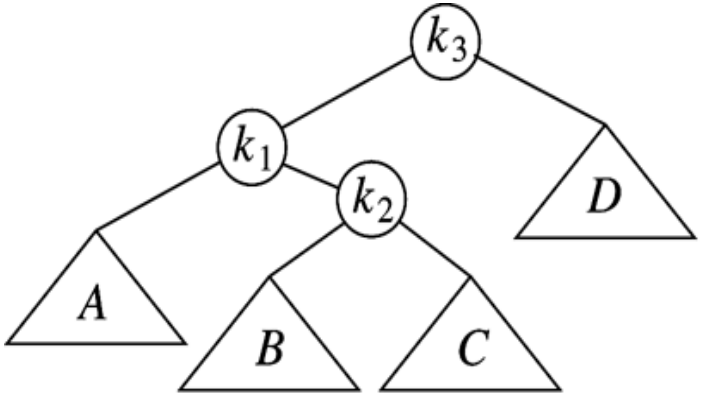



Height = h

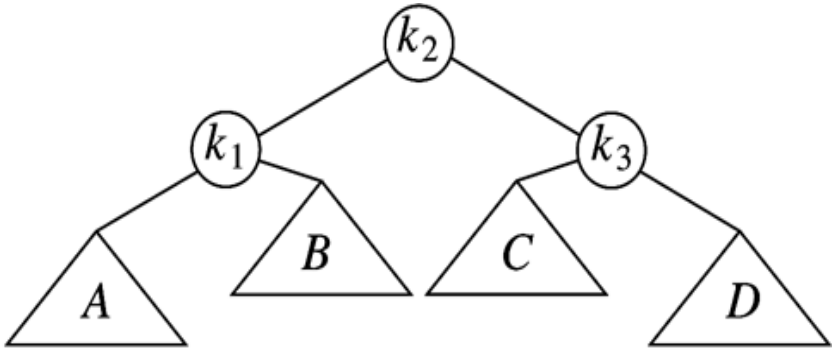
Delete




Determine all heights




After Delete




After double rotation