Sorting (Part II: Divide and Conquer)

CSE 373

Data Structures

Lecture 14

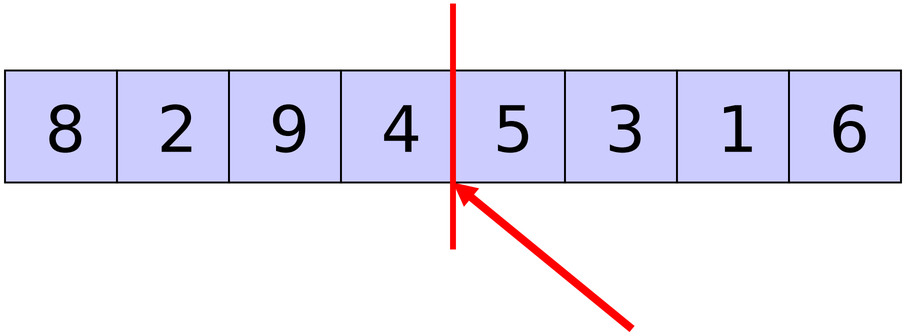
Readings

- Reading
  - Section 7.6, Mergesort
  - Section 7.7, Quicksort

“Divide and Conquer”

- *Very important strategy in computer science:*
  - *Divide problem into smaller parts*
  - *Independently solve the parts*
  - *Combine these solutions to get overall solution*
- **Idea 1** *: Divide array into two halves, recursively sort left and right halves, then merge two halves ⇒ Mergesort*
- **Idea 2 :** *Partition array into items that are “small” and items that are “large”, then recursively sort the two sets ⇒ Quicksort*

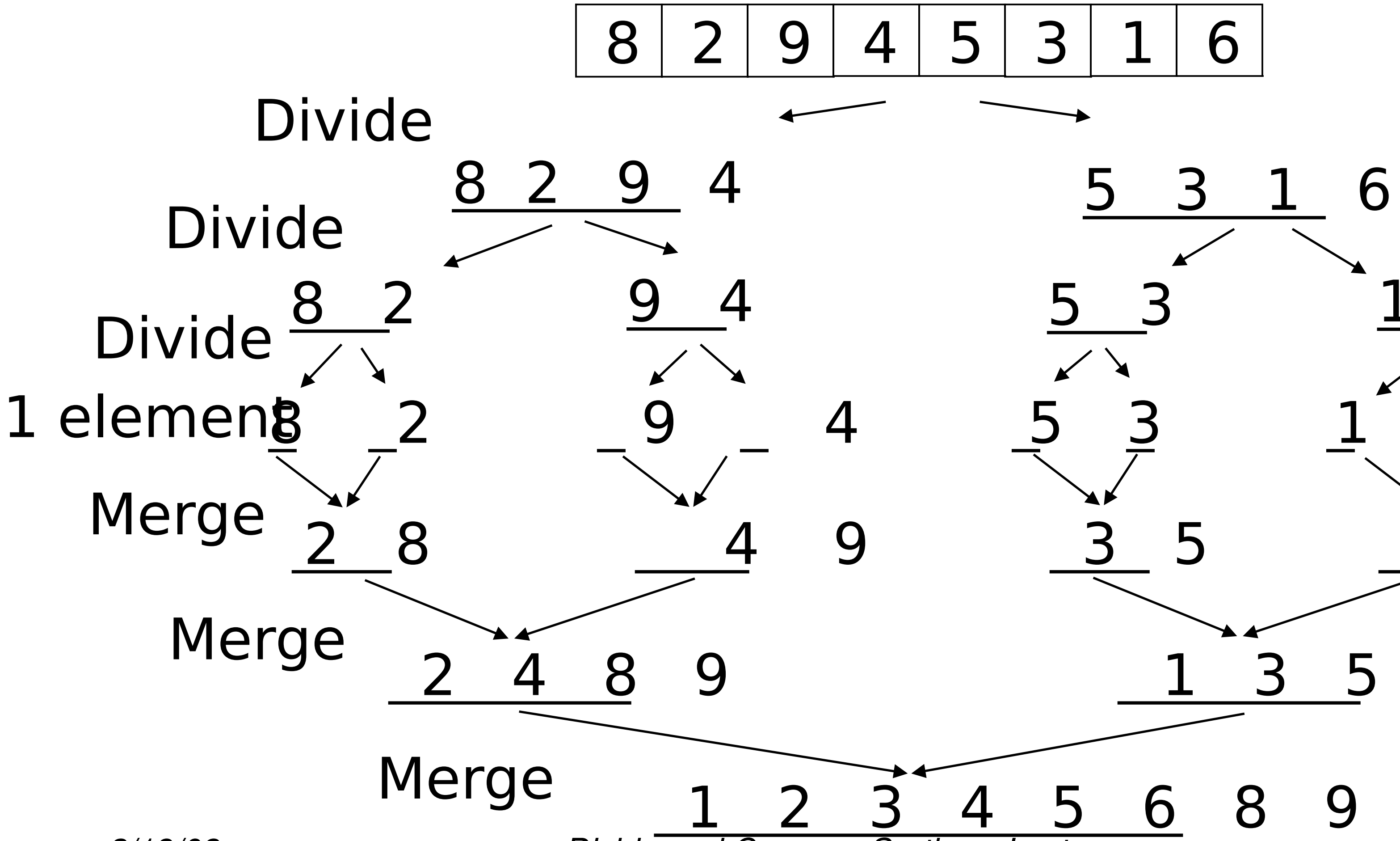
Mergesort



8	2	9	4	5	3	1	6

- *Divide it in two at the midpoint*
- *Conquer each side in turn (by recursively sorting)*
- *Merge two halves together*

Mergesort Example



8 2 9 4 5 3 1 6

Divide

8 2 9 4

5 3 1 6

Divide

8 2

9 4

5 3

1 6

Divide

1 element

8 2

9 4

5 3

1 6

Merge

2 8

4 9

3 5

1 6

Merge

2 4 8 9

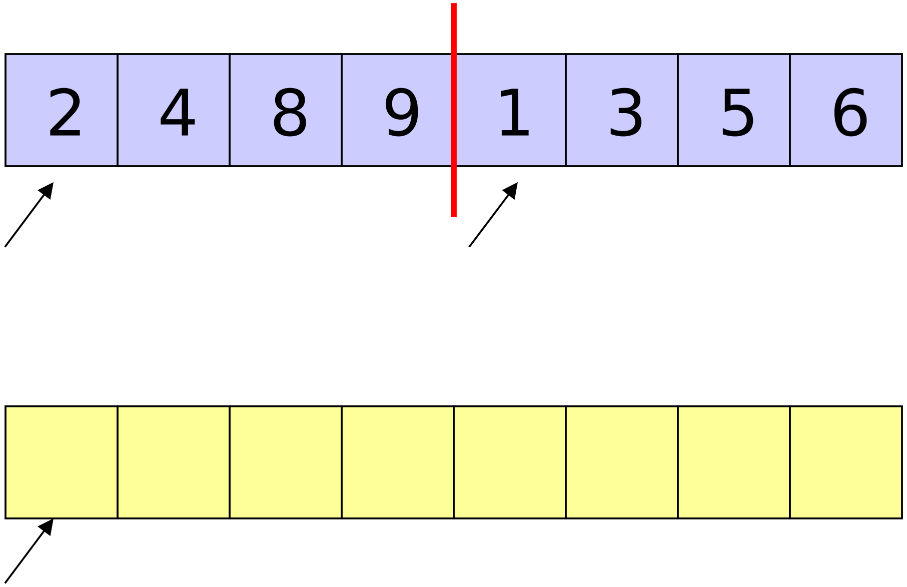
1 3 5 6

Merge

1 2 3 4 5 6 8 9

Auxiliary Array

- *The merging requires an auxiliary array.*

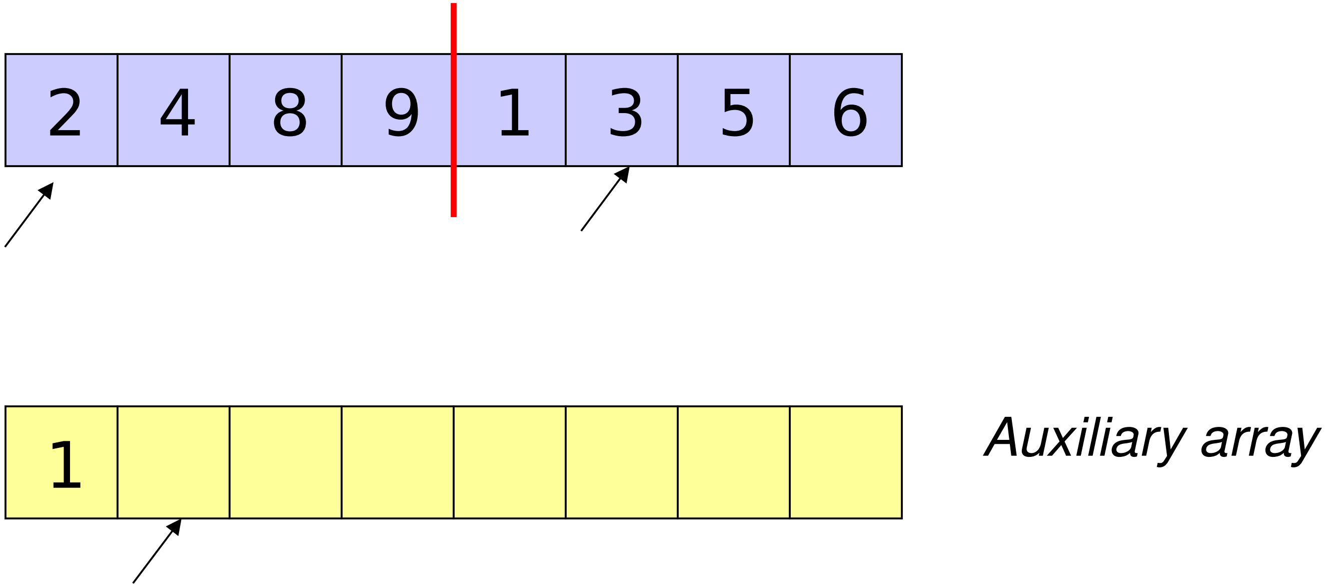


2	4	8	9	1	3	5	6

Auxiliary array

Auxiliary Array

- *The merging requires an auxiliary array.*



Auxiliary Array

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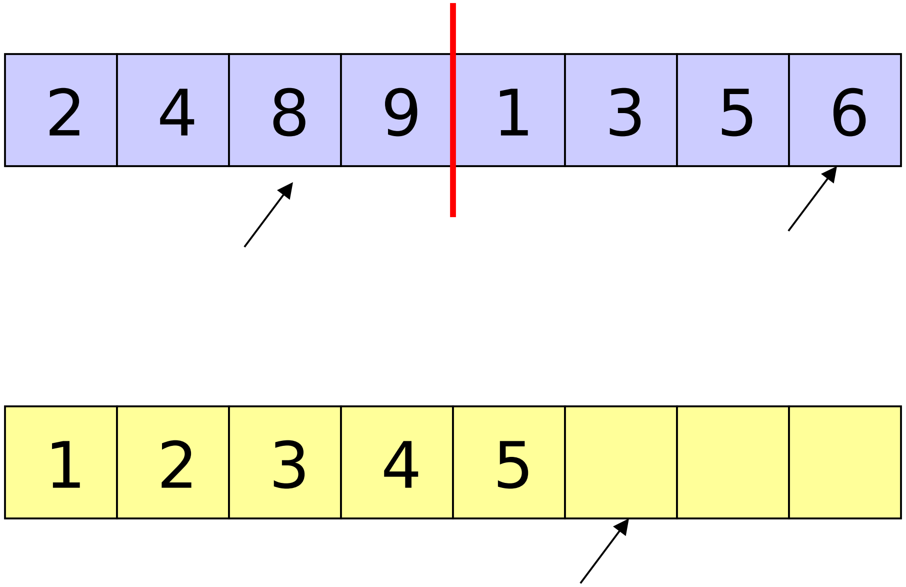
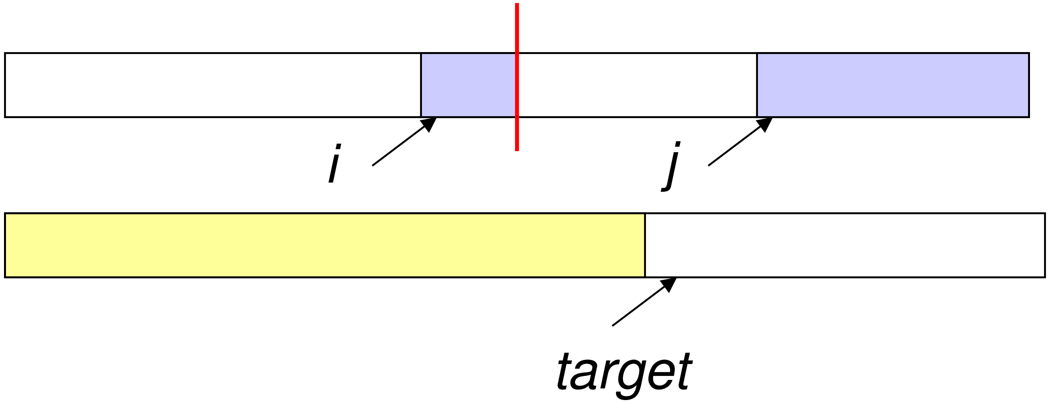


2	4	8	9	1	3	5	6

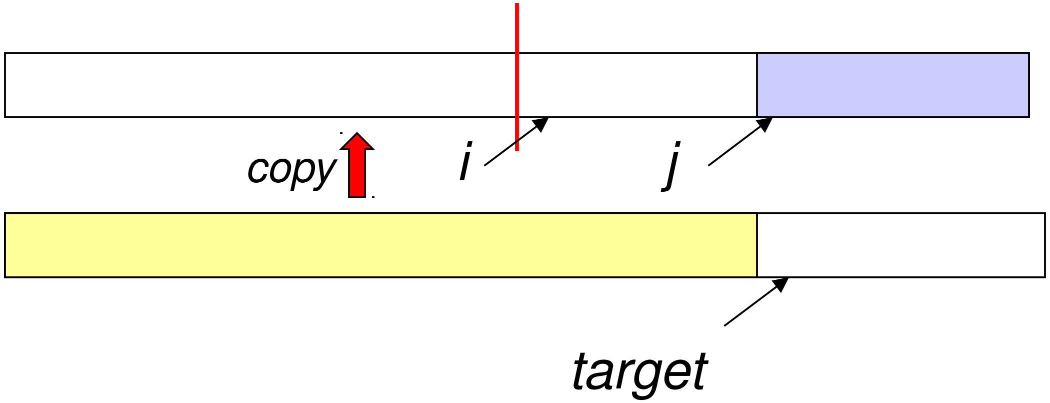
1	2	3	4	5			

Auxiliary array

Merging

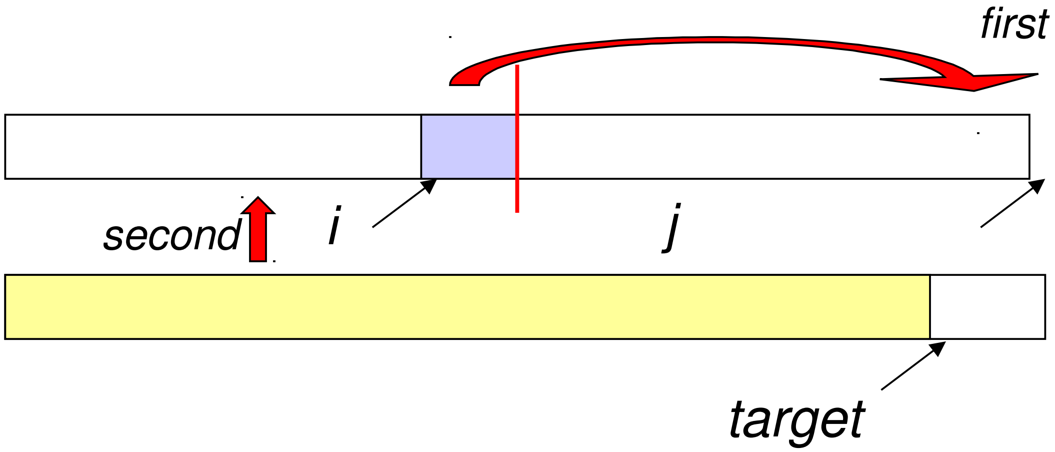


normal



Left completed first

Merging



Right completed first

Merging

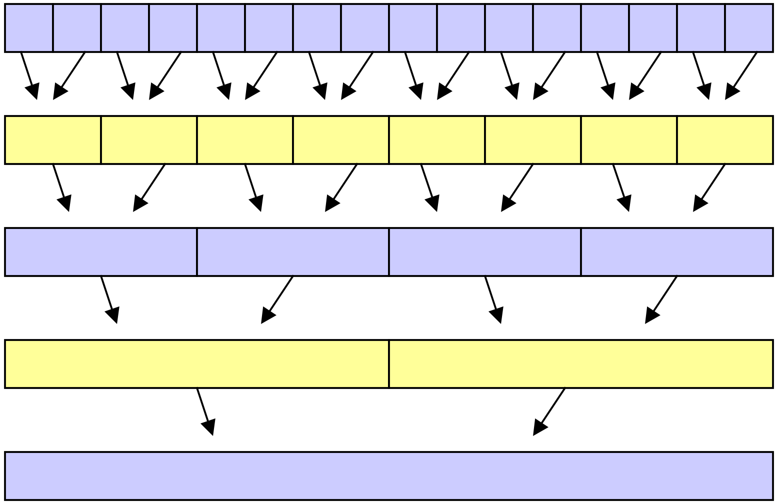
```
Merge(A[], T[] : integer array, left, right : integer) : {
  mid, i, j, k, l, target : integer;
  mid := (right + left)/2;
  i := left; j := mid + 1; target := left;
  while i ≤ mid and j ≤ right do
    if A[i] ≤ A[j] then T[target] := A[i] ; i:= i + 1;
      else T[target] := A[j]; j := j + 1;
    target := target + 1;
  if i > mid then //left completed//
    for k := left to target-1 do A[k] := T[k];
  if j > right then //right completed//
    k : = mid; l := right;
    while k ≥ i do A[l] := A[k]; k := k-1; l := l-1;
    for k := left to target-1 do A[k] := T[k];
}
```

Recursive Mergesort

```
Mergesort(A[], T[] : integer array, left, right : integer) : {
 if left < right then
  mid := (left + right)/2;
  Mergesort(A,T,left,mid);
  Mergesort(A,T,mid+1,right);
  Merge(A,T,left,right);
}

MainMergesort(A[1..n]: integer array, n : integer) : {
 T[1..n]: integer array;
 Mergesort[A,T,1,n];
}
```

Iterative Mergesort



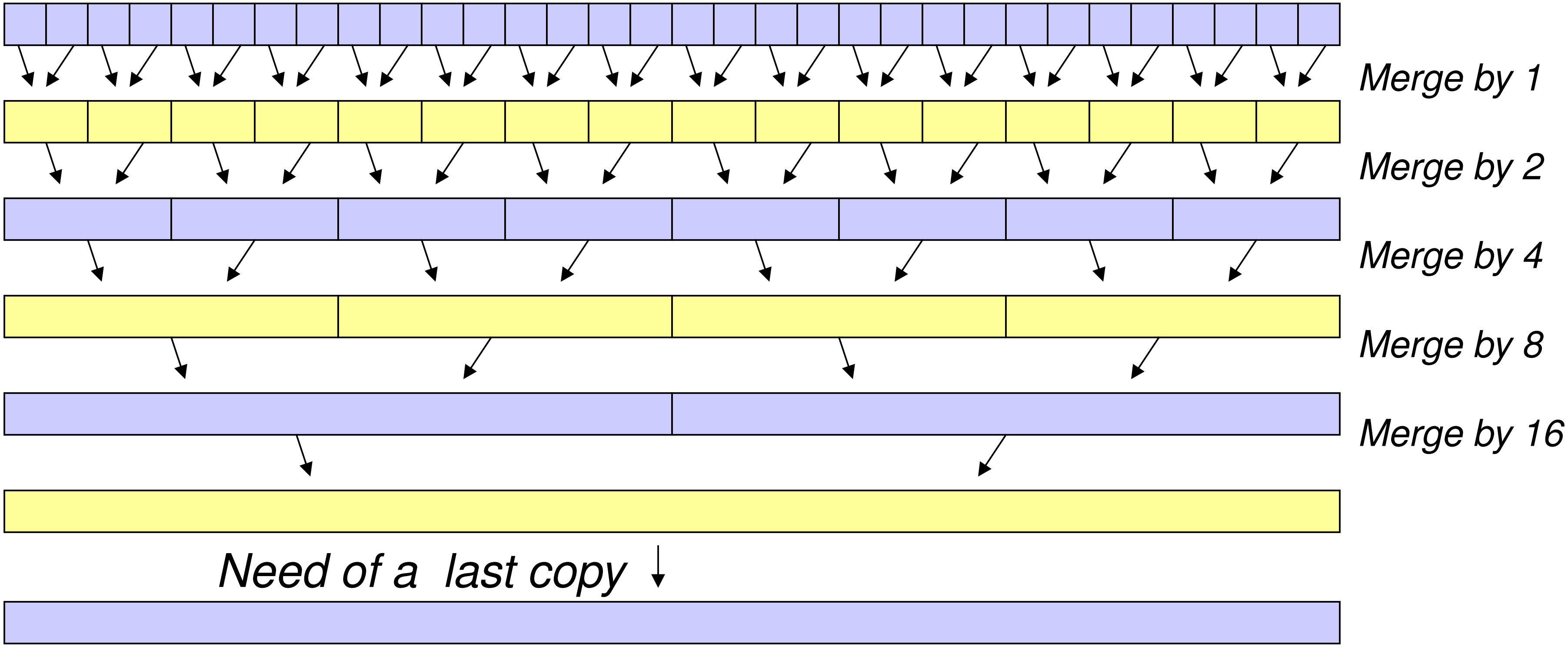
Merge by 1

Merge by 2

Merge by 4

Merge by 8

Iterative Mergesort



Merge by 1

Merge by 2

Merge by 4

Merge by 8

Merge by 16

Need of a last copy

Iterative Mergesort

```
IterativeMergesort(A[1..n]: integer array, n : integer) : {
//precondition: n is a power of 2//
  i, m, parity : integer;
  T[1..n]: integer array;
  m := 2; parity := 0;
  while m ≤ n do
    for i = 1 to n – m + 1 by m do
       if parity = 0 then Merge(A,T,i,i+m-1);
         else Merge(T,A,i,i+m-1);
    parity := 1 – parity;
    m := 2*m;
  if parity = 1 then
    for i = 1 to n do A[i] := T[i];
}
```

How do you handle non-powers of 2?
How can the final copy be avoided?

Mergesort Analysis

- *Let T(N) be the running time for an array of N elements*
- *Mergesort divides array in half and calls itself on the two halves. After returning, it merges both halves using a temporary array*
- *Each recursive call takes T(N/2) and merging takes O(N)*

Mergesort Recurrence Relation

- *The recurrence relation for T(N) is:*
  - *$T(1) \leq a$*
    - *base case: 1 element array constant time*
  - *$T(N) \leq 2T(N/2) + bN$*
    - *Sorting N elements takes*
      - *the time to sort the left half*
      - *plus the time to sort the right half*
      - *plus an O(N) time to merge the two halves*
- *$T(N) = O(n \log n)$ (see Lecture 5 Slide17)*

Properties of Mergesort

- Not in-place
  - Requires an auxiliary array (O(n) extra space)
- Stable
  - Make sure that left is sent to target on equal values.
- Iterative Mergesort reduces copying.

Quicksort

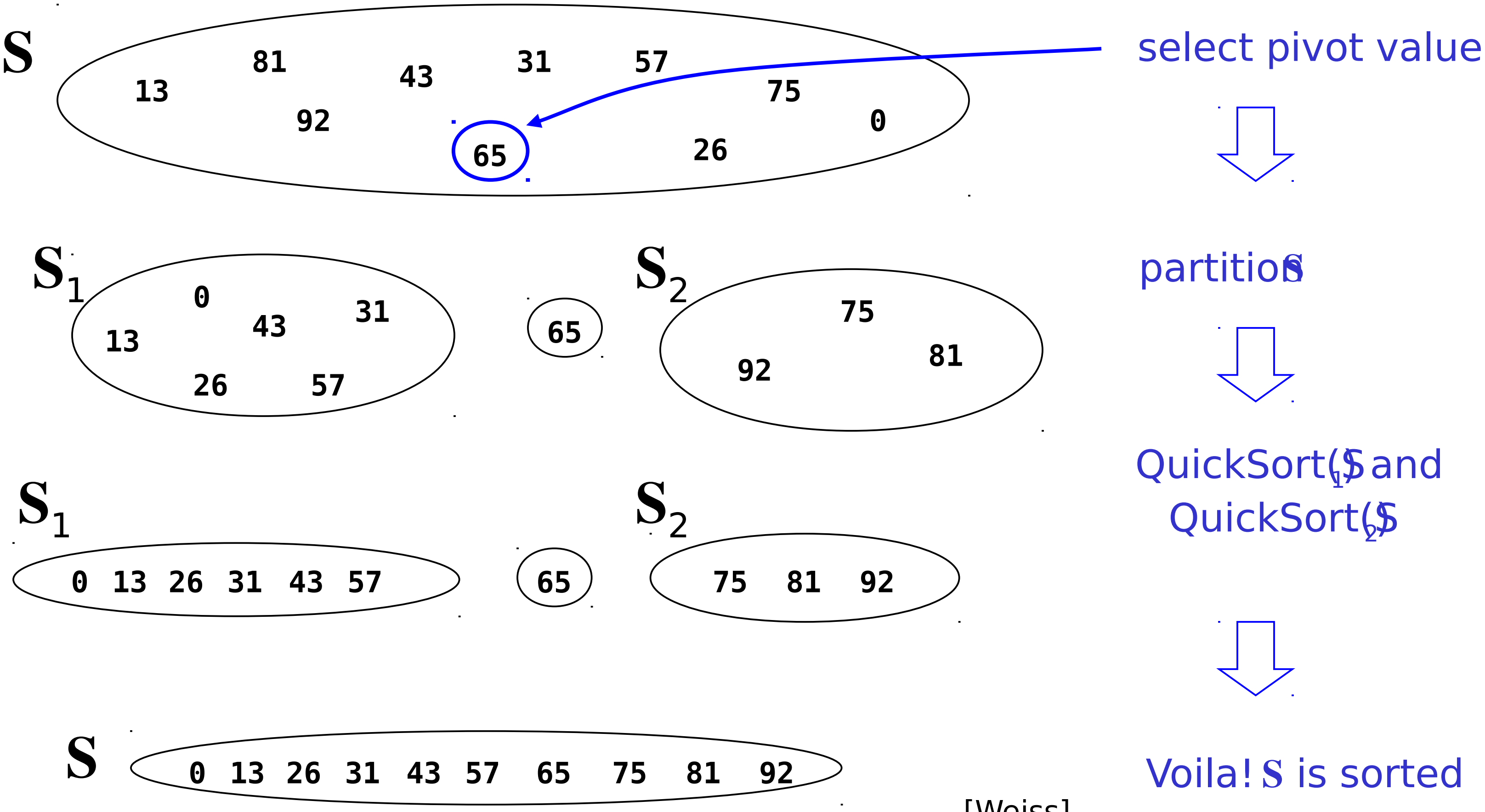
- *Quicksort uses a divide and conquer strategy, but does not require the O(N) extra space that MergeSort does*
  - *Partition array into left and right sub-arrays*
    - *Choose an element of the array, called pivot*
    - *the elements in left sub-array are all less than pivot*
    - *elements in right sub-array are all greater than pivot*
  - *Recursively sort left and right sub-arrays*
  - *Concatenate left and right sub-arrays in O(1) time*

“Four easy steps”

- *To sort an array* **S**

  1. *If the number of elements in* **S** *is 0 or 1, then return. The array is sorted.*
  2. *Pick an element v in* **S**. *This is the pivot value.*
  3. *Partition* **S***-{v} into two disjoint subsets,* $\mathbf{S}_1$ = *{all values* $x \le v$*}, and* $\mathbf{S}_2$ = *{all values* $x \ge v$*}.*
  4. *Return QuickSort(*$\mathbf{S}_1$*), v, QuickSort(*$\mathbf{S}_2$*)*

The steps of QuickSort



select pivot value

partition S

QuickSort(S_1) and QuickSort(S_2)

Voila! S is sorted

[Weiss]

Details, details

- *Implementing the actual partitioning*
- *Picking the pivot*
  - *want a value that will cause $|S_1|$ and $|S_2|$ to be non-zero, and close to equal in size if possible*
- *Dealing with cases where the element equals the pivot*

Quicksort Partitioning

- *Need to partition the array into left and right sub-arrays*
  - *the elements in left sub-array are □ pivot*
  - *elements in right sub-array are □ pivot*
- *How do the elements get to the correct partition?*
  - *Choose an element from the array as the pivot*
  - *Make one pass through the rest of the array and swap as needed to put elements in partitions*

Partitioning:Choosing the pivot

- One implementation (there are others)
  - median3 finds pivot and sorts left, center, right
    - Median3 takes the median of leftmost, middle, and rightmost elements
    - An alternative is to choose the pivot randomly (need a random number generator; “expensive”)
    - Another alternative is to choose the first element (but can be very bad. Why?)
  - Swap pivot with next to last element

Partitioning in-place

- Set pointers i and j to start and end of array
- Increment i until you hit element A[i] > pivot
- Decrement j until you hit elmt A[j] < pivot
- Swap A[i] and A[j]
- Repeat until i and j cross
- Swap pivot (at A[N-2]) with A[i]

Example

Choose the pivot as the median of three

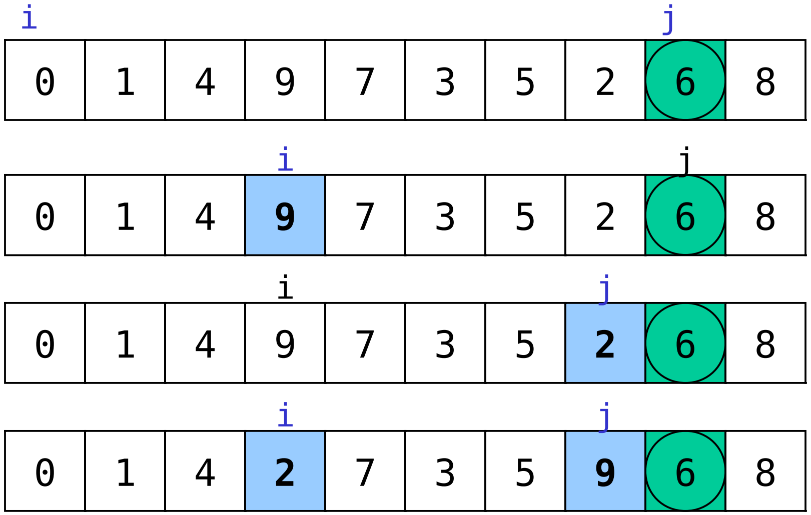
0	1	2	3	4	5	6	7	8	9
8	1	4	9	**0**	3	5	2	7	**6**

Median of 0, 6, 8 is 6. Pivot is 6

0	1	4	9	7	3	5	2	6	8
i								j	

Place the largest at the right
and the smallest at the left.
Swap pivot with next to last element.

Example



i								j	
0	1	4	9	7	3	5	2	6	8

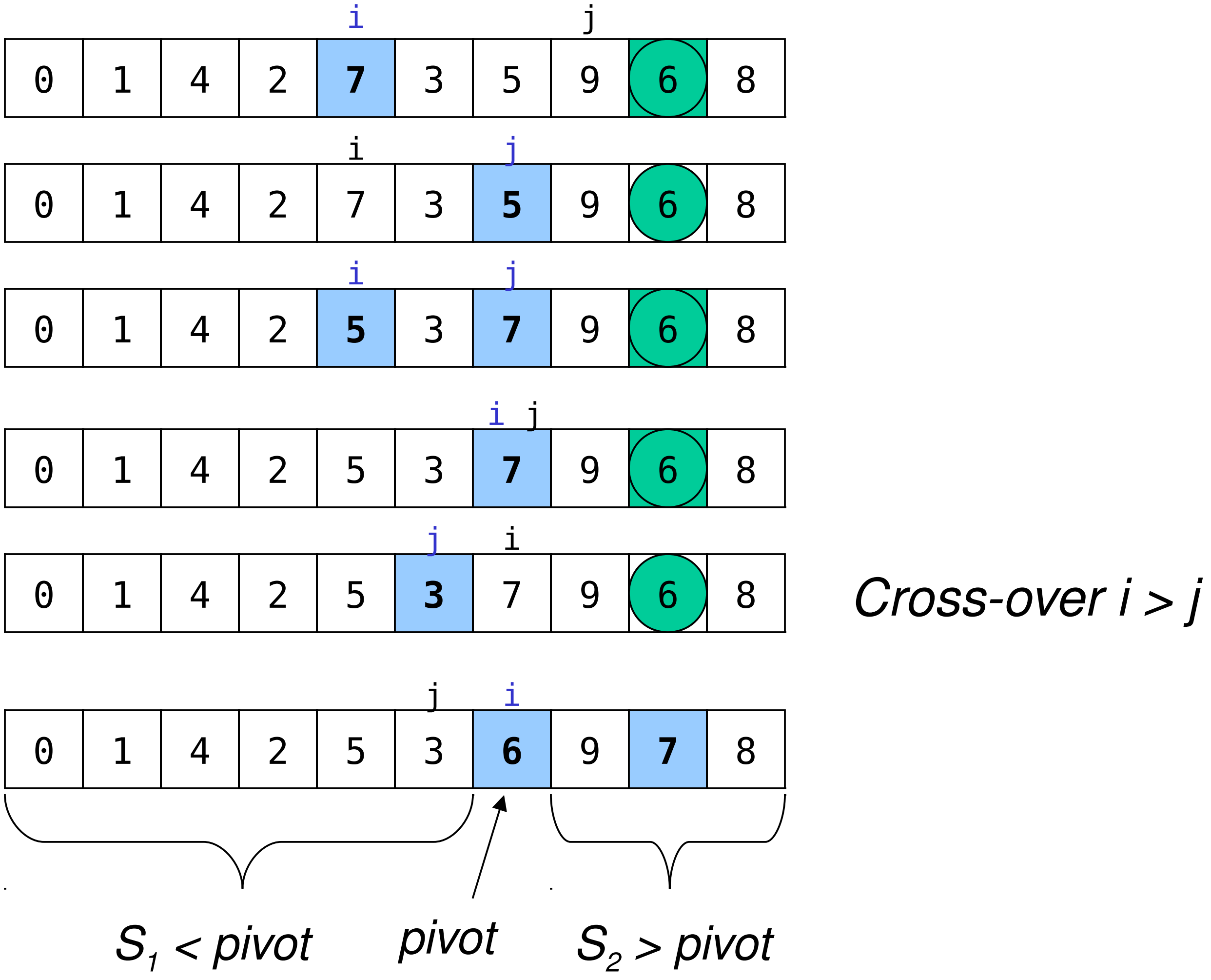
			i					j	
0	1	4	**9**	7	3	5	2	6	8

			i				j		
0	1	4	9	7	3	5	**2**	6	8

			i				j		
0	1	4	**2**	7	3	5	**9**	6	8

Move i to the right up to A[i] larger than pivot.
Move j to the left up to A[j] smaller than pivot.
Swap

Example



				i			j		
0	1	4	2	**7**	3	5	9	6	8

				i		j			
0	1	4	2	7	3	**5**	9	6	8

				i		j			
0	1	4	2	**5**	3	**7**	9	6	8

						i j			
0	1	4	2	5	3	**7**	9	6	8

					j	i			
0	1	4	2	5	**3**	7	9	6	8

Cross-over i > j

					j	i			
0	1	4	2	5	3	**6**	9	**7**	8

S_1 < pivot *pivot* *S_2 > pivot*

Recursive Quicksort

```
Quicksort(A[]: integer array, left,right : integer): {
pivotindex : integer;
if left + CUTOFF  right then
  pivot := median3(A,left,right);
  pivotindex := Partition(A,left,right-1,pivot);
  Quicksort(A, left, pivotindex – 1);
  Quicksort(A, pivotindex + 1, right);
else
  Insertionsort(A,left,right);
}
```

Don’t use quicksort for small arrays.
CUTOFF = 10 is reasonable.

Quicksort Best Case Performance

- *Algorithm always chooses best pivot and splits sub-arrays in half at each recursion*
  - *T(0) = T(1) = O(1)*
    - *constant time if 0 or 1 element*
  - *For N > 1, 2 recursive calls plus linear time for partitioning*
  - *T(N) = 2T(N/2) + O(N)*
    - *Same recurrence relation as Mergesort*
  - *T(N) = O(N log N)*

Quicksort Worst Case Performance

- Algorithm always chooses the worst pivot – one sub-array is empty at each recursion
  - T(N) □ a for N □ C
  - T(N) □ T(N-1) + bN
  - □ T(N-2) + b(N-1) + bN
  - □ T(C) + b(C+1)+ … + bN
  - □ a +b(C + (C+1) + (C+2) + … + N)
  - T(N) = $O(N^2)$
- Fortunately, average case performance is O(N log N) (see text for proof)

Properties of Quicksort

- *Not stable because of long distance swapping.*
- *No iterative version (without using a stack).*
- *Pure quicksort not good for small arrays.*
- *"In-place", but uses auxiliary storage because of recursive call (O(logn) space).*
- *O(n log n) average case performance, but O(n^2) worst case performance.*

Folklore

- "Quicksort is the best in-memory sorting algorithm."
- Truth
  - Quicksort uses very few comparisons on average.
  - Quicksort does have good performance in the memory hierarchy.
    - Small footprint
    - Good locality