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2.2 MERGESORT

- ***mergesort***
- ***bottom-up mergesort***
- ***sorting complexity***
- ***comparators***
- ***stability***

2/STR/MOT/DEF/1/A

Two classic sorting algorithms: mergesort and quicksort

Critical components in the world's computational infrastructure.

- **Full scientific understanding of their properties has enabled us to develop them into practical system sorts.**
- **Quicksort honored as one of top 10 algorithms of 20th century in science and engineering.**

Mergesort. **[this lecture]**

…

Quicksort. **[next lecture]**

…

Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

http://algs4.cs.princeton.edu

2.2 MERGESORT

- ***mergesort***
- ***bottom-up mergesort***
- ***sorting complexity***
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- ***stability***

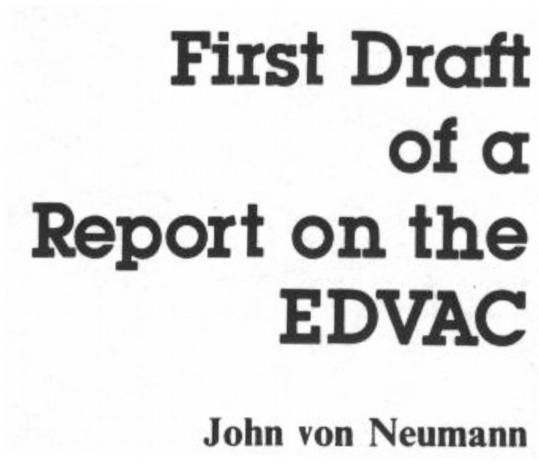
Mergesort

Basic plan.

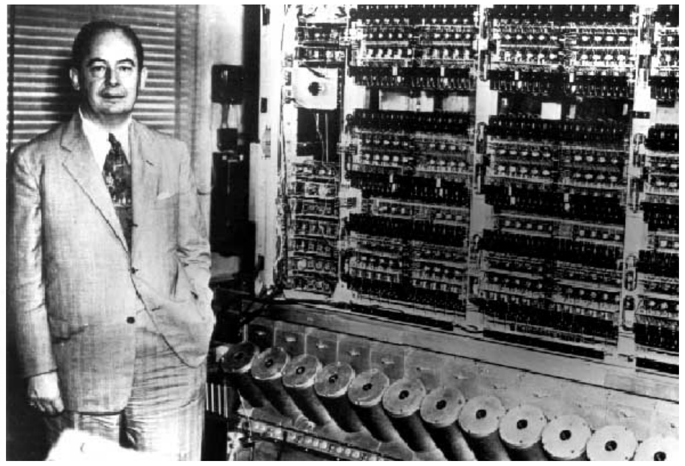
- **Divide array into two halves.**
- **Recursively sort each half.**
- **Merge two halves.**

input	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
sort left half	E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
sort right half	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
merge results	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Mergesort overview

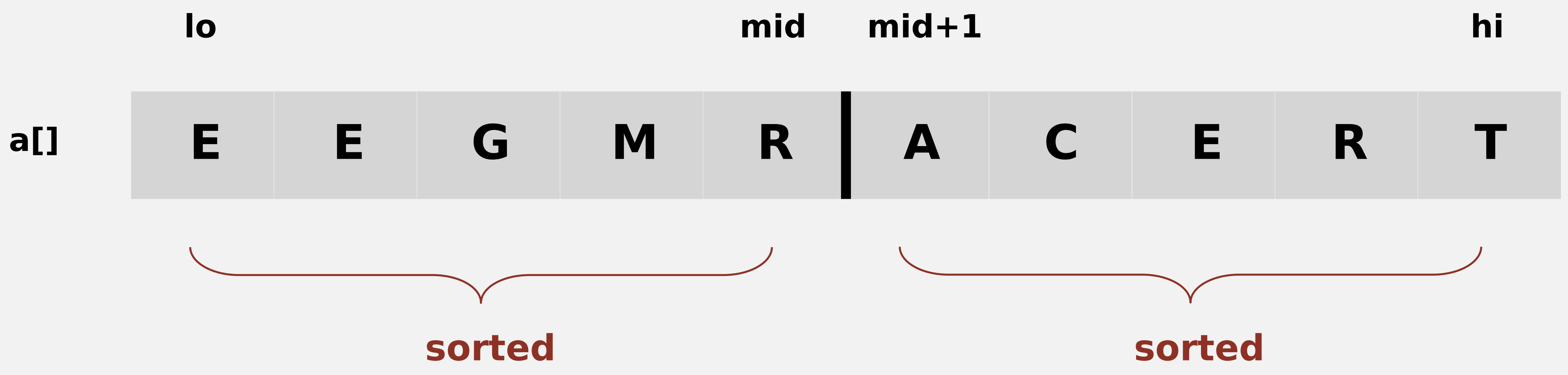

First Draft
of a
Report on the
EDVAC

John von Neumann



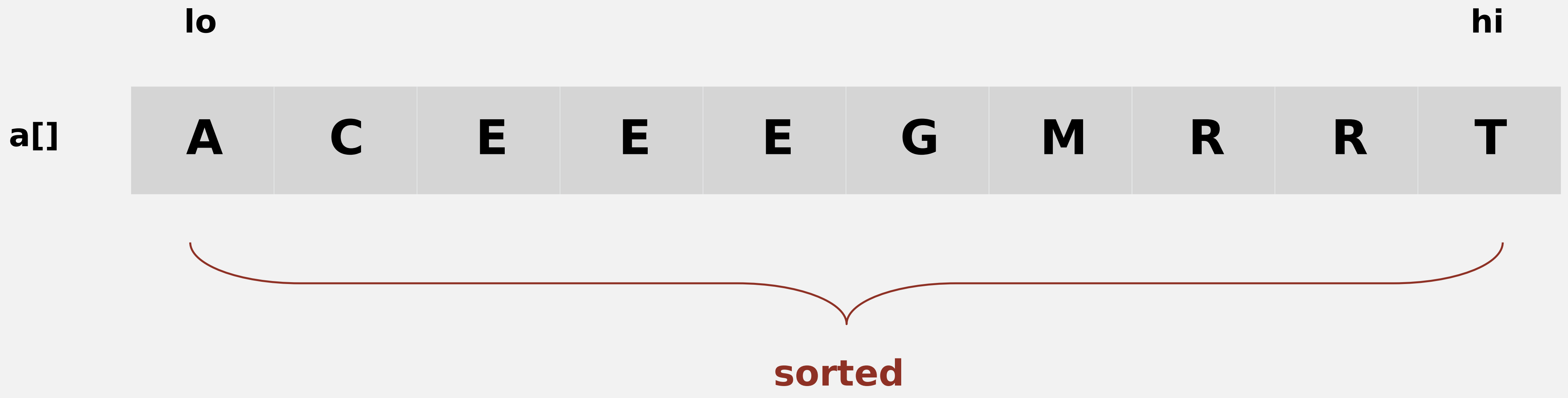
Abstract in-place merge demo

Goal. **Given two sorted subarrays a[lo] to a[mid] and a[mid+1] to a[hi], replace with sorted subarray a[lo] to a[hi].**



Abstract in-place merge demo

Goal. **Given two sorted subarrays a[lo] to a[mid] and a[mid+1] to a[hi], replace with sorted subarray a[lo] to a[hi].**

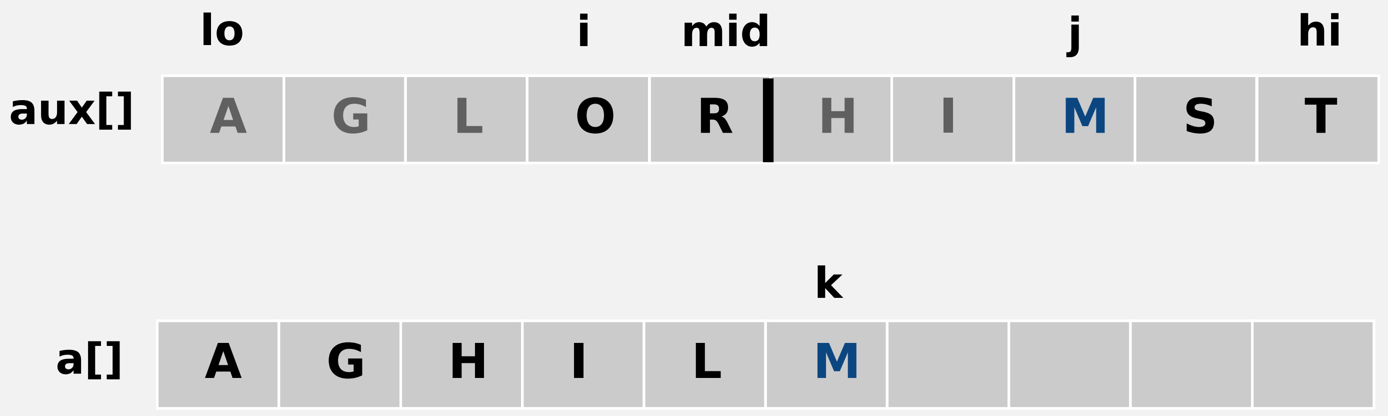


Merging: Java implementation

```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
{

  for (int k = lo; k <= hi; k++)
    aux[k] = a[k];

  int i = lo, j = mid+1;
  for (int k = lo; k <= hi; k++)
  {
    if      (i > mid)              a[k] = aux[j++];
    else if (j > hi)               a[k] = aux[i++];
    else if (less(aux[j], aux[i])) a[k] = aux[j++];
    else                           a[k] = aux[i++];
  }
}
```

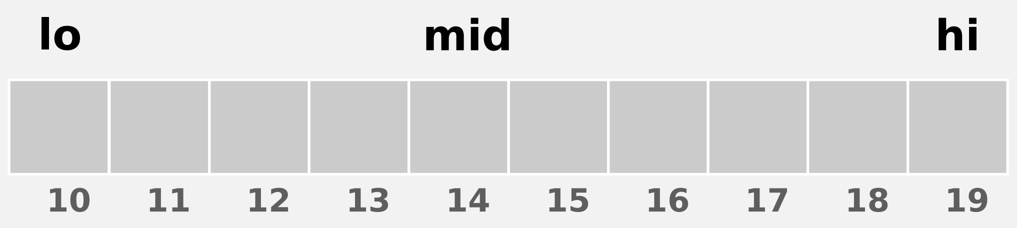
copy

merge



Mergesort: Java implementation

```
public class Merge
{
  private static void merge(...)
  {  /* as before */  }

  private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
  {
    if (hi <= lo) return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
  }

  public static void sort(Comparable[] a)
  {
    Comparable[] aux = new Comparable[a.length];
    sort(a, aux, 0, a.length - 1);
  }
}
```



Mergesort: trace

a[]

lo hi

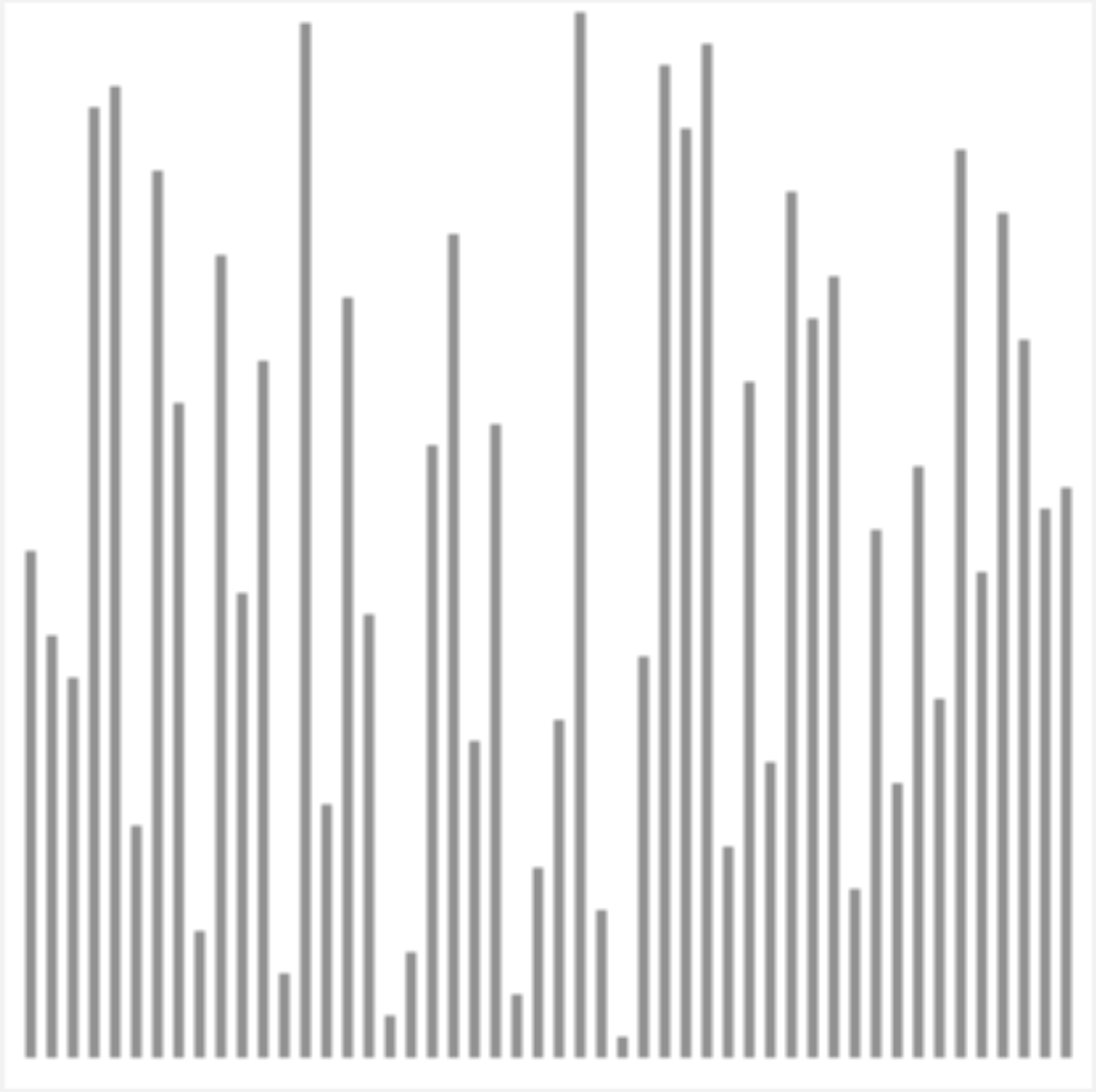
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	M	**E**	**R**	**G**	**E**	**S**	**O**	**R**	**T**	**E**	**X**	**A**	**M**	**P**	**L**	**E**
merge(a, aux, 0, 0, 1)	**E**	**M**	R	G	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 2, 2, 3)	E	M	**G**	**R**	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 0, 1, 3)	**E**	**G**	**M**	**R**	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 4, 4, 5)	E	G	M	R	**E**	**S**	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 6, 6, 7)	E	G	M	R	E	S	**O**	**R**	T	E	X	A	M	P	L	E
merge(a, aux, 4, 5, 7)	E	G	M	R	**E**	**O**	**R**	**S**	T	E	X	A	M	P	L	E
merge(a, aux, 0, 3, 7)	**E**	**E**	**G**	**M**	**O**	**R**	**R**	**S**	T	E	X	A	M	P	L	E
merge(a, aux, 8, 8, 9)	E	E	G	M	O	R	R	S	**E**	**T**	X	A	M	P	L	E
merge(a, aux, 10, 10, 11)	E	E	G	M	O	R	R	S	E	T	**A**	**X**	M	P	L	E
merge(a, aux, 8, 9, 11)	E	E	G	M	O	R	R	S	**A**	**E**	**T**	**X**	M	P	L	E
merge(a, aux, 12, 12, 13)	E	E	G	M	O	R	R	S	A	E	T	X	**M**	**P**	L	E
merge(a, aux, 14, 14, 15)	E	E	G	M	O	R	R	S	A	E	T	X	M	P	**E**	**L**
merge(a, aux, 12, 13, 15)	E	E	G	M	O	R	R	S	A	E	T	X	**E**	**L**	**M**	**P**
merge(a, aux, 8, 11, 15)	E	E	G	M	O	R	R	S	**A**	**E**	**E**	**L**	**M**	**P**	**T**	**X**
merge(a, aux, 0, 7, 15)	**A**	**E**	**E**	**E**	**E**	**G**	**L**	**M**	**M**	**O**	**P**	**R**	**R**	**S**	**T**	**X**

Trace of merge results for top-down mergesort

result after recursive call

Mergesort: animation

50 random items

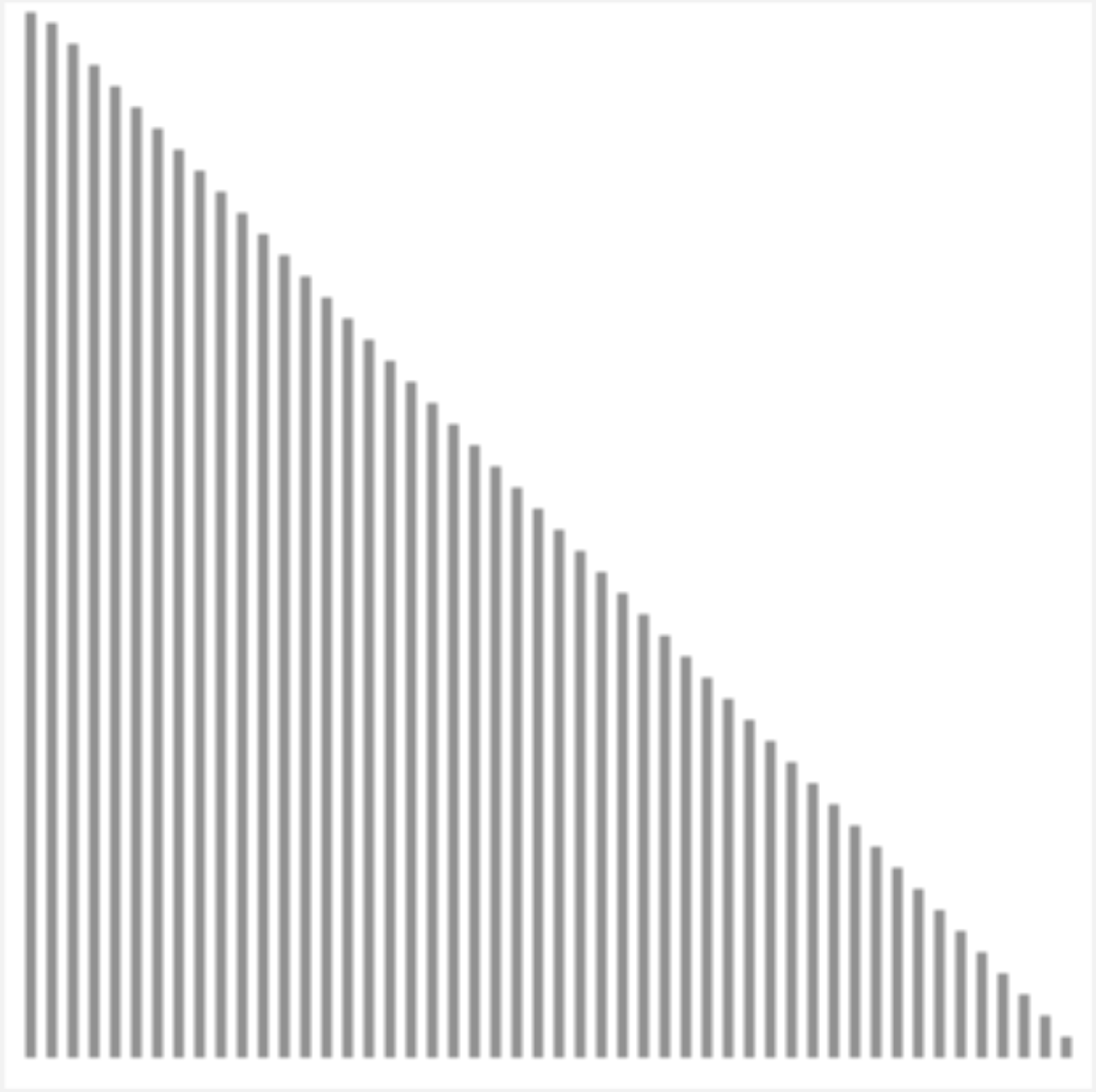


http://www.sorting-algorithms.com/merge-sort

▲	algorithm position
▬	in order
▬	current subarray
▬	not in order

Mergesort: animation

50 reverse-sorted items



http://www.sorting-algorithms.com/merge-sort

▲ algorithm position

▬ in order

▬ current subarray

▬ not in order

Mergesort: empirical analysis

Running time estimates:

- **Laptop executes 10^8 compares/second.**
- **Supercomputer executes 10^{12} compares/second.**

	insertion sort (N^2)			mergesort (N log N)		
computer	**thousand**	**million**	**billion**	**thousand**	**million**	**billion**
home	instant	2.8 hours	317 years	instant	1 second	18 min
super	instant	1 second	1 week	instant	instant	instant

Bottom line. **Good algorithms are better than supercomputers.**

Mergesort: number of compares

Proposition. **Mergesort uses $\le N \lg N$ compares to sort an array of length N.**

Pf sketch. **The number of compares $C(N)$ to mergesort an array of length N satisfies the recurrence:**

$$C(N) \le \underbrace{C(\lceil N/2 \rceil)}_{\text{left half}} + \underbrace{C(\lfloor N/2 \rfloor)}_{\text{right half}} + \underbrace{N}_{\text{merge}} \quad \textbf{for } N > 1\textbf{, with } C(1) = 0.$$

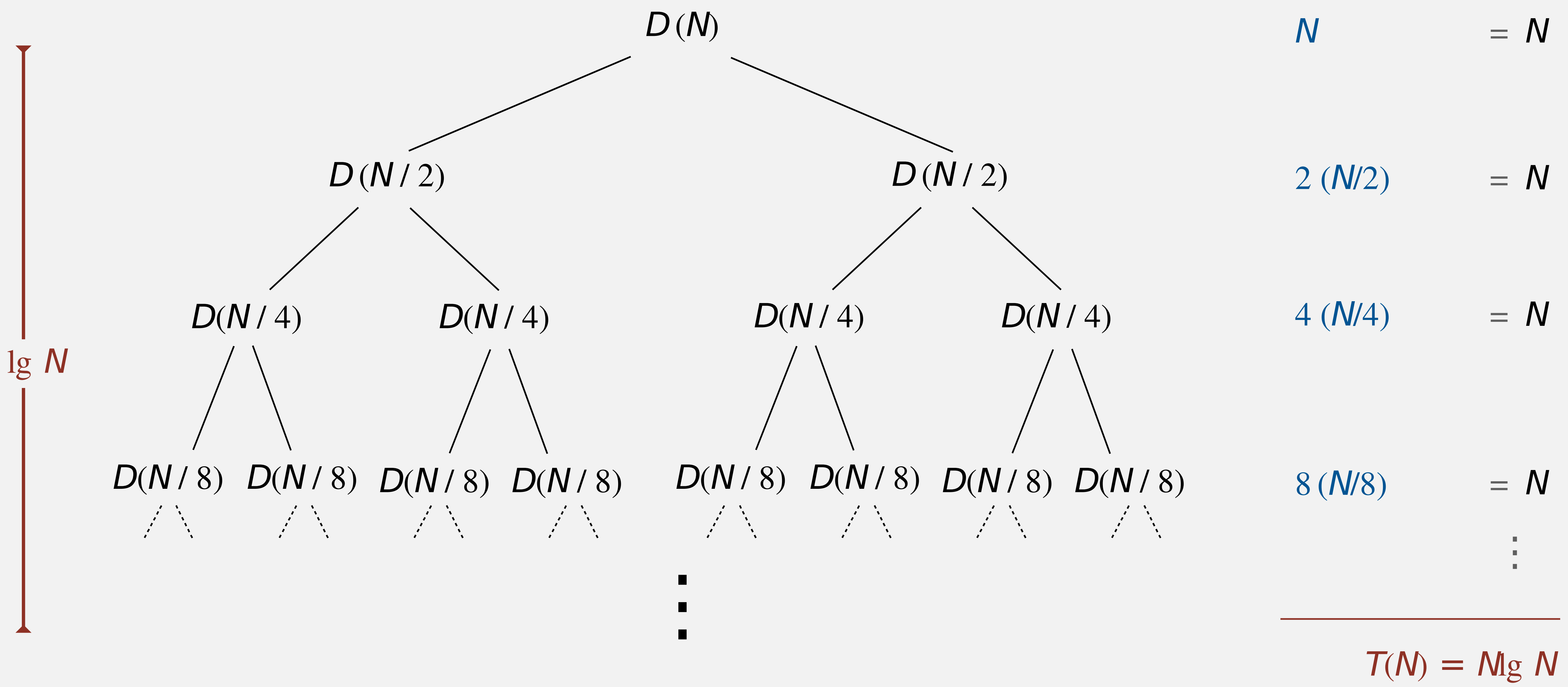
We solve the recurrence when N is a power of 2: result holds for all N (analysis cleaner in this case)

$$D(N) = 2\,D(N/2) + N, \textbf{ for } N > 1\textbf{, with } D(1) = 0.$$

Divide-and-conquer recurrence: proof by picture

Proposition. **If $D(N)$ satisfies $D(N) = 2\,D(N/2) + N$ for $N > 1$, with $D(1) = 0$, then $D(N) = N \lg N$.**

Pf 1. **[assuming N is a power of 2]**



Divide-and-conquer recurrence: proof by induction

Proposition. If $D(N)$ satisfies $D(N) = 2\,D(N/2) + N$ for $N > 1$, with $D(1) = 0$, then $D(N) = N \lg N$.

Pf 2. **[assuming N is a power of 2]**

- **Base case:** $N = 1$.
- **Inductive hypothesis:** $D(N) = N \lg N$.
- **Goal: show that** $D(2N) = (2N) \lg (2N)$.

$D(2N) = 2\,D(N) + 2N$	given
$= 2\,N \lg N + 2N$	inductive hypothesis
$= 2\,N\,(\lg (2N) - 1) + 2N$	algebra
$= 2\,N \lg (2N)$	QED

Mergesort: number of array accesses

Proposition. **Mergesort uses $6N \lg N$ array accesses to sort an array of length *N*.**

Pf sketch. **The number of array accesses $A(N)$ satisfies the recurrence:**

$$A(N) \le A(\lceil N/2 \rceil) + A(\lfloor N/2 \rfloor) + 6N$$ **for $N > 1$, with $A(1) = 0$.**

Key point. **Any algorithm with the following structure takes $N \lg N$ time:**

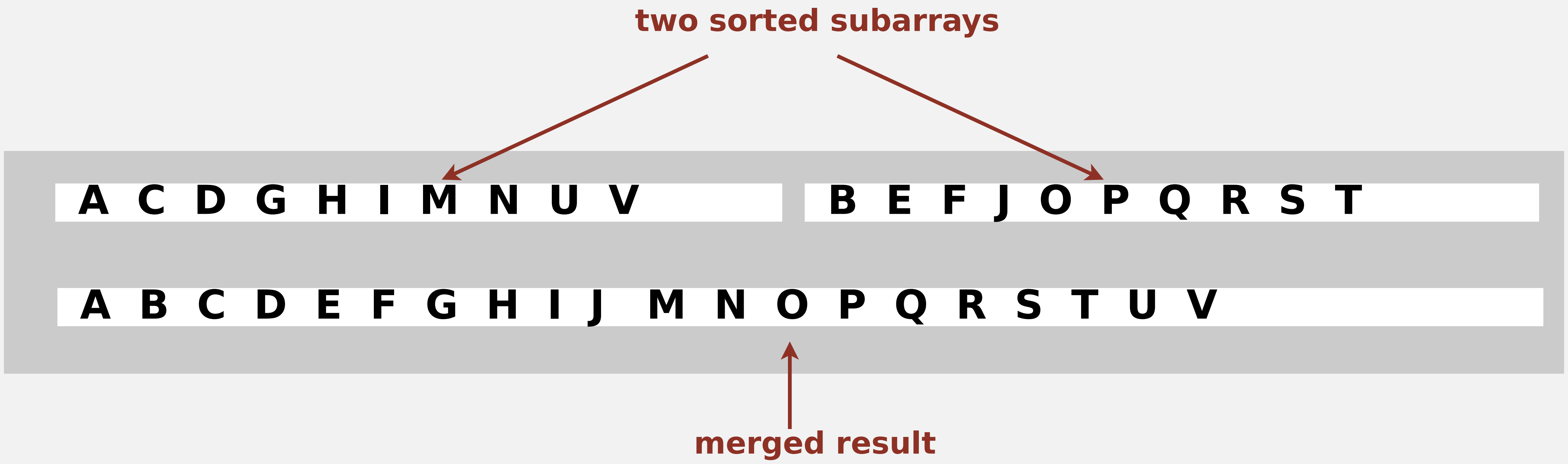
```
public static void linearithmic(int N)
{
   if (N == 0) return;
   linearithmic(N/2);        ← solve two problems
   linearithmic(N/2);        ← of half the size
   linear(N);                ← do a linear amount of work
}
```

Notable examples. **FFT, hidden-line removal, Kendall-tau distance, …**

Mergesort analysis: memory

Proposition. **Mergesort uses extra space proportional to N.**

Pf. **The array `aux[]` needs to be of length N for the last merge.**



Def. **A sorting algorithm is in-place if it uses $\le c \log N$ extra memory.**

Ex. **Insertion sort, selection sort, shellsort.**

Challenge 1 (not hard). **Use `aux[]` array of length $\sim \frac{1}{2} N$ instead of N.**

Challenge 2 (very hard). **In-place merge.** **[Kronrod 1969]**

Mergesort: practical improvements

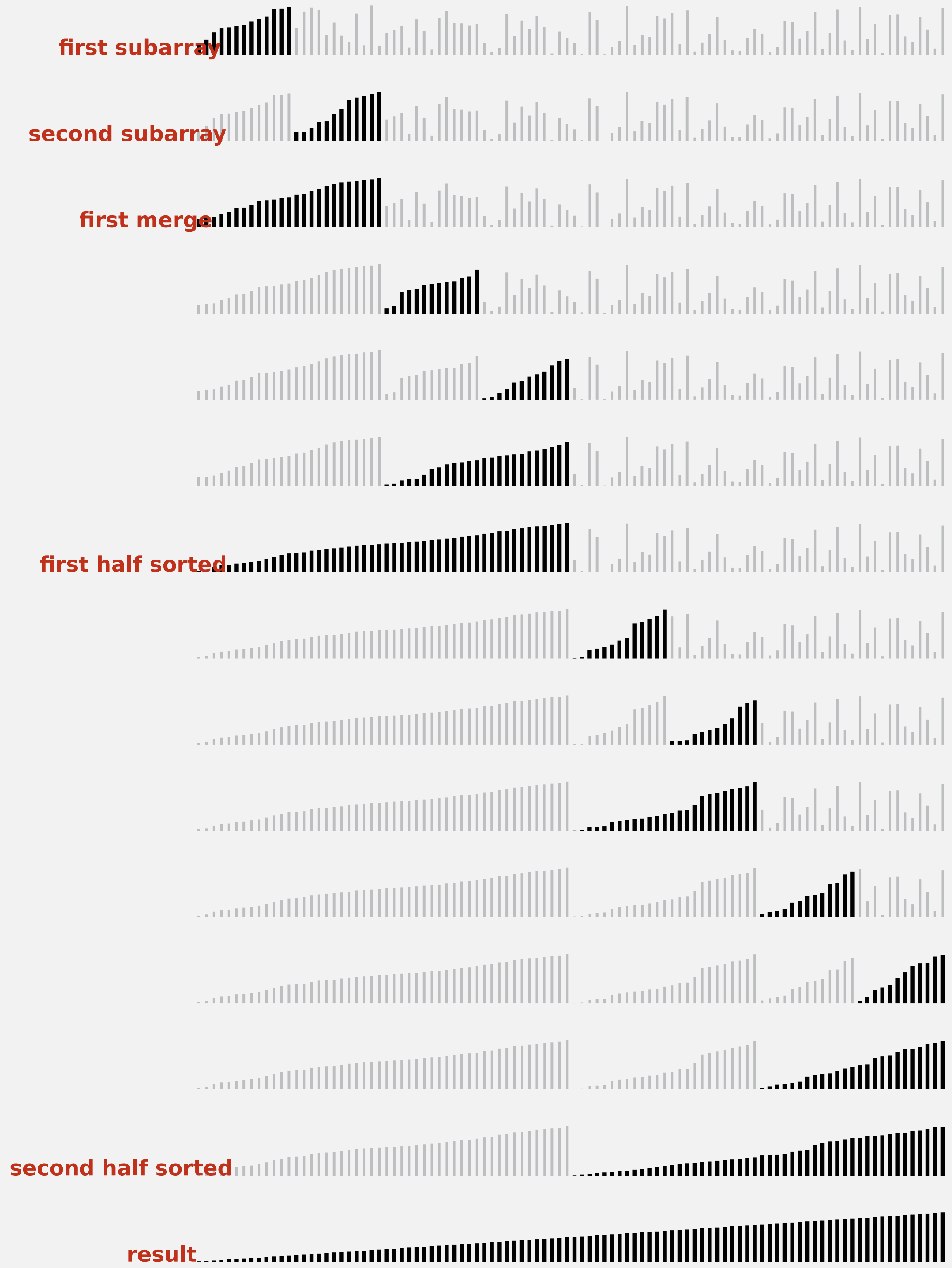
Use insertion sort for small subarrays.

Mergesort has too much overhead for tiny subarrays.

Cutoff to insertion sort for ≈ 10 items.

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
  if (hi <= lo + CUTOFF - 1)
  {
    Insertion.sort(a, lo, hi);
    return;
  }
  int mid = lo + (hi - lo) / 2;
  sort (a, aux, lo, mid);
  sort (a, aux, mid+1, hi);
  merge(a, aux, lo, mid, hi);
}
```

Mergesort with cutoff to insertion sort: visualization



Visual trace of top-down mergesort for with cutoff for small subarrays

Mergesort: practical improvements

Stop if already sorted.

- **Is largest item in first half ≤ smallest item in second half?**
- **Helps for partially-ordered arrays.**

A B C D E F G H I J   M N O P Q R S T U V

A B C D E F G H I J M N O P Q R S T U V

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo) return;
   int mid = lo + (hi - lo) / 2;
   sort (a, aux, lo, mid);
   sort (a, aux, mid+1, hi);
   if (!less(a[mid+1], a[mid])) return;
   merge(a, aux, lo, mid, hi);
}
```

Mergesort: practical improvements

Eliminate the copy to the auxiliary array. **Save time (but not space) by switching the role of the input and auxiliary array in each recursive**

```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
{
   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++)
   {
      if      (i > mid)          aux[k] = a[j++];
      else if (j > hi)           aux[k] = a[i++];
      else if (less(a[j], a[i])) aux[k] = a[j++];
      else                       aux[k] = a[i++];
   }
}

private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo) return;
   int mid = lo + (hi - lo) / 2;
   sort (aux, a, lo, mid);
   sort (aux, a, mid+1, hi);
   merge(a, aux, lo, mid, hi);
}
```

merge from a[] to aux[]

assumes aux[] is initialize to a[] once, before recursive calls

switch roles of aux[] and a[]

Java 6 system sort

Basic algorithm for sorting objects = mergesort.

- **Cutoff to insertion sort = 7.**
- **Stop-if-already-sorted test.**
- **Eliminate-the-copy-to-the-auxiliary-array trick.**

Arrays.sort(a)

http://www.java2s.com/Open-Source/Java/6.0-JDK-Modules/j2me/java/util/Arrays.java.html

Algorithms

Robert Sedgewick | Kevin Wayne

http://algs4.cs.princeton.edu

2.2 Mergesort

- mergesort
- **bottom-up mergesort**
- sorting complexity
- comparators
- stability

Bottom-up mergesort

Basic plan.

- **Pass through array, merging subarrays of size 1.**
- **Repeat for subarrays of size 2, 4, 8, ...**

```
                                          a[i]
                                 0  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15
   sz = 1                        M  E  R  G  E  S  O  R  T  E  X  A  M  P  L  E
 merge(a, aux,  0,  0,  1)       E  M  R  G  E  S  O  R  T  E  X  A  M  P  L  E
 merge(a, aux,  2,  2,  3)       E  M  G  R  E  S  O  R  T  E  X  A  M  P  L  E
 merge(a, aux,  4,  4,  5)       E  M  G  R  E  S  O  R  T  E  X  A  M  P  L  E
 merge(a, aux,  6,  6,  7)       E  M  G  R  E  S  O  R  T  E  X  A  M  P  L  E
 merge(a, aux,  8,  8,  9)       E  M  G  R  E  S  O  R  E  T  X  A  M  P  L  E
 merge(a, aux, 10, 10, 11)       E  M  G  R  E  S  O  R  E  T  A  X  M  P  L  E
 merge(a, aux, 12, 12, 13)       E  M  G  R  E  S  O  R  E  T  A  X  M  P  L  E
 merge(a, aux, 14, 14, 15)       E  M  G  R  E  S  O  R  E  T  A  X  M  P  E  L
   sz = 2
 merge(a, aux,  0,  1,  3)       E  G  M  R  E  S  O  R  E  T  A  X  M  P  E  L
 merge(a, aux,  4,  5,  7)       E  G  M  R  E  O  R  S  E  T  A  X  M  P  E  L
 merge(a, aux,  8,  9, 11)       E  G  M  R  E  O  R  S  A  E  T  X  M  P  E  L
 merge(a, aux, 12, 13, 15)       E  G  M  R  E  O  R  S  A  E  T  X  E  L  M  P
   sz = 4
 merge(a, aux,  0,  3,  7)       E  E  G  M  O  R  R  S  A  E  T  X  E  L  M  P
 merge(a, aux,  8, 11, 15)       E  E  G  M  O  R  R  S  A  E  E  L  M  P  T  X
   sz = 8
 merge(a, aux,  0,  7, 15)       A  E  E  E  E  G  L  M  M  O  P  R  R  S  T  X
```

Trace of merge results for bottom-up mergesort

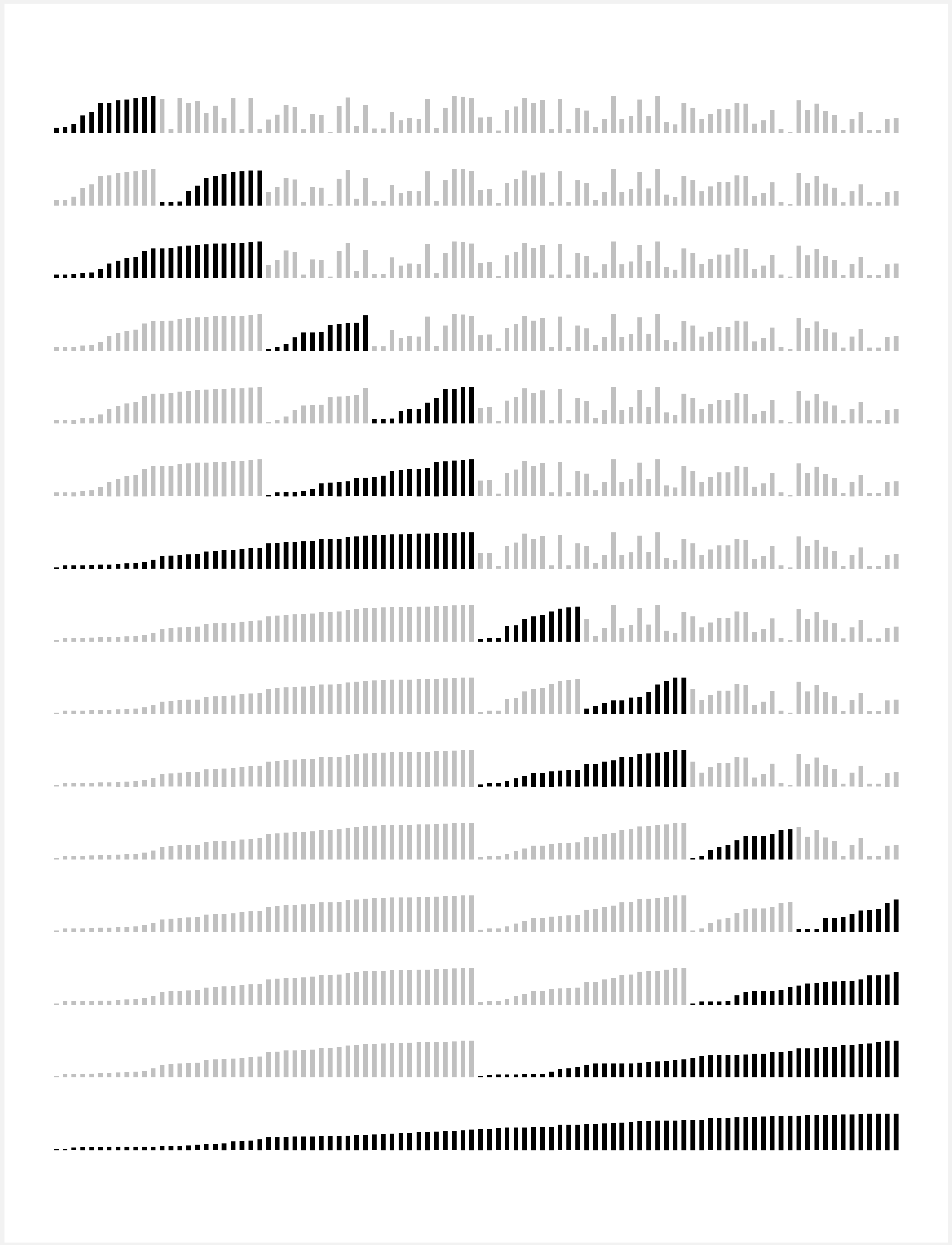
Bottom-up mergesort: Java implementation

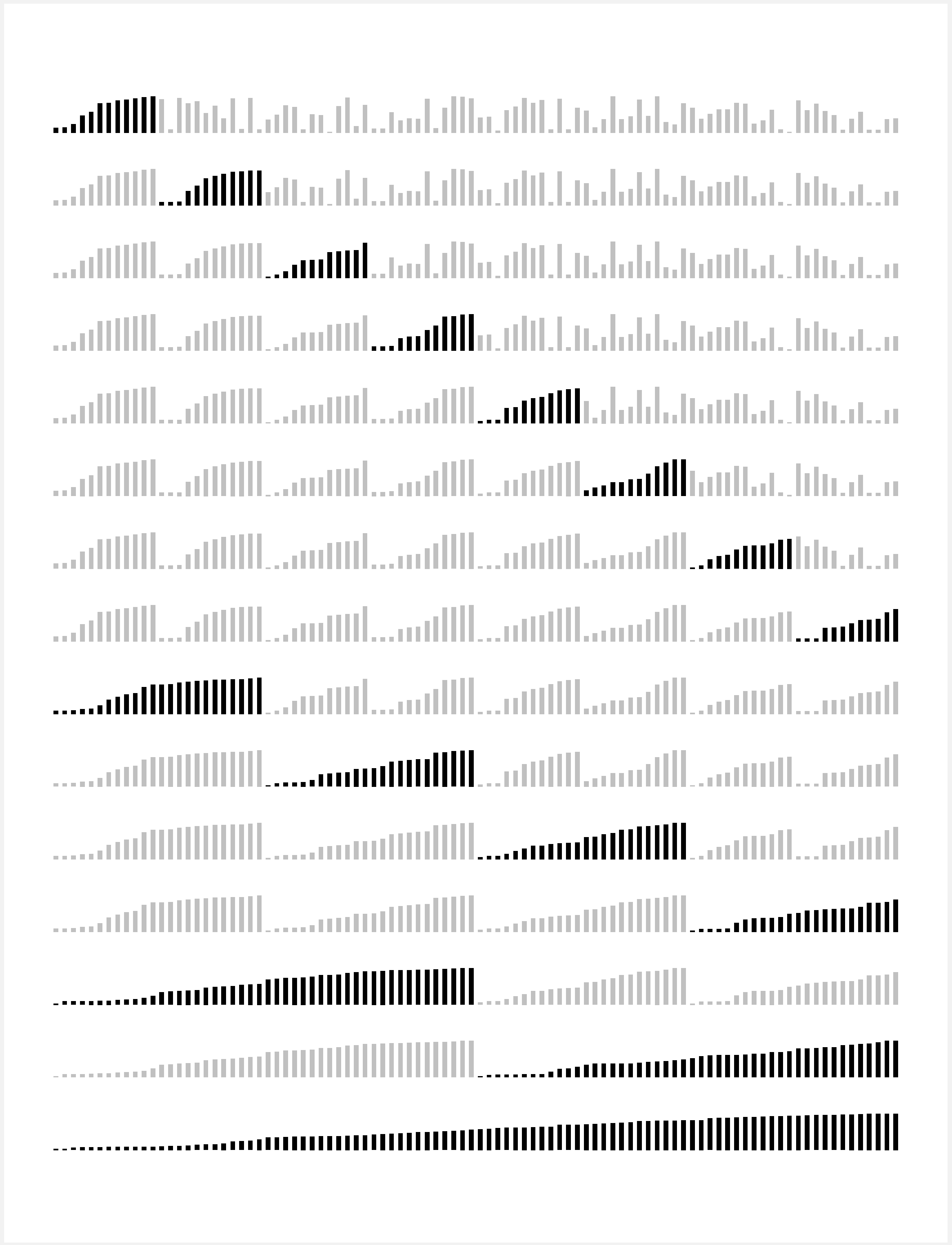
```
public class MergeBU
{
   private static void merge(...)
   {  /* as before */  }

   public static void sort(Comparable[] a)
   {
      int N = a.length;
      Comparable[] aux = new Comparable[N];
      for (int sz = 1; sz < N; sz = sz+sz)
         for (int lo = 0; lo < N-sz; lo += sz+sz)
            merge(a, aux, lo, lo+sz-1, Math.min(lo+sz+sz-1, N-1));
   }
}
```

but about 10% slower than recursive, top-down mergesort on typical systems

Bottom line. **Simple and non-recursive version of mergesort.**

Mergesort: visualizations


top-down mergesort (cutoff = 12)


bottom-up mergesort (cutoff = 12)

Natural mergesort

Idea. **Exploit pre-existing order by identifying naturally-occurring runs**

input

1	**5**	**10**	**16**	**3**	**4**	**23**	**9**	**13**	**2**	**7**	**8**	**12**	**14**

first run

1	**5**	**10**	**16**	3	4	23	9	13	2	7	8	12	14

second run

1	5	10	16	**3**	**4**	**23**	9	13	2	7	8	12	14

merge two runs

1	**3**	**4**	**5**	**10**	**16**	**23**	9	13	2	7	8	12	14

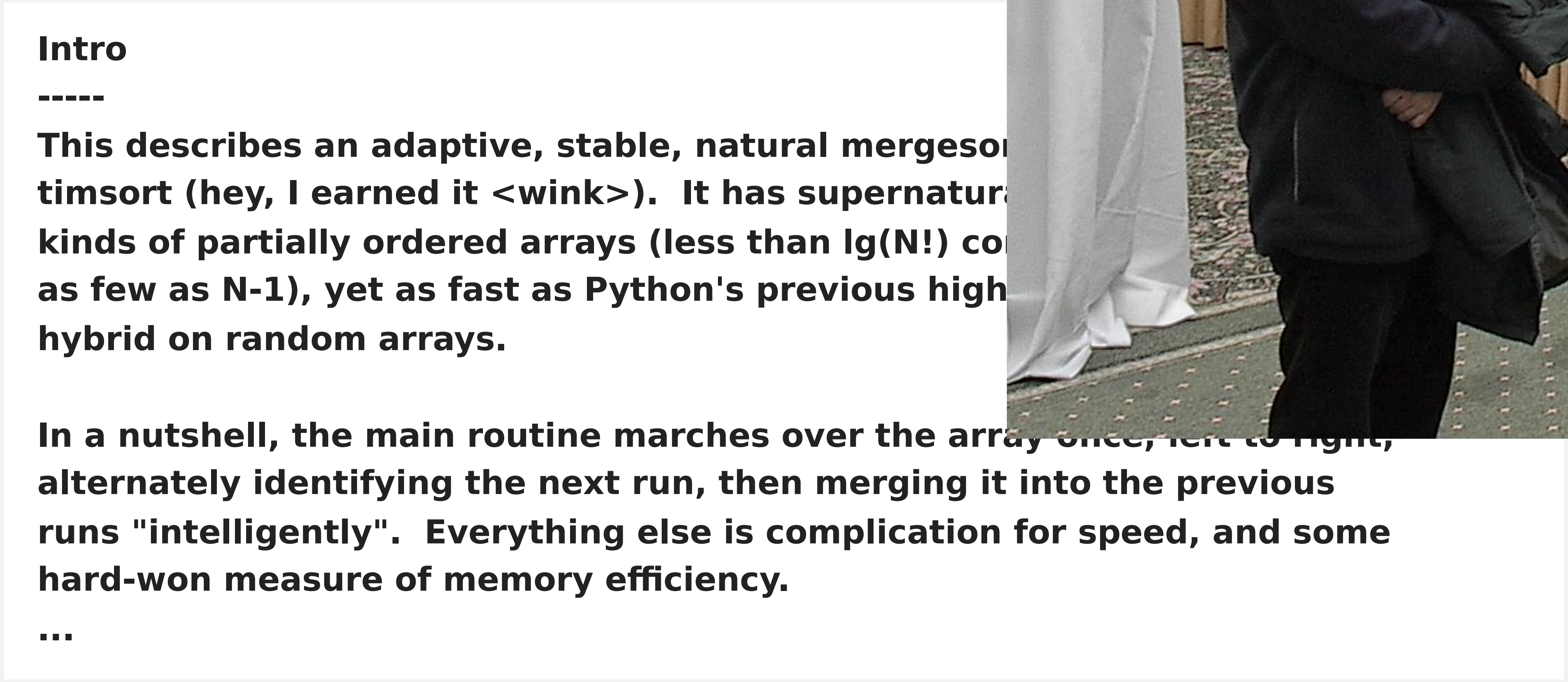
Tradeoff. **Fewer passes vs. extra compares per pass to identify runs.**

Timsort

Natural mergesort.

Use binary insertion sort to mak

A few more clever optimizations



Intro

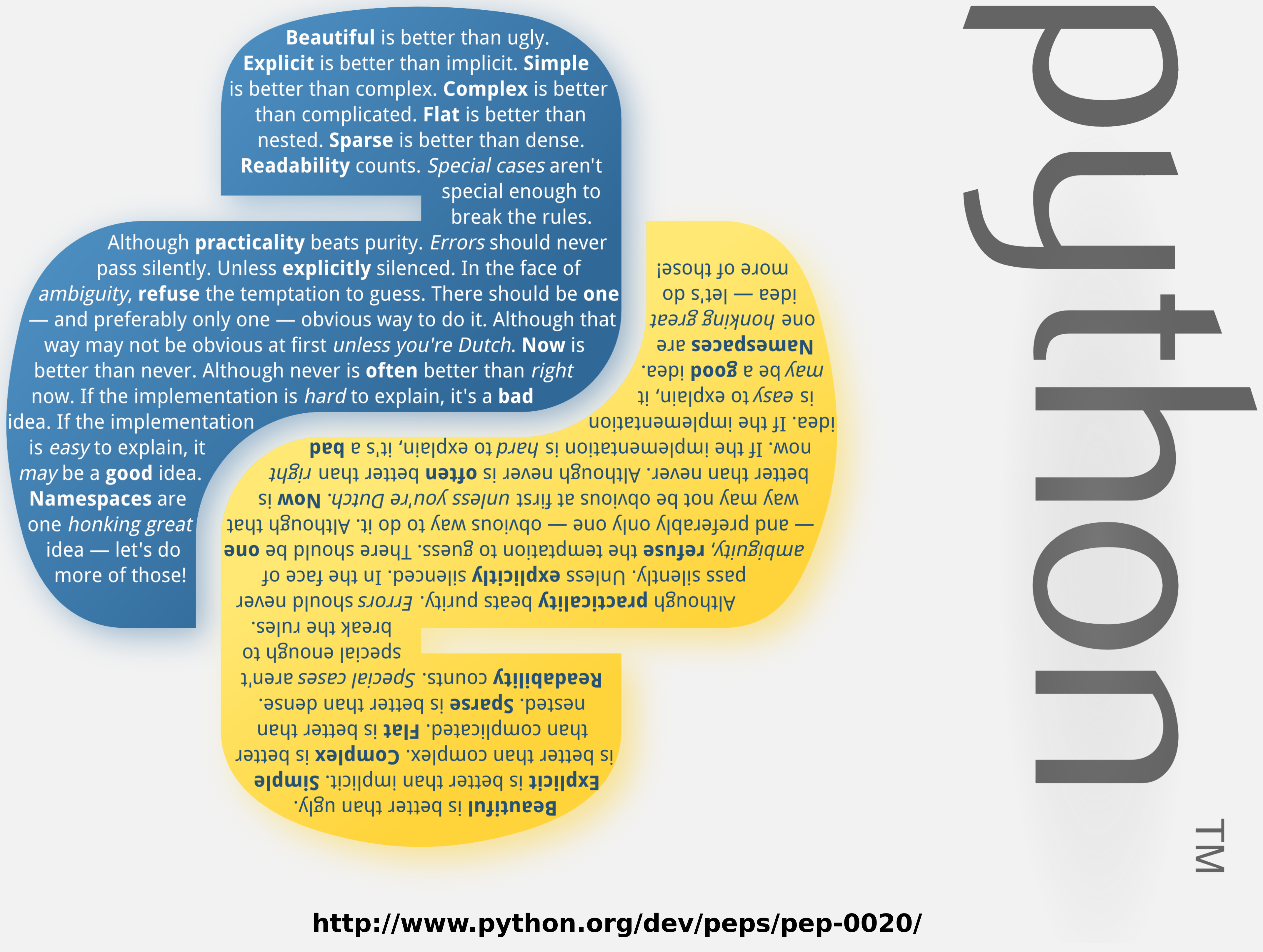
This describes an adaptive, stable, natural mergeso
timsort (hey, I earned it <wink>). It has supernatur
kinds of partially ordered arrays (less than lg(N!) co
as few as N-1), yet as fast as Python's previous high
hybrid on random arrays.

In a nutshell, the main routine marches over the arra
alternately identifying the next run, then merging it into the previous
runs "intelligently". Everything else is complication for speed, and some
hard-won measure of memory efficiency.
...

Consequence. **Linear time on many arrays with pre-existing order.**

Now widely used. **Python, Java 7, GNU Octave, Android, ….**

The Zen of Python


Beautiful is better than ugly. **Explicit** is better than implicit. **Simple** is better than complex. **Complex** is better than complicated. **Flat** is better than nested. **Sparse** is better than dense. **Readability** counts. *Special cases* aren't special enough to break the rules. Although **practicality** beats purity. *Errors* should never pass silently. Unless **explicitly** silenced. In the face of *ambiguity*, **refuse** the temptation to guess. There should be **one** — and preferably only one — obvious way to do it. Although that way may not be obvious at first *unless you're Dutch*. **Now** is better than never. Although never is **often** better than *right* now. If the implementation is *hard* to explain, it's a **bad** idea. If the implementation is *easy* to explain, it *may* be a **good** idea. **Namespaces** are one *honking great* idea — let's do more of those!

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http://www.python.org/dev/peps/pep-0020/

http://westmarch.sjsoft.com/2012/11/zen-of-python-poster/

http://algs4.cs.princeton.edu

2.2 Mergesort

- *mergesort*
- *bottom-up mergesort*
- ***sorting complexity***
- *comparators*
- *stability*

Complexity of sorting

Computational complexity. **Framework to study efficiency of algorithm for solving a particular problem X.**

Model of computation. **Allowable operations.**

Cost model. **Operation count(s).**

Upper bound. **Cost guarantee provided by some algorithm for X.**

Lower bound. **Proven limit on cost guarantee of all algorithms for X.**

Optimal algorithm. **Algorithm with best possible cost guarantee for X.**

lower bound ~ upper bound

Example: sorting.

Model of computation: decision tree. ← can access information only through compares (e.g., Java Comparable framework)

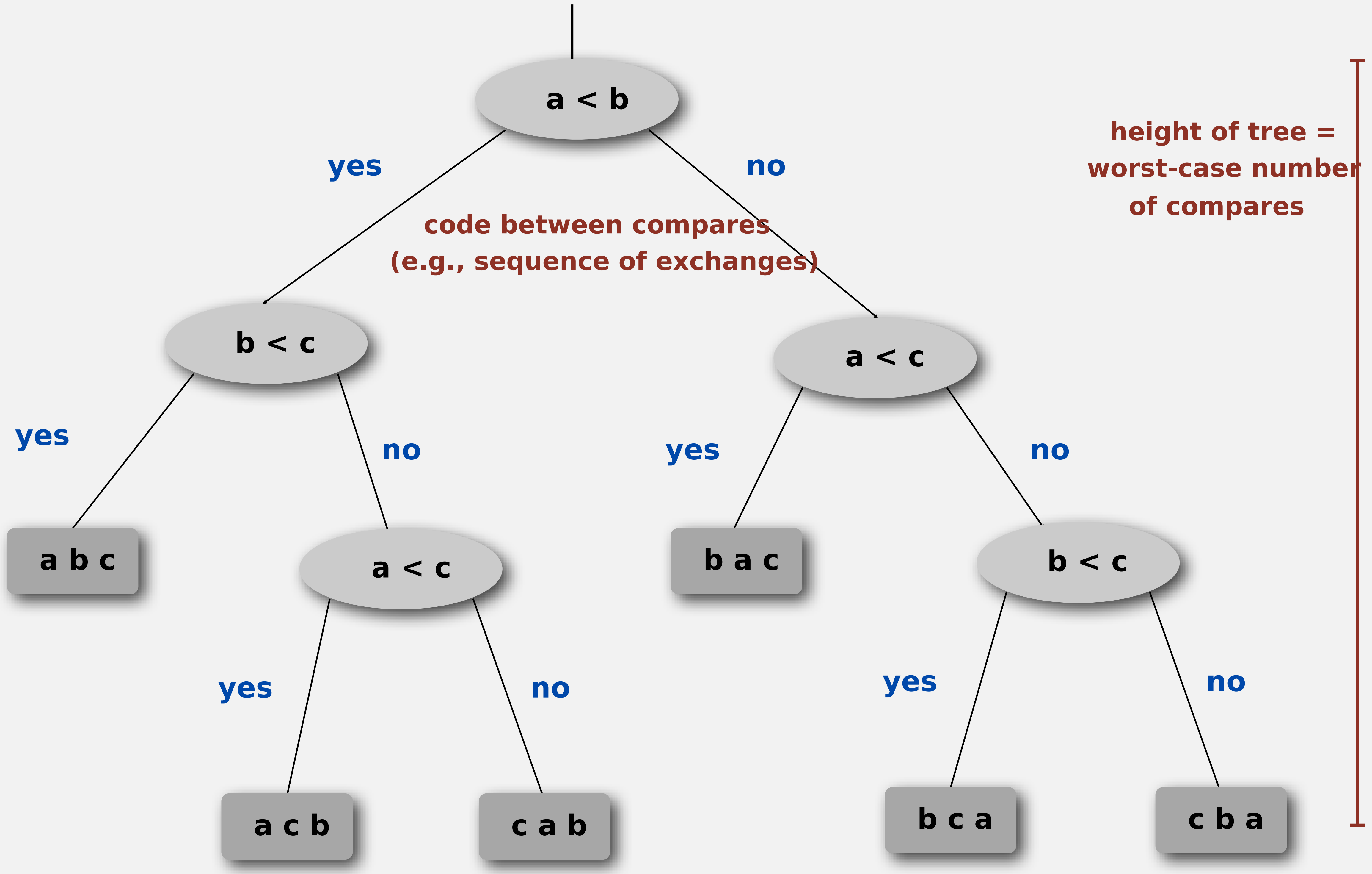
Cost model: # compares.

Upper bound: ~ $N \lg N$ from mergesort.

Lower bound:

Optimal algorithm:

Decision tree (for 3 distinct keys a, b, and c)



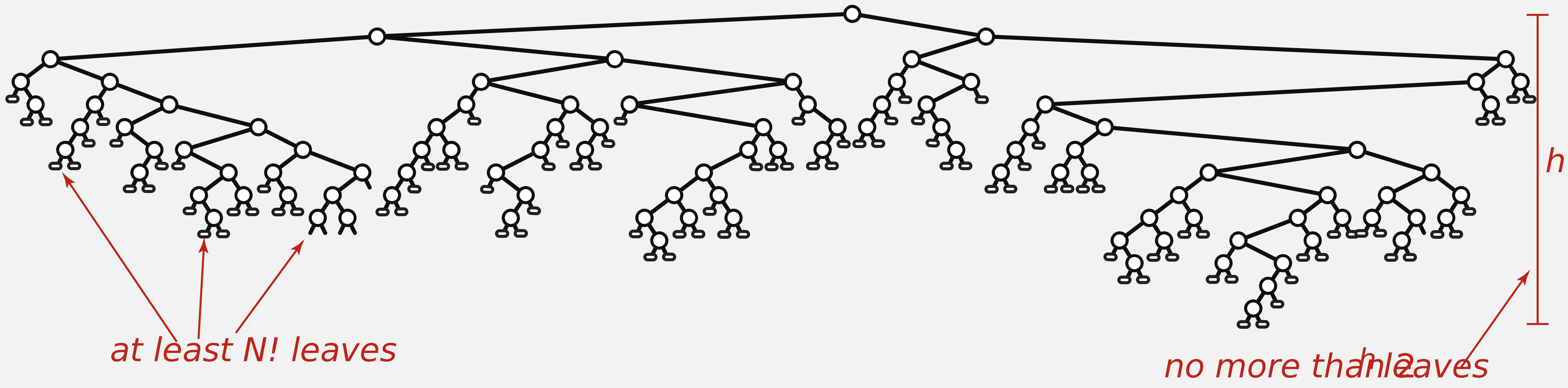
each leaf corresponds to one (and only one) ordering;
(at least) one leaf for each possible ordering

Compare-based lower bound for sorting

Proposition. Any compare-based sorting algorithm must use at least $\lg(N!) \sim N \lg N$ compares in the worst-case.

Pf.

- **Assume array consists of N distinct values a_1 through a_N.**
- **Worst case dictated by height h of decision tree.**
- **Binary tree of height h has at most 2^h leaves.**
- **$N!$ different orderings $\Rightarrow$ at least $N!$ leaves.**



Compare-based lower bound for sorting

Proposition. **Any compare-based sorting algorithm must use at least** $\lg(N!) \sim N \lg N$ **compares in the worst-case.**

Pf.

- **Assume array consists of** N **distinct values** a_1 **through** a_N.
- **Worst case dictated by height** h **of decision tree.**
- **Binary tree of height** h **has at most** 2^h **leaves.**
- $N!$ **different orderings** $\Rightarrow$ **at least** $N!$ **leaves.**

$$2^h \geq \#\text{ leaves} \geq N!$$

$$\Rightarrow\ h \geq \lg(N!) \sim N \lg N$$

Stirling's formula

Complexity of sorting

Model of computation. **Allowable operations.**

Cost model. **Operation count(s).**

Upper bound. **Cost guarantee provided by some algorithm for** X**.**

Lower bound. **Proven limit on cost guarantee of all algorithms for** X**.**

Optimal algorithm. **Algorithm with best possible cost guarantee for** X**.**

Example: sorting.

- **Model of computation: decision tree.**
- **Cost model: # compares.**
- **Upper bound:** $\sim N \lg N$ **from mergesort.**
- **Lower bound:** $\sim N \lg N$**.**
- **Optimal algorithm = mergesort.**

First goal of algorithm design: **optimal algorithms.**

Complexity results in context

Compares? **Mergesort is optimal with respect to number compares.**

Space? **Mergesort is not optimal with respect to space usage.**

Lessons. **Use theory as a guide.**

Ex. **Design sorting algorithm that guarantees $\frac{1}{2} N \lg N$ compares?**

Ex. **Design sorting algorithm that is both time- and space-optimal?**

Complexity results in context (continued)

Lower bound may not hold if the algorithm can take advantage of:

The initial order of the input.
Ex: insert sort requires only a linear number of compares on partia
sorted arrays.

The distribution of key values.
Ex: 3-way quicksort requires only a linear number of compares on
arrays with a constant number of distinct keys. [stay tuned]

The representation of the keys.
Ex: radix sort requires no key compares — it accesses the data via
character/digit compares.

Algorithms

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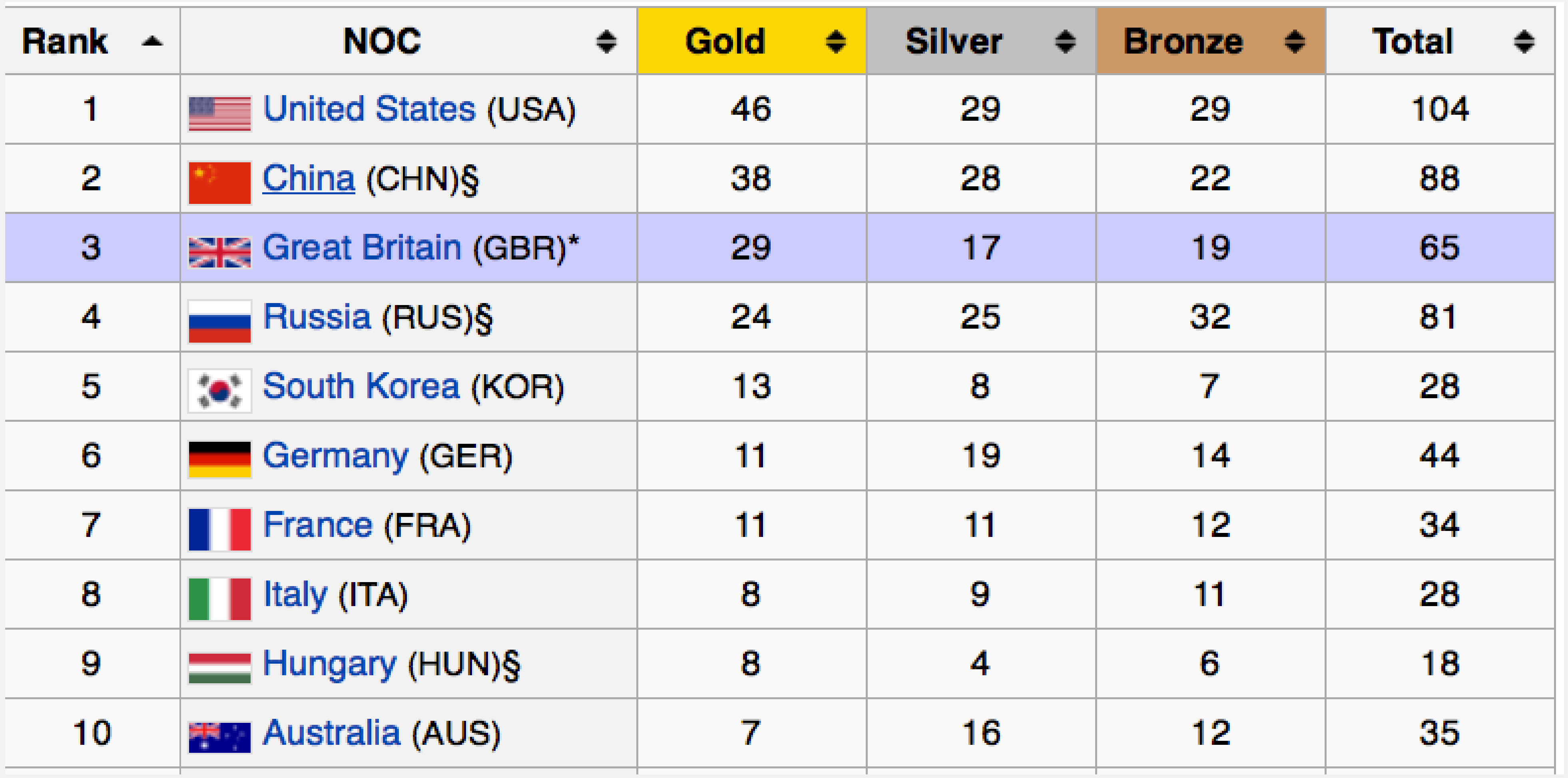
2.2 Mergesort

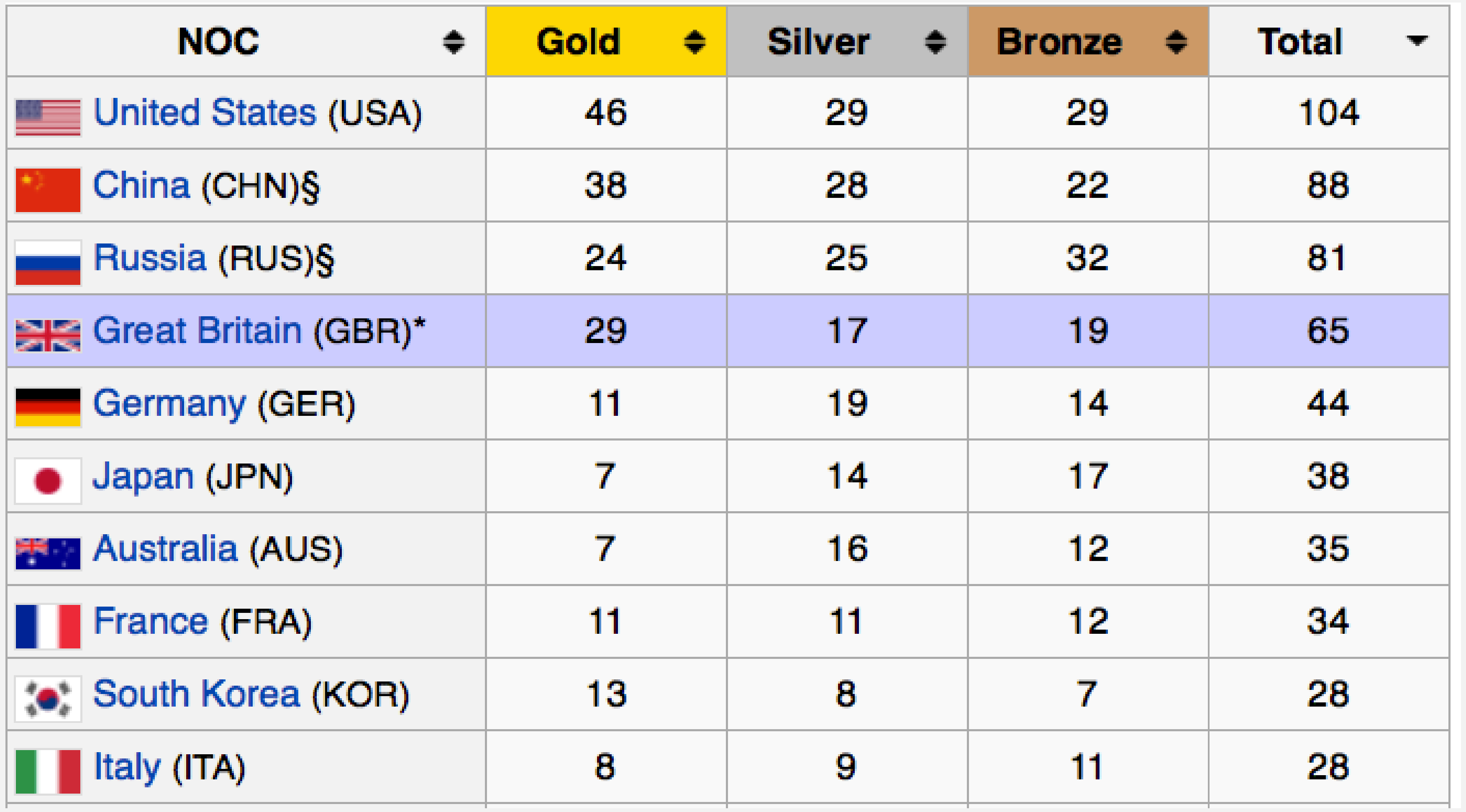
- mergesort
- bottom-up mergesort
- sorting complexity
- **comparators**
- stability

Sort countries by gold medals



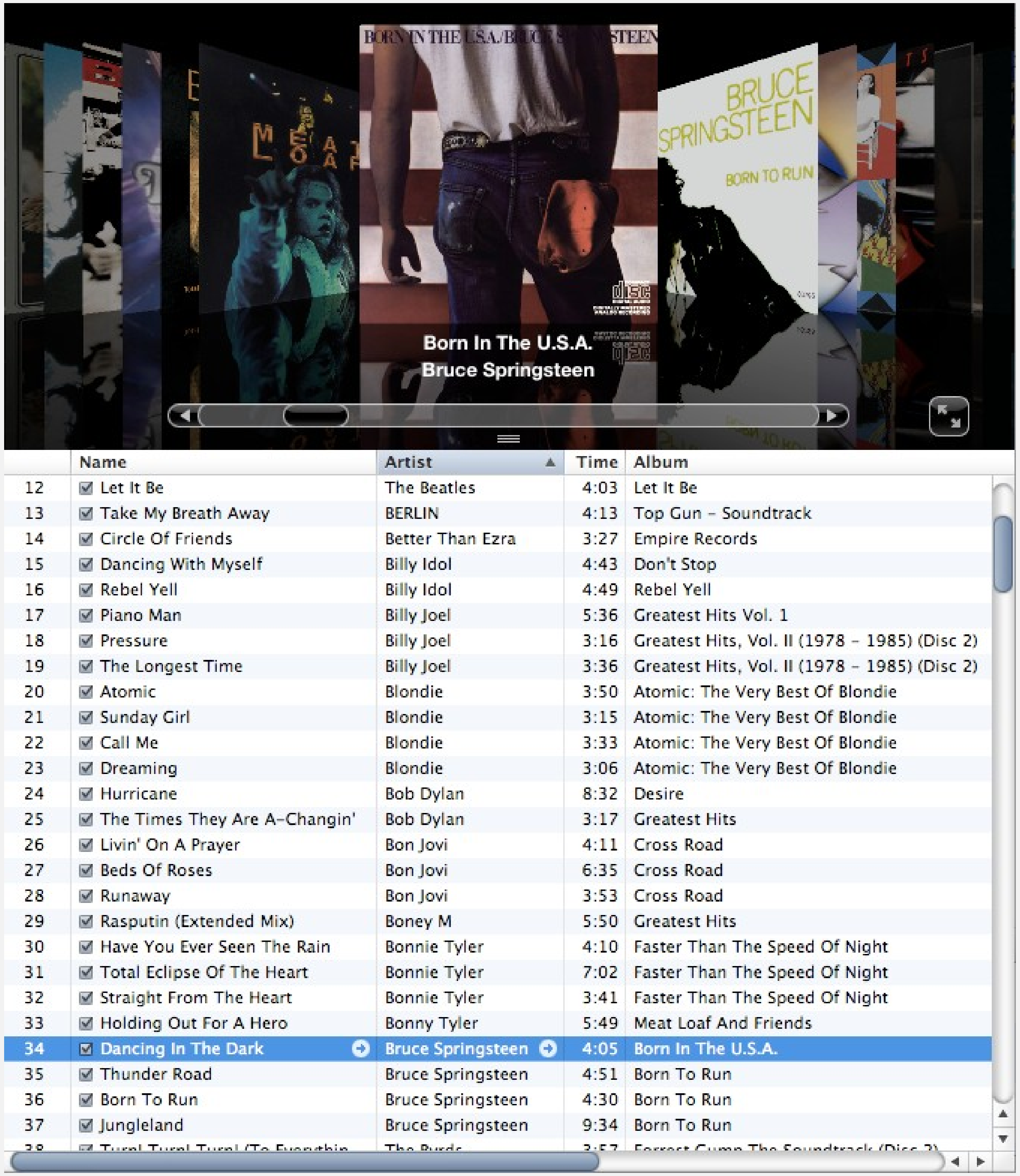
Rank ▲	NOC ⬍	Gold ⬍	Silver ⬍	Bronze ⬍	Total ⬍
1	United States (USA)	46	29	29	104
2	China (CHN)§	38	28	22	88
3	Great Britain (GBR)*	29	17	19	65
4	Russia (RUS)§	24	25	32	81
5	South Korea (KOR)	13	8	7	28
6	Germany (GER)	11	19	14	44
7	France (FRA)	11	11	12	34
8	Italy (ITA)	8	9	11	28
9	Hungary (HUN)§	8	4	6	18
10	Australia (AUS)	7	16	12	35

Sort countries by total medals



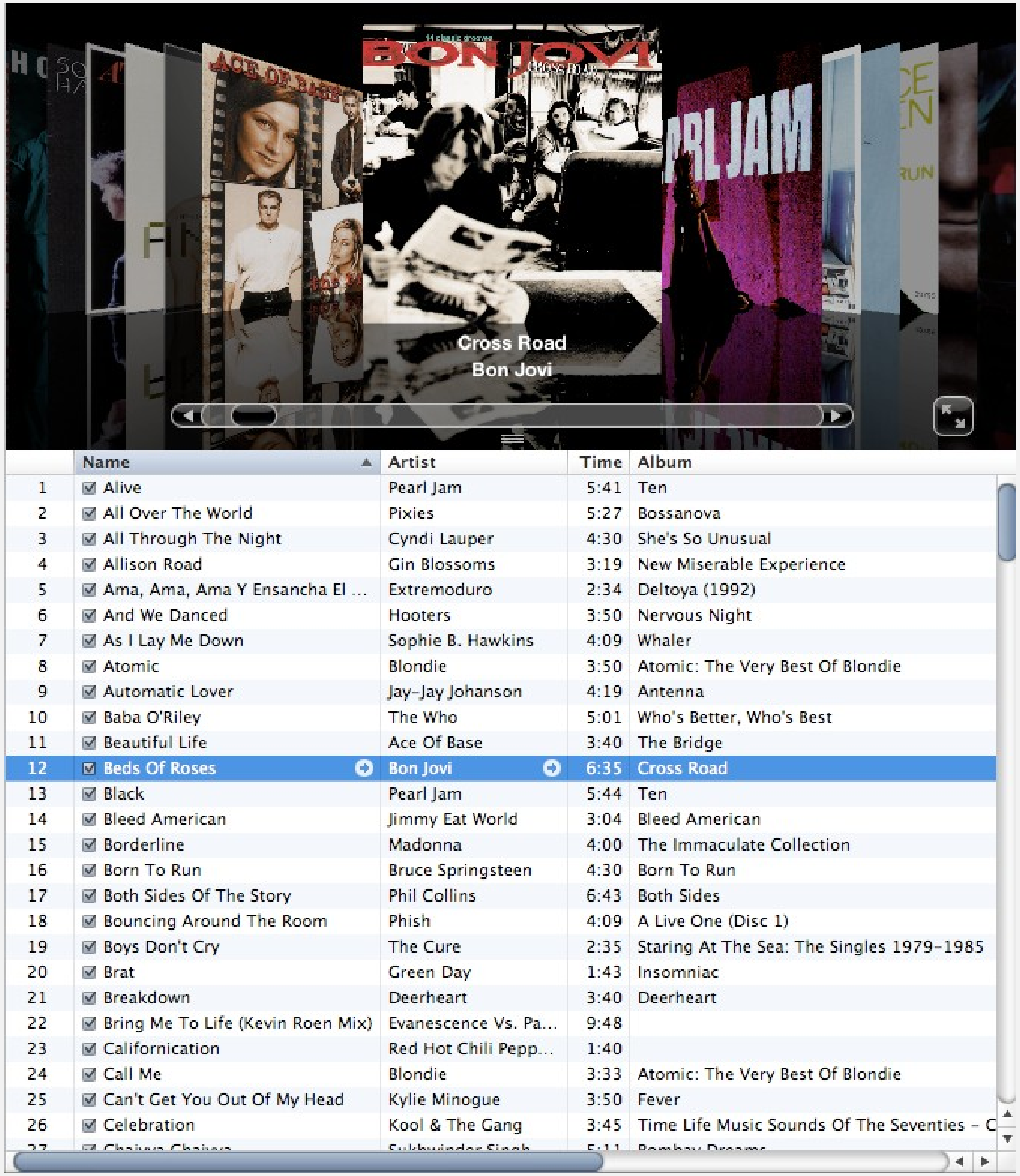
NOC	Gold	Silver	Bronze	Total
United States (USA)	46	29	29	104
China (CHN)§	38	28	22	88
Russia (RUS)§	24	25	32	81
Great Britain (GBR)*	29	17	19	65
Germany (GER)	11	19	14	44
Japan (JPN)	7	14	17	38
Australia (AUS)	7	16	12	35
France (FRA)	11	11	12	34
South Korea (KOR)	13	8	7	28
Italy (ITA)	8	9	11	28

Sort music library by artist



	Name	Artist ▲	Time	Album
12	☑ Let It Be	The Beatles	4:03	Let It Be
13	☑ Take My Breath Away	BERLIN	4:13	Top Gun - Soundtrack
14	☑ Circle Of Friends	Better Than Ezra	3:27	Empire Records
15	☑ Dancing With Myself	Billy Idol	4:43	Don't Stop
16	☑ Rebel Yell	Billy Idol	4:49	Rebel Yell
17	☑ Piano Man	Billy Joel	5:36	Greatest Hits Vol. 1
18	☑ Pressure	Billy Joel	3:16	Greatest Hits, Vol. II (1978 - 1985) (Disc 2)
19	☑ The Longest Time	Billy Joel	3:36	Greatest Hits, Vol. II (1978 - 1985) (Disc 2)
20	☑ Atomic	Blondie	3:50	Atomic: The Very Best Of Blondie
21	☑ Sunday Girl	Blondie	3:15	Atomic: The Very Best Of Blondie
22	☑ Call Me	Blondie	3:33	Atomic: The Very Best Of Blondie
23	☑ Dreaming	Blondie	3:06	Atomic: The Very Best Of Blondie
24	☑ Hurricane	Bob Dylan	8:32	Desire
25	☑ The Times They Are A-Changin'	Bob Dylan	3:17	Greatest Hits
26	☑ Livin' On A Prayer	Bon Jovi	4:11	Cross Road
27	☑ Beds Of Roses	Bon Jovi	6:35	Cross Road
28	☑ Runaway	Bon Jovi	3:53	Cross Road
29	☑ Rasputin (Extended Mix)	Boney M	5:50	Greatest Hits
30	☑ Have You Ever Seen The Rain	Bonnie Tyler	4:10	Faster Than The Speed Of Night
31	☑ Total Eclipse Of The Heart	Bonnie Tyler	7:02	Faster Than The Speed Of Night
32	☑ Straight From The Heart	Bonnie Tyler	3:41	Faster Than The Speed Of Night
33	☑ Holding Out For A Hero	Bonny Tyler	5:49	Meat Loaf And Friends
34	☑ Dancing In The Dark	Bruce Springsteen	4:05	Born In The U.S.A.
35	☑ Thunder Road	Bruce Springsteen	4:51	Born To Run
36	☑ Born To Run	Bruce Springsteen	4:30	Born To Run
37	☑ Jungleland	Bruce Springsteen	9:34	Born To Run
38	☑ Turn! Turn! Turn! (To Everythin	The Byrds	3:57	Forrest Gump The Soundtrack (Disc 2)

Sort music library by song name



		Name ▲	Artist	Time	Album
1	✓	Alive	Pearl Jam	5:41	Ten
2	✓	All Over The World	Pixies	5:27	Bossanova
3	✓	All Through The Night	Cyndi Lauper	4:30	She's So Unusual
4	✓	Allison Road	Gin Blossoms	3:19	New Miserable Experience
5	✓	Ama, Ama, Ama Y Ensancha El ...	Extremoduro	2:34	Deltoya (1992)
6	✓	And We Danced	Hooters	3:50	Nervous Night
7	✓	As I Lay Me Down	Sophie B. Hawkins	4:09	Whaler
8	✓	Atomic	Blondie	3:50	Atomic: The Very Best Of Blondie
9	✓	Automatic Lover	Jay-Jay Johanson	4:19	Antenna
10	✓	Baba O'Riley	The Who	5:01	Who's Better, Who's Best
11	✓	Beautiful Life	Ace Of Base	3:40	The Bridge
12	✓	Beds Of Roses	Bon Jovi	6:35	Cross Road
13	✓	Black	Pearl Jam	5:44	Ten
14	✓	Bleed American	Jimmy Eat World	3:04	Bleed American
15	✓	Borderline	Madonna	4:00	The Immaculate Collection
16	✓	Born To Run	Bruce Springsteen	4:30	Born To Run
17	✓	Both Sides Of The Story	Phil Collins	6:43	Both Sides
18	✓	Bouncing Around The Room	Phish	4:09	A Live One (Disc 1)
19	✓	Boys Don't Cry	The Cure	2:35	Staring At The Sea: The Singles 1979-1985
20	✓	Brat	Green Day	1:43	Insomniac
21	✓	Breakdown	Deerheart	3:40	Deerheart
22	✓	Bring Me To Life (Kevin Roen Mix)	Evanescence Vs. Pa...	9:48	
23	✓	Californication	Red Hot Chili Pepp...	1:40	
24	✓	Call Me	Blondie	3:33	Atomic: The Very Best Of Blondie
25	✓	Can't Get You Out Of My Head	Kylie Minogue	3:50	Fever
26	✓	Celebration	Kool & The Gang	3:45	Time Life Music Sounds Of The Seventies – C

Comparable interface: review

Comparable interface: sort using a type's natural order.

```
public class Date implements Comparable<Date>
{
   private final int month, day, year;

   public Date(int m, int d, int y)
   {
      month = m;
      day   = d;
      year  = y;
   }
      ...
   public int compareTo(Date that)
   {
      if (this.year  < that.year ) return -1;
      if (this.year  > that.year ) return +1;
      if (this.month < that.month) return -1;
      if (this.month > that.month) return +1;
      if (this.day   < that.day  ) return -1;
      if (this.day   > that.day  ) return +1;
      return 0;
   }
}
```

natural order

Comparator interface

Comparator interface: sort using an alternate order.

public interface Comparator<Key>		
int	compare(Key v, Key w)	*compare keys v and w*

Required property. Must be a total order.

string order	example
natural order	Now is the time
case insensitive	is Now the time
Spanish language	café cafetero cuarto churro nube ñoño
British phone book	McKinley Mackintosh

pre-1994 order for digraphs ch and ll and rr

Comparator interface: system sort

To use with Java system sort:

- **Create Comparator object.**
- **Pass as second argument to Arrays.sort().**

uses natural order

uses alternate order defined by Comparator<String> object

```
String[] a;
...
Arrays.sort(a);
...
Arrays.sort(a, String.CASE_INSENSITIVE_ORDER);
...
Arrays.sort(a, Collator.getInstance(new Locale("es")));
...
Arrays.sort(a, new BritishPhoneBookOrder());
...
```

Bottom line. **Decouples the definition of the data type from the definition of what it means to compare two objects of that type.**

Comparator interface: using with our sorting libraries

To support comparators in our sort implementations:

- Use `Object` instead of `Comparable`.
- Pass `Comparator` to `sort()` and `less()` and use it in `less()` .

insertion sort using a Comparator

```
public static void sort(Object[] a, Comparator comparator)
{
   int N = a.length;
   for (int i = 0; i < N; i++)
      for (int j = i; j > 0 && less(comparator, a[j], a[j-1]); j--)
         exch(a, j, j-1);
}

private static boolean less(Comparator c, Object v, Object w)
{  return c.compare(v, w) < 0;   }

private static void exch(Object[] a, int i, int j)
{  Object swap = a[i]; a[i] = a[j]; a[j] = swap;  }
```

Comparator interface: implementing

To implement a comparator:

- Define a (nested) class that implements the `Comparator` interface.
- Implement the `compare()` method.

```
public class Student
{
   private final String name;
   private final int section;
   ...

   public static class ByName implements Comparator<Student>
   {
     public int compare(Student v, Student w)
      {  return v.name.compareTo(w.name);  }
   }

   public static class BySection implements Comparator<Student>
   {
      public int compare(Student v, Student w)
      {  return v.section - w.section;  }
   }
}
```

this trick works here
since no danger of overflow

Comparator interface: implementing

To implement a comparator:

- **Define a (nested) class that implements the `Comparator` interface.**
- **Implement the `compare()` method.**

Arrays.sort(a, new Student.ByName());

Andrews	3	A	664-480-0023	097 Little
Battle	4	C	874-088-1212	121 Whitman
Chen	3	A	991-878-4944	308 Blair
Fox	3	A	884-232-5341	11 Dickinson
Furia	1	A	766-093-9873	101 Brown
Gazsi	4	B	766-093-9873	101 Brown
Kanaga	3	B	898-122-9643	22 Brown
Rohde	2	A	232-343-5555	343 Forbes

Arrays.sort(a, new Student.BySection());

Furia	1	A	766-093-9873	101 Brown
Rohde	2	A	232-343-5555	343 Forbes
Andrews	3	A	664-480-0023	097 Little
Chen	3	A	991-878-4944	308 Blair
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Gazsi	4	B	766-093-9873	101 Brown

Algorithms

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2.2 MERGESORT

- *mergesort*
- *bottom-up mergesort*
- *sorting complexity*
- *comparators*
- ***stability***

Stability

A typical application. **First, sort by name; then sort by section.**

Selection.sort(a, new Student.ByName());

Andrews	3	A	664-480-0023	097 Little
Battle	4	C	874-088-1212	121 Whitman
Chen	3	A	991-878-4944	308 Blair
Fox	3	A	884-232-5341	11 Dickinson
Furia	1	A	766-093-9873	101 Brown
Gazsi	4	B	766-093-9873	101 Brown
Kanaga	3	B	898-122-9643	22 Brown
Rohde	2	A	232-343-5555	343 Forbes

Selection.sort(a, new Student.BySection());

Furia	1	A	766-093-9873	101 Brown
Rohde	2	A	232-343-5555	343 Forbes
[redacted]	3	A	991-878-4944	308 Blair
[redacted]	3	A	884-232-5341	11 Dickinson
[redacted]	3	A	664-480-0023	097 Little
[redacted]	3	B	898-122-9643	22 Brown
Gazsi	4	B	766-093-9873	101 Brown
Battle	4	C	874-088-1212	121 Whitman

@#%&@! **Students in section 3 no longer sorted by name.**

A stable sort preserves the relative order of items with equal keys.

Stability

Q. Which sorts are stable?

A. Need to check algorithm (and implementation).

sorted by time	sorted by location (not stable)	sorted by location (stable)
Chicago 09:00:00	Chicago 09:25:52	Chicago 09:00:00
Phoenix 09:00:03	Chicago 09:03:13	Chicago 09:00:59
Houston 09:00:13	Chicago 09:21:05	Chicago 09:03:13
Chicago 09:00:59	Chicago 09:19:46	Chicago 09:19:32
Houston 09:01:10	Chicago 09:19:32	Chicago 09:19:46
Chicago 09:03:13	Chicago 09:00:00	Chicago 09:21:05
Seattle 09:10:11	Chicago 09:35:21	Chicago 09:25:52
Seattle 09:10:25	Chicago 09:00:59	Chicago 09:35:21
Phoenix 09:14:25	Houston 09:01:10	Houston 09:00:13
Chicago 09:19:32	Houston 09:00:13	Houston 09:01:10
Chicago 09:19:46	Phoenix 09:37:44	Phoenix 09:00:03
Chicago 09:21:05	Phoenix 09:00:03	Phoenix 09:14:25
Seattle 09:22:43	Phoenix 09:14:25	Phoenix 09:37:44
Seattle 09:22:54	Seattle 09:10:25	Seattle 09:10:11
Chicago 09:25:52	Seattle 09:36:14	Seattle 09:10:25
Chicago 09:35:21	Seattle 09:22:43	Seattle 09:22:43
Seattle 09:36:14	Seattle 09:10:11	Seattle 09:22:54
Phoenix 09:37:44	Seattle 09:22:54	Seattle 09:36:14

no longer sorted by time

still sorted by time

Stability when sorting on a second key

Stability: insertion sort

Proposition. Insertion sort is stable.

```
public class Insertion
{
   public static void sort(Comparable[] a)
   {
      int N = a.length;
      for (int i = 0; i < N; i++)
         for (int j = i; j > 0 && less(a[j], a[j-1]); j--)
            exch(a, j, j-1);
   }
}
```

i	j	0	1	2	3	4
0	0	B_1	A_1	A_2	A_3	B_2
1	0	A_1	B_1	A_2	A_3	B_2
2	1	A_1	A_2	B_1	A_3	B_2
3	2	A_1	A_2	A_3	B_1	B_2
4	4	A_1	A_2	A_3	B_1	B_2
		A_1	A_2	A_3	B_1	B_2

Pf. Equal items never move past each other.

Stability: selection sort

Proposition. **Selection sort is not stable.**

```
public class Selection
{
   public static void sort(Comparable[] a)
   {
      int N = a.length;
      for (int i = 0; i < N; i++)
      {
         int min = i;
         for (int j = i+1; j < N; j++)
            if (less(a[j], a[min]))
               min = j;
         exch(a, i, min);
      }
   }
}
```

i	min	0	1	2
0	2	B_1	B_2	A
1	1	A	B_2	B_1
2	2	A	B_2	B_1
		A	B_2	B_1

Pf by counterexample. **Long-distance exchange can move one equal it past another one.**

Stability: shellsort

Proposition. Shellsort sort is not stable.

```
public class Shell
{
  public static void sort(Comparable[] a)
  {
    int N = a.length;
    int h = 1;
    while (h < N/3) h = 3*h + 1;
    while (h >= 1)
    {
      for (int i = h; i < N; i++)
      {
        for (int j = i; j > h && less(a[j], a[j-h]); j -= h)
          exch(a, j, j-h);
      }
      h = h/3;
    }
  }
}
```

h	0	1	2	3	4
	B_1	B_2	B_3	B_4	A_1
4	A_1	B_2	B_3	B_4	B_1
1	A_1	B_2	B_3	B_4	B_1
	A_1	B_2	B_3	B_4	B_1

Pf by counterexample. Long-distance excha

Stability: mergesort

Proposition. Mergesort is stable.

```
public class Merge
{
  private static void merge(...)
  {  /* as before */  }

  private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
  {
    if (hi <= lo) return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
  }

  public static void sort(Comparable[] a)
  {  /* as before */  }
}
```

Pf. Suffices to verify that merge operation is stable.

Stability: mergesort

Proposition. **Merge operation is stable.**

```
private static void merge(...)
{
  for (int k = lo; k <= hi; k++)
     aux[k] = a[k];

  int i = lo, j = mid+1;
  for (int k = lo; k <= hi; k++)
  {
     if      (i > mid)             a[k] = aux[j++];
     else if (j > hi)              a[k] = aux[i++];
     else if (less(aux[j], aux[i])) a[k] = aux[j++];
     else                          a[k] = aux[i++];
  }
}
```

0	1	2	3	4	5	6	7	8	9	10
A_1	A_2	A_3	B	D	A_4	A_5	C	E	F	G

Pf. **Takes from left subarray if equal keys.**

Sorting summary

	inplace?	stable?	best	average	worst	remarks
selection	✔		$\frac{1}{2} N^2$	$\frac{1}{2} N^2$	$\frac{1}{2} N^2$	N exchanges
insertion	✔	✔	N	$\frac{1}{4} N^2$	$\frac{1}{2} N^2$	use for small N or partially ordered
shell	✔		$N \log_3 N$	?	$c\, N^{3/2}$	tight code; subquadratic
merge		✔	$\frac{1}{2} N \lg N$	$N \lg N$	$N \lg N$	$N \log N$ guarantee; stable
timsort		✔	N	$N \lg N$	$N \lg N$	improves mergesort when preexisting order
?	✔	✔	N	$N \lg N$	$N \lg N$	holy sorting grail