Lecture 18
Solving Shortest Path Problem: Dijkstra's Algorithm

October 23, 2009

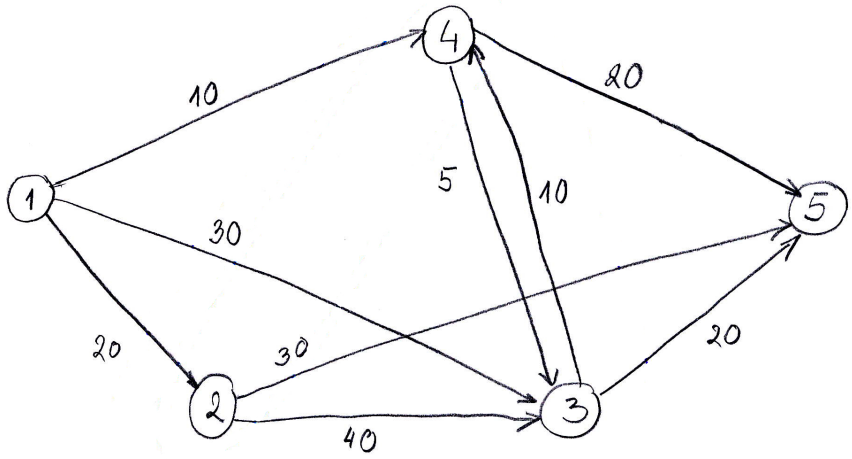
Outline

- Focus on Dijkstra's Algorithm
  - Importance: Where it has been used?
  - Algorithm's general description
  - Algorithm steps in detail
  - Example

One-To-All Shortest Path Problem

We are given a weighted network (V, E, C) with node set V, edge set E, and the weight set C specifying weights c_{ij} for the edges $(i, j) \in E$. We are also given a starting node $s \in V$. The **one-to-all shortest path** problem is the problem of determining the shortest path from node s to all the other nodes in the network.

1 Problem, definition, English /math



2 Problem, example, diagram

The weights on the links are also referred as **costs**.

Algorithms Solving the Problem

- Dijkstra's algorithm `3 Algorithm, analogies, list`
  - Solves only the problems with nonnegative costs, i.e.,
    $$c_{ij} \geq 0 \text{ for all } (i,j) \in E$$
- Bellman-Ford algorithm
  - Applicable to problems with arbitrary costs
- Floyd-Warshall algorithm
  - Applicable to problems with arbitrary costs
  - Solves a more general all-to-all shortest path problem

Floyd-Warshall and Bellman-Ford algorithm solve the problems on graphs that do not have a cycle with negative cost.

Importance of Dijkstra's algorithm

Many more problems than you might at first think can be cast as shortest path problems, making Dijkstra's algorithm a powerful and general tool.

For example: 4 Algorithm, applications, list/English

- Dijkstra's algorithm is applied to automatically find directions between physical locations, such as driving directions on websites like Mapquest or Google Maps.
- In a networking or telecommunication applications, Dijkstra's algorithm has been used for solving the min-delay path problem (which is the shortest path problem). For example in data network routing, the goal is to find the path for data packets to go through a switching network with minimal delay.
- It is also used for solving a variety of shortest path problems arising in plant and facility layout, robotics, transportation, and VLSI* design

*Very Large Scale Integration

5 Algorithm, idea, English / list/ pseudo

General Description

Suppose we want to find a shortest path from a given node s to other nodes in a network (one-to-all shortest path problem)

- Dijkstra's algorithm solves such a problem
  - It finds the shortest path from a given node s to all other nodes in the network
  - Node s is called a starting node or an initial node
- How is the algorithm achieving this?
  - Dijkstra's algorithm starts by assigning some initial values for the distances from node s and to every other node in the network
  - It operates in steps, where at each step the algorithm improves the distance values.
  - At each step, the shortest distance from node s to another node is determined

Formal Description

6 Algorithm, definition, list /English

The algorithm characterizes each node by its state

The state of a node consists of two features:

distance value and **status label**

- Distance value of a node is a scalar representing an estimate of the its distance from node s.
- Status label is an attribute specifying whether the distance value of a node is equal to the shortest distance to node s or not.
  - The status label of a node is **Permanent** if its distance value is equal to the shortest distance from node s
  - Otherwise, the status label of a node is **Temporary**

The algorithm maintains and step-by-step updates the states of the nodes

At each step one node is designated as **current**

Notation

In what follows:

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7 Algorithm, definition: partial , li
st
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- d_ℓ denotes the distance value of a node ℓ.

- p or t denotes the status label of a node, where p stand for permanent and t stands for temporary

- c_{ij} is the cost of traversing link (i, j) as given by the problem

The state of a node ℓ is the ordered pair of its distance value d_ℓ and its status label.

Algorithm Steps

8 Algorithm, example: emphasised , sequence/diagram/math

Step 1. *Initialization*

- Assign the zero distance value to node s, and label it as **Permanent**. [The state of node s is $(0, p)$.]
- Assign to every node a distance value of ∞ and label them as **Temporary**. [The state of every other node is (∞, t).]
- Designate the node s as the **current** node

Step 2. *Distance Value Update and Current Node Designation Update*
Let i be the index of the current node.

(1) Find the set J of **nodes with temporary labels** that can be reached from the current node i by a link (i, j). **Update the distance values of these nodes.**

- For each $j \in J$, the distance value d_j of node j is updated as follows

$$new \quad d_j = \min\{d_j, d_i + c_{ij}\}$$

where c_{ij} is the cost of link (i, j), as given in the network problem.

(2) Determine a node j that has the smallest distance value d_j among all nodes $j \in J$,

$$\text{find } j^* \text{ such that} \qquad \min_{j \in J} d_j = d_{j^*}$$

(3) Change the label of node j^* to **permanent** and designate this node as **the current node**.

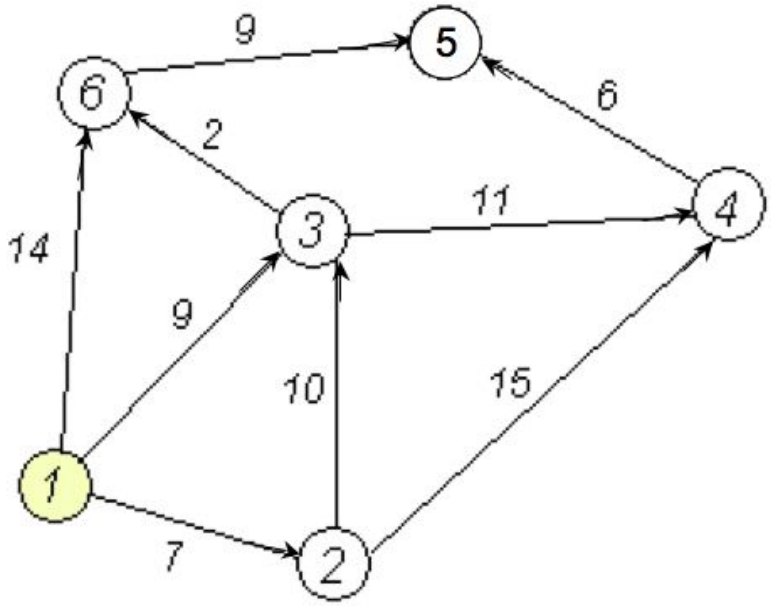
Step 3. *Termination Criterion*

If all nodes that can be reached from node s have been permanently labeled, then stop - we are done.

If we cannot reach any temporary labeled node from the current node, then all the temporary labels become permanent - we are done.

Otherwise, go to Step 2.

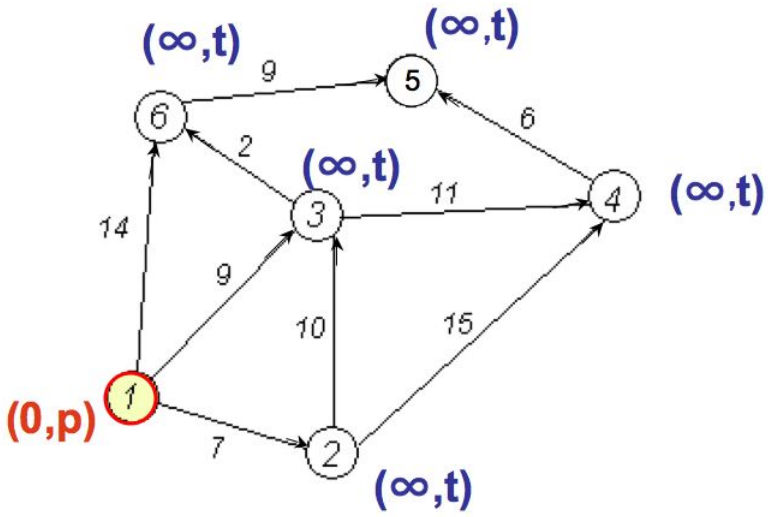
Dijkstra's Algorithm: Example



We want to find the shortest path from node 1 to all other nodes using Dijkstra's algorithm.

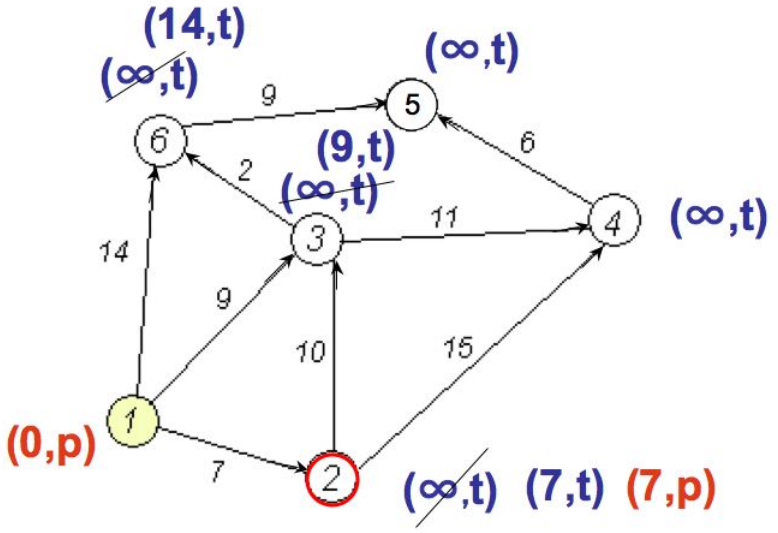
Initialization - Step 1

- Node 1 is designated as the current node
- The state of node 1 is $(0, p)$
- Every other node has state (∞, t)



Step 2

- Nodes 2, 3,and 6 can be reached from the current node 1
- Update distance values for these nodes

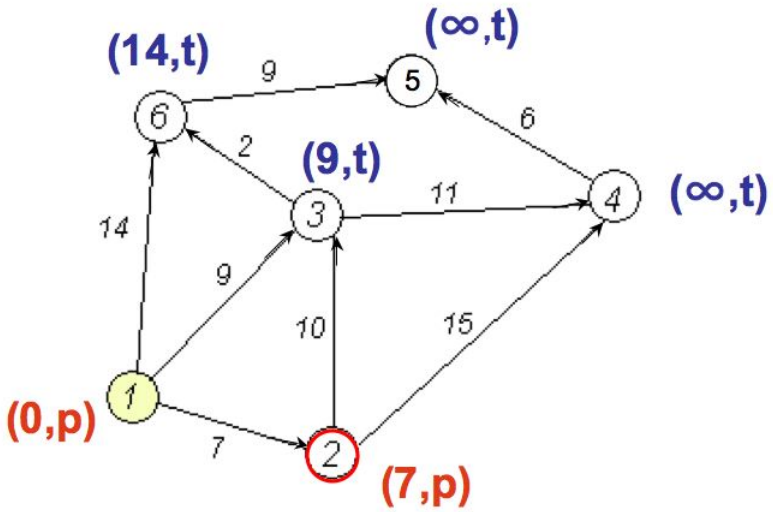


$$d_2 = \min\{\infty, 0 + 7\} = 7$$
$$d_3 = \min\{\infty, 0 + 9\} = 9$$
$$d_6 = \min\{\infty, 0 + 14\} = 14$$

- Now, among the nodes 2, 3, and 6, node 2 has the smallest distance value
- The status label of node 2 changes to permanent, so its state is $(7, p)$, while the status of 3 and 6 remains temporary
- Node 2 becomes the current node

Step 3

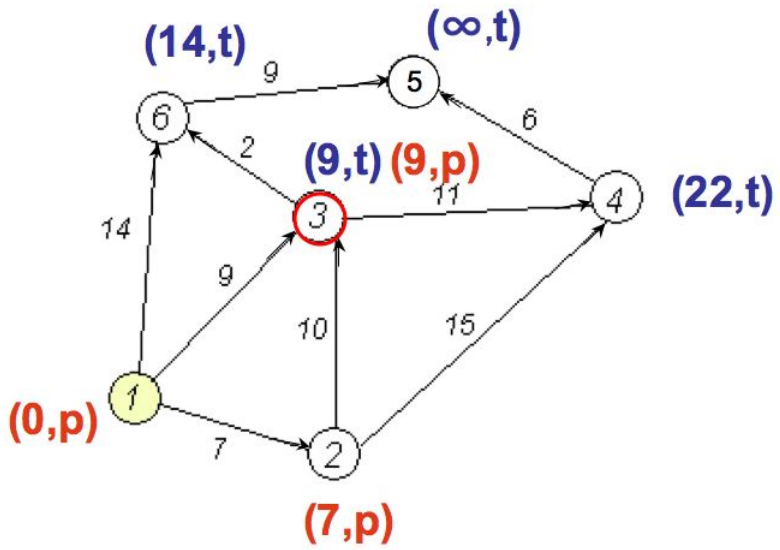
Graph at the end of Step 2



We are not done, not all nodes have been reached from node 1, so we perform another iteration (back to Step 2)

Another Implementation of Step 2

- Nodes 3 and 4 can be reached from the current node 2
- Update distance values for these nodes



$$d_3 = \min\{9, 7 + 10\} = 9$$
$$d_6 = \min\{\infty, 7 + 15\} = 22$$

- Now, between the nodes 3 and 4 node 3 has the smallest distance value
- The status label of node 3 changes to permanent, while the status of 6 remains temporary
- Node 3 becomes the current node

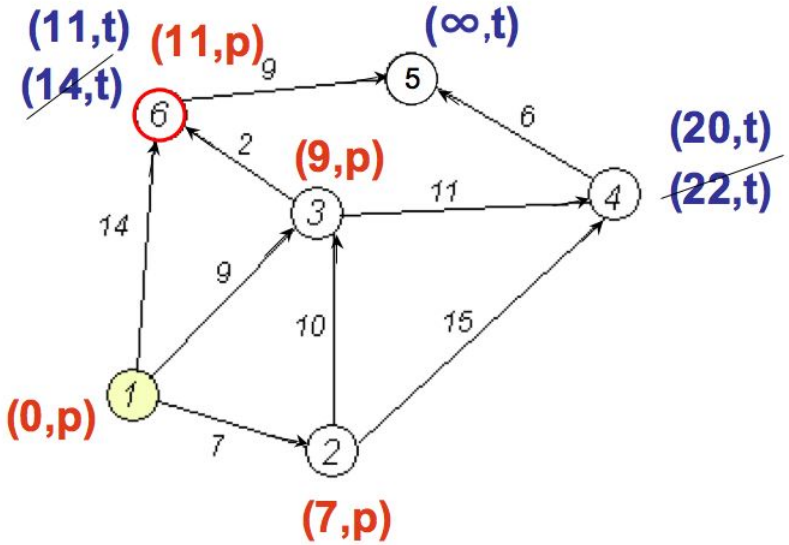
We are not done (Step 3 fails), so we perform another Step 2

9 Implementation, idea:partial, sequence/diagram

Another Step 2

- Nodes 6 and 4 can be reached from the current node 3
- Update distance values for them

$$d_4 = \min\{22, 9 + 11\} = 20$$
$$d_6 = \min\{14, 9 + 2\} = 11$$



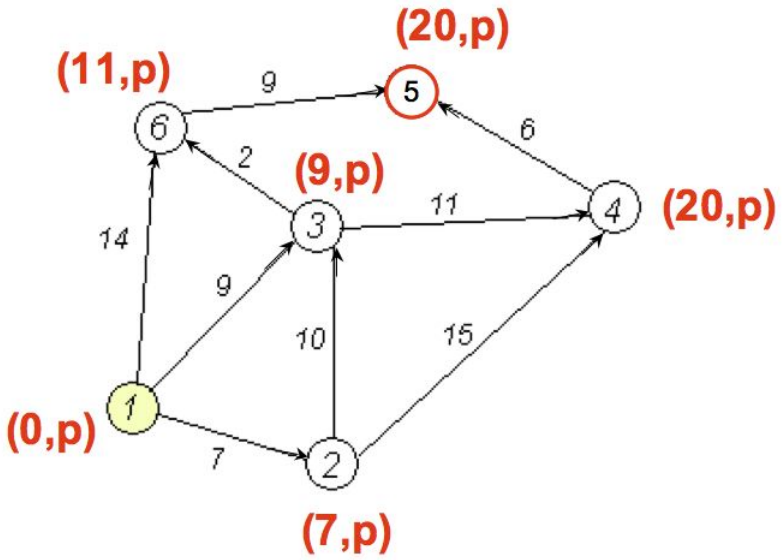
- Now, between the nodes 6 and 4 node 6 has the smallest distance value
- The status label of node 6 changes to permanent, while the status of 4 remains temporary
- Node 6 becomes the current node

We are not done (Step 3 fails), so we perform another Step 2

Another Step 2

- Node 5 can be reached from the current node 6
- Update distance value for node 5

$$d_5 = \min\{\infty, 11 + 9\} = 20$$



- Now, node 5 is the only candidate, so its status changes to permanent
- Node 5 becomes the current node

From node 5 we cannot reach any other node. Hence, node 4 gets permanently labeled and we are done.

Chapter 6.3.2 in your book has another example of the implementation of Dijkstra's algorithm