Inspectable Incrementality in Minimum Feedback Arc Set Solving

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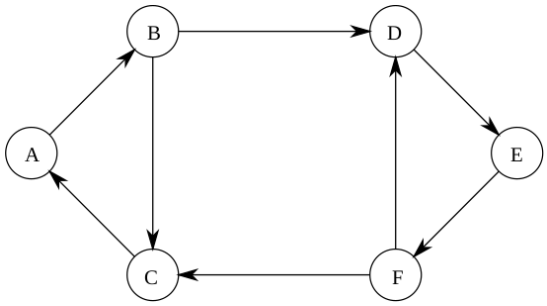
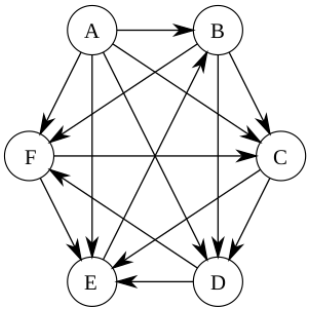


(a) A three cycle graph which demonstrates that greedily selecting the minimum weighted edge will not provide the minimum feedback arc set.



(b) A four cycle graph where each cycle depends on the edge (E, B).

ABSTRACT

Abstract here

KEYWORDS

incremental SAT solving, minimum feedback arc set, SMT solving

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1 INTRODUCTION

The minimum feedback arc set problem is a canonical NP-Complete problem given by Karp [5]. Worse still, Kann [4] showed that the problem is APX-hard. Despite these results finding the minimum feedback arc set of a directed graph is desirable for many domains: such as certain rank-choice voting systems, tournament ranking systems, and dependency graphs in general.

colin citation?

Solutions to the minimum feedback arc set problem are commonly implemented in widely used graph libraries yet all suffer from a distinct flaw. While each implementation provides a solution, the implementations only do so without allowing the user to inspect intermediate steps; which might contain useful domain information. For example, the graph Figure 1b displays a directed graph which encodes a single win tournament system. Each vertex is a player and each edge encodes a win over a contestant, for example we see that player E beat contestant B. Observe that the Figure 1b contains cycles which implies that there is not a linear ordering amongst the contestants of the tournament, and so any ranking of the contestants would violate a win of one contestant over another. Removing the minimum feedback arc set would remove the cycles, which yields a linear order in the tournament that violates the fewest number of wins. In this example, intermediate results are edges which compose the minimum feedback arc set, thus if one had access to the intermediate results one might choose to rematch the opponents rather than remove the win. Crucially, the result of the rematch may change the final minimum feedback arc set. Such a procedure could therefore increase the confidence in the results of the tournament.

To allow for *inspectable incrementality* we propose a novel direction in solving the minimum feedback arc set problem based on recent advances in satisfiability solving (SAT) and satisfiability-modulo theories (SMT) solving. Our approach is to utilize an SMT solver to detect cycles and minimize the weight of the feedback arc set. Incrementality in this approach is given through use of an *incremental* SMT solver and generation of *unsatisfiable cores*. A minimum unsatisfiable core is the minimum set of clauses in a SAT or SMT formula which prevent a SAT or SMT solver run from finding a satisfiable assignment. An incremental solver provides the user the ability to add or remove constraints and thereby direct the solver during runtime. Incrementality in our approach is crucial as it allows the user decision points to interact with the solution process. Thus, a user might observe a unsatisfiable core which corresponds to a cycle and deliberately resample *only* the edges in the discovered cycle.

Unfortunately, we find that incremental SMT for finding a minimum feedback arc set suffers from significant issues which seem

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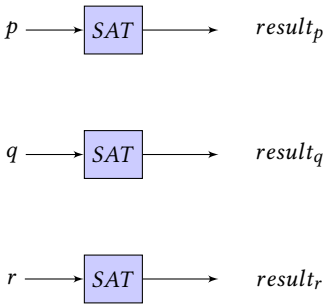
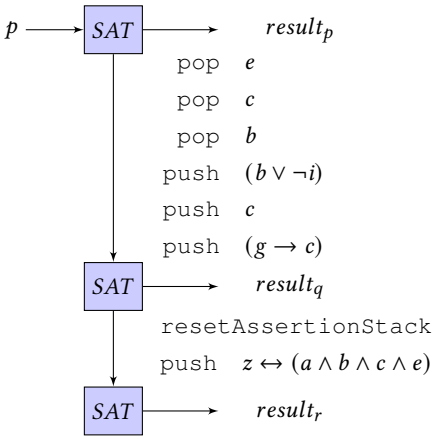
to be intractable. In addition to this result we make the following contributions:

(1) An identification of the most intractable problems for an incremental SMT encoding on the minimum feedback arc set problem. (section 3)
(2) An SMT encoding to find the minimum feedback arc set of an arbitrary directed and cyclic graph. (section 4)
(3) An empirical evaluation of the SMT encoding. (section 6)

2 BACKGROUND AND RELATED WORK



(a) Brute force procedure, no reuse between solver calls.



(b) Incremental procedure, reuse defined by POP and PUSH.

Figure 2: Comparison of incremental and non-incremental SAT procedures.

In this section, we provide background on the minimum feedback arc set problem, incremental SAT and SMT solving, and unsatisfiable cores. We begin with preliminaries on graphs and feedback arc sets. We close the section with a description of incremental SAT solving and SMT solving.

On a directed graph $G(V, E)$ a *feedback arc set* S is a set of edges in E, such that removing them from the graph G results in an acyclic graph. The minimum feedback arc set problem finds the S^* of the minimum size. For weighted graphs, one might desire S^* to have the minimum total weight of any S. Finding this S^* is the minimum weighted feedback arc set problem.

With the minimum feedback arc set problem defined, we provide the following background on incremental SAT solvers and assume knowledge of SMT solvers. Suppose, we have three related propositional formulas that we desire to solve.

$$p = a \wedge b \wedge c \wedge e$$
$$q = a \wedge (b \vee \neg i) \wedge c \wedge (g \rightarrow c)$$
$$r = z \leftrightarrow (a \wedge b \wedge c \wedge e)$$

p is simply a conjunction of variables. In q, relative to p, we see that the variables i and g are added, e is removed, and there are two new clauses: $(b \vee \neg i)$ and $(g \rightarrow c)$, both of which possibly affect the values of b and c. In r, the variables and constraints introduced in p are further constrained to a new variable, z.

Suppose one wants to find a model for each formula. Using a non-incremental SAT solver results in the procedure illustrated in Figure 2a; where SAT solving is a batch process and no information is reused. Alternatively, a procedure using an incremental SAT solver is illustrated in Figure 2b; in this scenario, all formulas are solved by single solver instance where terms are programmatically added or removed from the solver.

The ability to add and remove terms is enabled by manipulating the *assertion stack*, to add or remove levels on the stack. The incremental interface provides two commands: PUSH to create a new *scope* and add a level to the stack, and POP to remove the topmost level on the stack. Consider the following SMTLIB2 program which demonstrates three levels on the assertion stack. The SMTLIB2 standard [2] is an international standard that defines a general interface for SAT and SMT solvers. The program follows the procedure outlined in Figure 2b and solves p, q and r:

```
(declare-const a Bool)          ;; variable declarations for p
(declare-const b Bool)
(declare-const c Bool)
(declare-const e Bool)
(assert a)                      ;; a is shared between p and q
(push)                          ;; solve p
  (assert e)
  (assert c)
  (assert b)
  (check-sat)                   ;; check-sat on p
(pop)                           ;; remove e, c, and b assertions
(push)                          ;; solve for q
  (declare-const i Bool)        ;; new variables
  (declare-const g Bool)
  (assert (or b (not i)))       ;; new clause
  (assert c)                    ;; re-add c
  (assert (=> g c))             ;; new clause
  (check-sat)                   ;; check sat of q
(pop)                           ;; i and g out of scope
(reset)                         ;; reset the assertion stack
(declare-const a Bool)          ;; variable declarations for r
(declare-const b Bool)
(declare-const c Bool)
(declare-const e Bool)
(declare-const z Bool)
(assert (= z (and a (and b (and c (and e ))))))
(check-sat)                     ;; check-sat on r
```

We begin by defining p, and assert a outside of a new scope so that it can be reused for q. Internally, all levels on the assertions stack are conjoined and asserted when a CHECK-SAT command is issued. Thus, we reuse a by exploiting this conjunction behavior. Had we asserted

(and a (and b (and c (and e)))), then we would not be able to reuse only the assertion on a since it was created in conjunction with other variables. The first PUSH command enters a new level on the assertion stack, the remaining variables are asserted and we issue a CHECK-SAT call. After the POP command, all assertions and declarations from the previous level are removed. Thus, after we solve q the variables i and g cannot be referenced as they are no longer in scope. Similarly, after the first CHECK-SAT call and subsequent POP, e, c and b are no longer defined.

In an efficient process one would initially add as many *shared* terms as possible, such as a from p and then reuse that term as many times as needed. An efficient process should perform only enough manipulation of the assertion stack as required to reach the next SAT problem of interest from the current one. However, notice that doing so is not entirely straight forward; we were only able to reuse a from p in q because we defined p in a non-intuitive way by utilizing the internal behavior of the assertion stack. This situation is exacerbated by SAT problems such as r, where due to the equivalence between a new term and shared terms, we are forced to completely remove everything on the stack with a RESET command just to construct r. Thus incremental SAT solvers provide the primitive operations required to solve related SAT problems efficiently, yet writing the SMTLIB2 program to solve the set efficiently is not straightforward.

3 PROBLEMS WITH AN INCREMENTAL ENCODING

Our original conception for this study was to explore the use of incremental SAT and SMT solvers to solve the minimum feedback arc set problem. However, we found fundamental problems with this approach during several courses of experimentation. In this section, we review the problematic nature of the incremental approach thereby enriching the problem. We provide a working non-incremental SMT encoding in the next section.

The incremental approach was a combination of two SAT and SMT features. First, we desired to encode the problem in such a way that an UNSAT would be returned from the solver. With an UNSAT returned, we could then query for an *unsatisfiable core*, that is, the set of constraints which prevent the solver from unifying. Second, we sought to use *scopes* to add and remove clauses from the incremental solver via PUSH and POP calls. With scopes it is possible to refine the constraints *during* the solving routine, possibly simplifying the problem space. We hypothesized that by using scopes and unsatisfiable cores we could reveal domain information to the user and direct the solver to a solution in an efficient manner.

Unfortunately the incremental approach has numerous problems for finding the minimum feedback arc set, which appear to be intractable. First, in order to generate unsatisfiable cores constraints must be added to the solver instance *and watched*. In and of itself, this is not an intractable problem, the SMTLIB2 standard defines a function to track constraints called ASSERT_AND_TRACK which takes a constraint and a name. The name is then returned in the unsatisfiable core if the concomitant constraint is in the core. The issue becomes a constant time cost incurred by parsing the unsatisfiable core into a usable format for the solving routine to continue. The pattern becomes: query for an unsatisfiable core, parse the core, refine the constraints, and then repeat. Thus the constant time cost is exacerbated for every core that is generated. In addition, there is no guarantee that the unsatisfiable core itself is minimal, although some SAT/SMT solvers, such as z3 [3] expose a setting to produce minimal cores at the expense of performance.

The more serious issue is the use of scopes is problematic for tackling the minimum feedback arc set problem at all. While scopes would allow one to direct the solver to a solution, or explore possible solutions, it requires knowledge of the problem domain which is not readily available during the encoding process. In order to employ scopes, one must know before the encoding step which constraints are likely to change. This knowledge is required so that the constraints can be pushed onto the assertion stack *as late as possible*. When these constraints are on top of the assertion stack (and thus pushed last) they can be removed from the assertion stack *without* removing constraints that we do not wish to refine. Without this knowledge, there is a substantial risk that the we may need to remove all, or almost all constraints from the solver just to repack the solver and refine the query. In the worst case, this requires an additional complete traversal of the graph. The issue is further exacerbated by the minimum feedback arc set problem because it is not possible to know before hand which edges, and thus which constraints, will need to be refined. In fact finding the unique set of constraints that will need to be refined is itself an instance of finding the minimum feedback arc set!

4 APPROACH AND SMT ENCODING

Our approach is to translate an exact integer programming method by Baharev et al. [1] to an SMT problem. The translation to an SMT problem is straightforward, requiring no changes from linear program encoding to an SMT encoding.

To find the minimum feedback arc set of a graph G, the method requires a cycle matrix, a_{ij}. The cycle matrix tracks which edges are in which cycles in the graph. For some edge e_j, if e_j participates in cycle i then $a_{ij} = 1$ otherwise $a_{ij} = 0$, indicating that e_j *does not* participate in the ith cycle. For example, consider the edge (B, D) from Figure 1a and assume (B, D) has edge ID $j = 3$, if the 0th cycle in Figure 1a is composed of edges: (A, B), (B, C), and (C, A), then $a_{03} = 0$. Similarly, assuming (A, B) has edge ID $j = 0$, then $a_{00} = 1$ because (A, B) participates in cycle 0.

With the cycle matrix, the method is composed of c constraints, where c is the number of cycles and a minimization over the sum of edges. Each edge, (i, j) is encoded as a binary integer SMT variable e_j, where j is the edge's unique ID. The SMT variable e_j only ranges from 0 to 1, indicating whether the jth edge is in the minimum feedback arc set ($e_j = 1$) or not ($e_j = 0$). We express the constraints in the following equations:

$$\text{minimize} \sum_{j=1}^{|E|} e_j \tag{1}$$

$$(\sum_{j=1}^{|E|} a_{ij} e_j) \geq 1, \text{ for each } i = 0, 1, 2 \ldots, c \tag{2}$$

Conceptually, Equation 1 requires the SMT solver to consider all edges in the graph G and find the smallest number of edges which compose a feedback arc set. The encoding of a feedback arc set occurs in Equation 2. With the constraint $\sum(\ldots) \geq 1$ for the ith

Edge	ID
(A, B)	0
(B, C)	1
(C, A)	2
(B, D)	3
(D, E)	4
(E, F)	5
(F, C)	6
(F, D)	7

Cycle	Cycle edges
0	(A, B) → (B, C) → (C, A)
1	(D, E) → (E, F) → (F, D)
2	(A, B) → (B, D) → (D, E) (E, F) → (F, C) → (C, A)

Table 1: Table of Edge IDs and Cycle IDs for Figure 1a

cycle, Equation 2 states that for each cycle in the graph G pick one or more edges. By $\sum(a_{ij}e_j) \geq 1$, Equation 2 forces the SMT solver to consider edges that occur in most cycles. The logic is easiest to observe with an example, consider the graph shown in Figure 1a, with the edge and cycle IDs shown in Table 1: For this graph, the constraints generated by Equation 2 would be:

$$1e_0 + 1e_1 + 1e_2 + 0e_3 + 0e_4 + 0e_5 + 0e_6 + 0e_7 \geq 1 \quad \text{Cycle 0}$$
$$0e_0 + 0e_1 + 0e_2 + 0e_3 + 1e_4 + 1e_5 + 0e_6 + 1e_7 \geq 1 \quad \text{Cycle 1}$$
$$1e_0 + 0e_1 + 1e_2 + 1e_3 + 1e_4 + 1e_5 + 1e_6 + 0e_7 \geq 1 \quad \text{Cycle 2}$$

Where we see the edge e_0 (edge (A, B)) occurs in cycles 0, and 2 (as a_{00} and $a_{20} = 1$). Therefore, the easiest way for the SMT solver to satisfy these constraints *and* the minimization constraint is to set $e_0 = 1$. However, doing so will not satisfy the cycle constraints for cycle 1, thus the solver must pick either e_4 ((D, E)), e_5 ((E, F)), or e_7 ((F, C)). Again the simplest path is to pick e_4, or e_5 as setting these variables to 1 satisfies more than one constraint. Thus, we see that by this encoding the SMT solver is forced to minimize the number of edges to pick, and therefore maximize the number of cycle constraints satisfied by setting a given edge to 1, which corresponds to finding the minimum feedback arc set of a graph.

5 CASE STUDIES

To evaluate this approach, we construct a prototype SMT-enabled algorithm and assess the prototype on Erdős-Rényi graphs. Erdős-Rényi graphs are generated according to two parameters: p, a metric of connectedness for the graph, and k is the number of vertices in the graph. In addition, we add a parameter s which represents the size of the range of possible weights on edges in a graph.

With these parameters we employ random generation to construct sample graphs. We are interested in the individual effect k, p and s have on runtime, in addition to the interactions effects between each parameter. Consider the case where p and k are left unbound, yet s is set to 1. This is the specific case where solving the minimum weighted feedback arc set problem solves the minimum feedback arc set problem. Thus, by setting s to 1 we generate graphs to solve the minimum feedback arc set problem. Consider the case where s is larger than one. In this case, edges must possess positive weights and the weighted minimum feedback arc set may be different than the minimum feedback arc set.

Hi Mike. The rest of the section is describing each case individually and then research questions. We don't have these spelled out right now. What we hypothesize is that at some combination of k, p and s the solver runtime will become infeasible. Then we can make determinations on the viability of this method in different domains. For example, most tournaments will be a bounded number of participants, and score ranges, but may have maximum connectivity. Thus, we would be able to make a conclusion on the viability of our method for the tournament domain. As a stretch goal we wish to consider graphs of different distributions that Erdős-Rényi, e.g. Barabási-Albert or Watts and Strogatz.

6 RESULTS AND DISCUSSION

It was good enough to graduate

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A APPENDIX

Source code listings

```
"""- Module     : app.py
- Copyright : (c) Jeffrey M. Young
              ; (c) Colin Shea-Blymyer
- License   : BSD3
- Maintainer: youngjef@oregonstate.edu,
    sheablyc@oregonstate.edu
- Stability : experimental

Module communicates with the backend solver to
    solve minimum feedback arc set problems

You'll notice that we thread the solver instance
    and cache through several
functions. This is purposeful to get as close as
    possible to referential
transparency in python. Think of it as a Reader
    Monad carrying the cache and
solver handle.

"""

from pprint import pprint
from copy   import deepcopy
import z3   as z

import gadgets as g
import graphs  as gs
import utils   as u
```

```
def find_cycle(cache,s,graph):
    """Find a cycle in a graph. s is a solver
        object, graph is an adjacency list
    e.g. {0:[0,1,2,3,4,5], 1:[2,3] ...}. Given the
        graph iterate over the
    graph, convert the key and values to symbolic
        terms and check for a cycle.
    Returns the unsat core as a list of strings e.
        g. [1->2, 2->3, 3->4, 4->1]
    """
    for source, sinks in graph.items():
        for sink in sinks:
            ## convert to strings
            s_source = str(source)
            s_sink   = str(sink)

            ## create symbolics
            sym_from = u.make_sym(cache,source)
            sym_to   = u.make_sym(cache,sink)

            ## lets go, this is just a fold over
                the dict yielding ()
            g.cycle_check(s, sym_from, sym_to, u.
                make_name(s_source,s_sink))

    r = []
    if s.check() == z.unsat: r = s.unsat_core()
    return u.parse_core(r)

def relax(graph, unsat_core, strategy):
    """Take a graph, an unsat-core and a strategy.
        use the strategy to find the
    edge to relax, relatx the edge and return the
        graph

    This function mutates graph
    """
    source, sink = strategy(unsat_core)
    outgoing = deepcopy(graph[source])
    print(graph[source])
    print(outgoing)
    print(sink in outgoing)
    print(outgoing.remove(sink))
    graph[source] = outgoing.remove(sink)
    if graph[source] is None: graph[source] = []

def go (cache,s,graph):
    # flag to end loop
    done = False

    # kick off
    while not done:
        pprint(graph)

        # get the core
        core = find_cycle(cache, s, graph)
        pprint(core)

        # if the core is empty then we are done,
            if not then relax and recur
        if core:
            # relax an edge by some strategy
            relax(graph,core,lambda x : x[0])

            # this is good enough for now but in
                the future we should try to
            # remove only the right constraint,
                which will be much much faster
            s.reset()
        else:
            done = True


def main():
    # just use a heterogeneous map as a cache
    cache = {}

    # spin up the solver
    s = z.Solver()

    go(cache,s,gs.round_robin_graph)
```

```
"""
- Module     : graphs.py
- Copyright  : (c) Jeffrey M. Young
; (c) Colin Shea-Blymyer
- License    : BSD3
- Maintainer: youngjef@oregonstate.edu,
    sheablyc@oregonstate.edu
- Stability  : experimental

Module which defines sample graphs
"""


round_robin_graph = { "A": ["B","C","D","E","F"
    ],
                      "B": ["C","F"],
                      "C": ["D","E"],
                      "D": ["E","F"],
                      "E": ["B"],
                      "F": ["E","C"]
                    }

anti_greed_graph = { 0: [1],
                     1: [2,3],
                     2: [0],
                     3: [4],
                     4: [5],
                     5: [3,2]
}
```

```
"""
- Module     : utils.py
- Copyright  : (c) Jeffrey M. Young
             ; (c) Colin Shea-Blymyer
- License    : BSD3
- Maintainer: youngjef@oregonstate.edu,
    sheablyc@oregonstate.edu
- Stability  : experimental

Common utility functions
"""

from z3 import *

def make_name(frm,to): return frm + "->" + to

def parse_edge(edge): # return list(map(int,edge.
    __str__().split("->")))
    str_edge = edge.__str__()
    return list(map(lambda x: x,str_edge.split("->
        ")))

def parse_core(core):
    """Parse an unsat core. An unsat core is
        shallowly embedded as a list of z3
    BoolRef objects such as: [1->2, 2->3, 3->4, 4-
        >1], to operate on these we
    need to coerce them to a string parse the
        string and coerce the Ints out.

    Input:  List of strings, e.g.,      [1->2  , 2
        ->3  , 3->4  , 4->1  ]
    Output: List of List of Ints, e.g., [[1, 2],
        [2, 3], [3, 4], [4, 1]]

    """
    return list(map(lambda e: parse_edge(e), core)
        )

def make_sym(cache, new, ty=Int):
    """Create a new symbolic variable in the
        backend solver. We use a cache to
    avoid repeated calls to the solver.
        Furthermore, because we are naming
    constraints we must ensure that we don't
        accidental use a duplicate name in
    the solver or else it will throw an exception
    """

    if new not in cache:
        sym_new = ty(str(new))
        cache[new] = sym_new

    return cache, cache[new]
```