partitions-leanblueprint

Carter Hastings   Quinn Oveson

June 26, 2025

0.1 Definitions

Definition 1 (Sequence). A sequence, denoted a or $\{a_n\}$, is a function $a : \mathbb{N} \to \mathbb{R}$.

Definition 2 (Convergence). A sequence $\{a_n\}$ converges to $L \in \mathbb{R}$ if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - L| < \varepsilon$. We say $\{a_n\}$ converges if there exists $L \in \mathbb{R}$ such that $\{a_n\}$ converges to L.

Definition 3 (Bounded Above). A set $S \subseteq \mathbb{R}$ is bounded above if there exists an $M \in \mathbb{R}$ such that for all $s \in S$, $s \leq M$. We say that M is an upper bound for S.

Definition 4 (supremum). Let $S \subseteq \mathbb{R}$ be non-empty. We say that $y \in \mathbb{R}$ is the supremum of S if

(i) y is an upper bound for S
(ii) for any upper bound z of S, $y \leq z$

Definition 5 (continuous). A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if for all $x \in \mathbb{R}$ and for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $y \in \mathbb{R}$ with $|x - y| < \delta$, $|f(x) - f(y)| < \varepsilon$.

0.2 Theorems

Theorem 6 (Limit Laws).
    Let $C \in \mathbb{R}$. Suppose $\{a_n\}$ converges to L and $\{b_n\}$ converges to K. Then

(i) $\{Ca_n\}$ converges to CL
(ii) $\{a_n + b_n\}$ converges to $L + K$

Proof. Trivial. $\qquad\square$

Lemma 7.
    Suppose that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n \geq 0$. Then $\lim_{n\to\infty} a_n \geq 0$.

Theorem 8 (Order Limit Theorem).
    Let $\{a_n\}$ and $\{b_n\}$ be sequences. Suppose that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n \leq b_n$. Then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

Theorem 9 (Completeness Axiom). *Let $S \subseteq \mathbb{R}$ be non-empty and bounded above. Then there exists a $y \in \mathbb{R}$ such that y is the supremum of S.*

Theorem 10 (Intermediate Value Theorem).
    Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and let $a < b \in \mathbb{R}$. Suppose that $f(a) < f(b)$ and let $y \in [f(a), f(b)]$. Then there exists a $c \in [a, b]$ such that $f(c) = y$.