

partitions-leanblueprint

Carter Hastings Quinn Oveson

November 5, 2025

0.1 Definitions

Definition 1 (Modular Form). In lean, A modular form of weight $k \in \mathbb{N}$ is a function $f : \mathbb{C} \rightarrow \mathbb{C}$ such that :

- (1) f is holomorphic on \mathbb{H}
- (2) For all $z \in \mathbb{H}$, $f(z+1) = f(z)$
- (3) For all $z \in \mathbb{H}$, $f(-1/z) = z^k f(z)$
- (4) f is bounded as $\text{Im}(z) \rightarrow \infty$

Definition 2 (Integer Modular Form). An integer modular form of weight $k \in \mathbb{N}$ is a sequence $a : \mathbb{N} \rightarrow \mathbb{Z}$ such that $\sum_{n=0}^{\infty} a(n)q^n$ is a modular form of weight k , where $q = e^{2\pi iz}$.

Definition 3 (ModularFormMod ℓ). A modular form mod ℓ of weight $k \in \mathbb{Z}/(\ell-1)\mathbb{Z}$ is a sequence $a : \mathbb{N} \rightarrow \mathbb{Z}/\ell\mathbb{Z}$ such that there exists an integer modular form b of weight k' where $b \equiv a \pmod{\ell}$ and $k' \equiv k \pmod{\ell-1}$.

Definition 4 (Theta). Θ sends modular forms mod ℓ of weight k to weight $k+2$ by $(\Theta a)n = na(n)$.

Definition 5 (U Operator). The operator U sends modular forms mod ℓ of weight k to weight k by $(a|U)n = a(\ell n)$.

Definition 6 (hasWeight). A modular form mod ℓ called a has weight $j \in \mathbb{N}$ if there exists an integer modular form b of weight j such that $b \equiv a \pmod{\ell}$.

Definition 7 (Filtration). Let a be a of a modular form mod ℓ . The filtration of $a, \omega(a)$, is defined as the minimum natural number j such that a has weight j . The filtration of the zero function is 0.F

Definition 8 (multiplication and exponentiation). It's worth stated how multiplication and exponentiation are defined here, because they are not defined in the normal way. The multiplication of two modular forms mod ℓ called a and b is defined as

$$(a \cdot b)n = \sum_{x+y=n} a(x)b(y).$$

The exponentiation of a modular form mod ℓ called a to the power of $j \in \mathbb{N}$ is defined as

$$(a^j)n = \sum_{x_1+\dots+x_j=n} \prod_{i=1}^j a(x_i).$$

0.2 PowPrime

Definition 9 (permutational equivalence). Two functions $a, b : \text{Fin } n \rightarrow \mathbb{N}$, which can be thought of as tuples of n natural numbers, are permutationally equivalent if there exists a bijective function $\sigma : \text{Fin } n \rightarrow \text{Fin } n$ such that $a = b \circ \sigma$. This is an equivalence relation.

Lemma 10. If $x = (x_1, \dots, x_k)$ is not constant (i.e not all x_i are equal) then for any $n \in \mathbb{N}$,

$$k \mid \#\{y = (y_1, \dots, y_k) : \sum_{i=1}^k y_i = n \text{ and } x \text{ and } y \text{ are permutationally equivalent}\}$$

Proof.

□

Lemma 11. *If x and y are permutationally equivalent then $\prod_{i=1}^k a(x_i) = \prod_{i=1}^k a(y_i)$.*

Proof.

□

Lemma 12. *Let $x = (x_1, x_2, \dots, x_k)$ and $n \in \mathbb{N}$. Suppose that $\sum_{i=1}^k x_i = n$.*

(1) *If $k \nmid n$ then x is not constant.*

(2) *If $k \mid n$ and $x \neq (n/k, \dots, n/k)$ then x is not constant.*

Proof.

□

Theorem 13 (Pow Prime). *Let ℓ be a prime and a a modular form mod ℓ of any weight. Then*

$$a^\ell(n) = \begin{cases} a(n/\ell) & \text{if } \ell \mid n \\ 0 & \text{otherwise} \end{cases}$$

Proof.

□

0.3 Theorems

Theorem 14. *Let a be a modular form mod ℓ . Then $(a|U)^\ell = a - \Theta^{\ell-1}a$.*

Proof.

□

Lemma 15. *Let a be a modular form mod ℓ . If $\omega(a) = 0$ then a is constant, i.e. for all $n > 0, a(n) = 0$.*

Theorem 16. *Let a be a modular form mod ℓ and $i \in \mathbb{N}$. Then $\omega(a^i) = i\omega(a)$.*

Theorem 17. *Let a be a modular form mod ℓ of weight k . Then $\omega(a) \equiv k \pmod{\ell-1}$.*

Definition 18 (Delta).

Δ is the sequence obtained from $q(\prod_{n=1}^{\infty}(1-q^n))^{24}$. It is an integer modular form of weight 12. The modular form mod ℓ , also called Δ_ℓ , is defined to be the reduction of Δ .

Definition 19. For a prime $\ell \geq 5$, we define $\delta_\ell = \frac{\ell^2-1}{24} \in \mathbb{N}$. We define $f_\ell = \Delta^{\delta_\ell}$, which is an integer modular form of weight $12\delta_\ell$. We also define the reduction of f_ℓ as a modular form mod ℓ .

Lemma 20. $\omega(f_\ell) = 12\delta_\ell = \frac{\ell^2-1}{2}$.

Theorem 21. *This is part (1) of Lemma 2.1.*

Let a be a modular form mod ℓ . Then $\omega(\Theta a) \leq \omega(a) + \ell + 1$.

Theorem 22. *This is part (2) of Lemma 2.1.*

Let a be a modular form mod ℓ . Then $\omega(\Theta a) = \omega(a) + \ell + 1$ if and only if $\ell \nmid \omega(a)$.

Theorem 23. *This is Lemma 3.2.*

For all $m \in \mathbb{N}$, $\omega(\Theta^m f_\ell) \geq \omega(f_\ell) = \frac{\ell^2-1}{2}$.

Theorem 24. *This is part (1) of Lemma 3.3.*

If $\ell \nmid \omega(\Theta^{\ell-1} f_\ell)$ then $\omega(\Theta^{\ell-1} f_\ell) = \frac{\ell^2-1}{2}$.

Proof.

□

Theorem 25. *This is part (2) of Lemma 3.3.
If $\ell \mid \omega(\Theta^{\ell-1} f_\ell)$ then $\omega(f_\ell|U) > 0$.*

Proof. □

Lemma 26. *Let a be a modular form mod ℓ . If $\ell \mid \omega(a)$ then there exists an $\alpha \in \mathbb{N}$ such that $\omega(\Theta a) = \omega(a) + \ell + 1 - (\alpha + 1)(\ell - 1)$.*

Proof. □

Theorem 27. *If $(f_\ell|U) = 0$ then $\ell \mid \omega(\Theta^{\ell-2} f_\ell)$.*

Proof. □

Lemma 28. *For all $m \in \mathbb{N}$ with $m \leq \frac{\ell+1}{2}$, $\omega(\Theta^m f_\ell) = \frac{\ell^2-1}{2} + m(\ell + 1)$.*

Proof. Induction. □

Theorem 29. $\ell \mid \omega(\Theta^{\frac{\ell+1}{2}} f_\ell)$.

Proof. □

Definition 30. We define α to be the natural number such that $\omega(\Theta^{\frac{\ell+3}{2}} f_\ell) = \frac{\ell^2-1}{2} + \frac{\ell+3}{2}(\ell + 1) - (\alpha + 1)(\ell - 1)$.

Such an α exists, because $\ell \mid \omega(\Theta^{\frac{\ell+1}{2}} f_\ell) = \frac{\ell^2-1}{2} + \frac{\ell+1}{2}(\ell + 1)$.

Definition 31. We define j to be the least natural number such that $\ell \mid \omega(\Theta^{\frac{\ell+3}{2}+j} f_\ell)$. Such a j exists, because $\ell \mid \omega(\Theta^{\ell-2} f_\ell)$.

Note : This definition requires that $(f_\ell|U) = 0$. We will assume this fact from now on.

Lemma 32. $\alpha \leq \frac{\ell+3}{2}$.

Proof. □

Lemma 33. $j \leq \frac{\ell-7}{2}$.

Proof. □

Lemma 34. *For all $m \leq j$, $\omega(\Theta^{\frac{\ell+3}{2}+m} f_\ell) = \frac{\ell^2-1}{2} + (\frac{\ell+3}{2} + m)(\ell + 1) - (\alpha + 1)(\ell - 1)$.*

Proof. □

Lemma 35. $\ell \mid (j + 1) + (\alpha + 1)$.

Proof. □

Lemma 36. $\alpha + 1 = \frac{\ell+5}{2}$.

Proof. □

Theorem 37. $\omega(\Theta^{\frac{\ell+3}{2}} f_\ell) = \frac{\ell^2-1}{2} + 4$.

Proof. □

Lemma 38. $f_\ell(\delta_\ell + 1) = 1$.

Proof. □

Theorem 39. $(\Theta^{\frac{\ell+3}{2}} f_\ell) (\delta_\ell + 1) = 241(\delta_\ell + 1)^{\frac{\ell+3}{2}}$.

Theorem 40. If $\ell \geq 13$ is prime, then $(f_\ell|U) \neq 0$.

Proof. □

Definition 41. Let p be the partition function. We say that there is a Ramanujan congruence mod a prime ℓ if $\forall n \in \mathbb{N}$, $\ell \mid p(\ell n - \delta_\ell)$.

Theorem 42. If $n > 0$ and $m \geq n$, then $p(n)$ is equal to the n th coefficient in the product expansion of

$$\prod_{i=0}^m (1 - X^{i+1})^{-1}$$

Note : In lean, $p(0) = 0$.

Proof. This proof is almost entirely due to Archive.Wiedijk100Theorems.Partition □

Definition 43. Let α be a commutative semiring, and let $f, g : \mathbb{N} \rightarrow \alpha[[X]]$ be two sequences of power series with coefficients in α . We say that f and g are eventually equal, and write $f \rightarrow g$, if $\forall n \in \mathbb{N}$, $\exists m \in \mathbb{N}$, $\forall k \leq n, \forall j \geq m$, the k th coefficient of $f(j)$ is equal to the k th coefficient of $g(j)$. This is an equivalence relation.

Theorem 44. If $f \rightarrow f'$ and $g \rightarrow g'$, then $fg \rightarrow f'g'$.

Proof. □

Lemma 45. Let $n, N, \ell \in \mathbb{N}$ with $n < N$ and $\ell > 0$. Let $a : \mathbb{N} \rightarrow \alpha$ be a sequence with coefficients in a semiring α . Then the n th coefficient of

$$\sum_{i=0}^N a(n) X^{\ell i} = \begin{cases} a(n/\ell) & \text{if } \ell \mid n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. □

Lemma 46. Let $j, \ell, N, M \in \mathbb{N}$ with $\ell > 0$ and $N, M > \ell j$. Let $a, b : \mathbb{N} \rightarrow \alpha$. Then the ℓj th coefficient of

$$\sum_{i=0}^N a(i) X^i \sum_{i=0}^M b(i) X^{\ell i} = \sum_{x+y=j} a(\ell x) b(y)$$

Proof. □

Lemma 47. Let $m, N \in \mathbb{N}$. Let $f : \mathbb{N} \rightarrow \alpha[[X]]$ be a sequence with $f(0) = 0$, the zero power series. Then

$$X^m \sum_{i=0}^N f(i) X^i = \sum_{i=0}^{N+m} f(i-m) X^i$$

Proof. □

Lemma 48. Let $M, \ell, k \in \mathbb{N}$ with $\ell \nmid k$. Then the k th coefficient of

$$\prod_{i=0}^M (1 - X^{\ell(i+1)})^\ell = 0$$

Proof.

□

Lemma 49. *Let $\ell, K \in \mathbb{N}$ with $\ell > 0$. Then there exists a $c : \mathbb{N} \rightarrow \alpha$ and $M \in \mathbb{N}$ such that*

$$\prod_{i=0}^K (1 - X^{\ell(i+1)})^\ell = \sum_{i=0}^M c(i) X^{\ell i}$$

Proof.

□

Theorem 50. *The coefficients of the Delta product are eventually equal to the coefficients of the Delta function. In other words,*

$$X \prod_{i \leq} (1 - X^{i+1})^{24} \longrightarrow \sum_{i \leq} \Delta(i) X^i$$

Proof.

□

Theorem 51. *If $\ell \geq 5$ is prime, then the coefficients of the f_ℓ product are eventually equal to the coefficients of the f_ℓ function. In other words,*

$$(X^{\delta_\ell} \prod_{i \leq} (1 - X^{i+1})^{24\delta_\ell} \longrightarrow \sum_{i \leq} f_\ell(i) X^i)$$

Proof.

□

Theorem 52. *Let $\ell \geq 5$ be prime. If there is a Ramanujan congruence mod ℓ , then $(f_\ell|U) = 0$.*

Proof.

□

Theorem 53. *If $\ell \geq 13$ is prime, then there does not exist a Ramanujan congruence mod ℓ .*

Proof.

□