

# partitions-leanblueprint

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October 30, 2025

## 0.1 Definitions

**Definition 1** (Modular Form). In lean, A modular form of weight  $k \in \mathbb{N}$  is a function  $f : \mathbb{C} \rightarrow \mathbb{C}$  such that :

- (1)  $f$  is holomorphic on  $\mathbb{H}$
- (2) For all  $z \in \mathbb{H}$ ,  $f(z+1) = f(z)$
- (3) For all  $z \in \mathbb{H}$ ,  $f(-1/z) = z^k f(z)$
- (4)  $f$  is bounded as  $\text{Im}(z) \rightarrow \infty$

**Definition 2** (Integer Modular Form). An integer modular form of weight  $k \in \mathbb{N}$  is a sequence  $a : \mathbb{N} \rightarrow \mathbb{Z}$  such that  $\sum_{n=0}^{\infty} a(n)q^n$  is a modular form of weight  $k$ , where  $q = e^{2\pi iz}$ .

**Definition 3** (ModularFormMod  $\ell$ ). A modular form mod  $\ell$  of weight  $k \in \mathbb{Z}/(\ell-1)\mathbb{Z}$  is a sequence  $a : \mathbb{N} \rightarrow \mathbb{Z}/\ell\mathbb{Z}$  such that there exists an integer modular form  $b$  of weight  $k'$  where  $b \equiv a \pmod{\ell}$  and  $k' \equiv k \pmod{\ell-1}$ .

**Definition 4** (Theta).  $\Theta$  sends modular forms mod  $\ell$  of weight  $k$  to weight  $k+2$  by  $(\Theta a)n = na(n)$ .

**Definition 5** (U Operator). The operator  $U$  sends modular forms mod  $\ell$  of weight  $k$  to weight  $k$  by  $(a|U)n = a(\ell n)$ .

**Definition 6** (hasWeight). A modular form mod  $\ell$  called  $a$  has weight  $j \in \mathbb{N}$  if there exists an integer modular form  $b$  of weight  $j$  such that  $b \equiv a \pmod{\ell}$ .

**Definition 7** (Filtration). Let  $a$  be a modular form mod  $\ell$ . The filtration of  $a$ ,  $\omega(a)$ , is defined as the minimum natural number  $j$  such that  $a$  has weight  $j$ . The filtration of the zero function is 0.

**Definition 8** (multiplication and exponentiation). It's worth stated how multiplication and exponentiation are defined here, because they are not defined in the normal way. The multiplication of two modular forms mod  $\ell$  called  $a$  and  $b$  is defined as

$$(a \cdot b)n = \sum_{x+y=n} a(x)b(y).$$

The exponentiation of a modular form mod  $\ell$  called  $a$  to the power of  $j \in \mathbb{N}$  is defined as

$$(a^j)n = \sum_{x_1+\dots+x_j=n} \prod_{i=1}^j a(x_i).$$

## 0.2 PowPrime

**Definition 9** (permutational equivalence). Two functions  $a, b : \text{Fin } n \rightarrow \mathbb{N}$ , which can be thought of as tuples of  $n$  natural numbers, are permutationally equivalent if there exists a bijective function  $\sigma : \text{Fin } n \rightarrow \text{Fin } n$  such that  $a = b \circ \sigma$ . This is an equivalence relation.

**Lemma 10.** If  $x = (x_1, \dots, x_k)$  is not constant (i.e not all  $x_i$  are equal) then for any  $n \in \mathbb{N}$ ,

$$k \mid \#\{y = (y_1, \dots, y_k) : \sum_{i=1}^k y_i = n \text{ and } x \text{ and } y \text{ are permutationally equivalent}\}$$

*Proof.* □

**Lemma 11.** *If  $x$  and  $y$  are permutationally equivalent then  $\prod_{i=1}^k a(x_i) = \prod_{i=1}^k a(y_i)$ .*

*Proof.* □

**Lemma 12.** *Let  $x = (x_1, x_2, \dots, x_k)$  and  $n \in \mathbb{N}$ . Suppose that  $\sum_{i=1}^k x_i = n$ .*

*(1) If  $k \nmid n$  then  $x$  is not constant.*

*(2) If  $k \mid n$  and  $x \neq (n/k, \dots, n/k)$  then  $x$  is not constant.*

*Proof.* □

**Theorem 13** (Pow Prime). *Let  $\ell$  be a prime and  $a$  a modular form mod  $\ell$  of any weight. Then*

$$a^\ell(n) = \begin{cases} a(n/\ell) & \text{if } \ell \mid n \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* □

### 0.3 Theorems

**Theorem 14.** *Let  $a$  be a modular form mod  $\ell$ . Then  $(a|U)^\ell = a - \Theta^{\ell-1}a$ .*

*Proof.* □

**Lemma 15.** *Let  $a$  be a modular form mod  $\ell$ . If  $\omega(a) = 0$  then  $a$  is constant, i.e. for all  $n > 0$ ,  $a(n) = 0$ .*

**Theorem 16.** *Let  $a$  be a modular form mod  $\ell$  and  $i \in \mathbb{N}$ . Then  $\omega(a^i) = i\omega(a)$ .*

**Theorem 17.** *Let  $a$  be a modular form mod  $\ell$  of weight  $k$ . Then  $\omega(a) \equiv k \pmod{\ell-1}$ .*

**Definition 18** (Delta).

$\Delta$  is the sequence obtained from  $q(\prod_{n=1}^{\infty} (1 - q^n))^{24}$ . It is an integer modular form of weight 12. The modular form mod  $\ell$ , also called  $\bar{\Delta}$ , is defined to be the reduction of  $\Delta$ .

**Definition 19.** For a prime  $\ell \geq 5$ , we define  $\delta_\ell = \frac{\ell^2-1}{24} \in \mathbb{N}$ . We define  $f_\ell = \Delta^{\delta_\ell}$ , which is an integer modular form of weight  $12\delta_\ell$ . We also define the reduction of  $f_\ell$  as a modular form mod  $\ell$

**Lemma 20.**  $\omega(f_\ell) = 12\delta_\ell = \frac{\ell^2-1}{2}$ .

**Theorem 21.** *This is part (1) of Lemma 2.1.*

*Let  $a$  be a modular form mod  $\ell$ . Then  $\omega(\Theta a) \leq \omega(a) + \ell + 1$ .*

**Theorem 22.** *This is part (2) of Lemma 2.1.*

*Let  $a$  be a modular form mod  $\ell$ . Then  $\omega(\Theta a) = \omega(a) + \ell + 1$  if and only if  $\ell \nmid \omega(a)$ .*

**Theorem 23.** *This is Lemma 3.2.*

*For all  $m \in \mathbb{N}$ ,  $\omega(\Theta^m f_\ell) \geq \omega(f_\ell) = \frac{\ell^2-1}{2}$ .*

**Theorem 24.** *This is part (1) of Lemma 3.3.*

*If  $\ell \nmid \omega(\Theta^{\ell-1} f_\ell)$  then  $\omega(\Theta^{\ell-1} f_\ell) = \frac{\ell^2-1}{2}$ .*

*Proof.* □

**Theorem 25.** *This is part (2) of Lemma 3.3.  
If  $\ell \mid \omega(\Theta^{\ell-1}f_\ell)$  then  $\omega(f_\ell|U) > 0$ .*

*Proof.* □

**Lemma 26.** *Let  $a$  be a modular form mod  $\ell$ . If  $\ell \mid \omega(a)$  then there exists an  $\alpha \in \mathbb{N}$  such that  $\omega(\Theta a) = \omega(a) + \ell + 1 - (\alpha + 1)(\ell - 1)$ .*

*Proof.* □

**Theorem 27.** *If  $(f_\ell|U) = 0$  then  $\ell \mid \omega(\Theta^{\ell-2}f_\ell)$ .*

*Proof.* □

**Lemma 28.** *For all  $m \in \mathbb{N}$  with  $m \leq \frac{\ell+1}{2}$ ,  $\omega(\Theta^m f_\ell) = \frac{\ell^2-1}{2} + m(\ell + 1)$ .*

*Proof.* Induction. □

**Theorem 29.**  $\ell \mid \omega(\Theta^{\frac{\ell+1}{2}} f_\ell)$ .

*Proof.* □

**Definition 30.** We define  $\alpha$  to be the natural number such that  $\omega(\Theta^{\frac{\ell+3}{2}} f_\ell) = \frac{\ell^2-1}{2} + \frac{\ell+3}{2}(\ell + 1) - (\alpha + 1)(\ell - 1)$ .

Such an  $\alpha$  exists, because  $\ell \mid \omega(\Theta^{\frac{\ell+1}{2}} f_\ell) = \frac{\ell^2-1}{2} + \frac{\ell+1}{2}(\ell + 1)$ .

**Definition 31.** We define  $j$  to be the least natural number such that  $\ell \mid \omega(\Theta^{\frac{\ell+3}{2}+j} f_\ell)$ . Such a  $j$  exists, because  $\ell \mid \omega(\Theta^{\ell-2} f_\ell)$ .

Note : This definition requires that  $(f_\ell|U) = 0$ . We will assume this fact from now on.

**Lemma 32.**  $\alpha \leq \frac{\ell+3}{2}$ .

*Proof.* □

**Lemma 33.**  $j \leq \frac{\ell-7}{2}$ .

*Proof.* □

**Lemma 34.** *For all  $m \leq j$ ,  $\omega(\Theta^{\frac{\ell+3}{2}+m} f_\ell) = \frac{\ell^2-1}{2} + (\frac{\ell+3}{2} + m)(\ell + 1) - (\alpha + 1)(\ell - 1)$ .*

*Proof.* □

**Lemma 35.**  $\ell \mid (j + 1) + (\alpha + 1)$ .

*Proof.* □

**Lemma 36.**  $\alpha + 1 = \frac{\ell+5}{2}$ .

*Proof.* □

**Theorem 37.**  $\omega(\Theta^{\frac{\ell+3}{2}} f_\ell) = \frac{\ell^2-1}{2} + 4$ .

*Proof.* □

**Lemma 38.**  $f_\ell(\delta_\ell + 1) = 1$ .

*Proof.* □

**Theorem 39.**  $(\Theta^{\frac{\ell+3}{2}} f_\ell) (\delta_\ell + 1) = 241(\delta_\ell + 1)^{\frac{\ell+3}{2}}.$

**Theorem 40.** *If  $\ell \geq 13$  is prime, then  $(f_\ell|U) \neq 0$ .*

*Proof.* □

**Definition 41.** Let  $p$  be the partion function. We say that there is a Ramanujan congruence mod a prime  $\ell$  if  $\forall n \in \mathbb{N}, \ell \mid p(\ell n - \delta_\ell)$ .

**Theorem 42.** *If  $n > 0$  and  $m \geq n$ , then  $p(n)$  is equal to the  $n$ th coefficient in the product exapansion of*

$$\prod_{i=0}^m (1 - X^{i+1})^{-1}$$

*Note :* In lean,  $p(0) = 0$ .

*Proof.* This proof is almost entirely due to Archive.Wiedijk100Theorems.Partition □

**Definition 43.** Let  $\alpha$  be a field, and let  $f, g : \mathbb{N} \rightarrow \alpha[[X]]$  be two sequences of power series with coefficients in  $\alpha$ . We say that  $f$  and  $g$  are eventually equal, and write  $f \rightarrow g$ , if  $\forall n \in \mathbb{N}, \exists m \in \mathbb{N}, \forall k \leq n, \forall j \geq m$ , the  $k$ th coefficient of  $f(j)$  is equal to the  $k$ th coefficient of  $g(j)$ . This is an equivalence relation.

**Theorem 44.** *If  $f \rightarrow f'$  and  $g \rightarrow g'$ , then  $fg \rightarrow f'g'$ .*

*Proof.* □

**Lemma 45.** *Let  $n, N, \ell \in \mathbb{N}$  with  $n < N$  and  $\ell > 0$ . Let  $a : \mathbb{N} \rightarrow \alpha$  be a sequence with coefficients in a field  $\alpha$ . Then the  $n$ th coefficient of*

$$\sum_{i=0}^N a(n) X^{\ell i} = \begin{cases} a(n/\ell) & \text{if } \ell \mid n, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* □

**Lemma 46.** *Let  $j, \ell, N, M \in \mathbb{N}$  with  $\ell > 0$  and  $N, M > \ell j$ . Let  $a, b : \mathbb{N} \rightarrow \alpha$ . Then the  $\ell j$  th coefficient of*

$$\sum_{i=0}^N a(i) X^i \sum_{i=0}^M b(i) X^{\ell i} = \sum_{x+y=j} a(\ell x) b(y)$$

*Proof.* □

**Lemma 47.** *Let  $m, N \in \mathbb{N}$ . Let  $f : \mathbb{N} \rightarrow \alpha[[X]]$  be a sequence with  $f(0) = 0$ , the zero power series. Then*

$$X^m \sum_{i=0}^N f(i) X^i = \sum_{i=0}^{N+m} f(i-m) X^i$$

*Proof.* □

**Lemma 48.** *Let  $M, \ell, k \in \mathbb{N}$  with  $\ell \nmid k$ . Then the  $k$ th coefficient of*

$$\prod_{i=0}^M (1 - X^{\ell(i+1)})^\ell = 0$$

*Proof.*

□

**Lemma 49.** *Let  $\ell, K \in \mathbb{N}$  with  $\ell > 0$ . Then there exists a  $c : \mathbb{N} \rightarrow \alpha$  and  $M \in \mathbb{N}$  such that*

$$\prod_{i=0}^K (1 - X^{\ell(i+1)})^\ell = \sum_{i=0}^M c(i) X^{\ell i}$$

*Proof.*

□

**Theorem 50.** *The coefficients of the Delta product are eventually equal to the coefficients of the Delta function. In other words,*

$$X \prod_{i \leq \cdot} (1 - X^{i+1})^{24} \longrightarrow \sum_{i \leq \cdot} \Delta(i) X^i$$

*Proof.*

□

**Theorem 51.** *If  $\ell \geq 5$  is prime, then the coefficients of the  $f_\ell$  product are eventually equal to the coefficients of the  $f_\ell$  function. In other words,*

$$(X \prod_{i \leq \cdot} (1 - X^{i+1})^{24})^{\delta_\ell} \longrightarrow \sum_{i \leq \cdot} f_\ell(i) X^i$$

*Proof.*

□

**Theorem 52.** *Let  $\ell \geq 5$  be prime. If there is a ramanujan congruence mod  $\ell$ , then  $(f_\ell|U) = 0$ .*

*Proof.*

□

**Theorem 53.** *If  $\ell \geq 13$  is prime, then there does not exist a ramanujan congruence mod  $\ell$ .*

*Proof.*

□