

partitions-leanblueprint

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0.1 Definitions

Definition 1 (Modular Form). In lean, A modular form of weight $k \in \mathbb{N}$ is a function $f : \mathbb{C} \rightarrow \mathbb{C}$ such that :

- (1) f is holomorphic on \mathbb{H}
- (2) For all $z \in \mathbb{H}$, $f(z+1) = f(z)$
- (3) For all $z \in \mathbb{H}$, $f(-1/z) = z^k f(z)$
- (4) f is bounded as $\text{Im}(z) \rightarrow \infty$

Definition 2 (Integer Modular Form). An integer modular form of weight $k \in \mathbb{N}$ is a sequence $a : \mathbb{N} \rightarrow \mathbb{Z}$ such that $\sum_{n=0}^{\infty} a(n)q^n$ is a modular form of weight k , where $q = e^{2\pi iz}$.

Definition 3 (ModularFormMod ℓ). A modular form mod ℓ of weight $k \in \mathbb{Z}/(\ell-1)\mathbb{Z}$ is a sequence $a : \mathbb{N} \rightarrow \mathbb{Z}/\ell\mathbb{Z}$ such that there exists an integer modular form b of weight k' where $b \equiv a \pmod{\ell}$ and $k' \equiv k \pmod{\ell-1}$.

Definition 4 (Theta). Θ sends modular forms mod ℓ of weight k to weight $k+2$ by $(\Theta a)n = na(n)$.

Definition 5 (U Operator). The operator U sends modular forms mod ℓ of weight k to weight k by $(a|U)n = a(\ell n)$.

Definition 6 (hasWeight). A modular form mod ℓ called a has weight $j \in \mathbb{N}$ if there exists an integer modular form b of weight j such that $b \equiv a \pmod{\ell}$.

Definition 7 (Filtration). Let a be a modular form mod ℓ . The filtration of a , $\omega(a)$, is defined as the minimum natural number j such that a has weight j . The filtration of the zero function is 0.

Definition 8 (multiplication and exponentiation). It's worth stated how multiplication and exponentiation are defined here, because they are not defined in the normal way. The multiplication of two modular forms mod ℓ called a and b is defined as

$$(a \cdot b)n = \sum_{x+y=n} a(x)b(y).$$

The exponentiation of a modular form mod ℓ called a to the power of $j \in \mathbb{N}$ is defined as

$$(a^j)n = \sum_{x_1+\dots+x_j=n} \prod_{i=1}^j a(x_i).$$

0.2 PowPrime

Definition 9 (permutational equivalence). Two functions $a, b : \text{Fin } n \rightarrow \mathbb{N}$, which can be thought of as tuples of n natural numbers, are permutationally equivalent if there exists a bijective function $\sigma : \text{Fin } n \rightarrow \text{Fin } n$ such that $a = b \circ \sigma$. This is an equivalence relation.

Lemma 10. If $x = (x_1, x_2, \dots, x_k)$ is not constant (i.e not all x_i are equal) then for any $n \in \mathbb{N}$,

$$k \mid \#\{y = (y_1, y_2, \dots, y_k) : \sum_{i=1}^k y_i = n \text{ and } x \text{ and } y \text{ are permutationally equivalent}\}$$

Proof. □

Lemma 11. *If x and y are permutationally equivalent then $\prod_{i=1}^k a(x_i) = \prod_{i=1}^k a(y_i)$.*

Proof. □

Lemma 12. *Let $x = (x_1, x_2, \dots, x_k)$ and $n \in \mathbb{N}$. Suppose that $\sum_{i=1}^k x_i = n$.*

(1) If $k \nmid n$ then x is not constant.

(2) If $k \mid n$ and $x \neq (n/k, \dots, n/k)$ then x is not constant.

Proof. □

Theorem 13 (Pow Prime). *Let ℓ be a prime and a a modular form mod ℓ of any weight. Then*

$$(a^\ell)_n = \begin{cases} a(n/\ell) & \text{if } \ell \mid n \\ 0 & \text{otherwise} \end{cases}$$

Proof. □

0.3 Theorems

Theorem 14. *Let a be a modular form mod ℓ . Then $(a|U)^\ell = a - \Theta^{\ell-1}a$.*

Proof. □

Lemma 15. *Let a be a modular form mod ℓ . If $\omega(a) = 0$ then a is constant, i.e. for all $n > 0$, $a(n) = 0$.*

Theorem 16. *Let a be a modular form mod ℓ and $i \in \mathbb{N}$. Then $\omega(a^i) = i\omega(a)$.*

Theorem 17. *Let a be a modular form mod ℓ of weight k . Then $(\omega(a) \equiv k \pmod{\ell-1})$.*

Definition 18 (Delta).

Δ is the sequence obtained from $q(\prod_{n=1}^{\infty} (1 - q^n))^{24}$. It is a modular form mod ℓ of weight 12.

Definition 19. For a prime $\ell \geq 5$, we define $\delta_\ell = \frac{\ell^2-1}{24} \in \mathbb{N}$. We define $f_\ell = \Delta^{\delta_\ell}$, which is a modular form mod ℓ of weight $12\delta_\ell$.

Lemma 20. $\omega(f_\ell) = 12\delta_\ell = \frac{\ell^2-1}{2}$.

Theorem 21. *This is part (1) of Lemma 2.1.*

Let a be a modular form mod ℓ . Then $\omega(\Theta a) \leq \omega(a) + \ell + 1$.

Theorem 22. *This is part (2) of Lemma 2.1.*

Let a be a modular form mod ℓ . Then $\omega(\Theta a) = \omega(a) + \ell + 1$ if and only if $\ell \nmid \omega(a)$.

Theorem 23. *This is Lemma 3.2.*

For all $m \in \mathbb{N}$, $\omega(\Theta^m f_\ell) \geq \omega(f_\ell) = \frac{\ell^2-1}{2}$.

Theorem 24. *This is part (1) of Lemma 3.3.*

If $\ell \nmid \omega(\Theta^{\ell-1} f_\ell)$ then $\omega(\Theta^{\ell-1} f_\ell) = \frac{\ell^2-1}{2}$.

Proof. □

Theorem 25. *This is part (2) of Lemma 3.3.
If $\ell \mid \omega(\Theta^{\ell-1}f_\ell)$ then $\omega(f_\ell|U) > 0$.*

Proof.

□

Lemma 26. *Let a be a modular form mod ℓ . If $\ell \mid \omega(a)$ then there exists an $\alpha \in \mathbb{N}$ such that $\omega(\Theta a) = \omega(a) + \ell + 1 - (\alpha + 1)(\ell - 1)$.*

Proof.

□

Theorem 27. *If $(f_\ell|U) = 0$ then $\ell \mid \omega(\Theta^{\ell-2}f_\ell)$.*

Proof.

□

Lemma 28. *For all $m \in \mathbb{N}$ with $m \leq \frac{\ell+1}{2}$, $\omega(\Theta^m f_\ell) = \frac{\ell^2-1}{2} + m(\ell+1)$.*

Proof. Induction.

□

Theorem 29. $\ell \mid \omega(\Theta^{\frac{\ell+1}{2}} f_\ell)$.

Proof.

□

Definition 30. We define α to be the natural number such that $\omega(\Theta^{\frac{\ell+3}{2}} f_\ell) = \frac{\ell^2-1}{2} + \frac{\ell+3}{2}(\ell+1) - (\alpha+1)(\ell-1)$.

Such an α exists, because $\ell \mid \omega(\Theta^{\frac{\ell+1}{2}} f_\ell) = \frac{\ell^2-1}{2} + \frac{\ell+1}{2}(\ell+1)$.

Definition 31. We define j to be the least natural number such that $\ell \mid \omega(\Theta^{\frac{\ell+3}{2}+j} f_\ell)$. Such a j exists, because $\ell \mid \omega(\Theta^{\ell-2} f_\ell)$.

Note : This definition requires that $(f_\ell|U) = 0$. We will assume this fact from now on.

Lemma 32. $\alpha \leq \frac{\ell+3}{2}$.

Proof.

□

Lemma 33. $j \leq \frac{\ell-7}{2}$.

Proof.

□

Lemma 34. *For all $m \leq j$, $\omega(\Theta^{\frac{\ell+3}{2}+m} f_\ell) = \frac{\ell^2-1}{2} + (\frac{\ell+3}{2} + m)(\ell+1) - (\alpha+1)(\ell-1)$.*

Proof.

□

Lemma 35. $\ell \mid (j+1) + (\alpha+1)$.

Proof.

□

Lemma 36. $\alpha + 1 = \frac{\ell+5}{2}$.

Proof.

□

Theorem 37. $\omega(\Theta^{\frac{\ell+3}{2}} f_\ell) = \frac{\ell^2-1}{2} + 4$.

Proof.

□

Lemma 38. $f_\ell(\delta_\ell + 1) = 1$.

Proof.

□