# partitions-leanblueprint

Carter Hastings Quinn Oveson

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## 0.1 Definitions

**Definition 1** (Modular Form)**.** In lean, A modular form of weight $k \in \mathbb{N}$ is a function $f : \mathbb{C} \to \mathbb{C}$ such that :

(1) $f$ is holomorphic on $\mathbb{H}$
(2) For all $z \in \mathbb{H}, f(z+1) = f(z)$
(3) For all $z \in \mathbb{H}, f(-1/z) = z^k f(z)$
(4) $f$ is bounded as $\mathrm{Im}(z) \to \infty$

**Definition 2** (Integer Modular Form)**.** An integer modular form of weight $k \in \mathbb{N}$ is a sequence $a : \mathbb{N} \to \mathbb{Z}$ such that $\sum_{n=0}^{\infty} a(n)q^n$ is a modular form of weight $k$, where $q = e^{2\pi i z}$.

**Definition 3** (ModularFormMod $\ell$)**.** A modular form mod $\ell$ of weight $k \in \mathbb{Z}/(\ell-1)\mathbb{Z}$ is a sequence $a : \mathbb{N} \to \mathbb{Z}/\ell\mathbb{Z}$ such that there exists an integer modular form $b$ of weight $k'$ where $b \equiv a \mod \ell$ and $k' \equiv k \mod (\ell - 1)$.

**Definition 4** (Theta)**.** $\Theta$ sends modular forms mod $\ell$ of weight $k$ to weight $k+2$ by
$(\Theta a)n = na(n)$.

**Definition 5** (U Operator)**.** The operator $U$ sends modular forms mod $\ell$ of weight $k$ to weight $k$ by
$(a|U)n = a(\ell n)$.

**Definition 6** (hasWeight)**.** A modular form mod $\ell$ called $a$ has weight $j \in \mathbb{N}$ if there exists an integer modular form $b$ of weight $j$ such that $b \equiv a \mod \ell$.

**Definition 7** (Filtration)**.** Let a be a of a modular form mod $\ell$. The filtration of $a, \omega(a)$, is defined as the minimum natural number $j$ such that $a$ has weight $j$. The filtration of the zero function is 0.

**Definition 8** (multiplication and exponentiation)**.** It's worth stated how multiplication and exponentiation are defined here, because they are not defined in the normal way. The multiplication of two modular forms mod $\ell$ called $a$ and $b$ is defined as

$$(a \cdot b)n = \sum_{x+y=n} a(x)b(y).$$

The exponentiation of a modular form mod $\ell$ called $a$ to the power of $j \in \mathbb{N}$ is defined as

$$(a^j)n = \sum_{x_1+\ldots+x_j=n} \prod_{i=1}^{j} a(x_i).$$

## 0.2 PowPrime

**Definition 9** (permutational equivalence)**.** Two functions $a, b : \mathrm{Fin}\ n \to \mathbb{N}$, which can be thought of as tuples of n natural numbers, are permutationally equivalent if there exists a bijective function $\sigma : \mathrm{Fin}\ n \to \mathrm{Fin}\ n$ such that $a = b \circ \sigma$. This is an equivalence relation.

**Lemma 10.** *If $x = (x_1, ..., x_k)$ is not constant (i.e not all $x_i$ are equal) then for any $n \in \mathbb{N}$,*

$$k \mid \#\{y = (y_1, ..., y_k) : \sum_{i=1}^{k} y_i = n \textit{ and } x \textit{ and } y \textit{ are permutationally equivalent}\}$$

*Proof.* $\square$

**Lemma 11.** *If $x$ and $y$ are permutationally equivalent then $\prod_{i=1}^k a(x_i) = \prod_{i=1}^k a(y_i)$.*

*Proof.* $\square$

**Lemma 12.** *Let $x = (x_1, x_2, ..., x_k)$ and $n \in \mathbb{N}$. Suppose that $\sum_{i=1}^k x_i = n$.*
*(1) If $k \nmid n$ then $x$ is not constant.*
*(2) If $k \mid n$ and $x \neq (n/k, ..., n/k)$ then $x$ is not constant.*

*Proof.* $\square$

**Theorem 13** (Pow Prime)**.** *Let $\ell$ be a prime and $a$ a modular form mod $\ell$ of any weight. Then*

$$a^\ell(n) = \begin{cases} a(n/\ell) & \text{if } \ell \mid n \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* $\square$

## 0.3 Theorems

**Theorem 14.** *Let $a$ be a modular form mod $\ell$. Then $(a|U)^\ell = a - \Theta^{\ell-1}a$.*

*Proof.* $\square$

**Lemma 15.** *Let $a$ be a modular form mod $\ell$. If $\omega(a) = 0$ then $a$ is constant, i.e. for all $n > 0, a(n) = 0$.*

**Theorem 16.** *Let $a$ be a modular form mod $\ell$ and $i \in \mathbb{N}$. Then $\omega(a^i) = i\omega(a)$.*

**Theorem 17.** *Let $a$ be a modular form mod $\ell$ of weight $k$. Then $(\omega(a) \equiv k \mod (\ell - 1)$.*

**Definition 18** (Delta)**.**
$\Delta$ is the sequence obtained from $q(\prod_{n=1}^\infty (1-q^n))^{24}$. It is an integer modular form of weight 12. The modular form mod $\ell$, also called $\Delta$, is defined to the the reduction of $\Delta$.

**Definition 19.** For a prime $\ell \geq 5$, we define $\delta_\ell = \frac{\ell^2-1}{24} \in \mathbb{N}$. We define $f_\ell = \Delta^{\delta_\ell}$, which is an integer modular form of weight $12\delta_\ell$. We also define the reduction of fl as an modular form mod $\ell$

**Lemma 20.** $\omega(f_\ell) = 12\delta_\ell = \frac{\ell^2-1}{2}$.

**Theorem 21.** *This is part (1) of Lemma 2.1.*
*Let $a$ be a modular form mod $\ell$. Then $\omega(\Theta a) \leq \omega(a) + \ell + 1$.*

**Theorem 22.** *This is part (2) of Lemma 2.1.*
*Let $a$ be a modular form mod $\ell$. Then $\omega(\Theta a) = \omega(a) + \ell + 1$ if and only if $\ell \nmid \omega(a)$.*

**Theorem 23.** *This is Lemma 3.2.*
*For all $m \in \mathbb{N}$, $\omega(\Theta^m f_\ell) \geq \omega(f_\ell) = \frac{\ell^2-1}{2}$.*

**Theorem 24.** *This is part (1) of Lemma 3.3.*
*If $\ell \nmid \omega(\Theta^{\ell-1} f_\ell)$ then $\omega(\Theta^{\ell-1} f_\ell) = \frac{\ell^2-1}{2}$.*

*Proof.* $\square$

**Theorem 25.** *This is part (2) of Lemma 3.3.*
*If* $\ell \mid \omega(\Theta^{\ell-1} f_\ell)$ *then* $\omega(f_\ell | U) > 0$.

*Proof.* □

**Lemma 26.** *Let* $a$ *be a modular form mod* $\ell$. *If* $\ell \mid \omega(a)$ *then there exists an* $\alpha \in \mathbb{N}$ *such that* $\omega(\Theta a) = \omega(a) + \ell + 1 - (\alpha + 1)(\ell - 1)$.

*Proof.* □

**Theorem 27.** *If* $(f_\ell | U) = 0$ *then* $\ell \mid \omega(\Theta^{\ell-2} f_\ell)$.

*Proof.* □

**Lemma 28.** *For all* $m \in \mathbb{N}$ *with* $m \leq \frac{\ell+1}{2}$, $\omega(\Theta^m f_\ell) = \frac{\ell^2-1}{2} + m(\ell+1)$.

*Proof.* Induction. □

**Theorem 29.** $\ell \mid \omega(\Theta^{\frac{\ell+1}{2}} f_\ell)$.

*Proof.* □

**Definition 30.** We define $\alpha$ to be the natural number such that $\omega(\Theta^{\frac{\ell+3}{2}} f_\ell) = \frac{\ell^2-1}{2} + \frac{\ell+3}{2}(\ell + 1) - (\alpha + 1)(\ell - 1)$.
Such an $\alpha$ exists, because $\ell \mid \omega(\Theta^{\frac{\ell+1}{2}} f_\ell) = \frac{\ell^2-1}{2} + \frac{\ell+1}{2}(\ell+1)$.

**Definition 31.** We define $j$ to be the least natural number such that $\ell \mid \omega(\Theta^{\frac{\ell+3}{2}+j} f_\ell)$. Such a $j$ exists, because $\ell \mid \omega(\Theta^{\ell-2} f_\ell)$.

Note : This definition requires that $(f_\ell | U) = 0$. We will assume this fact from now on.

**Lemma 32.** $\alpha \leq \frac{\ell+3}{2}$.

*Proof.* □

**Lemma 33.** $j \leq \frac{\ell-7}{2}$.

*Proof.* □

**Lemma 34.** *For all* $m \leq j$, $\omega(\Theta^{\frac{\ell+3}{2}+m} f_\ell) = \frac{\ell^2-1}{2} + (\frac{\ell+3}{2} + m)(\ell+1) - (\alpha+1)(\ell-1)$.

*Proof.* □

**Lemma 35.** $\ell \mid (j+1) + (\alpha+1)$.

*Proof.* □

**Lemma 36.** $\alpha + 1 = \frac{\ell+5}{2}$.

*Proof.* □

**Theorem 37.** $\omega(\Theta^{\frac{\ell+3}{2}} f_\ell) = \frac{\ell^2-1}{2} + 4$.

*Proof.* □

**Lemma 38.** $f_\ell(\delta_\ell + 1) = 1$.

*Proof.* □

**Theorem 39.** $(\Theta^{\frac{\ell+3}{2}} f_\ell)\,(\delta_\ell + 1) = 241(\delta_\ell + 1)^{\frac{\ell+3}{2}}$.

**Theorem 40.** *If $\ell \geq 13$ is prime, then $(f_\ell|U) \neq 0$.*

*Proof.* □

**Definition 41.** Let $p$ be the partion function. We say that there is a Ramanujan congruence mod a prime $\ell$ if $\forall n \in \mathbb{N}, \ell \mid p(\ell n - \delta_\ell)$.

**Theorem 42.** *If $n > 0$ and $m \geq n$, then $p(n)$ is equal to the nth coefficient in the product exapansion of*

$$\prod_{i=0}^{m}(1 - X^{i+1})^{-1}$$

*Note : In lean, $p(0) = 0$.*

*Proof.* This proof is almost entirely due to Archive.Wiedijk100Theorems.Partition □

**Definition 43.** Let $\alpha$ be a field, and let $f, g : \mathbb{N} \to \alpha[[X]]$ be two sequences of power series with coefficients in $\alpha$. We say that $f$ and $g$ are eventually equal, and write $f \longrightarrow g$, if $\forall n \in \mathbb{N}, \exists m \in \mathbb{N}, \forall k \leq n, \forall j \geq m$, the kth coefficient of $f(j)$ is equal to the kth coefficient of $g(j)$. This is an equivalence relation.

**Theorem 44.** *If $f \longrightarrow f'$ and $g \longrightarrow g'$, then $fg \longrightarrow f'g'$.*

*Proof.* □

**Lemma 45.** *Let $n, N, \ell \in \mathbb{N}$ with $n < N$ and $\ell > 0$. Let $a : \mathbb{N} \to \alpha$ be a sequence with coefficients in a field $\alpha$. Then the nth coefficient of*

$$\sum_{i=0}^{N} a(n)X^{\ell i} = \begin{cases} a(n/\ell) & \textit{if } \ell \mid n, \\ 0 & \textit{otherwise.} \end{cases}$$

*Proof.* □

**Lemma 46.** *Let $j, \ell, N, M \in \mathbb{N}$ with $\ell > 0$ and $N, M > \ell j$. Let $a, b : \mathbb{N} \to \alpha$. Then the $\ell j$ th coefficient of*

$$\sum_{i=0}^{N} a(i)X^{i} \sum_{i=0}^{M} b(i)X^{\ell i} = \sum_{x+y=j} a(\ell x)b(y)$$

*Proof.* □

**Lemma 47.** *Let $m, N \in \mathbb{N}$. Let $f : \mathbb{N} \to \alpha[[X]]$ be a sequence with $f(0) = 0$, the zero power series. Then*

$$X^m \sum_{i=0}^{N} f(i)X^i = \sum_{i=0}^{N+m} f(i-m)X^i$$

*Proof.* □

**Lemma 48.** *Let $M, \ell, k \in \mathbb{N}$ with $\ell \nmid k$. Then the kth coefficient of*

$$\prod_{i=0}^{M}(1 - X^{\ell(i+1)})^{\ell} = 0$$

*Proof.* ∎

**Lemma 49.** *Let $\ell, K \in \mathbb{N}$ with $\ell > 0$. Then there exists a $c : \mathbb{N} \to \alpha$ and $M \in \mathbb{N}$ such that*

$$\prod_{i=0}^{K}(1 - X^{\ell(i+1)})^{\ell} = \sum_{i=0}^{M} c(i) X^{\ell i}$$

*Proof.* ∎

**Theorem 50.** *The coefficients of the Delta product are eventually equal to the coefficients of the Delta function. In other words,*

$$X \prod_{i \le \cdot}(1 - X^{i+1})^{24} \longrightarrow \sum_{i \le \cdot} \Delta(i) X^{i}$$

**Theorem 51.** *If $\ell \geq 5$ is prime, then the coefficients of the $f_\ell$ product are eventually equal to the coefficients of the $f_\ell$ function. In other words,*

$$(X \prod_{i \le \cdot}(1 - X^{i+1})^{24})^{\delta_\ell} \longrightarrow \sum_{i \le \cdot} f_\ell(i) X^{i}$$

*Proof.* ∎

**Theorem 52.** *Let $\ell \geq 5$ be prime. If there is a ramanujan congruence mod $\ell$, then $(f_\ell | U) = 0$.*

*Proof.* ∎

**Theorem 53.** *If $\ell \geq 13$ is prime, then there does not exists a ramanujan congruence mod $\ell$.*

*Proof.* ∎