partitions-leanblueprint

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0.1 Definitions

Definition 1 (Modular Form)**.** In lean, A modular form of weight $k \in \mathbb{N}$ is a function $f : \mathbb{C} \to \mathbb{C}$ such that :

(1) f is holomorphic on $\mathbb{H}$
(2) For all $z \in \mathbb{H}, f(z) = f(z+1)$
(3) For all $z \in \mathbb{H}, f(z) = z^{-k} f(-1/z)$
(4) f is bounded as $\mathrm{Re}(z) \to \infty$

Definition 2 (Integer Modular Form)**.** An integer modular form of weight $k \in \mathbb{N}$ is a sequence $a : \mathbb{N} \to \mathbb{Z}$ such that $\sum_{n=0}^{\infty} a(n) q^n$ is a modular form of weight k, where $q = e^{2\pi i z}$.

Definition 3 (ModularFormMod ℓ)**.** A modular form mod ℓ of weight $k \in \mathbb{Z}/(\ell-1)\mathbb{Z}$ is a sequence $a : \mathbb{N} \to \mathbb{Z}/\ell\mathbb{Z}$ such that there exists an integer modular form b of weight k' where $b \equiv a$ mod ℓ and $k' \equiv k \mod (\ell-1)$.

Definition 4 (Theta)**.** Θ sends modular forms mod ℓ of weight k to weight $k+2$ by
$(\Theta a)n = na(n)$.

Definition 5 (U Operator)**.** The operator U sends modular forms mod ℓ of weight k to weight k by
$(a|U)n = a(\ell n)$.

Definition 6 (hasWeight)**.** A modular form mod ℓ called a has weight $j \in \mathbb{N}$ if there exists an integer modular form b of weight j such that $b \equiv a \mod \ell$.

Definition 7 (Filtration)**.** Let a be a of a modular form mod ℓ. The filtration of $a, \omega(a)$, is defined as the minimum natural number j such that a has weight j. The filtration of the zero function is 0.

Definition 8 (multiplication and exponentiation)**.** It's worth stated how multiplication and exponentiation are defined here, because they are not defined in the normal way. The multiplication of two modular forms mod ℓ called a and b is defined as

$$(a * b)n = \sum_{x+y=n} a(x)b(y).$$

The exponentiation of a modular form mod ℓ called a to the power of $j \in \mathbb{N}$ is defined as

$$(a^j)n = \sum_{x_1 + \ldots + x_j = n} \prod_{i=1}^{j} a(x_i).$$

0.2 PowPrime

Definition 9 (permutational equivalence)**.** Two functions $a, b : \mathrm{Fin}\, n \to \mathbb{N}$, which can be thought of as tuples of n natural numbers, are permutationally equivalent if there exists a bijective function $\sigma : \mathrm{Fin}\, n \to \mathrm{Fin}\, n$ such that $a = b \circ \sigma$. This is an equivalence relation.

Lemma 10. *If $x = (x_1, x_2, ..., x_k)$ is not constant (i.e not all x_i are equal) then for any $n \in \mathbb{N}$,*

$$k \mid \#\{y = (y_1, y_2, ..., y_k) : \sum_{i=1}^{k} y_i = n \text{ and } x \text{ and } y \text{ are permutationally equivalent}\}$$

Proof. □

Lemma 11. *If x and y are permutationally equivalent then $\prod_{i=1}^{k} a(x_i) = \prod_{i=1}^{k} a(y_i)$.*

Proof. □

Lemma 12. *Let $x = (x_1, x_2, ..., x_k)$ and $n \in \mathbb{N}$. Suppose that $\sum_{i=1}^{k} x_i = n$.*
(1) If $k \nmid n$ then x is not constant.
(2) If $k \mid n$ and $x \neq (n/k, ..., n/k)$ then x is not constant.

Proof. □

Theorem 13 (Pow Prime)**.** *Let ℓ be a prime and a be a modular form mod ℓ of any weight. Then*

$$(a^\ell)n = \begin{cases} a(n/\ell) & \text{if } \ell \mid n \\ 0 & \text{otherwise} \end{cases}$$

Proof. □

0.3 Theorems

Theorem 14. *Let a be a modular form mod ℓ. Then $(a|U)^\ell = a - \Theta^{\ell-1}a$.*

Proof. □

Lemma 15. *Let a be a modular form mod ℓ. If $\omega(a) = 0$ then a is constant, i.e. for all $n > 0, a(n) = 0$.*

Theorem 16. *Let a be a modular form mod ℓ and $i \in \mathbb{N}$. Then $\omega(a^i) = i\omega(a)$.*

Definition 17 (Delta)**.** Δ *is the sequence obtained from $q(\prod_{n=1}^{\infty}(1 - q^n))^{24}$. It is a modular form mod ℓ of weight 12.*

Definition 18. *For a prime $\ell \geq 5$, we define $\delta_\ell = \frac{\ell^2-1}{24} \in \mathbb{N}$. We define $f_\ell = \Delta^{\delta_\ell}$, which is a modular form mod ℓ of weight $12\delta_\ell$.*

Lemma 19. $\omega(f_\ell) = 12\delta_\ell = \frac{\ell^2-1}{2}$.

Theorem 20. *This is part (1) of Lemma 2.1.*
Let a be a modular form mod ℓ. Then $\omega(\Theta a) \leq \omega(a) + \ell + 1$.

Theorem 21. *This is part (2) of Lemma 2.1.*
Let a be a modular form mod ℓ. Then $\omega(\Theta a) = \omega(a) + \ell + 1$ if and only if $\ell \nmid \omega(a)$.

Theorem 22. *This is Lemma 3.2.*
For all $m \in \mathbb{N}$, $\omega(\Theta^m f_\ell) \geq \omega(f_\ell) = \frac{\ell^2-1}{2}$.

Theorem 23. *This is part (1) of Lemma 3.3.*
If $\ell \nmid \omega(\Theta^{\ell-1} f_\ell)$ then $\omega(\Theta^{\ell-1} f_\ell) = \frac{\ell^2-1}{2}$.

Proof. □

Theorem 24. *This is part (2) of Lemma 3.3.*
If $\ell \mid \omega(\Theta^{\ell-1} f_\ell)$ then $\omega(f_\ell|U) > 0$.

Proof. □