partitions-leanblueprint

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0.1 Definitions

Definition 1 (Modular Form)**.** In lean, A modular form of weight $k \in \mathbb{N}$ is a function $f : \mathbb{C} \to \mathbb{C}$ such that :

(1) f is holomorphic on $\mathbb{H}$
(2) For all $z \in \mathbb{H}, f(z) = f(z+1)$
(3) For all $z \in \mathbb{H}, f(z) = z^{-k} f(-1/z)$
(4) f is bounded as $\mathrm{Re}(z) \to \infty$

Definition 2 (Integer Modular Form)**.** An integer modular form of weight $k \in \mathbb{N}$ is a sequence $a : \mathbb{N} \to \mathbb{Z}$ such that $\sum_{n=0}^{\infty} a(n)q^n$ is a modular form of weight k, where $q = e^{2\pi i z}$.

Definition 3 (ModularFormMod ℓ)**.** A modular form mod ℓ of weight $k \in \mathbb{Z}/(\ell-1)\mathbb{Z}$ is a sequence $a : \mathbb{N} \to \mathbb{Z}/\ell\mathbb{Z}$ such that there exists an integer modular form b of weight k' where $b \equiv a \mod \ell$ and $k' \equiv k \mod (\ell - 1)$.

Definition 4 (Theta)**.** Θ sends modular forms mod ℓ of weight k to weight $k+2$ by
$(\Theta a)n = na(n)$.

Definition 5 (U Operator)**.** The operator U sends modular forms mod ℓ of weight k to weight k by
$(a|U)n = a(\ell n)$.

Definition 6 (hasWeight)**.** A modular form mod ℓ called a has weight $j \in \mathbb{N}$ if there exists an integer modular form b of weight j such that $b \equiv a \mod \ell$.

Definition 7 (Filtration)**.** The filtration of a modular form mod ℓ called a is defined as the minimum natural number j such that a has weight j. The filtration of the zero function is 0.

Definition 8 (multiplication and exponentiation)**.** It's worth stated how multiplication and exponentiation are defined here, because they are not defined in the normal way. The multiplication of two modular forms mod ℓ called a and b is defined as

$$(a * b)n = \sum_{x+y=n} a(x)b(y).$$

The exponentiation of a modular form mod ℓ called a to the power of $k \in \mathbb{N}$ is defined as

$$(a^j)n = \sum_{x_1+\ldots+x_j=n} \prod_{i=1}^{j} a(x_i).$$

0.2 PowPrime

Definition 9 (permutational equivalence)**.** Two functions $a, b : \text{Fin } n \to \mathbb{N}$, which can be thought of as tuples of n natural numbers, are permutationally equivalent if there exists a bijective function $\sigma : \text{Fin } n \to \text{Fin } n$ such that $a = b \circ \sigma$. This is an equivalence relation.

Lemma 10. *If $x = (x_1, x_2, ..., x_k)$ is not constant (i.e not all x_i are equal) then for any $n \in \mathbb{N}$,*

$$k \mid \#\{y = (y_1, y_2, ..., y_k) : \sum_{i=1}^{k} y_i = n \textit{ and } x \textit{ and } y \textit{ are permutationally equivalent}\}$$

Lemma 11. *If x and y are permutationally equivalent then* $\prod_{i=1}^k a(x_i) = \prod_{i=1}^k a(y_i)$.

Lemma 12. *Let* $x = (x_1, x_2, ..., x_k)$ *and* $n \in \mathbb{N}$. *Suppose that* $\sum_{i=1}^k x_i = n$.
(1) If $k \nmid n$ *then* x *is not constant.*
(2) If $k \mid n$ *and* $x \neq (n/k, ..., n/k)$ *then* x *is not constant.*

Theorem 13. *Let* ℓ *be a prime and* a *be a modular form mod* ℓ *of any weight. Then*

$$(a^\ell)n = \begin{cases} a(n/\ell) & \text{if } \ell \mid n \\ 0 & \text{otherwise} \end{cases}$$