

# partitions-leanblueprint

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## 0.1 Definitions

**Definition 1** (Sequence). A sequence, denoted  $a$  or  $\{a_n\}$ , is a function  $a : \mathbb{N} \rightarrow \mathbb{R}$ .

**Definition 2** (Convergence). A sequence  $\{a_n\}$  converges to  $L \in \mathbb{R}$  if for all  $\varepsilon > 0$  there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|a_n - L| < \varepsilon$ . We say  $\{a_n\}$  converges if there exists  $L \in \mathbb{R}$  such that  $\{a_n\}$  converges to  $L$ .

**Definition 3** (Bounded Above). A set  $S \subseteq \mathbb{R}$  is bounded above if there exists an  $M \in \mathbb{R}$  such that for all  $s \in S$ ,  $s \leq M$ . We say that  $M$  is an upper bound for  $S$ .

**Definition 4** (supremum). Let  $S \subseteq \mathbb{R}$  be non-empty. We say that  $y \in \mathbb{R}$  is the supremum of  $S$  if

- (i)  $y$  is an upper bound for  $S$
- (ii) for any upper bound  $z$  of  $S$ ,  $y \leq z$

**Definition 5** (continuous). A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous if for all  $x \in \mathbb{R}$  and for all  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for all  $y \in \mathbb{R}$  with  $|x - y| < \delta$ ,  $|f(x) - f(y)| < \varepsilon$ .

## 0.2 Theorems

**Theorem 6** (Limit Laws).

*Let  $C \in \mathbb{R}$ . Suppose  $\{a_n\}$  converges to  $L$  and  $\{b_n\}$  converges to  $K$ . Then*

- (i)  $\{Ca_n\}$  converges to  $CL$
- (ii)  $\{a_n + b_n\}$  converges to  $L + K$

*Proof.* Trivial. □

**Lemma 7.**

*Suppose that there exists an  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $a_n \geq 0$ . Then  $\lim_{n \rightarrow \infty} a_n \geq 0$ .*

**Theorem 8** (Order Limit Theorem).

*Let  $\{a_n\}$  and  $\{b_n\}$  be sequences. Suppose that there exists an  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $a_n \leq b_n$ . Then  $\lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n$ .*

**Theorem 9** (Completeness Axiom). *Let  $S \subseteq \mathbb{R}$  be non-empty and bounded above. Then there exists a  $y \in \mathbb{R}$  such that  $y$  is the supremum of  $S$ .*

**Theorem 10** (Intermediate Value Theorem).

*Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous and let  $a < b \in \mathbb{R}$ . Suppose that  $f(a) < f(b)$  and let  $y \in [f(a), f(b)]$ . Then there exists a  $c \in [a, b]$  such that  $f(c) = y$ .*

*Proof.* Sketch:

Let  $K = \{x \in [a, b] | f(x) \leq y\}$ . Then  $K$  is non-empty and bounded above by  $b$ , so it has a supremum  $c \in \mathbb{R}$ . If  $f(c) < y$ , then by continuity of  $f$ , there exists a  $\delta > 0$  such that for all  $x \in (c - \delta, c + \delta)$ ,  $f(x) < y$ . Choose  $x = c + \delta/2$ . Then  $x \in K$  and  $x > c$ , so  $c$  is not an upper bound for  $K$ . If  $f(c) > y$ , then by continuity of  $f$ , there exists a  $\delta > 0$  such that for all  $x \in (c - \delta, c + \delta)$ ,  $f(x) > y$ . Choose  $x = c - \delta/2$ . Then  $x$  is an upper bound for  $K$  and  $x < c$ , so  $c$  is not the supremum of  $K$ . Thus,  $f(c) = y$ . □

## 0.3 Modular Forms

**Definition 11** (Modular Form). In lean, A modular form of weight  $k \in \mathbb{N}$  is a function  $f : \mathbb{C} \rightarrow \mathbb{C}$  such that :

- (1)  $f$  is holomorphic on  $\mathbb{H}$
- (2) For all  $z \in \mathbb{H}$ ,  $f(z) = f(z + 1)$
- (3) For all  $z \in \mathbb{H}$ ,  $f(z) = z^{-k} f(-1/z)$
- (4)  $f$  is bounded as  $\text{Re}(z) \rightarrow \infty$

**Definition 12** (Integer Modular Form). An integer modular form of weight  $k \in \mathbb{N}$  is a sequence  $a : \mathbb{N} \rightarrow \mathbb{Z}$  such that  $\sum_{n=0}^{\infty} a(n)q^n$  is a modular form of weight  $k$ , where  $q = e^{2\pi iz}$ .