partitions-leanblueprint

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0.1 Definitions

Definition 1 (Sequence)**.** A sequence, denoted a or $\{a_n\}$, is a function $a : \mathbb{N} \to \mathbb{R}$.

Definition 2 (Convergence)**.** A sequence $\{a_n\}$ converges to $L \in \mathbb{R}$ if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - L| < \varepsilon$. We say $\{a_n\}$ converges if there exists $L \in \mathbb{R}$ such that $\{a_n\}$ converges to L.

Definition 3 (Bounded Above)**.** A set $S \subseteq \mathbb{R}$ is bounded above if there exists an $M \in \mathbb{R}$ such that for all $s \in S$, $s \leq M$. We say that M is an upper bound for S.

Definition 4 (supremum)**.** Let $S \subseteq \mathbb{R}$ be non-empty. We say that $y \in \mathbb{R}$ is the supremum of S if

(i) y is an upper bound for S
(ii) for any upper bound z of S, $y \leq z$

Definition 5 (continuous)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if for all $x \in \mathbb{R}$ and for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $y \in \mathbb{R}$ with $|x - y| < \delta$, $|f(x) - f(y)| < \varepsilon$.

0.2 Theorems

Theorem 6 (Limit Laws)**.**
Let $C \in \mathbb{R}$. Suppose $\{a_n\}$ converges to L and $\{b_n\}$ converges to K. Then

(i) $\{Ca_n\}$ converges to CL
(ii) $\{a_n + b_n\}$ converges to $L + K$

Proof. Trivial. □

Lemma 7.
Suppose that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n \geq 0$. Then $\lim_{n\to\infty} a_n \geq 0$.

Theorem 8 (Order Limit Theorem)**.**
Let $\{a_n\}$ and $\{b_n\}$ be sequences. Suppose that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n \leq b_n$. Then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

Theorem 9 (Completeness Axiom)**.** *Let $S \subseteq \mathbb{R}$ be non-empty and bounded above. Then there exists a $y \in \mathbb{R}$ such that y is the supremum of S.*

Theorem 10 (Intermediate Value Theorem)**.**
Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and let $a < b \in \mathbb{R}$. Suppose that $f(a) < f(b)$ and let $y \in [f(a), f(b)]$. Then there exists a $c \in [a, b]$ such that $f(c) = y$.

Proof. Sketch:
Let $K = \{x \in [a, b] | f(x) \leq y\}$. Then K is non-empty and bounded above by b, so it has a supremum $c \in \mathbb{R}$. If $f(c) < y$, then by continuity of f, there exists a $\delta > 0$ such that for all $x \in (c-\delta, c+\delta)$, $f(x) < y$. Choose $x = c+\delta/2$.Then $x \in K$ and $x > c$, so c is not an upper bound for K. If $f(c) > y$, then by continuity of f, there exists a $\delta > 0$ such that for all $x \in (c-\delta, c+\delta)$, $f(x) > y$. Choose $x = c - \delta/2$. Then x is an upper bound for K and $x < c$, so c is not the supremum of K. Thus, $f(c) = y$.

□