Quiz 1 via Lockdown Browser

When: (Week 5) Tuesday, **February 11 at 4:30pm**

Where: **LT2A, LT1**

Refer to "Rules & Quiz Details" under "Assignments-Quiz" in NTULearn for more info.

Important: **You must do the Practice Quiz (TBA) at least once** before the actual Quiz.

If you have problem doing the online quiz with your own equipment, email me by **Friday, February 7**.

Ch 5. Probability

- Basic Concepts – Review of Set Theory and Venn Diagram
- Counting – Ordered, Unordered, Sampling With and Without Replacement.
- Probability Theory
- Base Rate: Bayes' Theorem

Set Theory - Review

A set is a collection of elements.

Eg: $A = \{Head, Tail\}$, $Head \in A$, $Tail \in A$

$B = \{1, 2, 3, 4, 5, 6\}$

$C = \{1, 3, 5\}$

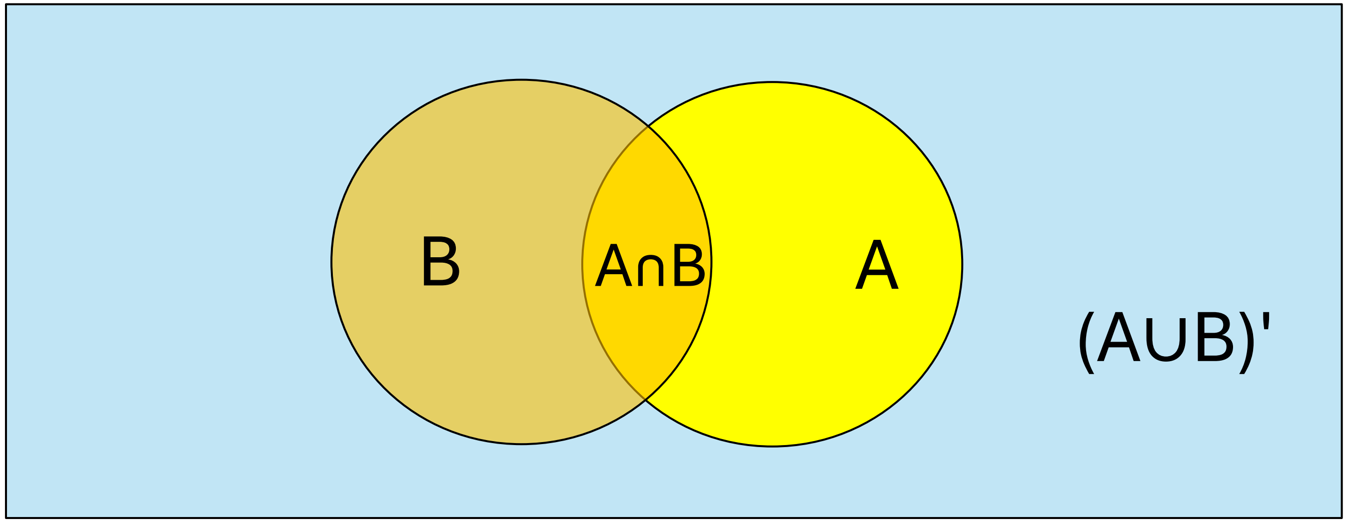
Set C is a subset of Set B since all elements in C are also in B. Notation: $C \subset B$

The set of all elements in a given context is known as an universal set $\boldsymbol{\Omega}$.

Null set $\emptyset$ is a set with no element.

- # Venn Diagram

universal set $\boldsymbol{\Omega}$



Set A $\subset$ $\boldsymbol{\Omega}$. Complement of A, denoted by A', consists of all elements in $\boldsymbol{\Omega}$ that are not in A.
Eg: $\boldsymbol{\Omega}$ = {1,2,3,4,5,6}, A={1,2}, A'={3,4,5,6}

Union of A and B consists of all elements in A or B. Notation: A ∪ B.
Eg: {1,2,3} ∪ {2,3,4} = {1,2,3,4}

Intersection of A and B consists of all elements in both A and B. Notation: A ∩ B.

Eg: {1,2,3} ∩ {2,3,4} = {2,3}

Consider rolling a die. Let the outcome set $A=\{1,2,3\}$ and set $B=\{2,4,6\}$. Match the following:
https://app.wooclap.com/AGLILC?from=instruction-slide

- Counting

When solving probability problems, we may involve counting.

Eg: Calculating the probability of guessing the correct 4 digits numeric code.

Each digit is sampled from {0,1,2,...,9}.

X1	X2	X3	X4

$Xi \in \{0,1,2,...,9\}$

- Counting

Different scenarios for consideration:

- sampling with replacement

3	6	7	9	11	14

Sample: 7,6,11,6,3…

- sampling without replacement

Sample: 7,6,11,3…

3	6	7	9	11	14

- digits drawn are ordered

$(7,6,11,3)\neq(6,7,3,11)$

- digits drawn are unordered

$\{7,6,11,3\}=\{6,7,3,11\}$

▪ Counting – general scenario

Sampling	Eg: Pick 3 numbers (one at a time) from a set of 10 digits. Arrangement can be ordered or unordered	Two options: with or without replacement. After picking each number from the set, the number can be replaced or no replacement.

- Counting – ordered sampling with replacement

Eg: Calculate the total number of different 4-digit numeric codes. Each digit is independently selected from {0,1,2,...,9}.

X1	X2	X3	X4

10 x 10 x 10 x 10 = 10^4

- Counting – ordered sampling with replacement

Ordered sampling k elements from a set of n elements with replacement.

slot	1	2	3	...			k
cases	n	n	n	...			n

No. of possible arrangements = n^k

Eg: Calculate the total number of different 4-digit numeric codes. Each digit is independently selected from {0,1,2,...,9}.

10 possible numbers for each digit

Total arrangements = 10 x 10 x 10 x 10 = 10^4

- Counting – ordered sampling without replacement

slot	1	2	3	…			k
cases	n	n-1	n-2	…			n-k+1

Ordered sampling k elements from a set of n elements without replacement.

No. of possible arrangements =

n x (n – 1) x (n – 2) x (n – 3) x … x (n – k +1)

Number of k permutations of n elements:

$$^{n}\mathrm{P}_{k} = \frac{n!}{(n-k)!}$$

Where n!=n*(n-1)*…*2*1

Eg: Calculate the total number of different 4-digit numeric codes. Each digit is sampled from $\{0,1,2,\ldots,9\}$ without replacement.

1^{st} digit: 10 possible numbers

2^{nd} digit: 9 possible numbers

3^{rd} digit: 8 possible numbers and so on.

Total arrangements = 10 x 9 x 8 x 7

i.e. there are ${}^{10}P_4 = \frac{n!}{(n-k)!} = \frac{10!}{(10-4)!}$ arrangements

Eg: 4 boys and 2 girls sit in a row. Find the following:

(i) No. of ways of putting these 6 (distinct) people in a row.

ways=6*5*4*3*2*1=6!=720

(ii) No. of ways such that each girl has a boy to her left and to her right.

slot	1	2	3	4	5	6
B/G		G			G	
		G		G		
			G		G	

#ways=3*2!*4!=3*6*24=432

Order doesn't matter

#ways=3

- Counting – unordered sampling without replacement

We choose k elements from a set of n elements and the ordering does not matter. → Divide by k!

Number of arrangements equals to k combinations of n elements

$$= \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Note: $\binom{n}{k}$

$$= \frac{n!}{k!\,(n-k)!}$$

$$= \binom{n}{n-k}$$

Eg: How many combinations of two numbers between 1 and 6 are there:

Ans: $\frac{6!}{2!\,(6-2)!} = \frac{6 \times 5}{2 \times 1} = 15$

- Counting – unordered sampling with replacement

Pick k elements from a set of n elements, one at a time with replacement and the ordering does not matter.

Eg: Pick two numbers from {1, 2, 3}, we have

6 arrangements with n=3 and k=2.

i.e. (1,1), (1,2), (1,3), (2,2), (2,3) and (3,3)

*|****|*****|**||*|****
k samples: "*"
n-1 dividers: "|"

In general:

No. of arrangements = $\binom{n+k-1}{k}$

Equal to #solutions of
X1+...+Xn=k

Xi: number of time element i is sampled

Unordered Sampling with Replacement | Samples | Counting Methods

Eg: A 6-bit code is made up of 4 1’s and 2 0’s.
Calculate:

(i) the no. of distinct codewords in the code?

- Equal to #subsets of length 2: $\binom{6}{2}$ = 15

(i) the no. of codewords such that a ‘0’ has a ‘1’ to its left and to its right.

slot	1	2	3	4	5	6
0/1	1	0	1	1	0	1
	1	0	1	0	1	1
	1	1	0	1	0	1

#codewords = 3

- Probability Theory

In a random experiment, the outcome of the experiment occurs with a certain probability.

Eg: We toss a coin. If the coin is unbiased, then the chances of getting Heads is 50%.

Eg: We roll a fair die two times. The outcome $\in$ {11, 12, 13, 14, 15, 16, 21, 22, … , 65, 66}.

Total of 36 possible outcomes.

Probability of getting a “1” followed by another “1” is 1/36.

Probability of getting 1st no. = 2nd no. is 6/36.

Definitions

- A sample space is the set S of all possible outcomes of an experiment.
- An event is a set of one or more (favorable) outcomes in the sample space.
- Two events are mutually exclusive if they have no outcomes in common.
- They are exhaustive if they cover all possible outcomes.
- Two events are independent if the probability that one occurs is NOT affected by whether or not the other has occurred.

Ch 5. Probability

- Basic Concepts – Review of Set Theory and Venn Diagram
- Counting – Ordered, Unordered, Sampling With and Without Replacement.
- Probability Theory
- Base Rate: Bayes' Theorem

Axioms of Probability:

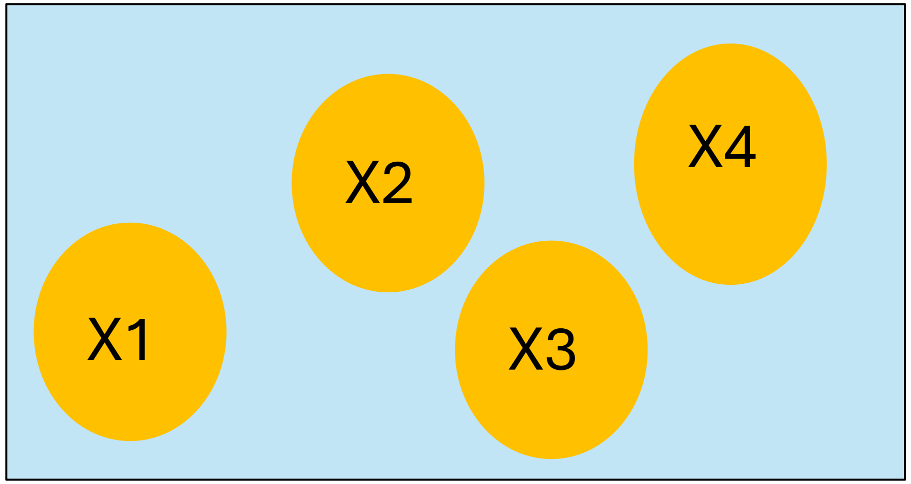
(1) Probability of any outcome or event X is a non-negative:

$$P(X) \geq 0$$

(2) Probability of the sample space S is 1:

$$P(S) = 1$$

Venn diagram



(3) If X_1, X_2, X_3, ... are mutually exclusive events, then:

$$P(X_1 \text{ or } X_2 \text{ or } X_3 \ldots) = P(X_1) + P(X_2) + P(X_3) + \ldots$$

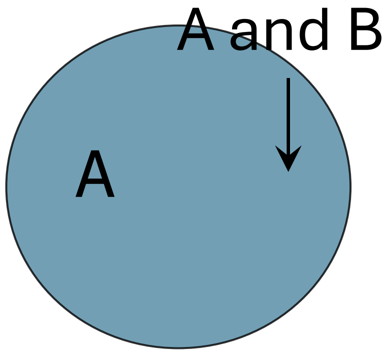
Probabilities and Events (mutually exclusive)

Eg: A company has decided that in the next 5 years, 40% of their new employees will be men, 30% will be Singaporean, and 35% will be foreigner women. What percentage of new employees will be Singaporean men?

	Men	Women	Total
Foreigner		0.35	
Singaporean	$0.3-0.25$ =0.05	$0.6-0.35$ =0.25	0.3
Total	0.4	$1-0.4$ =0.6	

Probabilities and Events (non-mutually exclusive)

Eg: A certain kind of fruit is grown in 2 districts, A and B. Both areas sometimes get fruitflies. Suppose the probabilities are P(A)=0.1, P(B)=0.05 and P(A and B)=0.02, what is the probability that one or other (or both) districts are infected at a given time?

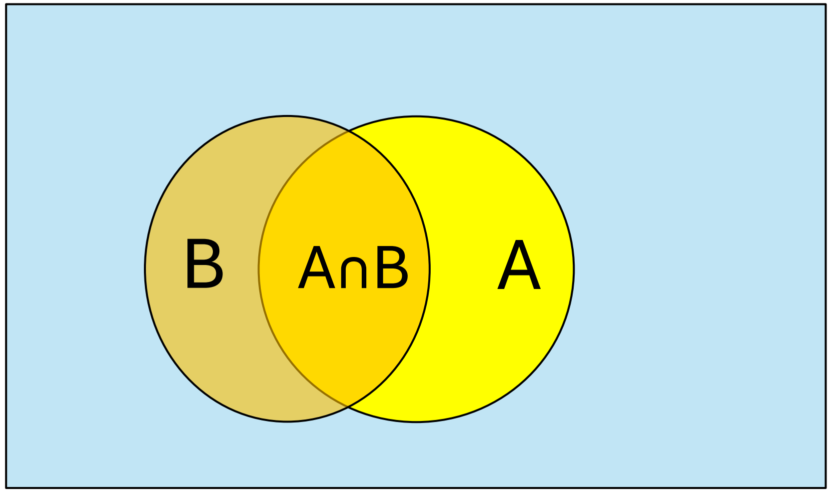


P(A or B) = P(A) + P(B) – P(A and B)
= 0.1 + 0.05 – 0.02
= 0.13

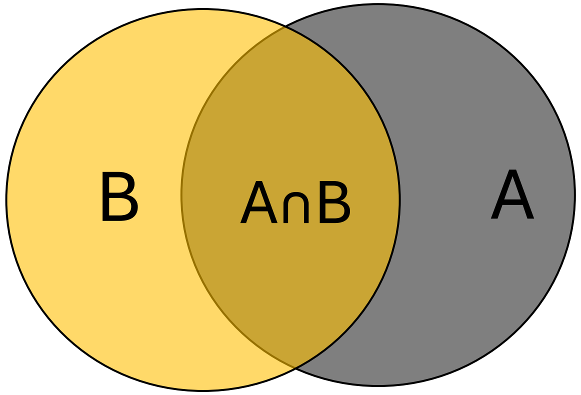
Probabilities and Events (independent)

Eg: A company has 2 guards. Each carries a pager activated by sensors. Guard 1 and guard 2 respond to pager alert 80% and 50% of the time respectively. They independently report any alert. What is the probability that at least one will report an alert?

universal set $\boldsymbol{\Omega}$



universal set $\boldsymbol{\Omega}'$=B



$P(A)/P(\Omega)=P(A \text{ and } B)/P(B)$

$P(A \text{ and } B) = P(A) * P(B) = 0.8*0.5=0.4$

Worked Examples considered so far:

Probabilities and Events (mutually exclusive)

Probabilities and Events (non-mutually exclusive)

Probabilities and Events (independent)

Next: Consider Conditional Probabilities

i.e. Probability of an event A given that an event B has occurred, denoted as $P(A|B)$

Conditional Probabilities

Assuming that 10 percent of the days are rainy in a certain city.

$$P(\text{rain}) = 0.1$$

The probability that it rains given that it is cloudy might be, say,

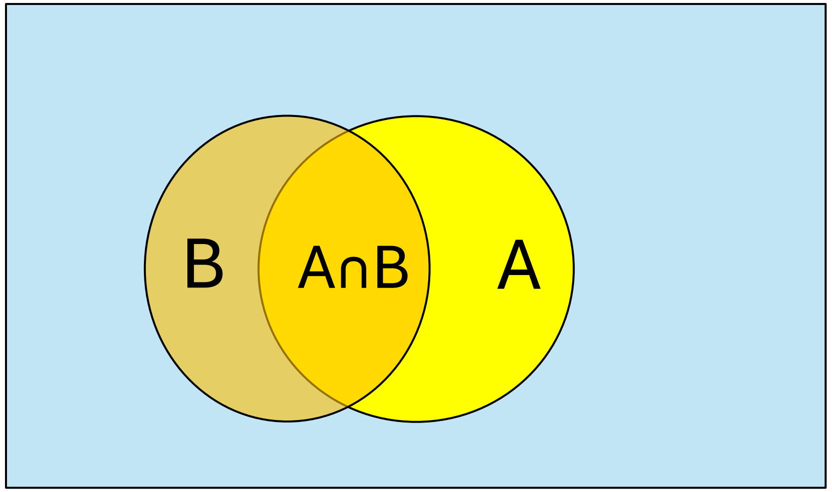
$$P(\text{rain}|\text{cloudy}) = 0.8$$

This is known as conditional probability: the probability of event A given that event B has occurred.

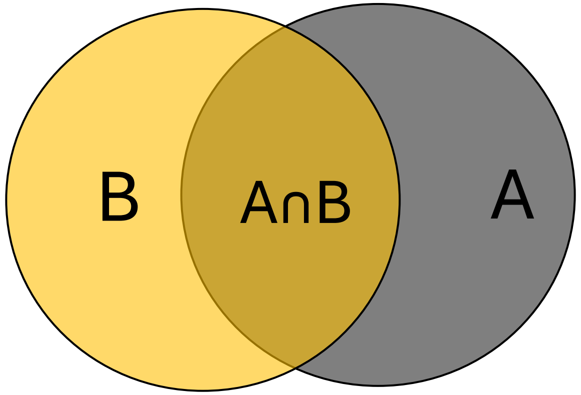
$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{where } B > 0$$

Independent events: P(A|B)=P(A)

universal set $\boldsymbol{\Omega}$



universal set $\boldsymbol{\Omega}$'=B



P(A|B)=P(A ∩ B)/P(B)

Conditional Probabilities

Given P(A)=0.5 and P(B)=0.8, determine the following.

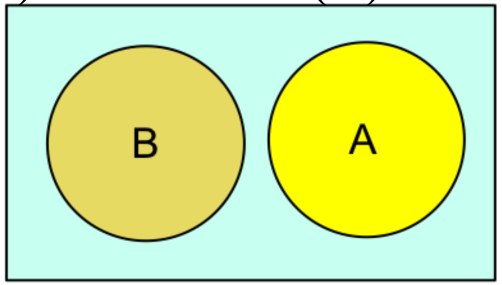


Fig 1.

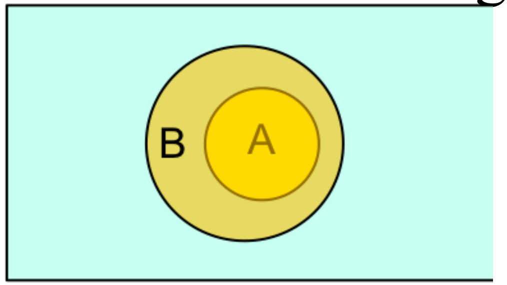


Fig 2.

(1) $P(A|B)$ for Fig 1. → 0

(2) $P(A|B)$ for Fig 2. → 0.5/0.8=0.625

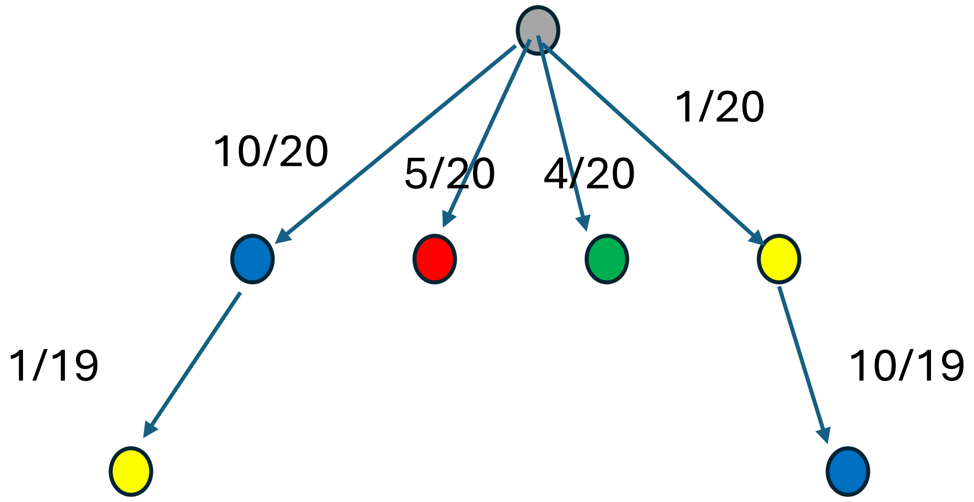
(3) $P(B|A)$ for Fig 2. → 1

Conditional Probabilities

Eg: A jar contains 10 blue, 5 red, 4 green and 1 yellow marbles. Two marbles are randomly picked. What is the probability that one will be blue and the other yellow?

$P(A)=(10/20)*(1/19)+(1/20)*(10/19)=20/380=1/19$

Illustration using a tree diagram:

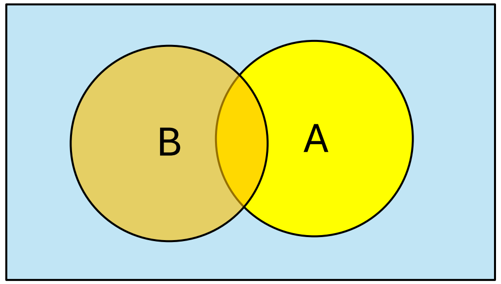


Bayes' Theorem

Given two events A and B, where P(A) > 0, we have:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A|B)P(B)}{P(A)}$$

Note: $P(A) = P(A \cap B) + P(A \cap B')$



Similarly:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$$

Bayes' Theorem

Eg: A test correctly identifies a disease in 95% of people who have it. It correctly identifies no disease in 94% of people who do not have it. In the population, 3% of the people have the disease. What is the probability that one has the disease if tested positive?

33%

Event A: A given patient has the disease
Event B: The test is positive for the patient

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$$

=0.95*0.03/0.0867~= 0.33

- $P(B|A)=0.95$
- $P(B'|A')=0.94$
- $P(A)=0.03$

- $P(B|A')=0.06$
- $P(A')=0.97$

- $P(B)=P(B \cap A)+P(B \cap A')=$
  $=P(A)*P(B|A)+P(A')*P(B|A')$
  $= 0.03*0.95+0.97*0.06$
  $=0.0285+0.0582 = 0.0867$

Summary For Probability Calculations

Are X and Y mutually exclusive?

Yes

1. P(X and Y) = 0 because event (X and Y) is impossible
2. P(X or Y) = P(X) + P(Y)

No

1. P(X and Y) > 0
2. P(X or Y) = P(X) + P(Y) – P(X and Y)

Are X and Y independent?

Yes

1. $P(X|Y) = P(X)$
2. $P(X \text{ and } Y) = P(X)\,P(Y)$

No

1. $P(X|Y) = \dfrac{P(X \text{ and } Y)}{P(Y)}$
2. $P(X \text{ and } Y) = P(X|Y)\,P(Y)$
   $= P(Y|X)\,P(X)$