# Quiz 2 practice questions solutions

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## 1 Solutions/answers

**Question 1** The number of iphones sold per day is a Poisson random variable. What is the average number of iphone sold per day if the probability of selling at least 1 iphone per day is a?

The p.m.f of a Poisson r.v. $X$ with mean $\mu$ is:

$$P(X = x) = \frac{e^{-\mu} \cdot \mu^x}{x!}$$

**solution/answer** We have that: $P(X \geq 1) = 1 - P(X = 0) = a$.
Thus,

$$1 - \frac{e^{-\mu} \cdot \mu^0}{0!} = a$$
$$e^{-\mu} = 1 - a$$
$$\mu = \ln\left(\frac{1}{1-a}\right) = \ln\left(\frac{1}{1-a}\right)$$

**Question 2** The circumference of a circle is cut at 2 points which are selected uniformly at random. What is the probability that the longer arc is at least [N] times the length of the shorter arc?

**solution/answer** Fix the location of the first point and let the circumference of the circle be $\ell$. The 2 points split the circle into 2 arcs of length $\ell_1, \ell_2$, such that $\ell_1 + \ell_2 = \ell$. Without loss of generality, suppose that $\ell_1 < \ell_2$. We have that:

$$\ell_2 > N \cdot \ell_1$$
$$\ell_1 + \ell_2 > (N + 1) \cdot \ell_1$$
$$\ell_1 < \frac{\ell}{N + 1}$$

Thus, the length $\ell_1$ of the arc between the 2 points has to be a most $\frac{\ell}{N+1}$, which happens with probability $\frac{2}{N+1}$ (since the second point can be either to the left or to the right of the first one).

**Question 3** The probability that a student fails a certain test is [p]. Ten students take the test. Determine the probability that at most 1 student will fail the test.

**solution/answer** The number of students who fail the test is distributed according to the Binomial distribution $B(p, 10)$, which has the following probability mass function:

$$P(X = i) = \binom{10}{i} p^i (1-p)^{10-i}$$

Thus, we have:

$$P(X \leq 1) = P(X = 0) + P(X = 1) = (1-p)^{10} + \binom{10}{1} p(1-p)^9$$

**Question 4** Consider a 90 minutes soccer match between teams A and B. The number of goals scored by each team is modeled by an independent Poisson process. Given that the average number of goals per minute scored by team A and B are [a] and [b] respectively, what is the probability that Team A won the match with the final score 2-0?

**solution/answer** Let $G_A$ and $G_B$ be the amount of goals scored in the game by team A and B respectively. Since the above variables follow the Poisosn process we have that:

$$P(G_A = k) = \frac{(a \cdot t)^k \cdot e^{-a \cdot t}}{k!}$$

and

$$P(G_B = k) = \frac{(b \cdot t)^k \cdot e^{-b \cdot t}}{k!}$$

We need to compute the following probability:

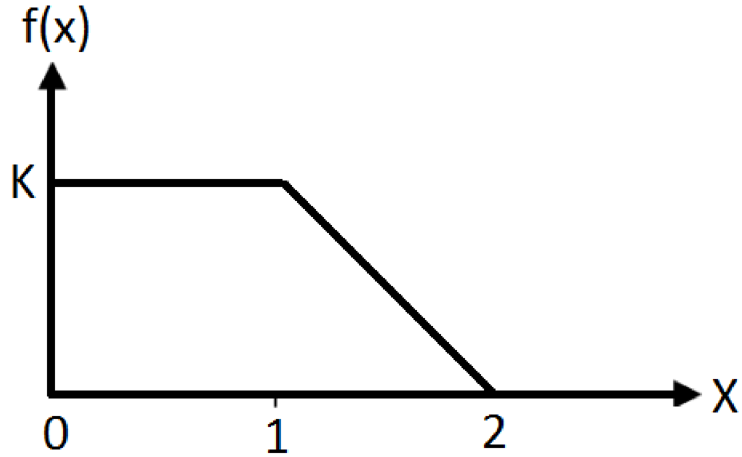
$$P(\{G_A = 2\} \cap \{G_B = 0\}) = P(\{G_A = 2\}) \cdot P(\{G_B = 0\})$$

So, for $t = 90$, we get:

$$P(\{G_A = 2\} \cap \{G_B = 0\}) = \frac{(90 \cdot a)^2 \cdot e^{-90 \cdot a}}{2!} \cdot \frac{(90 \cdot b)^0 \cdot e^{-90 \cdot b}}{0!}$$

$$P(\{G_A = 2\} \cap \{G_B = 0\}) = \frac{(90 \cdot a)^2 \cdot e^{-90 \cdot a}}{2!} \cdot e^{-90 \cdot b}$$

**Question 5** Figure below shows the probability density function f(x) of the continuous random variable X that takes values between 0 and 2.




In text form:
$f(x) = K$ for $0 \leq x \leq 1$, where K is a constant
$f(x) = 2K - Kx$ for $1 \leq x \leq 2$
Determine the following:
(1) The value of K is [A]
(2) The probability that X = 1.5 is [C]

**solution/answer** To determine the value of $K$, note that the area below the graph of the probability density function is equal to 1. The shape is a trapezoid and can be split into a rectangle for $0 \leq x \leq 1$ with area $1 \cdot K = K$ and a triangle for $1 \leq x \leq 2$ with area $\frac{1 \cdot K}{2} = K/2$. Thus, we conclude that:

$$K + K/2 = 3K/2 = 1$$

$$K = 2/3 = 0.667$$

For the probability of $X = 1.5$, we know that $P(X = 1.5) \int_{1.5}^{1.5} f(x)dx = 0$, since the event contains a single value.

**Question 6** Consider a random variable X with variance = [a]. Given Y = [b] X + 5, determine the Cov(X,Y).

**solution/answer** We have the following:

$$Cov(X,Y) = E[(X - \mu_X) \cdot (Y - \mu_Y)] = E[(X - \mu_X) \cdot (b \cdot X + 5 - (b \cdot \mu_X + 5))]$$

$$Cov(X,Y) = E[b \cdot (X - \mu_X)^2] = b \cdot E[(X - \mu_X)^2] = b \cdot Var(X) = b \cdot a$$

**Question 7** A component factory has six Class $A$ and four Class $B$ machines that produce certain transistors. Each transistor is independently produced by the machines. The probability that a Class $A$ machine produces a defective transistor is $p_A$. For Class $B$ machine, the probability of getting a defective transistor is $p_B$.

A machine is randomly selected and 10 transistors produced by the machine are inspected. Out of the 10 transistors inspected, 2 are found to be defective. What is the probability that the machine selected is a Class $A$ machine?

**solution/answer** We define the events:

- A: "The transistors were produced by a class A machine"
- B: "The transistors were produced by a class B machine"

and the random variable:

- D: "The number of defective transistors produced"

Since there are 6 class $A$ machines and 4 class $B$ machines, we have for the initial machine selection that: $P(A) = \frac{6}{10} = 0.6$ and $P(B) = \frac{4}{10} = 0.4$.

We now need to compute: $P(A|D = 2)$. By applying Bayes theorem, we have the following:

$$P(A|D = 2) = \frac{P(D = 2|A) \cdot P(A)}{P(D = 2)} = \frac{\binom{10}{2} p_A^2 (1 - p_A)^8 \cdot 0.6}{P(D = 2)}$$

where

$$P(D = 2) = P(\{D = 2\} \cap A) + P(\{D = 2\} \cap B)$$

by applying the formula $P(X|Y) = \frac{P(X \cap Y)}{P(Y)}$ for each term, we get:

$$P(D = 2) = P(D = 2|A) \cdot P(A) + P(D = 2|B) \cdot P(B)$$

$$P(D = 2) = \binom{10}{2} p_A^2 (1 - p_A)^8 \cdot 0.6 + \binom{10}{2} p_B^2 (1 - p_B)^8 \cdot 0.4$$

Substituting above, we get:

$$P(A|D = 2) = \frac{\binom{10}{2} p_A^2 (1 - p_A)^8 \cdot 0.6}{\binom{10}{2} p_A^2 (1 - p_A)^8 \cdot 0.6 + \binom{10}{2} p_B^2 (1 - p_B)^8 \cdot 0.4}$$

**Question 8** Two boys play a game where each rolls a fair die concurrently and repeatedly until the sum of the 2 numbers is more than 8. What is the probability that they roll the dice exactly [A] times?

**solution/answer** The number of times follows a geometric distribution: $A \sim G(p)$, where $p$ is the probability of success in each pair of concurrent rolls. To compute $p$, we count the number of successful pairs of rolls (having a sum $> 8$) out of the 36 possible ones.

We observe that the following 10 pairs of rolls satisfy the requirement: $(3,6),(4,5),(5,4),(6,3),(4,6),(5,5),(6,4),(5,6),(6,5),(6,6)$. Thus, $p = 10/36$ and the probability is as follows:

$$P(X = A) = (1-p)^{A-1} \cdot p = \left(\frac{26}{36}\right)^{A-1} \cdot \frac{10}{36}$$

**Question 9** The probability density function f(x) of a continuous random variable X that takes values between 0 and 5 is given by:

$$f(x) = \begin{cases} \frac{2x}{25}, & \text{for } 0 < x < 5 \\ 0, & \text{otherwise} \end{cases}$$

Determine the [K]th percentile of X.

**solution/answer** To compute the $K$-th percentile ($P_K$), we write:

$$\int_{-\infty}^{P_K} f(x) = \int_0^{P_K} \frac{2x}{25} = \left[\frac{x^2}{25}\right]_0^{P_K} = \frac{K}{100}$$

Thus, we get:

$$\frac{P_K^2}{25} = \frac{K}{100}$$
$$P_K = \frac{\sqrt{K}}{2}$$

**Question 10** A biased coin of which the probability of getting a Head equals [p] is repeatedly flipped until either Head or Tail has occurred twice. Determine the expected number of flips.

**solution/answer** Let F be the discrete random variable describing the number of flips. We observe that after 3 flips, the condition of either Heads or Tails having appeared twice will be satisfied (pigeonhole principle). Thus,

$$P(F \leq 3) = 1$$

The only way the number of flips is less than 3, is when the first 2 flips are either both HEADS or both TAILS. Thus,

$$P(F = 2) = p^2 + (1-p)^2$$

Also, observe that: $P(F = 1) = 0$.

Therefore, we have:

$$E(F) = 2 \cdot P(F=2) + 3 \cdot P(F=3) = 2 \cdot (p^2 + (1-p)^2) + 3 \cdot (1 - p^2 - (1-p)^2)$$
$$E(F) = 3 - p^2 - (1-p)^2$$

**Question 11** A bag contains 2 fair coins and 1 biased coin of which the probability of getting the outcome "Head" is 0.75.

(1) You randomly pick a coin and toss it. The probability of getting a "Head" is [A].

(2) You randomly pick a coin and toss it, and get a "Head". The probability that it is the biased coin is [B].

**solution/answer**

1. We have: $P(HEADS) = P(HEADS|Fair)P(Fair) + P(HEADS|Biased)P(Biased)$. Thus,

$$P(HEADS) = \frac{1}{2} \cdot \frac{2}{3} + 0.75 \cdot \frac{1}{3} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} = 0.583$$

2. Using Bayes theorem, we get:

$$P(Biased|HEADS) = \frac{P(HEADS|Biased) \cdot P(Biased)}{P(HEADS)}$$

$$P(Biased|HEADS) = \frac{0.75 \cdot \frac{1}{3}}{7/12} = \frac{12 \cdot 0.25}{7} = \frac{3}{7} = 0.428$$

**Question 12** The probability density function of X is

$$f(x) = \begin{cases} e^x, & \text{for } x < 0 \\ 0, & \text{otherwise} \end{cases}$$

Determine the [K]th percentile of X.

**solution/answer** To compute the $K$-th percentile ($P_K$), we write:

$$\int_{-\infty}^{P_K} f(x) = \int_{-\infty}^{P_K} e^x = [e^x]_{-\infty}^{P_K} = e^{P_K}$$

Thus, we get:

$$e^{P_K} = \frac{K}{100}$$
$$P_K = \ln\left(\frac{K}{100}\right)$$

**Question 13** Two fair dice are rolled. What is the probability that the sum of the two outcomes is divisible by 3?

**solution/answer** The possible outcomes that sum to a multiple of 3 are: $3, 6, 9, 12$ and the possible pairs or rolls:

$$(1,2),(2,1),(1,5),(2,4),(3,3),(4,2),(5,1),(3,6),(4,5),(5,4),(6,3),(6,6)$$

Thus the probability is $\frac{12}{36} = 0.333$.

**Question 14** The cumulative distribution function F(x) of a random variable X is given by:

$$F(x) = \begin{cases} 1 - e^{-\frac{x-\alpha}{2}}, & \text{for } x > \alpha \\ 0, & \text{otherwise} \end{cases}$$

Determine the mean of X.

**solution/answer** For the mean of $X$, we need to compute the following integral:

$$\int_{-\infty}^{\infty} x f(x) dx =$$

where $f(x)$ is the probability density function, which is obtained by taking the derivative of $F(x)$ as follows:

$$f(x) = \frac{d}{dx} F(x) = \begin{cases} \frac{1}{2} e^{-\frac{x-\alpha}{2}}, & \text{for } x > \alpha \\ 0, & \text{otherwise} \end{cases}$$

We observe that the random variable $Y = X - \alpha$ has the following density function:

$$g(y) = \begin{cases} \frac{1}{2} e^{-\frac{y}{2}}, & \text{for } y > 0 \\ 0, & \text{otherwise} \end{cases}$$

which is an exponential distribution with parameter $\lambda = \frac{1}{2}$. Thus,

$$E[Y] = \frac{1}{\lambda} = 2$$

$$E[X] = E[Y + \alpha] = E[Y] + \alpha = 2 + \alpha$$

**Question 15** Given that X is a continuous random variable with probability density function $f_X(x) > 0$ for any $-\infty < x < \infty$, what is the probability of $X = b$ for some $b \in R$

**solution/answer** The probability is $\int_b^b f(x)dx = 0$, since the event only contains a single value rather than an inteval.

**Question 16** Given that a continuous random variable X is uniformly distributed between [A] and [B], what is the variance of X?

**solution/answer** The probability density function is the following:

$$f(x) = \begin{cases} \frac{1}{B-A}, & \text{for } A < x < B \\ 0, & \text{otherwise} \end{cases}$$

The variance is given by the following integral

$$Var(X) = E[X^2] - (E[X])^2$$

We also have that:

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_A^B \frac{x^2}{B-A} = \frac{1}{B-A}\left[\frac{x^3}{3}\right]_A^B = \frac{A^2+AB+B^2}{3}$$

while

$$E[X] = \int_{\infty}^{\infty} xf(x)dx = \int_A^B \frac{x}{B-A} = \frac{1}{B-A}\left[\frac{x^2}{2}\right]_A^B = \frac{1}{B-A}\frac{B^2-A^2}{2} = \frac{A+B}{2}$$

Thus, we have:

$$Var(X) = E[X^2]-(E[X])^2 = \frac{A^2+AB+B^2}{3} - \frac{(A+B)^2}{4} = \frac{4A^2+4AB+4B^2}{12} - \frac{3(A^2+2AB+B^2)}{12}$$

$$Var(X) = \frac{A^2-2AB+B^2}{12} = \frac{(A-B)^2}{12}$$

**Question 17** Given that $X \sim U(0,10)$ and $Y = 0.5X + [b]$. Determine $P(X > 1|Y < [c])$.

**solution/answer** The variable $X$ has the following pdf:

$$f(x) = \begin{cases} \frac{1}{10}, & \text{for } 0 < x < 10 \\ 0, & \text{otherwise} \end{cases}$$

We have:

$$P(X > 1|Y < c) = P(X > 1|0.5X + b < c) = P(X > 1|X < 2(c-b))$$

Since $P(X < 10) = 1$, if $2(c-b) > 10$, we get that:

$$P(X > 1|Y < c) = P(X > 1|X < 2(c-b)) = P(X > 1) = (10-1)\cdot\frac{1}{10} = 0.9$$

If $2(c-b) > 10$, we get that:

$$P(X > 1|Y < c) = P(X > 1|X < 2(c-b)) = \frac{P(\{X > 1\} \cap \{X < 2(c-b)\})}{P(X < 2(c-b))}$$

$$P(X > 1|Y < c) = \frac{P(1 < X < 2(c-b))}{P(0 < X < 2(c-b))} = \frac{2(c-b)-1}{2(c-b)}$$

**Question 18** In a certain district, there are [a] households with 1 member, [b] households with 2 members, [c] households with 3 members and [d] households with 4 members. A person in the district is randomly selected. Let X be the number of people in the household where the selected person lives. Determine the expected value of X.

**solution/answer** The total number of people in the district is: $\alpha \cdot 1 + b \cdot 2 + c \cdot 3 + d \cdot 4 = a + 2b + 3c + 4d$ people. The expected value is given by the following formula:

$$E[X] = \sum_{i=1}^{N} p_i x_i$$

$$E[X] = 1 \cdot \frac{a}{a+2b+3c+4d} + 2 \cdot \frac{2b}{a+2b+3c+4d} + 3 \cdot \frac{3c}{a+2b+3c+4d} + 4 \cdot \frac{4d}{a+2b+3c+4d}$$

**Question 19** Given that X is a normally distributed random variable, $X \sim N(3, 2)$, what is the probability that X is within 40% of a standard deviation from the mean?

You are required to use the following standard normal distribution table for $P(Z < z), 0 \leq z \leq 2.08$ to determine your answer.

| $z$ | 0.00 | 0.02 | 0.04 | 0.06 | 0.08 |
|---|---|---|---|---|---|
| 0.00 | 0.5000 | 0.5080 | 0.5160 | 0.5239 | 0.5319 |
| 0.20 | 0.5793 | 0.5871 | 0.5948 | 0.6026 | 0.6103 |
| 0.40 | 0.6554 | 0.6628 | 0.6700 | 0.6772 | 0.6844 |
| 0.60 | 0.7257 | 0.7324 | 0.7389 | 0.7454 | 0.7517 |
| 0.80 | 0.7881 | 0.7939 | 0.7995 | 0.8051 | 0.8106 |
| 1.00 | 0.8413 | 0.8461 | 0.8508 | 0.8554 | 0.8599 |
| 1.20 | 0.8849 | 0.8888 | 0.8925 | 0.8962 | 0.8997 |
| 1.40 | 0.9192 | 0.9222 | 0.9251 | 0.9279 | 0.9306 |
| 1.60 | 0.9452 | 0.9474 | 0.9495 | 0.9515 | 0.9535 |
| 1.80 | 0.9641 | 0.9656 | 0.9671 | 0.9686 | 0.9699 |
| 2.00 | 0.9772 | 0.9783 | 0.9793 | 0.9803 | 0.9812 |

**solution/answer** Regardless of the parameters of the given distribution, The probability its value is within 40% of its standard daviation from the mean is equal to: $P(-0.4 < Z < 0.4)$ for $Z \sim N(0, 1)$ following the standard normal distribution. We have that:

$$P(-0.4 < Z < 0.4) = P(Z < 0.4) - P(Z < -0.4) = P(Z < 0.4) - P(Z > 0.4)$$

$$P(-0.4 < Z < 0.4) = P(Z < 0.4) - (1 - P(Z < 0.4)) = 2P(Z < 0.4) - 1$$

From the table, we can see that $P(Z < 0.4) = 0.6554$. Thus,

$$P(-0.4 < Z < 0.4) = 2 \cdot P(Z < 0.4) - 1 = 2 \cdot 0.6554 - 1 = 0.3108$$

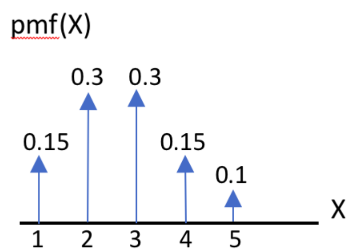
**Question 20** For each of the following probability distributions, determine the expected value of X:

(1) Binomial distribution: $X \sim B(10, 0.225)$, the value of E[X] is [A].
(2) Uniform distribution: $X \sim U(1.3, 4.5)$, the value of E[X] is [B].
(3) Geometric distribution: $X \sim G(0.32)$, the value of E[X] is [C].

**solution/answer** We compute the expected value for each case as follows:

1. $E[X] = 10 \cdot 0.225 = 2.25$
2. $E[X] = \frac{4.5+1.3}{2} = 2.9$
3. $E[X] = \frac{1}{0.32} = 3.125$

**Question 21** Figure below shows the probability mass function (pmf) of the random variable X.




i.e.$P(X = 1) = 0.15, P(X = 2) = 0.3, P(X = 3) = 0.3, P(X = 4) = 0.15$ and $P(X = 5) = 0.1$

Determine the following:

(1) The expected value of $X^2$ , $E[X^2] = [A]$

(2) The variance of X , Var(X) = [B]

(3) Probability that X is less than 3 , $P(X < 3)$ = [C]

**solution/answer** We calculate as follows:

1. $E[X^2] = \sum_{i=1}^{5} p_i \cdot X_i^2 = 0.15 \cdot 1^2 + 0.3 \cdot 2^2 + 0.3 \cdot 3^2 + 0.15 \cdot 4^2 + 0.1 \cdot 5^2 = 8.95$
2. $Var(X) = E[X^2] - (E[X])^2$
   Since $E[X] = \sum_{i=1}^{5} p_i \cdot X_i = 0.15 \cdot 1 + 0.3 \cdot 2 + 0.3 \cdot 3 + 0.15 \cdot 4 + 0.1 \cdot 5 = 2.75$, we get:
   $$Var(X) = E[X^2] - (E[X])^2 = 8.95 - 2.75^2 = 1.387$$
3. $P(X < 3) = P(X = 1) + P(X = 2) = 0.15 + 0.3 = 0.45$