CX2100 Prob & Stat
Tutorial #2 Solutions

Probability

1. A jar contains four coins: a nickel (5¢), a dime (10¢), a quarter (25¢), and a half-dollar (50¢). Three coins are randomly selected without replacement from the jar.

   (a) List all possible outcomes in sample space S.

   $$\begin{aligned} S = \{&(5,10,25),(10,25,5),(25,5,10),(5,25,10),(10,5,25),(25,10,5),\\ &(5,10,50),(10,50,5),(50,5,10),(5,50,10),(10,5,50),(50,10,5),\\ &(5,25,50),(25,50,5),(50,5,25),(5,50,25),(25,5,50),(50,25,5),\\ &(10,25,50),(25,50,10),(50,10,25),(10,50,25),(25,10,50),(50,25,10)\}\end{aligned}$$

   There are $\binom{4}{3} = 4$ combinations of three coins and each can be ordered in $3! = 6$ ways.

   (b) What is the probability that the total amount drawn will equal to 60¢ or more?
   The only way to get a total amount of less than 60¢ is when we have a nickel (5¢), a dime (10¢) and a quarter (25¢). There are 6 ways to order this. Hence, the probability is $\frac{24-6}{24} = \frac{3}{4}$.

2. A mother prepares nine popsicles of different flavours: three of orange, three of cherry and three of grape, for a party of four children. If every child is allowed to choose a popsicle of his/her favourite flavour, what is the probability that all of them will get their choices?

   $$Pr(\text{all get choice}) = 1 - Pr(\text{one does not get choice}),$$

   since there are three of each flavour and the most extreme case is when all four children have the same favourite flavour.

   $$\begin{aligned} Pr(\text{one does not get choice}) &= Pr(\text{the other three choose the same flavour})\\ &= \frac{1}{3}\times\frac{1}{3}\times\frac{1}{3}\\ &= \frac{1}{27}\end{aligned}$$

   So:

   $$Pr(\text{all get choice}) = \frac{26}{27}$$

3. When two events are mutually exclusive, they cannot both happen when the experiment is performed. Once event B has occurred, event A cannot occur, i.e. $P(A|B) = 0$ or $P(A \cap B) = 0$, and vice versa. The occurrence of event B certainly affects the probability of occurrence of event A. Therefore, mutually exclusive events must be dependent.

$P(A)$	$P(B)$	Conditions	$P(A\|B)$	$P(A \cap B)$	$P(A \cup B)$
0.3	0.4	mutually exclusive	0	0	0.7
0.3	0.4	independent	0.3	0.12	0.58
0.1	0.5	mutually exclusive	0	0	0.6
0.2	0.5	independent	0.2	0.1	0.6

When two events are independent, the occurrence of event B does not affect the probability of occurrence of event A, i.e. $P(A|B) = P(A)$ or $P(A \cap B) = P(A)P(B)$, and vice versa. Event A may still occur even if event B has occurred. Therefore, independent events cannot be mutually exclusive. Use the relationships above to fill in the table below:

- First row: mutually exclusive so $P(A \cap B) = 0$. Use $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- Second row: independent so $P(A|B) = P(A)$ and $P(A \cap B) = P(A)P(B)$. Use $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- Third row: mutually exclusive since $P(A \cup B) = P(A) + P(B)$. $P(A \cap B) = 0 \Rightarrow P(A|B) = \frac{P(A \cap B)}{P(B)} = 0$.
- Fourth row: independent since $P(A \cap B) = P(A)P(B)$. Hence, $P(A|B) = P(A)$. Use $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

4. A blood disease is found in 2% of the persons in a certain population. A new blood test will correctly identify 96% of the persons with the disease and 94% of the persons without the disease.

   (a) What is the probability that a person who is called positive by the blood test actually has the disease?

   Let T be the event of a positive test and D be the event that the person has the disease.

$$
\begin{aligned}
&P(\mathrm{D|T}) \\
=&\frac{P(\mathrm{D} \cap \mathrm{T})}{P(\mathrm{T})} \\
=&\frac{P(\mathrm{T|D})P(\mathrm{D})}{P(\mathrm{T|D})P(\mathrm{D}) + P(\mathrm{T|D'})P(\mathrm{D'})} \\
=&\frac{0.96 \times 0.02}{0.96 \times 0.02 + (1 - 0.94) \times (1 - 0.02)} \\
=&0.246 \text{ (to 3.d.p.)}
\end{aligned}
$$

   by Bayes' rule (definition of conditional probability) and the Law of Total Probability.

(b) What is the probability that a person who is called negative by the blood test actually does not have the disease?

$$
\begin{aligned}
&P(\text{D'}|\text{T'}) \\
=&\frac{P(\text{D'} \cap \text{T'})}{P(\text{T'})} \\
=&\frac{P(\text{T'}|\text{D'})P(\text{D'})}{1-(P(\text{T}|\text{D})P(\text{D})+P(\text{T}|\text{D'})P(\text{D'}))} \\
=&\frac{0.94 \times (1-0.02)}{1-(0.96 \times 0.02 + (1-0.94) \times (1-0.02))} \\
=&0.999 \text{ (to 3.d.p.)}
\end{aligned}
$$

(c) Comment on the results obtained in part (a) & (b).

High false positive rate of $100\% - 24.6\% = 75.4\%$ and low false negative rate of $100\% - 99.9\% = 0.01\%$. Probably good to be on the conservative side since the consequences of missing the blood disease is much higher than a false alarm.

5. (a) A magician has in his pocket a fair coin and a doctored coin where both sides are heads. If he randomly picks a coin to flip, and obtains a head, what is the probability that he picks the fair coin?

$$
\begin{aligned}
&P(\text{Fair coin}|\text{Head}) \\
=&\frac{P(\text{Head} \cap \text{Fair coin})}{P(\text{Head})} \\
=&\frac{P(\text{Head}|\text{Fair coin})P(\text{Fair coin})}{P(\text{Head})} \\
=&\frac{0.5 \times 0.5}{0.75} \\
=&\frac{1}{3}
\end{aligned}
$$

(b) If he flips the same coin the second time and obtains a head again, what is the probability that it is a fair coin?

$$
\begin{aligned}
&P(\text{Fair coin}|\text{2 Heads}) \\
=&\frac{P(\text{2 Heads} \cap \text{Fair coin})}{P(\text{2 Heads})} \\
=&\frac{P(\text{2 Heads}|\text{Fair coin})P(\text{Fair coin})}{P(\text{2 Heads})} \\
=&\frac{P(\text{2 Heads}|\text{Fair coin})P(\text{Fair coin})}{P(\text{2 Heads}|\text{Fair coin})P(\text{Fair coin}) + P(\text{2 Heads}|\text{Doctored coin})P(\text{Doctored coin})} \\
=&\frac{0.5 \times 0.5 \times 0.5}{0.5 \times 0.5 \times 0.5 + 1 \times 1 \times 0.5} \\
=&\frac{1}{5}
\end{aligned}
$$