CE2100/CZ2100
Probability and Statistics for Computing

Tutorial 3 - Large-Sample Tests of Hypotheses

In the following questions, it is assumed that the sample sizes are large enough for applying central limit theorem.

Problem 1

A four-sided (tetrahedral) die is tossed 1000 times, and 290 fours are observed. Is there evidence to conclude that the die is biased, that is, say, that more fours than expected are observed? (Use $\alpha = 0.05$)

Solution:

The null hypothesis is $H_0 : p = 0.25$, and the alternative hypothesis is $H_a : p > 0.25$. We can calculate the test statistic by

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.29 - 0.25}{\sqrt{\frac{0.25(0.75)}{1000}}} = 2.92$$

Thus, we reject the null hypothesis since $z = 2.92 > 1.645$.

Problem 2

Let p equal the proportion of drivers who use a seat belt in a state that does not have a mandatory seat belt law. It was claimed that $p = 0.14$. An advertising campaign was conducted to increase this proportion. Two months after the campaign, $y = 104$ out of a random sample of $n = 590$ drivers were wearing seat belts. Was the campaign successful? (Use $\alpha = 0.01$)

Solution:

The null hypothesis is $H_0 : p = 0.14$, and the alternative hypothesis is $H_a : p > 0.14$. The observed sample proportion is:

$$\hat{p} = \frac{104}{590} = 0.176$$

We then calculate the test statistic:

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.176 - 0.14}{\sqrt{\frac{0.14(0.86)}{590}}} = 2.52$$

If we use a significance level of $\alpha = 0.01$, we reject the null hypothesis and favour the alternative hypothesis since $z = 2.52 > 2.326$.

Problem 3

Boys of a certain age are known to have a mean weight of $\mu = 85$ pounds. A complaint is made that the boys living in a municipal children's home are underfed. As one bit of evidence, $n = 25$ boys (of the same age) are weighed and found to have a mean weight of $\bar{x} = 80.94$ pounds. It is known that the population standard deviation σ is 11.6 pounds. Based on the available data, what should be concluded concerning the complaint using the p-value approach? It is assumed that weight follows a normal distribution. (Use $\alpha = 0.05$)

Solution:

The null hypothesis is $H_0 : \mu = 85$, and the alternative hypothesis is $H_a : \mu < 85$. In general, we know that if the weights are normally distributed, then:

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

follows the standard normal $N(0, 1)$ distribution. Under the null hypothesis, we have:

$$z = \frac{80.94 - 85}{11.6/\sqrt{25}} = -1.75$$

For this one-tailed test, we reject the null hypothesis at the $\alpha = 0.05$ if the p-value is smaller than 0.05. In this case, the p-value is

$$P(Z < -1.75) = 0.0401 < \alpha = 0.05$$

Therefore, we reject the null hypothesis.

Problem 4

To compare customer satisfaction levels of two competing cable television companies, 174 customers of Company 1 and 355 customers of Company 2 were randomly selected and were asked to rate their cable companies on a five-point scale, with 1 being least satisfied and 5 most satisfied. The survey results are summarized in the following table

Company 1	Company 2
$n_1 = 174$	$n_2 = 355$
$\bar{x}_1 = 3.51$	$\bar{x}_2 = 3.24$
$s_1 = 0.51$	$s_2 = 0.52$

Test at the 1% level of significance whether the data provide sufficient evidence to conclude that Company 1 has a higher mean satisfaction rating than Company 2.

Solution:

The null hypothesis is $H_0 : \mu_1 - \mu_2 = 0$, and the alternative hypothesis is $H_a : \mu_1 - \mu_2 > 0$. Since the samples are independent and both are large the test statistic is

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{(3.51 - 3.24) - 0}{\sqrt{\frac{0.51^2}{174} + \frac{0.52^2}{355}}} = 5.684$$

Thus, we reject the null hypothesis since $z = 5.684 > 2.326$.

Additional Questions (Do not discuss in the tutorial)

Problem 5

The melting point of each of 16 samples of a certain brand of hydrogenated vegetable oil was determined, resulting in $\bar{x} = 94.32$. Assume that the distribution of the melting point is normal with $\sigma = 1.20$. Test $H_0 : \mu = 95$ versus $H_a : \mu \neq 95$ using a two-tailed level 0.01 test.

Solution:

We can calculate the p-value as

$$p\text{-value} = 2 \cdot \left[1 - \Phi\left(\left|\frac{94.32 - 95}{1.20/\sqrt{16}}\right|\right)\right] = 2 \cdot [1 - \Phi(2.27)] = 0.0232 > \alpha = 0.01$$

where $\Phi(x)$ is the CDF of standard normal distribution at x. Thus, we do not reject the null hypothesis.

Problem 6

The desired percentage of SiO_2 in a certain type of aluminous cement is 5.5. To test whether the true average percentage is 5.5 for a particular production facility, 16 independently obtained samples are analyzed. Suppose that the percentage of SiO_2 in a sample is normally distributed with $\sigma = 0.3$ and that $\bar{x} = 5.25$. Assuming that 16 samples are enough to apply CTL, does the above information indicate conclusively that the true average percentage differs from 5.5 under common significance levels?

Solution:

Note that the hypotheses are $H_0 : \mu = 5.5$ and $H_a : \mu \neq 5.5$. The p-value is

$$p\text{-value} = 2 \cdot \left[1 - \Phi\left(\left|\frac{5.25 - 5.5}{0.3/\sqrt{16}}\right|\right)\right] = 0.0008$$

which is smaller than any reasonable significance level ($\alpha = 0.1, 0.05, 0.01, 0.001$). Thus, we reject H_0.

Problem 7

A statistical statement appeared in *The Guardian* on Friday January 4, 2002:

> When spun on edge 250 times, a Belgian one-euro coin came up heads 140 times and tails 110. ‘It looks very suspicious to me’, said Barry Blight, a statistics lecturer at the London School of Economics. ‘If the coin were unbiased the chance of getting a result as extreme as that would be less than 6%’.

(a) Let θ be the probability of coming up heads. Consider the null hypothesis that the coin is fair, $H_0 : \theta = 0.5$. Carefully explain how the 6% figure arises. What term describes this value in hypothesis testing? Does it correspond to a one-sided or two-sided test?

(b) Would you reject H_0 at a significance level of $\alpha = 0.1$? What about $\alpha = 0.05$? (Using two-sided test)

(c) How many heads would you need to have observed out of 250 spins to reject at a significance of $\alpha = 0.01$?

Solution:

(a) Consider $H_0 : \theta = 0.5$, and $H_a : \theta > 0.5$ for one-sided test or $\theta \neq 0.5$ for two-sided test. The sample portion is

$$\hat{p} = \frac{140}{250}$$

and the test statistic is

$$z = \frac{\hat{p} - 0.5}{\sqrt{\frac{0.5 \times 0.5}{250}}} \sim N(0, 1)$$

Then we can compute the p-value

- one-sided: $p = P(z > \frac{0.56-0.5}{\sqrt{1/1000}}) \approx 0.02889$
- two-sided: $p = P(|z| > \frac{0.56-0.5}{\sqrt{1/1000}}) \approx 0.05778$

So the figure of 6% is the two-sided p-value.

(b) We saw in part (a) that the quoted 6% was a two-sided p-value. So for the rest of this problem, we'll use two-sided tests. The exact p-value was $p = 0.05778$. Since $0.05 < p < 0.1$ we reject H_0 at significance $\alpha = 0.1$ and don't reject at $\alpha = 0.05$.

(c) We consider the case that $x \geq 125$. The case that $x \leq 125$ can be done in a similar manner. We compute the p-value as

$$p = 2 \cdot P(x \geq n) = 2 \cdot P(z \geq \frac{\frac{n}{250} - 0.5}{\sqrt{1/1000}}) \leq 0.01$$

or

$$P(z \geq \frac{\frac{n}{250} - 0.5}{\sqrt{1/1000}}) \leq 0.005$$

When $\frac{\frac{n}{250}-0.5}{\sqrt{1/1000}} > 2.58$, the inequality will be valid and $n \geq 145.3$. Similarly, we can solve n as $n \leq 104.6$ for the case that $x \leq 125$. Therefore, we reject for greater than or equal to 146 heads or less than or equal to 104 heads.