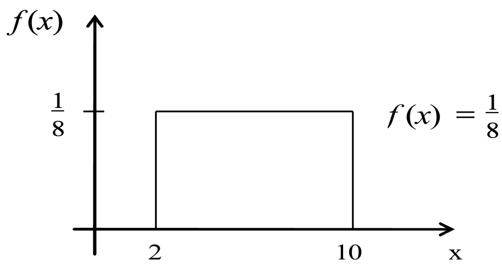
CX2100 Prob & Stat
Tutorial #4 Solutions

Continuous Probability Distribution

1. The figure below shows the graph of the uniform continuous distribution of a random variable that takes on values on the interval from 2 to 10. Find:



(a) $P(X < 7)$

$$\begin{aligned} P(X < 7) &= (7-2) \times \frac{1}{8} \\ &= 0.625 \end{aligned}$$

Instead of using the area of the rectangle (base × height), you can also take ratios of the lengths: $P(X < 7) = \frac{7-2}{10-2}$.

(b) $E(X)$

$$\begin{aligned} E(X) &= \int x f(x) \mathrm{d}x \\ &= \int_2^{10} x \frac{1}{8} \mathrm{d}x \\ &= \frac{1}{8} \left[\frac{x^2}{2} \right]_2^{10} \\ &= \frac{[100-4]}{8 \times 2} \\ &= 6 \end{aligned}$$

This can also be seen from the midpoint of the range in the figure.

(c) Var(X)

$$\begin{aligned}
\text{Var}(X) &= \int x^2 f(x)\mathrm{d}x - E(X)^2 \\
&= \int_2^{10} x^2 \frac{1}{8}\mathrm{d}x - 36 \\
&= \frac{1}{8}\left[\frac{x^3}{3}\right]_2^{10} - 36 \\
&= \frac{[1000-8]}{8\times 3} - 36 \\
&= \frac{124-108}{3} \\
&= \frac{16}{3}
\end{aligned}$$

2. The waiting time for one to be served in a queueing system is a random variable having an exponential distribution with an average of 4 minutes.

(a) Determine the variance of the waiting time.

An Exp(λ) random variable has a mean of $\frac{1}{\lambda}$ so we are looking at an Exp$\left(\frac{1}{4}\right)$ random variable.

$$\begin{aligned}
\text{Var}(X) &= \frac{1}{\lambda^2} \\
&= 16
\end{aligned}$$

(b) What is the probability that one has to wait for at least 10 minutes before being served?

$$\begin{aligned}
Pr(X \geq 10) &= \int_{10}^{\infty} 0.25e^{-0.25x}\mathrm{d}x \\
&= \left[-e^{-0.25x}\right]_{10}^{\infty} \\
&= e^{-2.5}
\end{aligned}$$

3. The cumulative distribution function of the r.v. X is given below:

$$F(x) = \begin{cases} 0, & x < 1 \\ 1 - x^{-3}, & x \geq 1 \end{cases}$$

(a) Determine the probability density function of X.

$$
\begin{aligned}
f(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(1 - x^{-3}\right) \text{ for } x \geq 1 \\
&= 3x^{-4} \text{ for } x \geq 1,
\end{aligned}
$$

and 0 otherwise.
We can write this as:

$$
f(x) = \begin{cases} 0, & x < 1 \\ 3x^{-4}, & x \geq 1 \end{cases}
$$

(b) Calculate $E[X]$ and $\mathrm{Var}[X]$.

$$
\begin{aligned}
E[X] &= \int x f(x) \mathrm{d}x \\
&= \int_1^\infty x \times 3x^{-4} \mathrm{d}x \\
&= \int_1^\infty 3x^{-3} \mathrm{d}x \\
&= \left[-\frac{3x^{-2}}{2}\right]_1^\infty \\
&= \frac{3}{2}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}[X] &= \int x^2 f(x) \mathrm{d}x - E[X]^2 \\
&= \int_1^\infty x^2 \times 3x^{-4} \mathrm{d}x - \frac{9}{4} \\
&= \int_1^\infty 3x^{-2} \mathrm{d}x - \frac{9}{4} \\
&= \left[-3x^{-1}\right]_1^\infty - \frac{9}{4} \\
&= \frac{3}{4}
\end{aligned}
$$

4. Given a r.v. having the normal distribution with $\mu = 16.2$ and $\sigma^2 = 1.5625$, find the probabilities that it will take on a value (use the standard normal distribution table)

   (a) greater than 16.8

   Let $X \sim N(16.2, 1.5625)$.

$$\begin{aligned} P(X > 16.8) &= P\left(Z > \frac{16.8 - 16.2}{\sqrt{1.5625}}\right) \\ &= P\left(Z > 0.48\right) \\ &= 1 - P\left(Z \leq 0.48\right) \\ &= 1 - 0.6844 \\ &= 0.3156 \end{aligned}$$

   (b) between 13.6 and 18.8

$$\begin{aligned} P(13.6 < X < 18.8) &= P\left(\frac{13.6 - 16.2}{\sqrt{1.5625}} < Z < \frac{18.8 - 16.2}{\sqrt{1.5625}}\right) \\ &= P\left(-2.08 < Z < 2.08\right) \\ &= 1 - 2 \times P\left(Z \geq 2.08\right) \\ &= 1 - 2 \times (1 - 0.9812) \\ &= 0.9624 \end{aligned}$$

5. Studies have shown that 22% of all patients taking a certain antibiotic will get a headache. Use the normal approximation to the binomial distribution to find the probability that among 50 patients taking this antibiotic

   (a) at least 10 will get a headache

   Let X_B be the number of patients getting a headache.
   This has a Binomial(50, 0.22) distribution.
   Note that $E[X_B] = np = 11 > 5$ and $nq = 39 > 5$ so we can approximate X_B with $X_N \sim N(11, 58.58)$, where $\text{Var}(X_N) = \text{Var}(X_B) = npq = 8.58$.
   With this,

$$\begin{aligned} P(X_B \geq 10) &= P\left(Z \geq \frac{9.5 - 11}{\sqrt{8.58}}\right) \text{ using continuity correction} \\ &= P\left(Z \geq -0.5120916\right) \\ &= P\left(Z \leq 0.5120916\right) \\ &\approx P\left(Z \leq 0.51\right) \text{ using the closest value in the Normal Table} \\ &= 0.6950 \end{aligned}$$

   Note that for the continuity correction, we chose 9.5 instead of 10.5 because we want to include 10 in the calculation.

(b) at most 15 will get a headache

$$
\begin{aligned}
P(X_B \leq 15) &= P\left(Z \leq \frac{15.5 - 11}{\sqrt{8.58}}\right) \text{ using continuity correction} \\
&= P\left(Z \leq 1.536275\right) \\
&\approx P\left(Z \leq 1.54\right) \text{ using the closest value in the Normal Table} \\
&= 0.9382
\end{aligned}
$$

Note that for the continuity correction, we chose 15.5 instead of 14.5 because we want to include 15 in the calculation.