CX2100 Prob & Stat
Tutorial #3 Solutions

Discrete Probability Distribution

1. Suppose that the probabilities are 0.2, 0.4, 0.3 and 0.1 that the number of wills filed on any day at Kusu Island will be 0, 1, 2, or 3.

   (a) What is the probability of having at least 2 wills filed per day?
   Let X be the number of wills filed on a day. Since the event of a total of 2 wills being filed on a day and the event of a total of 3 wills being filed on a day are mutually exclusive, i.e. $P(\{X = 2\} \cap \{\mathrm{X} = 3\}) = 0$,

   $$\begin{aligned} P(\{\mathrm{X} = 2\} \cup \{\mathrm{X} = 3\}) &= P(\{\mathrm{X} = 2\}) + P(\{\mathrm{X} = 3\}) \\ &= 0.3 + 0.1 \\ &= 0.4 \end{aligned}$$

   (b) Find the expected number of wills filed per day.

   $$\begin{aligned} E(X) &= \sum_{x=0}^{3} x \times P(X = x) \\ &= 0 \times 0.2 + 1 \times 0.4 + 2 \times 0.3 + 3 \times 0.1 \\ &= 1.3 \end{aligned}$$

   (c) Find the variance of the number of wills filed per day.

   $$\begin{aligned} \mathrm{Var}(X) &= \sum_{x=0}^{3} (x - E(X))^2 \times P(X = x) \\ &= (0 - 1.3)^2 \times 0.2 + (1 - 1.3)^2 \times 0.4 + (2 - 1.3)^2 \times 0.3 + (3 - 1.3)^2 \times 0.1 \\ &= 0.81 \end{aligned}$$

   Alternatively,

   $$\begin{aligned} \mathrm{Var}(X) &= \left(\sum_{x=0}^{3} x^2 \times P(X = x) \right) - E(X)^2 \\ &= \left(0^2 \times 0.2 + 1^2 \times 0.4 + 2^2 \times 0.3 + 3^2 \times 0.1\right) - 1.3^2 \\ &= 0.81 \end{aligned}$$

   Both methods give the same result.

2. Given that $f(x) = k/2^x$, is a discrete probability function for a r.v. that can take on the values x =0, 1, 2, 3 and 4. Find k and tabulate the cumulative probability $P(X \leq x)$.

x	$P(X \leq x)$
0	$\frac{16}{31}$
1	$\frac{16}{31} + \frac{16}{31\times 2} = \frac{16}{31} + \frac{8}{31} = \frac{24}{31}$
2	$\frac{24}{31} + \frac{16}{31\times 4} = \frac{24}{31} + \frac{4}{31} = \frac{28}{31}$
3	$\frac{28}{31} + \frac{16}{31\times 8} = \frac{28}{31} + \frac{2}{31} = \frac{30}{31}$
4	1

Since the total probability over all the possible values of X has to be 1,

$$\sum_{x=0}^{4} \frac{k}{2^x} = 1$$

$$\Rightarrow k \times \left(\frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}\right) = 1$$

$$\Rightarrow k \times \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}\right) = 1$$

$$\Rightarrow \frac{16+8+4+2+1}{16} k = 1$$

$$\Rightarrow k = \frac{16}{31}$$

3. A biased die is rolled 50 times and the number of twos appeared is 10. If the die is rolled for another 10 times, determine the following:

   (a) the probability that we get a two exactly 3 times.

   From the 50 past rolls, the probability we get a two in one roll can be determined by the relative frequency: $\frac{10}{50} = 0.2$. Thus, assuming that the 10 rolls are independent, we can calculate the probability of getting a two exactly 3 times by (1) calculating the probability of getting a sequence of 10 rolls with exactly 3 '2's (2) multiplying this by the number of such sequences (the number of ways to choose 3 out of the 10 rolls):

   $$\begin{aligned} & P(\text{Get a two exactly 3 times in 10 rolls}) \\ &= \#of sequences \times Pr(\text{Getting a sequence of 10 rolls with exactly 3 `2's}) \\ &= \binom{10}{3} (0.2)^3 (0.8)^7 \\ &= 0.20133 (\text{to 5.d.p.}) \end{aligned}$$

   This corresponds to $Pr(X = 3)$ where X is a Binomial(10, 0.2) random variable.

   (b) the expected number of twos.

   $$E(X) = n \times p = 10 \times 0.2 = 2$$

   (c) the variance of the number of twos.

   $$\text{Var}(X) = n \times p \times (1-p) = 10 \times 0.2 \times 0.8 = 1.6$$

Note that we have used the formulae for the mean and variance of a Binomial(10, 0.2) random variable for (b) and (c).

4. The number of calls coming per minute into a hotel reservation center is a Poisson random variable with mean 3.

   (a) Find the probability that no calls come in a given 1 minute period.

   Let X be the number of calls coming per minute. Then, $X \sim$ Poisson(3).

$$\begin{aligned} P(X = 0) &= \frac{\lambda^x}{x!} e^{-\lambda} \\ &= \frac{3^0}{0!} e^{-3} \\ &= e^{-3} \end{aligned}$$

   (b) Assume that the number of calls arriving in two different minutes are independent. Find the probability that at least two calls will arrive in a given two minutes period.

   Let X_1 and X_2 be the number of calls coming in the first and second minutes.

$$\begin{aligned} & P(X_1 + X_2 \geq 2) \\ =& 1 - P(X_1 + X_2 \leq 1) \\ =& 1 - (P(X_1 = 0, X_2 = 1) + P(X_1 = 1, X_2 = 0) + P(X_1 = 0, X_2 = 0)) \\ =& 1 - \left(e^{-3} \times \frac{3^1}{1!} e^{-3} + \frac{3^1}{1!} e^{-3} \times e^{-3} + e^{-3} \times e^{-3} \right) \\ =& 1 - 7e^{-6} \end{aligned}$$

   Note that we can multiply the probabilities for X_1 and X_2 because they are independent.
   Alternatively, we can also consider $Y \sim Poisson(6)$ to be the number of calls coming within two minutes and compute $P(Y \geq 2)$.

$$\begin{aligned} P(Y \geq 2) =& 1 - P(Y \leq 1) \\ =& 1 - (P(Y = 0) + P(Y = 1)) \\ =& 1 - e^{-6} - 6e^{-6} \\ =& 1 - 7e^{-6} \end{aligned}$$

5. The probability that a student fails Subject A exam is 0.05. If the student failed the subject, he will have to re-take it the following semester. Let X be the number of times he attempted to pass the subject.

   (a) Determine and name the probability distribution of X.

   The probability mass function is:

$$P(X = x) = (0.05)^{x-1}(1 - 0.05) = (0.05)^{x-1}(0.95),$$

since we calculate the probability of failing $x - 1$ times before finally passing. This corresponds to a Geometric(0.95) distribution where the success probability $p = 0.95$.

(b) Find the probability that a student will pass the subject with no more than 2 attempts.

$$P(X = 1) + Pr(X = 2) = (0.05)^{1-1}(0.95) + (0.05)^{2-1}(0.95) = 0.9975,$$

(c) Find the average number of attempts to pass the subject.

$$E(X = x) = \frac{1}{p} = \frac{1}{0.95} = 1.0526 \text{ (to 4.d.p.)}$$