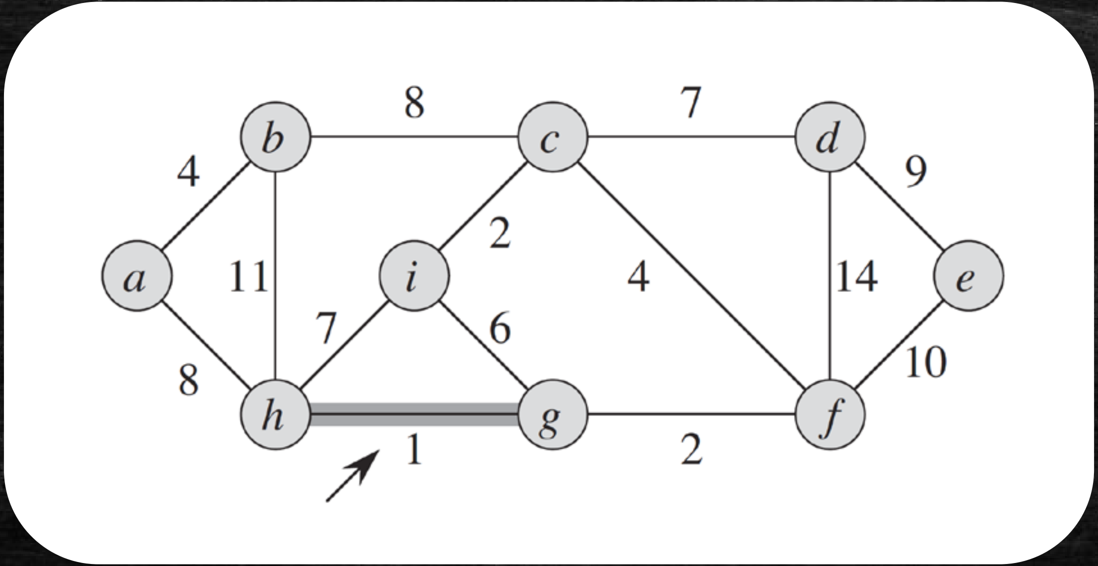
CS 457, Fall 2019

Drexel University, Department of Computer Science

Lecture 18

Minimum Spanning Tree of a Weighted Graph

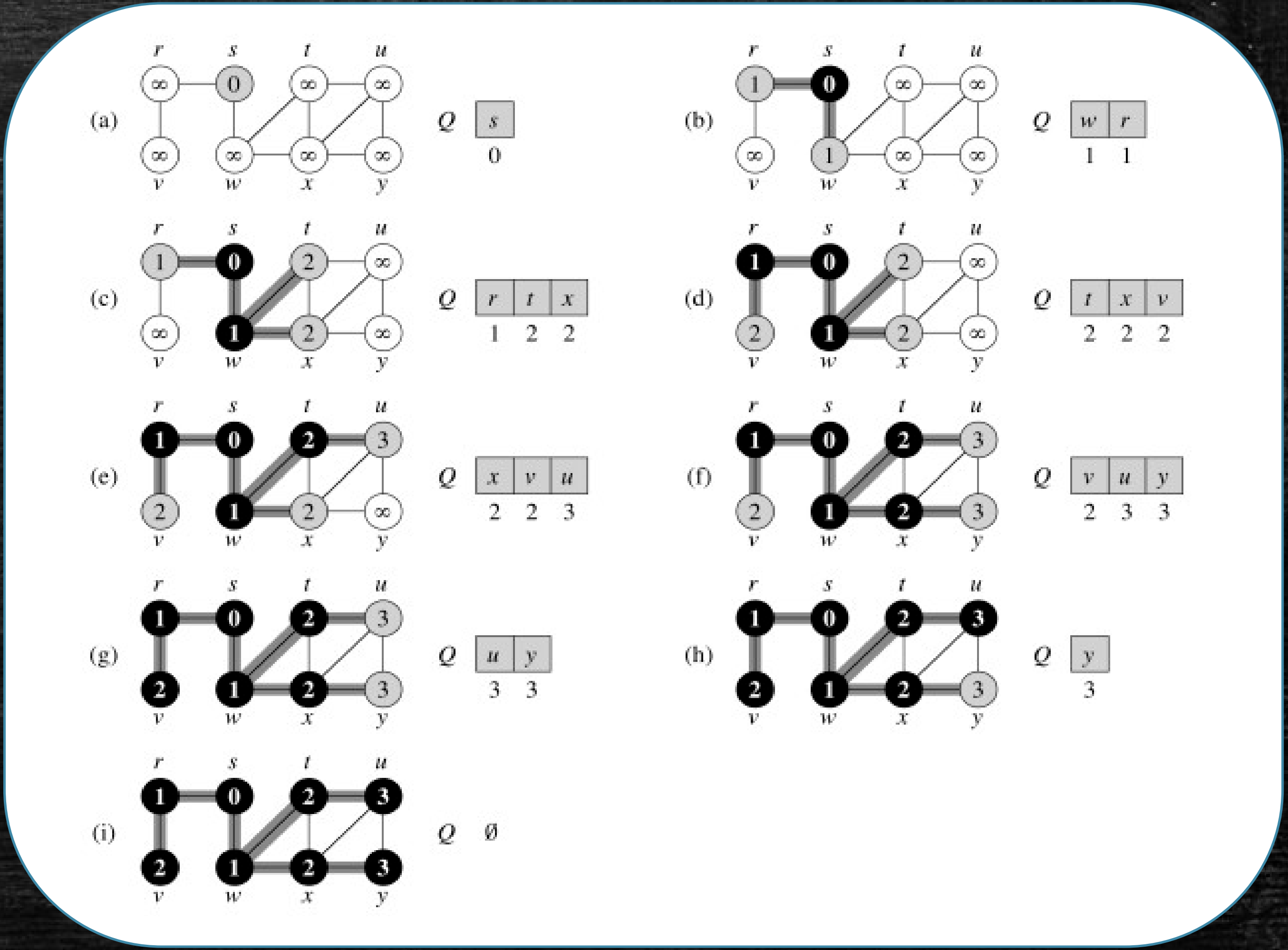
- Let G=(V,E) be a graph on n vertices, m edges, and a weight w on edges in E.
- Sub-graph T=(V,E') with $E' \subseteq E$ with no cycles is a spanning tree
- The weight of T is the sum of the weights of its edges: $w(T) = \sum_{(u,v) \in E'} w(u,v)$



BFS Algorithm

BFS (G, s)

```
for each u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```



DFS Algorithm

DFS (G)

```
for each vertex u ∈ G.V
        u.color = WHITE
        u.π = NIL
time = 0
for each vertex u ∈ G.V
        if u.color = WHITE
                DFS-Visit(G,u)
```

DFS-Visit(G,u)

```
time = time + 1
u.d = time
u.color = GRAY
for each vertex v ∈ G.Adj[u]
        if v.color == WHITE
            v.π = u
            DFS-Visit(G, v)
u.color = BLACK
time = time + 1
u.f = time
```

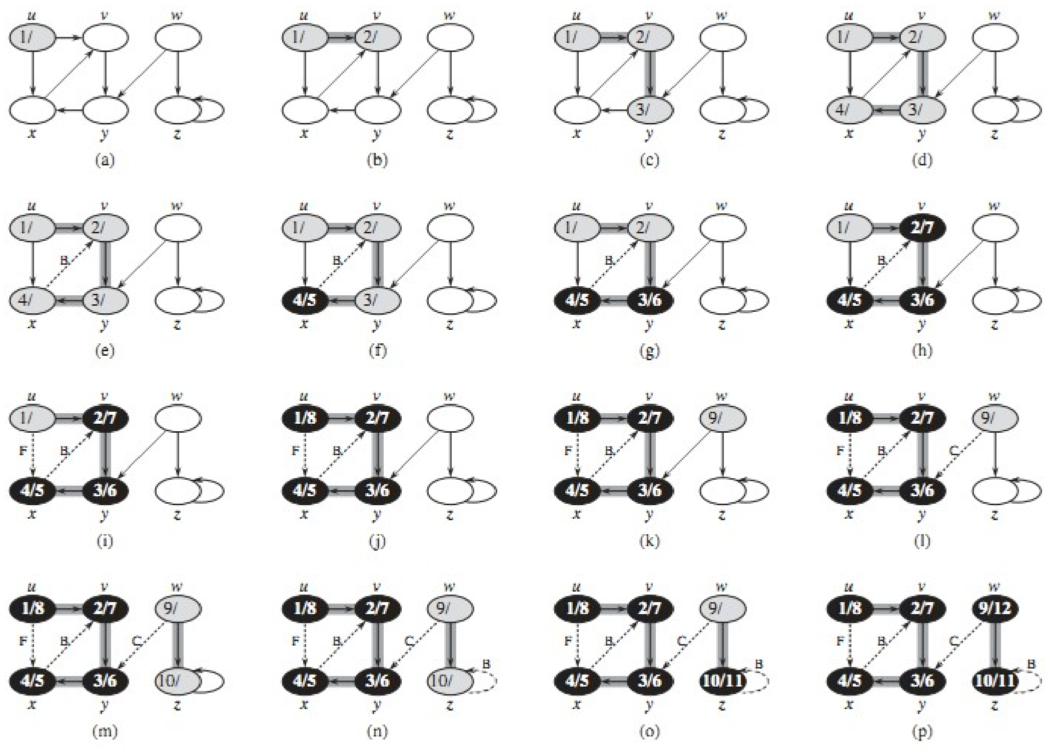
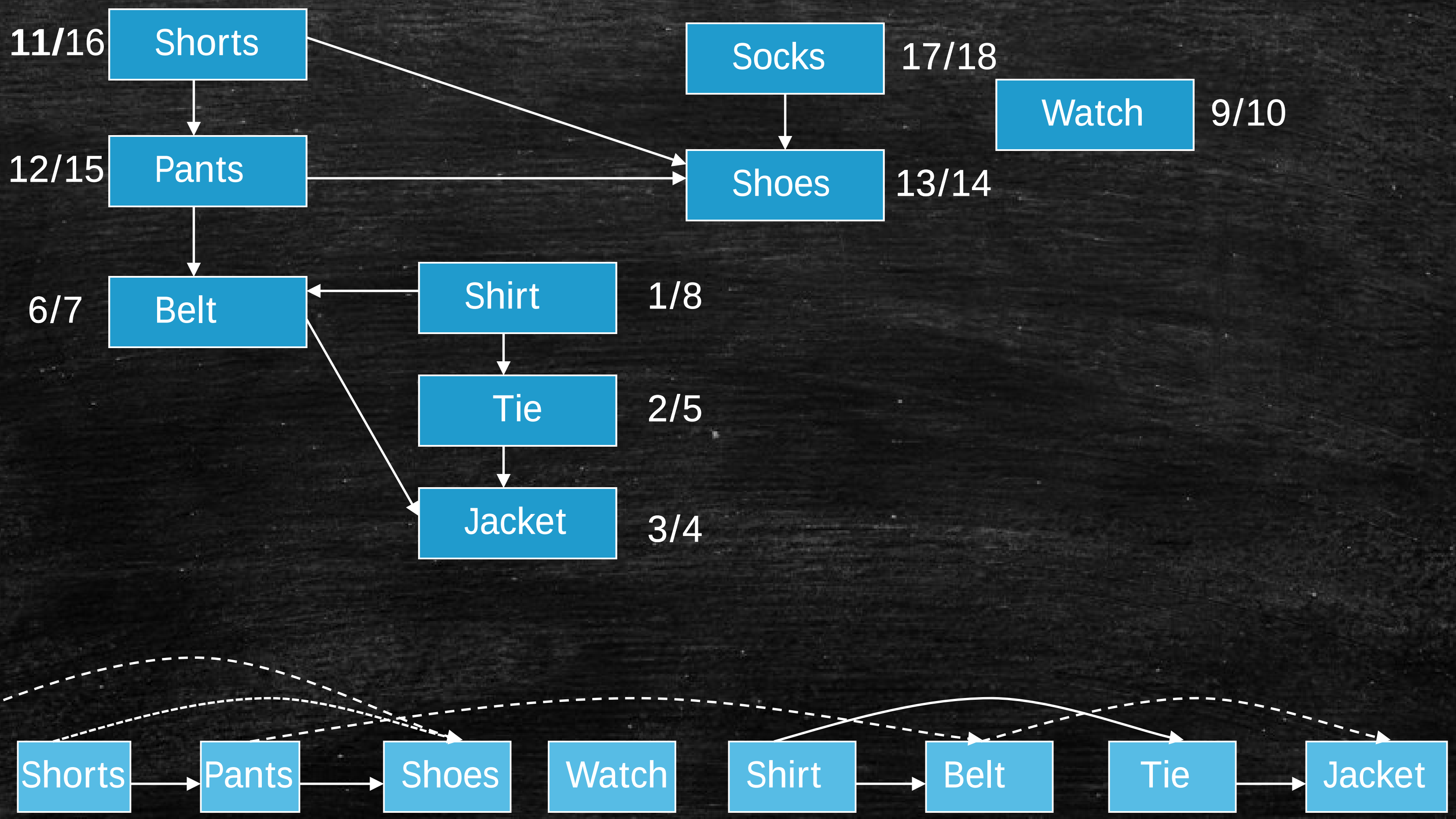


Figure 22.4 The progress of the depth-first-search algorithm DFS on a directed graph. As edges are explored by the algorithm, they are shown as either shaded (if they are tree edges) or dashed (otherwise). Nontree edges are labeled B, C, or F according to whether they are back, cross, or forward edges. Timestamps within vertices indicate discovery time/finishing times.

Example



11/16
Shorts
Socks
17/18
Watch
9/10
12/15
Pants
Shoes
13/14
6/7
Belt
Shirt
1/8
Tie
2/5
Jacket
3/4
Socks
Shorts
Pants
Shoes
Watch
Shirt
Belt
Tie
Jacket

Topological Sort Algorithm

TopologicalSort(G)

1. call DFS(G) to compute finishing times $v.f$ for each vertex v
2. as each vertex is finished, insert it onto the front of a linked list
3. return the linked list of vertices

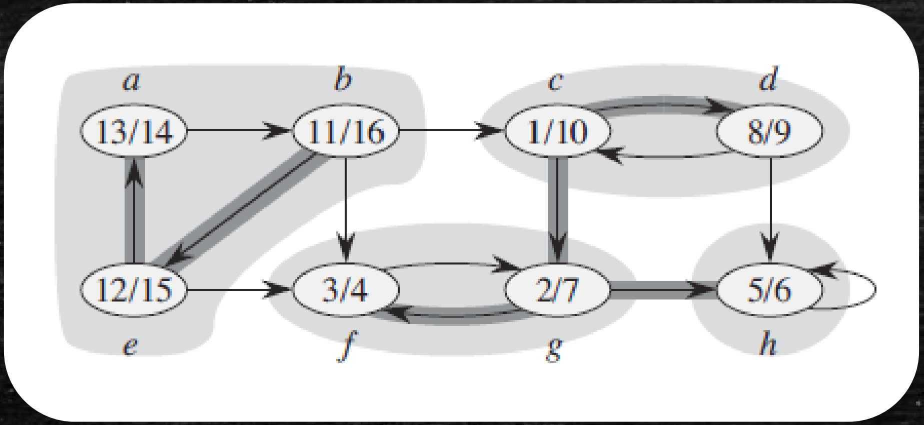
Running time

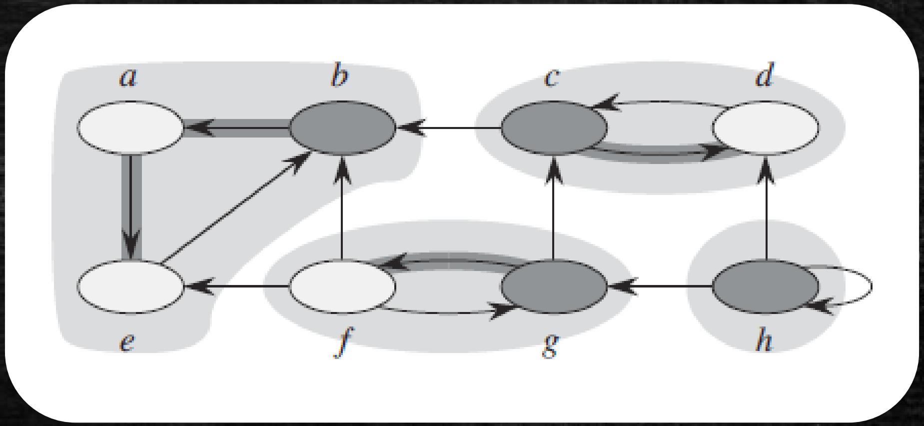
- DFS: $\Theta(n+m)$
- Insertion to linked list: $\Theta(n)$

Strongly Connected Components

Strongly Connected Componenets(G)

1. call DFS(G) to compute finishing times $v.f$ for each vertex v
2. compute G^T
3. call DFS(G^T), but in the main loop of DFS, use decreasing order w.r.t. $v.f$
4. return the vertices of each tree in the depth-first forest formed in line 3 as a separate strongly connected component

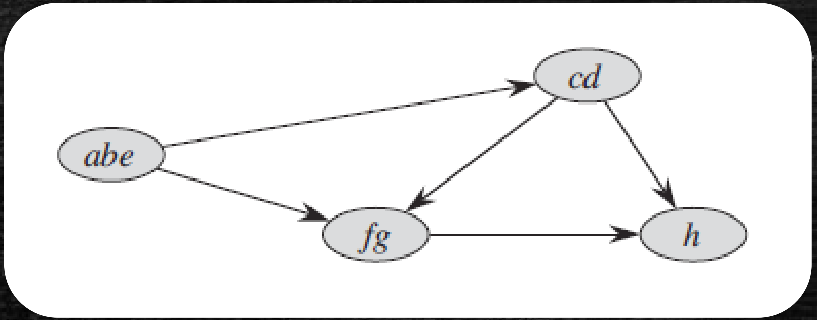


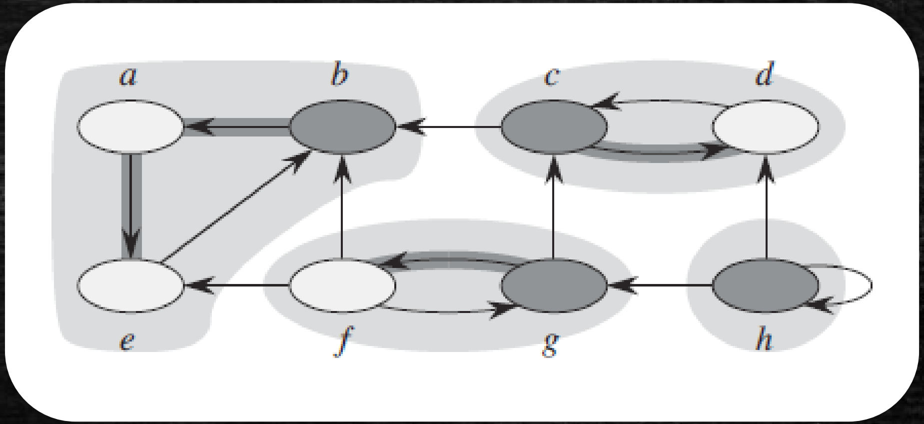


Strongly Connected Components

Strongly Connected Componenets(G)

1. call DFS(G) to compute finishing times $v.f$ for each vertex v
2. compute G^T
3. call DFS(G^T), but in the main loop of DFS, use decreasing order w.r.t. $v.f$
4. return the vertices of each tree in the depth-first forest formed in line 3 as a separate strongly connected component





Practice Problems

- A root of a DAG is a vertex r such that every other vertex of the DAG can be reached from r using a directed path. Give an algorithm that determines whether a given DAG has a root.

- Provide an algorithm that, given a directed acyclic graph $G = (V, E)$ and two vertices $s, t \in V$, returns the number of simple paths from s to t in G.

Activity-Selection problem

- You are given a set of n activities $S = \{a_1, a_2, \dots, a_n\}$, with each activity a_i associated with a start time $s_i > 0$ and a finish time $f_i > s_i$
- Two activities a_i and a_j are compatible if the intervals $[s_i, f_i)$ and $[s_j, f_j)$ do not overlap, i.e., $s_i \geq f_j$ or $s_j \geq f_i$
- Assume that these activities are sorted so that $f_1 \leq f_2 \leq \cdots \leq f_n$
- Select a maximum-size subset of mutually compatible activities

i	1	2	3	4	5	6	7	8	9	10	11
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16

- Dynamic programming solution:
  - Let S_{ij} be the set of activities that start after activity a_i ends and end before activity a_j starts
  - Let A_{ij} be a maximum set of mutually compatible activities in S_{ij} and $c[i, j]$ be its size

$$c[i, j] = \begin{cases} 0 & \text{if } S_{ij} = \emptyset\,, \\ \max\limits_{a_k \in S_{ij}} \{c[i, k] + c[k, j] + 1\} & \text{if } S_{ij} \neq \emptyset\,. \end{cases}$$

Activity-Selection problem

- You are given a set of n activities $S = \{a_1, a_2, \dots, a_n\}$, with each activity a_i associated with a start time s_i>0 and a finish time $f_i > s_i$
- Two activities a_i and a_j are compatible if the intervals $[s_i, f_i)$ and $[s_j, f_j)$ do not overlap, i.e., $s_i \geq f_j$ or $s_j \geq f_i$
- Assume that these activities are sorted so that $f_1 \leq f_2 \leq \cdots \leq f_n$
- Select a maximum-size subset of mutually compatible activities

i	1	2	3	4	5	6	7	8	9	10	11
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16

- Greedy solution:
  - Intuition: we should choose an activity that leaves the resource as available as possible
  - Choose the first activity to finish and repeat

Single-Source Shortest Paths

- Given a weighted, directed graph $G = (V, E)$
- Weight function $w: E \rightarrow \mathbb{R}$ mapping edges to real-valued weights
- The weight $w(p)$ of a path $p = \langle v_0, v_1, \dots, v_k \rangle$ is $\sum_{i=1}^{k} w(v_{i-1}, v_i)$
- The shortest-path weight $\delta(u, v)$ from u to v is

$$\delta(u,v) = \begin{cases} \min\{w(p) : u \overset{p}{\rightsquigarrow} v\} & \text{if there is a path from } u \text{ to } v\ , \\ \infty & \text{otherwise}\ . \end{cases}$$

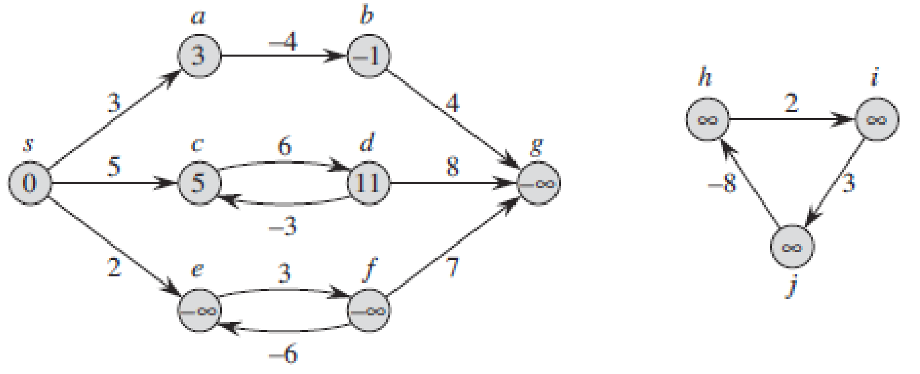
- A shortest-path from u to v is any path p with weight $w(p) = \delta(u, v)$
- Problem: find a shortest path from a given source vertex $s \in V$ to each $v \in V$

Single-Source Shortest Paths

- This problem exhibits optimal substructure, which is an indicator that the dynamic programming and the greedy methods may apply

Lemma 24.1 (Subpaths of shortest paths are shortest paths)
Given a weighted, directed graph $G = (V, E)$ with weight function $w : E \to \mathbb{R}$, let $p = \langle v_0, v_1, \ldots, v_k \rangle$ be a shortest path from vertex v_0 to vertex v_k and, for any i and j such that $0 \le i \le j \le k$, let $p_{ij} = \langle v_i, v_{i+1}, \ldots, v_j \rangle$ be the subpath of p from vertex v_i to vertex v_j. Then, p_{ij} is a shortest path from v_i to v_j.

- Negative-weight edges:
  - Do there exist negative-weight edges?
  - Do there exist negative-weight cycles?
  - Are these cycles reachable from the source?

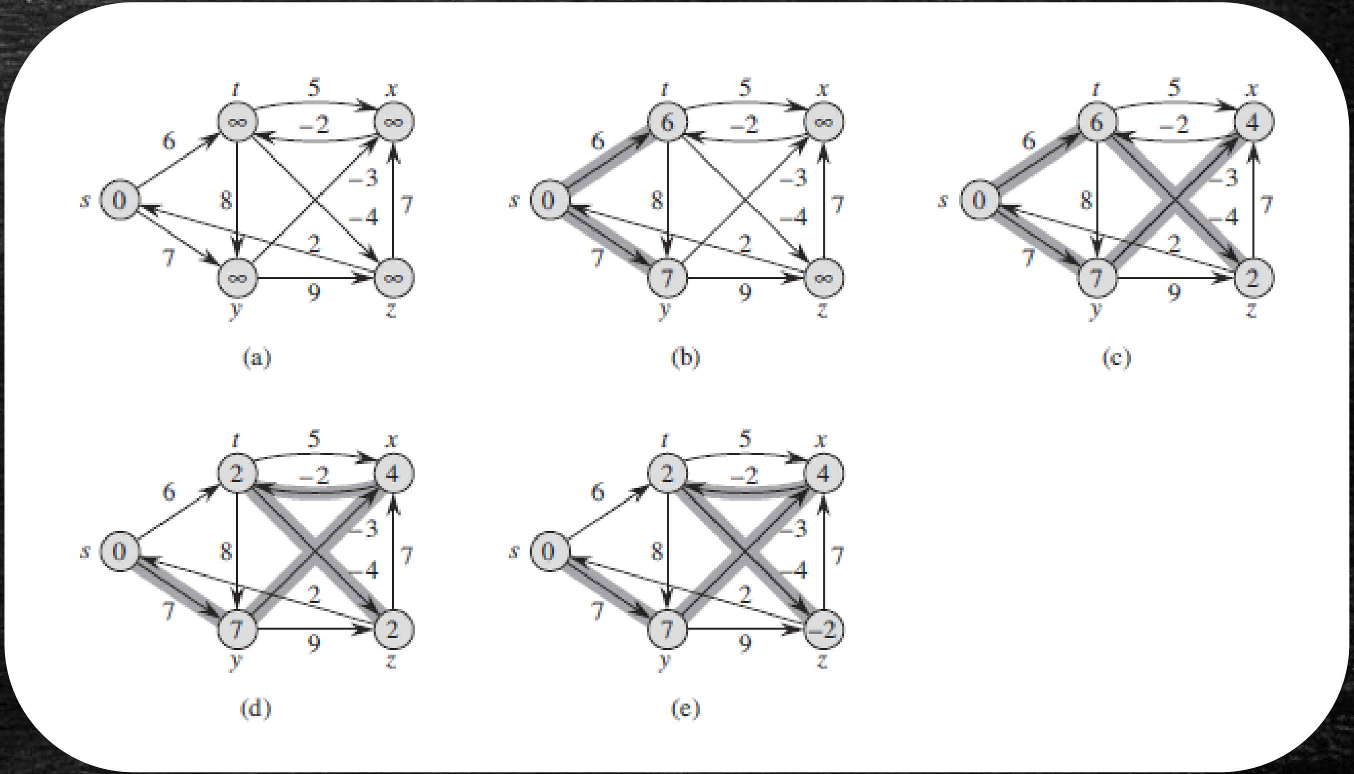


Bellman-Ford Algorithm: time $O(nm)$

```
INITIALIZE-SINGLE-SOURCE(G, s)
  for each vertex v ∈ G.V
      v.d = ∞
      v.π = NIL
  s.d = 0
```

```
BELLMAN-FORD(G, w, s)
  INITIALIZE-SINGLE-SOURCE(G, s)
  for i = 1 to |G.V| − 1
      for each edge (u, v) ∈ G.E
          RELAX(u, v, w)
  for each edge (u, v) ∈ G.E
      if v.d > u.d + w(u, v)
          return FALSE
  return TRUE
```

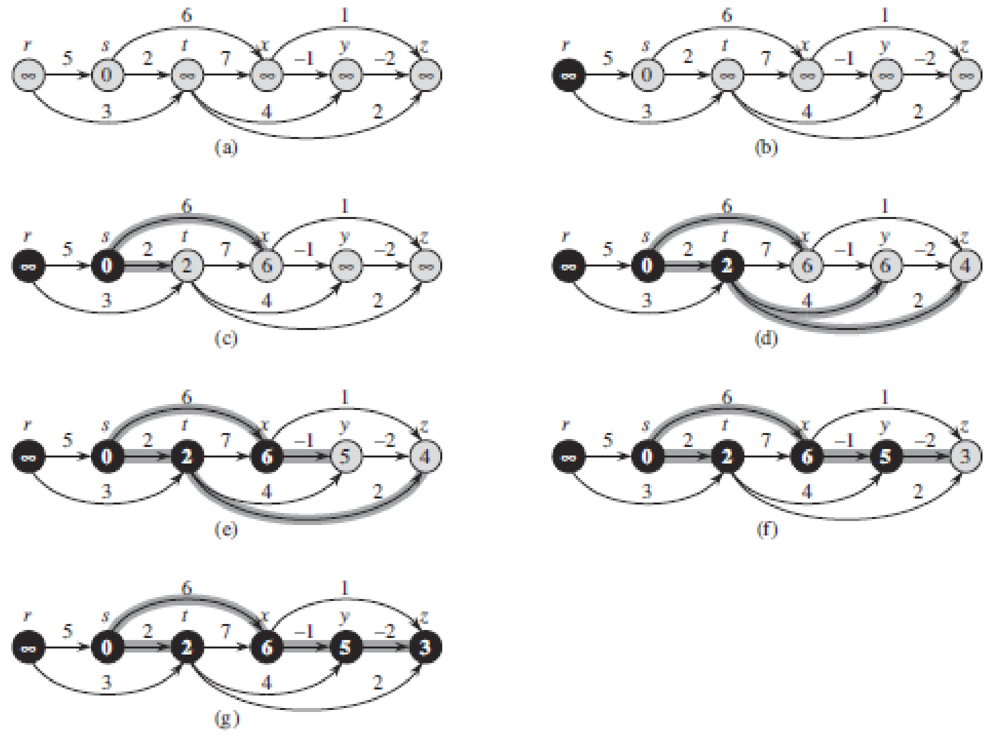
```
RELAX(u, v, w)
  if v.d > u.d + w(u, v)
      v.d = u.d + w(u, v)
      v.π = u
```



SSSP in Directed Acyclic Graphs

- Solve the problem of single-source shortest paths for DAGs:

```
DAG-SHORTEST-PATHS(G, w, s)
topologically sort the vertices of G
INITIALIZE-SINGLE-SOURCE(G, s)
for each vertex u, taken in topologically sorted order
    for each vertex v ∈ G.Adj[u]
        RELAX(u, v, w)
```

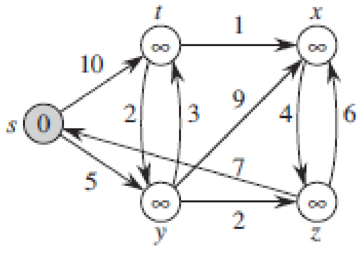


- Running time?
  - $O(n+m)$

Dijkstra’s Algorithm

- Dijkstra’s algorithm assumes nonnegative weights!

```
DIJKSTRA(G, w, s)
INITIALIZE-SINGLE-SOURCE(G, s)
S = ∅
Q = G.V
while Q ≠ ∅
    u = EXTRACT-MIN(Q)
    S = S ∪ {u}
    for each vertex v ∈ G.Adj[u]
        RELAX(u, v, w)
```



(a)

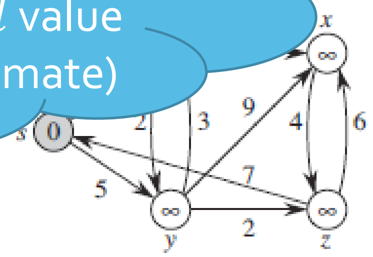
- Min-priority queue matters
- Time: $O(n^2 + m) = O(n^2)$
- This is a greedy algorithm

Dijkstra's Algorithm

- Dijkstra's algorithm assumes nonnegative weights!

```
DIJKSTRA(G
INITIALIZE
S = ∅
Q = G.V
while Q ≠ ∅
    u = EXTRACT-MIN(Q)
    S = S ∪ {u}
    for each vertex v ∈ G.Adj[u]
        RELAX(u, v, w)
```

Extract vertex $v \in Q$
with minimum $v.d$ value
(shortest path estimate)



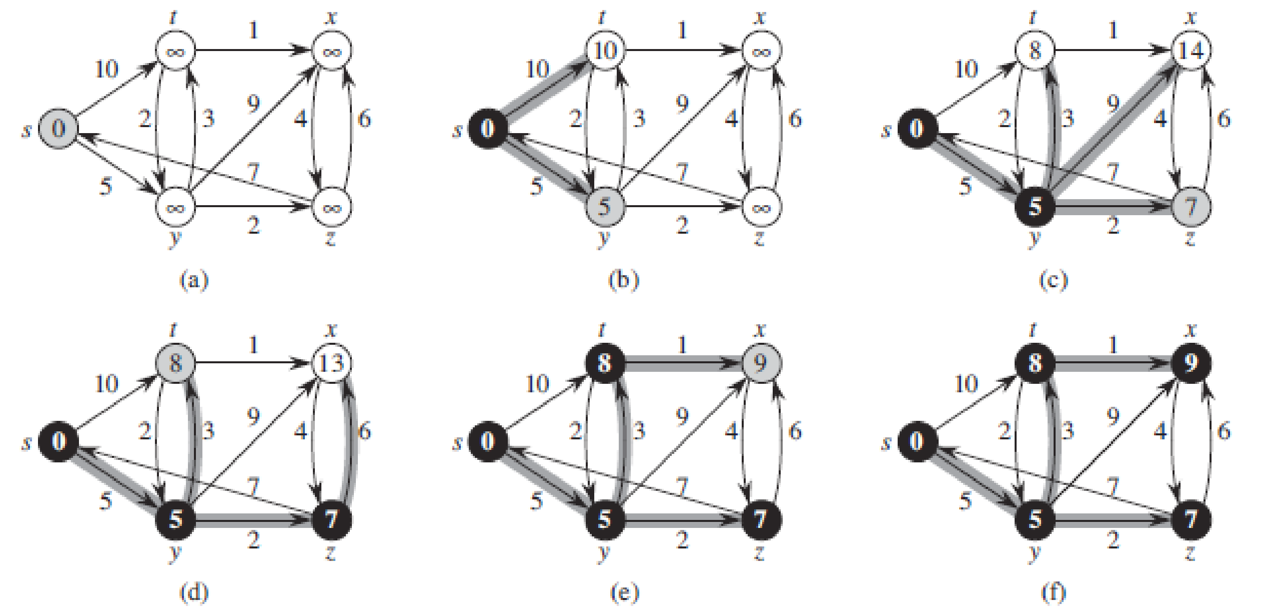
(a)

- Min-priority queue matters
- Time: $O(n^2 + m) = O(n^2)$
- This is a greedy algorithm

Dijkstra’s Algorithm

- Dijkstra’s algorithm assumes nonnegative weights!

```
DIJKSTRA(G, w, s)
    INITIALIZE-SINGLE-SOURCE(G, s)
    S = ∅
    Q = G.V
    while Q ≠ ∅
        u = EXTRACT-MIN(Q)
        S = S ∪ {u}
        for each vertex v ∈ G.Adj[u]
            RELAX(u, v, w)
```



- Min-priority queue matters
- Time: $O(n^2 + m) = O(n^2)$
- This is a greedy algorithm

Practice Problem

24-3 Arbitrage

Arbitrage is the use of discrepancies in currency exchange rates to transform one unit of a currency into more than one unit of the same currency. For example, suppose that 1 U.S. dollar buys 49 Indian rupees, 1 Indian rupee buys 2 Japanese yen, and 1 Japanese yen buys 0.0107 U.S. dollars. Then, by converting currencies, a trader can start with 1 U.S. dollar and buy $49 \times 2 \times 0.0107 = 1.0486$ U.S. dollars, thus turning a profit of 4.86 percent.

Suppose that we are given n currencies $c_1, c_2, \ldots, c_n$ and an $n \times n$ table R of exchange rates, such that one unit of currency c_i buys $R[i, j]$ units of currency c_j.

a. Give an efficient algorithm to determine whether or not there exists a sequence of currencies $\langle c_{i_1}, c_{i_2}, \ldots, c_{i_k} \rangle$ such that

$$R[i_1, i_2] \cdot R[i_2, i_3] \cdots R[i_{k-1}, i_k] \cdot R[i_k, i_1] > 1 .$$

Analyze the running time of your algorithm.

b. Give an efficient algorithm to print out such a sequence if one exists. Analyze the running time of your algorithm.