RECURRENCE EQUATIONS AND RANDOMIZED ANALYSIS

Recitation Week 3

WARM-UP RECURRENCE EQUATION

- Solve the following recurrence using any method

$$T(n) = T(n-1) + \log n$$

- We can't use the master theorem for this recurrence since it does not have the correct form
- We can solve this using the recursion tree or with substitution. We'll do both!
- Remember our steps:
- GUESS
- ASSUME
- SHOW

Recurrence Equations – Change of Variable

- What should we pick for our change of variables?

$$T(n) = 3T(\sqrt{n}) + \log n$$

$T(n)$
$= 3 \cdot T(n^{0.5}) + \log n$

- The "b" that we would like to use for the master method appears in the exponent
- We want it to be multiplying our recursive variable

$$S(m) = 3S(m/2) + m$$

$$T(n) = T(2^m) = S(m) = \Theta(m^{\log_2 3}) = \Theta(\log n^{\log_2 3}).$$

CHAPTER 5 – PROBABILISTIC ANALYSIS AND RANDOMIZED ALGORITHMS

WARM-UP: WHY STUDY "RANDOMIZED" ALGORITHMS?

- "Rock-paper-scissors"

  Input: A sequence of one of the three choices

  Propose a deterministic algorithm to play rock-paper-scissors against an "unknown" sequence of inputs and analyze its worst-case success rate and then propose a randomized algorithm that you think would be better

Warm-up: Why study randomized algorithms?

- Randomized algorithms show up in research and in practice
- "Rock-paper-scissors"

  Input: A sequence of one of the three choices

  Propose a deterministic algorithm to play rock-paper-scissors against an "unknown" sequence of inputs and analyze its worst-case success rate and then propose a randomized algorithm that you think would be better

- There are two different approaches to a randomized algorithms
  - Monte Carlo algorithms – guarantee running time but may give the wrong answer
  - Las Vegas algorithms – guarantee the right answer but have expected running time
- Both these techniques appear in practice for "hard" problems

PRIMER

- Probabilistic analysis and randomized algorithms help in three main ways
  - Randomized algorithms can have better expected running time or performance
  - Deterministic algorithms can be much harder to analyze and have larger constants
  - Analyzing worst-case for problems with "random" input can be too pessimistic
- Probabilistic analysis vs. Randomized algorithms
  - In probabilistic analysis we take the average performance across the distribution of random inputs leading to an average-case running time
  - In randomized algorithms we make random decisions in our algorithm and take the expectation of running times across all decisions leading to an expected running time

Rock-paper-scissors

Median select!

What we'll see today

THE ASSISTANT-HIRING PROBLEM

- SUPPOSE THAT A COMPANY IS LOOKING TO HIRE A NEW EMPLOYEE AND THERE ARE *N* CANDIDATES RANDOMLY ORDERED TO BE INTERVIEWED. THE FOLLOWING HIRING SCHEME IS PROPOSED:

Probabilistic analysis

```
HIRE-ASSISTANT(n)
best = 0              // candidate 0 is a least-qualified dummy candidate
for i = 1 to n
    interview candidate i
    if candidate i is better than candidate best
        best = i
        hire candidate i
```

- LET'S SAY THE ONLY COSTLY OPERATION IS HIRING A CANDIDATE (WITH COST O(1)).
- WHAT'S A WORST-CASE INSTANCE FOR THIS PROBLEM AND WHAT IS THE COST?
- WE'RE GOING TO FIND THE AVERAGE-CASE COST

INDICATOR RANDOM VARIABLES

- Indicator random variables are binary random variables which equal 1 if an event occurs and 0 otherwise

$$\mathrm{I}\{A\} = \begin{cases} 1 & \text{if } A \text{ occurs}\,, \\ 0 & \text{if } A \text{ does not occur}\,. \end{cases}$$

- For example, $X_{WJ} = I\{Dan\ is\ wearing\ jeans\}$ would be a random variable with realized value 1
- Calculating expectations is often easier when using indicator random Variables
- Example: Flipping fair coins
  - Use exhaustive checking and axioms of probability to calculate the expected number of heads when flipping 4 fair coins (i.e., $4 * \Pr(4\ heads) + 3 * \Pr(3\ heads) + 2 * \Pr(2\ heads) + 1 * \Pr(1\ head)$)
  - Calculate the expected number of heads using an indicator random variable and Linearity of expectation

Flipping Fair Coins

- Use exhaustive checking and axioms of probability to calculate the expected number of heads when flipping 4 fair coins (i.e., $4 \ * \Pr(4\ heads) + 3 * \Pr(3\ heads) + 2 * \Pr(2\ heads) + 1 * \Pr(1\ head)$)
- $\left(4 * \binom{4}{4} * \frac{1}{2^4}\right) + \left(3 * \binom{4}{3} * \frac{1}{2^4}\right) + \left(2 * \binom{4}{2} * \frac{1}{2^4}\right) + \left(1 * \binom{4}{1} * \frac{1}{2^4}\right) = \frac{4}{16} + \frac{12}{16} + \frac{12}{16} + \frac{4}{16} = 2$
- Calculate the expected number of heads using an indicator random variable and Linearity of expectation
- Let X_L be the indicator variable saying that coin L is heads. The number of heads is the sum of the indicator variables.
- $E\left[\sum_{l=1}^{4} X_l\right] = \sum_{l=1}^{4} E[X_l] = 4 * \frac{1}{2} = 2$

Linearity of expectation is very useful and powerful. It even works for dependent random variables

BACK TO THE HIRING ASSISTANT PROBLEM

```
HIRE-ASSISTANT(n)
1  best = 0                  // candidate 0 is a least-qualified dummy candidate
2  for i = 1 to n
3      interview candidate i
4      if candidate i is better than candidate best
5          best = i
6          hire candidate i
```

- Remember that we are looking for the number of candidates hired
- When picking a set of indicator random variables, use two cues:
  - Where does uncertainty come up? In other words, what are the events?
  - Can I combine my choice of random variables to find a solution?
- $X_i = I\{candidate\ i\ is\ hired\}$
- $\sum_{i=1}^{n} X_i = the\ number\ of\ hired\ candidates$

Calculating the Expected Number of Hires

```
HIRE-ASSISTANT(n)
best = 0          // candidate 0 is a least-qualified dummy candidate
for i = 1 to n
    interview candidate i
    if candidate i is better than candidate best
        best = i
        hire candidate i
```

- We're looking for the expected number of candidates hired
- We've assumed that all candidates have different quality and they are presented in random order (How many different orders?)
- Let's calculate the expected value of the indicator variable X_i
- When is candidate i hired?
  - What is the probability that candidate i is the best among the i total interviewees?

$$E[X_i] = \frac{1}{i}$$

Calculating the Expected Number of Hires

- We are looking for $\mathrm{E}[\sum_{i=1}^{n} X_i]$
- We know that $E[X_i] = \frac{1}{i}$
- We use linearity of expectation!
    - $\mathrm{E}[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$
- $\sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \frac{1}{i}$
- Therefore we hire $O(\log n)$ candidates on average

From Probabilistic Analysis to Randomized Algorithms

- We assumed that our input order for the hiring problem was randomly arranged
- We can easily convert our deterministic algorithm into a randomized algorithm and find the Expected cost

```
RANDOMIZED-HIRE-ASSISTANT(n)
randomly permute the list of candidates
best = 0            // candidate 0 is a least-qualified dummy candidate
for i = 1 to n
    interview candidate i
    if candidate i is better than candidate best
        best = i
        hire candidate i
```

- The text gives a way of randomly permuting arrays in linear time (pages 124-130)

One Additional Problem

- A group of n students are standing in a circle. Each simultaneously and independently uniformly at random point to either the student on their left or their right. Find the expected number of students who do not have anyone pointing at them.