CS 458, Data Structures and Algorithms II
Homework 4

June 2, 2020

Due on 11:00am June 9th, 2020. Collaboration is not allowed

Problem 1 (40 points)

Consider the following integer linear program that generalizes vertex cover

$$\begin{aligned} \text{minimize} \quad & c_1x_1 + \ldots + c_mx_m & \\ \text{subject to} \quad & a_{i,1}x_1 + \ldots + a_{i,m}x_m \geq b_i, & \text{for all } i \in 1, 2, \ldots, n, \\ & x_j \in \{0, 1\} & \text{for all } j \in 1, 2, \ldots, m. \end{aligned}$$

We will assume that the parameters satisfy the following conditions

- $a_{i,j}, b_i, c_j$ are all non-negative integers
- $b_i, c_j > 0$
- a feasible solution exists, namely $\sum_j a_{ij} \geq b_i$
- and let $f = \max_{i=1,\ldots,n}(\sum_{j=1}^{m} a_{i,j})$.

a. **(15 points)** State how the integer linear program generalizes vertex cover by defining $a_{i,j}, c_j, b_i$ and f in terms of a specific instance of vertex cover.

b . **(25 points)** Give a polynomial-time approximation algorithm for any problem captured by this general integer linear programming formulation with approximation ratio less than or equal to f (you need to provide an upper bound). In other words given any linear program that satisfies the conditions outlined above give an algorithm that has approximation ratio f.

Problem 2 (30 points)

Consider the *maximum cardinality matching problem on general graphs*: We seek to find a maximum cardinality subset of edges $M \subseteq E$ such that no edges $e, e' \in M$ share an endpoint. The integer linear programming formulation is the following:

$$\begin{aligned} \text{maximize} \quad & \sum_{e \in E} x_e & \\ \text{subject to} \quad & \sum_{e \text{ incident to } v} x_e \leq 1, & \forall v \in V \\ & x_e \in \{0, 1\}, & \forall e \in E \end{aligned}$$

Write its linear programming relaxation and show that the integrality gap is greater than or equal to 3/2 (you need to provide a lower bound)

Problem 3 (30 points)

Given an undirected graph $G = (V, E)$, suppose we want to remove a **minimum number** of edges $F \subseteq E$ such that the remaining graph $G' = (V, E - F)$ contains no triangle. (I.e., there are no three vertices u, v, w such that edges $(u, v), (v, w), (w, u)$ are all in G'.)

a . **(20 points)** Give a polynomial-time approximation algorithm to solve this problem with approximation ratio <u>less than or equal to</u> 3 (you need to provide an upper bound).

b . **(10 points)** Provide a lower bound that matches the upper bound you proved in part (a).