Homework 8
ECES 511

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Problem 1

Find the singular values of the matrix $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

Singular values are the square roots of the eigenvalues of $A^T A$ in decreasing order.

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \implies A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 & 1 \\ 2 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}$$

Now we find the eigenvalues of $A^T A$ expanding along the 3rd column.

$$det(A^T A - \lambda I) \implies \begin{vmatrix} 2-\lambda & 2 & 0 & 1 \\ 2 & 2-\lambda & 0 & 1 \\ 0 & 0 & -\lambda & 0 \\ 1 & 1 & 0 & 2-\lambda \end{vmatrix} = -\lambda \begin{vmatrix} 2-\lambda & 2 & 1 \\ 2 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = \lambda^4 - 6\lambda^3 + 6\lambda^2$$

$$\lambda^2(\lambda^2 - 6\lambda + 6) = 0$$

$$\lambda_1 = 0 \quad \lambda_2 = 0 \quad \lambda_3 = 3 + \sqrt{3} \quad \lambda_4 = 3 - \sqrt{3}$$

To get our singular values we compute $\sigma_i = \sqrt{\lambda_i}$ and order the σ's in decreasing order.

$$\sigma_1 = \sqrt{3 + \sqrt{3}} \qquad \sigma_2 = \sqrt{3 - \sqrt{3}} \quad \sigma_3 = 0 \quad \sigma_4 = 0$$

Problem 2

Find the SVD of the matrix $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

First we find the Σ matrix by finding the singular values as we did in problem 1.

$$A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \implies A^T A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Now we find the eigenvalues of $A^T A$ expanding along the 3rd column.

$$det(A^T A - \lambda I) \implies \begin{vmatrix} 1-\lambda & 0 & 1 & 0 \\ 0 & 1-\lambda & 0 & 1 \\ 1 & 0 & 1-\lambda & 0 \\ 0 & 1 & 0 & 1-\lambda \end{vmatrix} = \lambda^4 - 4\lambda^3 + 4\lambda^2$$

$$\lambda^2(\lambda - 2)^2 = 0$$

$$\lambda_1 = 0 \quad \lambda_2 = 0 \quad \lambda_3 = 2 \quad \lambda_4 = 2$$

To get our singular values we compute $\sigma_i = \sqrt{\lambda_i}$ and order the σ's in decreasing order.

$$\sigma_1 = \sqrt{2} \qquad \sigma_2 = \sqrt{2} \quad \sigma_3 = 0 \quad \sigma_4 = 0$$

To compute Σ we place the entries on the diagonal of a matrix with the same size of A

$$\Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix}$$

Next we look to find the V^T matrix which is just any orthonormal basis of the eigenvectors of A^TA. So first we find the eigenvectors of A^TA.

$$\lambda_1 = 0 \implies \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} x = 0 \implies q_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

And by inspection we can also find the second eigenvector associated with $\lambda = 0$

$$q_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad q_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda = 2 \implies \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} x = 0 \implies q_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

And by inspection we can also find the second eigenvector associated with $\lambda = 2$

$$q_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad q_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

We can see by inspection that all the eigenvectors are orthogonal to each other. Now they can be normalized by dividing by their magnitude.

$$q_1 = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \qquad q_2 = \begin{bmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix} \qquad q_3 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \qquad q_4 = \begin{bmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

Reordering them to match the singular values we get

$$V = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

Lastly we find U whos columns can be found by computing $\sigma_i^{-1}Av_i$ where i indicates the column. For any extra columns we need just arbitrarily extend the columns to be orthonormal using gram schmidt orthogonalization.

$$u_1 = \sigma_1^{-1}Av_1 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$$

$$u_2 = \sigma_2^{-1}Av_2 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

Normalizing each column we get

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Putting all of this together we can check the solution by checking the following relationship $A = U\Sigma V^T$

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix} \qquad V = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Problem 3

Given matrix $X = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 4 \\ 4 & 3 \end{bmatrix}$

1) Find the eigenvalues and eigenvectors φ_1 and φ_2 of the matrix.
   Singular values are the square roots of the eigenvalues of $A^T A$ in decreasing order.

$$A^T = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{bmatrix} \implies A^T A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 4 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 30 & 28 \\ 28 & 30 \end{bmatrix}$$

   Now we find the eigenvalues of $A^T A$ expanding along the 3rd column.

$$det(A^T A - \lambda I) \implies \begin{vmatrix} 30-\lambda & 28 \\ 28 & 30-\lambda \end{vmatrix} = (30-\lambda)(30-\lambda) - 28^2 = \lambda^2 - 60\lambda + 116$$

$$(\lambda - 58)(\lambda - 2) = 0$$

$$\lambda_1 = 58 \quad \lambda_2 = 2$$

   Finding the eigenvectors associated with λ_1 and λ_2

$$\lambda_1 = 58 \implies \begin{bmatrix} -28 & 28 \\ 28 & -28 \end{bmatrix} x = 0 \implies q_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_1 = 2 \implies \begin{bmatrix} 28 & 28 \\ 28 & 28 \end{bmatrix} x = 0 \implies q_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\lambda_1 = 58 \quad \lambda_2 = 28 \quad \varphi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \varphi_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

2) Calculate $Z = X[\varphi_1, \varphi_2]$, what is the relationship between vectors in Z and vectors in X?

$$Z = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 3 & -1 \\ 7 & 1 \\ 7 & -1 \end{bmatrix}$$

   The matrix $[\varphi_1 \varphi_2]$ equals a rotation matrix of 90°. If the matrix was normalized then the vectors in Z would represent the vectors but rotated 90 degrees. Since it isn't normalized they are also scaled by a factor of $\sqrt{2}$