Homework 2
ECES 511

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Problem 1

Find the basis of the range spaces and null spaces of the matricies below.

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{A}_2 = \begin{bmatrix} 4 & 1 & -1 \\ 3 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \qquad \mathbf{A}_3 = \begin{bmatrix} 1 & 2 & 4 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Basis($\mathbf{A}_1$) trivially independent:

$$q_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad q_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Null space($\mathbf{A}_1$):

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$$

With this we get $n_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ as the basis of our null space so our null space($\mathbf{A}_1$) = span$\{n_1\}$.

Basis($\mathbf{A}_2$): If the determinant is nonzero then all columns are independent of each other since the null space only contains 0. With this fact and the Dimension - rank = nullity, we know that the rank is 3 so we have 3 independent basis vectors.

$$\det \mathbf{A}_2 = \begin{vmatrix} 4 & 1 & -1 \\ 3 & 2 & 0 \\ 1 & 1 & 0 \end{vmatrix} = -1 \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} - 0 \begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} + 0 \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} = -1$$

Therefore a set of basis vectors are:

$$q_1 = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} \qquad q_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad q_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

Since our rank is 3 and our dimension is also 3 (full rank) then our null space must be empty (only contains the 0 vector).

Basis($\mathbf{A}_3$): by inspection $2 \cdot q_2 = q_3$ and the rest are independent

$$q_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad q_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \qquad q_3 = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

Null space($\mathbf{A}_3$):

$$\begin{bmatrix} 1 & 2 & 4 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a & 2b & 4c & 4d \\ 0 & -b & -2c & 2d \\ 0 & 0 & 0 & 1d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 0 \\ b \\ -\frac{b}{2} \\ 0 \end{bmatrix}$$

With this we get $n_1 = \begin{bmatrix} 0 \\ 1 \\ -\frac{1}{2} \\ 0 \end{bmatrix}$ as the basis of our null space so the null space($\mathbf{A}_3$) = span$\{n_1\}$.

Problem 2

Let $\mathbf{v_1} = \begin{bmatrix}1\\2\\1\\-2\\3\end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix}2\\5\\-1\\3\\-2\end{bmatrix}$, $\mathbf{v_3} = \begin{bmatrix}1\\3\\-2\\5\\-5\end{bmatrix}$, $\mathbf{v_4} = \begin{bmatrix}3\\1\\2\\-4\\1\end{bmatrix}$, $\mathbf{v_5} = \begin{bmatrix}5\\6\\1\\-1\\-1\end{bmatrix}$

Let $\mathbf{W} : \{\mathbf{w} \in \mathbb{R}^5 \quad | \quad \mathbf{w} = a_1\mathbf{v_1} + a_2\mathbf{v_2} + a_3\mathbf{v_3} + a_4\mathbf{v_4} + a_5\mathbf{v_5}\}$

1. Find basis $\mathbf{Q}$ of $\mathbf{W}$
   By inspection we can see that $q_1 + q_3 = q_2$ and $q_1 + q_3 + q_4 = q_5$ the rest are linearly independent. So we keep remove 2 of the linearly dependent, I'm choosing q_2, q_5 and keeping q_1, q_3, q_4

$$\mathbf{Q} = \begin{bmatrix}1 & 1 & 3\\2 & 3 & 1\\1 & -2 & 2\\-2 & 5 & -4\\3 & -5 & 1\end{bmatrix}$$

2. What is the dimension of $\mathbf{W}$
   $dim(\mathbf{W}) = 3$

3. What is the representation of vector $\mathbf{x} = \begin{bmatrix}-5\\6\\-5\\11\\-1\end{bmatrix}$ with respect to the basis $\mathbf{Q}$
   We can find the coefficients by gauss jordan reduction

$$\begin{bmatrix}1 & 1 & 3\\2 & 3 & 1\\1 & -2 & 2\\-2 & 5 & -4\\3 & -5 & 1\end{bmatrix}\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} = \begin{bmatrix}-5\\6\\-5\\11\\-1\end{bmatrix} \Leftrightarrow \begin{bmatrix}1 & 1 & 3 & -5\\2 & 3 & 1 & 6\\1 & -2 & 2 & -5\\-2 & 5 & -4 & 11\\3 & -5 & 1 & -1\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & 3 & -5\\2 & 3 & 1 & 6\\1 & -2 & 2 & -5\\-2 & 5 & -4 & 11\\3 & -5 & 1 & -1\end{bmatrix} \Longrightarrow \begin{bmatrix}1 & 1 & 3 & -5\\0 & 1 & -5 & 16\\0 & -3 & -1 & 0\\0 & 7 & 2 & 1\\0 & -8 & -8 & 14\end{bmatrix} \Longrightarrow \begin{bmatrix}1 & 1 & 3 & -5\\0 & 1 & -5 & 16\\0 & 0 & -48 & 142\\0 & 0 & 37 & -111\\0 & 0 & -16 & 48\end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix}1 & 1 & 3 & -5\\0 & 1 & -5 & 16\\0 & 0 & -48 & 142\\0 & 0 & 0 & -\frac{37}{24}\\0 & 0 & 0 & \frac{2}{3}\end{bmatrix} \Longrightarrow \begin{bmatrix}1 & 1 & 3 & -5\\0 & 1 & -5 & 16\\0 & 0 & -48 & 142\\0 & 0 & 0 & -\frac{37}{24}\\0 & 0 & 0 & 0\end{bmatrix} \Longrightarrow \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix}$$

   Since the last column is dependent on itself there is no soultion for $\mathbf{x}$ with respect to the basis $\mathbf{Q}$.

4. Orthogonalize $\mathbf{Q}$ using Gram-Schmidt orthogonalization

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{bmatrix}1\\2\\1\\-2\\3\end{bmatrix}$$

$$\mathbf{u}_2 = \mathbf{v}_3 - \frac{\mathbf{u}_1^T\mathbf{v}_3}{\sqrt{\mathbf{u}_1\mathbf{u}_1}}\mathbf{u}_1 = \begin{bmatrix}1\\3\\-2\\5\\-5\end{bmatrix} - \frac{1+6-2-10-15}{\sqrt{1+4+1+4+3}}\begin{bmatrix}1\\2\\1\\-2\\3\end{bmatrix} = \begin{bmatrix}3\\1\\2\\-4\\1\end{bmatrix} - \frac{-20}{\sqrt{13}}\begin{bmatrix}1\\2\\1\\-2\\3\end{bmatrix} \approx \begin{bmatrix}2.0526\\5.1053\\-0.9474\\2.8947\\-1.8421\end{bmatrix}$$

$$\mathbf{u}_2 = \mathbf{v}_4 - \frac{\mathbf{u}_1^T\mathbf{v}_4}{\sqrt{\mathbf{u}_1\mathbf{u}_1}}\mathbf{u}_1 - \frac{u_2^T v_4}{\sqrt{\mathbf{u}_2\mathbf{u}_2}}\mathbf{u}_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -4 \\ 1 \end{bmatrix} - \frac{3+2+2+8+3}{\sqrt{13}} \begin{bmatrix} 1 \\ 2 \\ 1 \\ -2 \\ 3 \end{bmatrix} - \frac{\mathbf{u}_2^T\mathbf{v}_4}{6.5534} \begin{bmatrix} 2.0526 \\ 5.1053 \\ -0.9474 \\ 2.8947 \\ -1.8421 \end{bmatrix} \approx \begin{bmatrix} 2.2462 \\ -0.4130 \\ 0.9632 \\ -1.8321 \\ -2.0159 \end{bmatrix}$$

So we have an orthonormal basis: $\mathbf{U} = \begin{bmatrix} 1 & 2.0526 & 2.2462 \\ 2 & 5.1053 & -0.4130 \\ 1 & -0.9474 & 0.9632 \\ -2 & 2.8947 & -1.8321 \\ 3 & -1.8421 & -2.0159 \end{bmatrix}$

Normalizing the basis we get: $\mathbf{e} = \begin{bmatrix} 0.2294 & 0.3132 & 0.6099 \\ 0.4588 & 0.7790 & -0.1121 \\ 0.2294 & -0.1446 & 0.2615 \\ -0.4588 & 0.4417 & -0.4974 \\ 0.6882 & -0.2811 & -0.5474 \end{bmatrix}$

5. Let $\mathbf{B} = \begin{bmatrix} 1 & 2 & 1 & 3 & 5 \\ 2 & 5 & 3 & 1 & 6 \\ 1 & -2 & -2 & 2 & 1 \\ -2 & 3 & 5 & -4 & -1 \\ 3 & -2 & -5 & 1 & -1 \end{bmatrix}$, find the range($\mathbf{B}$), the column space of $\mathbf{B}$, the rank of $\mathbf{B}$ and the null space of $\mathbf{B}$

We saw from above that q_2 and q_5 are linearly dependent on the other columns so the range will be the span of the remaining vecotrs.

range($\mathbf{B}$)=span($\begin{bmatrix} 1 \\ 2 \\ 1 \\ -2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -2 \\ 5 \\ -5 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \\ -4 \\ 1 \end{bmatrix}$)

The column space of $\mathbf{B}$ is the set of linearly independent vectors $\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ -2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -2 \\ 5 \\ -5 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \\ -4 \\ 1 \end{bmatrix} \}$

Since we have 3 linearly independent vectors we have rank($\mathbf{B}$)=3

Finding the null space:

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 5 \\ 2 & 5 & 3 & 1 & 6 \\ 1 & -1 & -2 & 2 & 1 \\ -2 & 3 & 5 & -4 & -1 \\ 3 & -2 & -5 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 1 & 2 & 1 & 3 & 5 & 0 \\ 2 & 5 & 3 & 1 & 6 & 0 \\ 1 & -1 & -2 & 2 & 1 & 0 \\ -2 & 3 & 5 & -4 & -1 & 0 \\ 3 & -2 & -5 & 1 & -1 & 0 \end{bmatrix}$$

Row reduce with gaussian elimination:

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Free variables c and e which gives us $n_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $n_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

So the null space is the span(n_1, n_2)