Homework 4
ECES 511

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Problem 1

Given

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

find $\mathbf{A}^{10}$, $\mathbf{A}^{103}$,and $e^{\mathbf{A}t}$

- $\mathbf{A}^{10}$
  Let $f(\lambda) = \lambda^{10}$

  $$\Delta(A) = (\lambda - 1)(\lambda(\lambda - 1)) = \lambda(\lambda - 1)^2$$

  $$\lambda_1 = 0 \qquad \lambda_2 = 1 \qquad \lambda_3 = 1$$

  Let $h(\lambda) = \beta_0 + \beta_1\lambda + \beta_2\lambda^2$
  With λ_1 and knowing it's multiplicity is 1 we have one equation

  $$h(\lambda_1) = f(\lambda_1) \implies \beta_0 + \beta_1(0) + \beta_2(0) = 0^{10} \implies \beta_0 = 0$$

  With λ_2 and knowing it's multiplicity is 2 we have two equations

  $$h(\lambda_2) = f(\lambda_2) \implies \beta_0 + \beta_1(1) + \beta_2(1)^2 = 1^{10} \implies \beta_1 + \beta_2 = 1$$

  $$h'(\lambda_2) = f'(\lambda_2) \implies \beta_1 + 2\beta_2(1) = (10)1^9 \implies \beta_1 + 2\beta_2 = (10)$$

  $$\beta_0 = 0 \qquad \beta_1 = -8 \qquad \beta_2 = 9$$

  $$\mathbf{A}^{10} = f(\mathbf{A}) = h(\mathbf{A}) = \beta_2\mathbf{A}^2 + \beta_1\mathbf{A} = 9\mathbf{A}^2 - 8\mathbf{A}$$

  $$\mathbf{A}^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

  $$\mathbf{A}^{10} = \begin{bmatrix} 9 & 9 & 9 \\ 0 & 0 & 9 \\ 0 & 0 & 9 \end{bmatrix} - \begin{bmatrix} 8 & 8 & 0 \\ 0 & 0 & 8 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

  $$\mathbf{A}^{10} = \begin{bmatrix} 1 & 1 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

- $\mathbf{A}^{103}$
  Let $f(\lambda) = \lambda^{103}$

  $$\Delta(A) = (\lambda - 1)(\lambda(\lambda - 1)) = \lambda(\lambda - 1)^2$$

  $$\lambda_1 = 0 \qquad \lambda_2 = 1 \qquad \lambda_3 = 1$$

  Let $h(\lambda) = \beta_0 + \beta_1\lambda + \beta_2\lambda^2$
  With λ_1 and knowing it's multiplicity is 1 we have one equation

  $$h(\lambda_1) = f(\lambda_1) \implies \beta_0 + \beta_1(0) + \beta_2(0) = 0^{103} \implies \beta_0 = 0$$

With λ_2 and knowing it's multiplicity is 2 we have two equations

$$h(\lambda_2) = f(\lambda_2) \implies \beta_0 + \beta_1(1) + \beta_2(1)^2 = 1^{103} \implies \beta_1 + \beta_2 = 1$$

$$h'(\lambda_2) = f'(\lambda_2) \implies \beta_1 + 2\beta_2(1) = (103)1^{102} \implies \beta_1 + 2\beta_2 = (103)$$

$$\beta_0 = 0 \qquad \beta_1 = -101 \qquad \beta_2 = 102$$

$$\mathbf{A}^{10} = f(\mathbf{A}) = h(\mathbf{A}) = \beta_2\mathbf{A}^2 + \beta_1\mathbf{A} = 102\mathbf{A}^2 - 101\mathbf{A}$$

$$\mathbf{A}^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}^{103} = \begin{bmatrix} 102 & 102 & 102 \\ 0 & 0 & 102 \\ 0 & 0 & 102 \end{bmatrix} - \begin{bmatrix} 101 & 101 & 0 \\ 0 & 0 & 101 \\ 0 & 0 & 101 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 102 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}^{103} = \begin{bmatrix} 1 & 1 & 102 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

- $e^{\mathbf{A}t}$
  Let $f(\lambda) = e^{\lambda t}$

$$\Delta(A) = (\lambda - 1)(\lambda(\lambda - 1)) = \lambda(\lambda - 1)^2$$

$$\lambda_1 = 0 \qquad \lambda_2 = 1 \qquad \lambda_3 = 1$$

Let $h(\lambda) = \beta_0 + \beta_1\lambda + \beta_2\lambda^2$
With λ_1 and knowing it's multiplicity is 1 we have one equation

$$h(\lambda_1) = f(\lambda_1) \implies \beta_0 + \beta_1(0) + \beta_2(0) = e^{0t} \implies \beta_0 = 1$$

With λ_2 and knowing it's multiplicity is 2 we have two equations

$$h(\lambda_2) = f(\lambda_2) \implies \beta_0 + \beta_1(1) + \beta_2(1)^2 = e^t \implies \beta_0 + \beta_1 + \beta_2 = e^t$$

$$h'(\lambda_2) = f'(\lambda_2) \implies \beta_1 + 2\beta_2(1) = te^t \implies \beta_0 + \beta_1 + 2\beta_2 = te^t$$

$$\beta_0 = 1 \qquad \beta_1 = 2e^t - te^t - 1 \qquad \beta_2 = te^t - e^t$$

$$e^{\mathbf{A}t} = f(\mathbf{A}) = h(\mathbf{A}) = \beta_2\mathbf{A}^2 + \beta_1\mathbf{A} + \beta_0 = (te^t - e^t)\mathbf{A}^2 + (2e^t - te^t - 1)\mathbf{A} + \mathbf{I}$$

$$\mathbf{A}^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{\mathbf{A}t} = (te^t - e^t)\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + (2e^t - te^t - 1)\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & e^t - 1 & te^t - e^t + 1 \\ 0 & 1 & e^t - 1 \\ 0 & 0 & e^t \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^t & e^t - 1 & te^t - e^t + 1 \\ 0 & 1 & e^t - 1 \\ 0 & 0 & e^t \end{bmatrix}$$

Problem 2

Find the least square solutions of $\mathbf{A}x = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

What is the quantity being minimized?

We can manipulate the original equation to get

$$x = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$$

This equation looks to minimize $(\hat{\mathbf{b}} - \mathbf{b})^2$. Where $\hat{\mathbf{b}} = A(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$ or what our mapping projects $\mathbf{A}$ to.

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$$

$$(\mathbf{A}^T\mathbf{A})^{-1} = \frac{1}{|\mathbf{A}^T\mathbf{A}|} adj(\mathbf{A}^T\mathbf{A}) = \frac{1}{25-1} \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$$

$$x = \frac{1}{24} \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{24} \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \frac{1}{24} \begin{bmatrix} 8 \\ -8 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$$

How close are we

$$\mathbf{A}x = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}$$

So our total error is $\frac{1}{3}^2 + (-\frac{2}{3})^2 + (-\frac{1}{3})^2 = \frac{2}{3}$

Problem 3

Suppose that we have measured three data points

$$(0, 6), \quad (1, 0), \quad (2, 0)$$

and our model is linear, compute the line of best fit by the method of least squares.

Create a system of equations to find a solution for our line which has a form $y = ax + b$

$$\mathbf{A} \begin{bmatrix} a \\ b \end{bmatrix} = \mathbf{b}$$

Each row in $\mathbf{A}$ and $\mathbf{b}$ will correspond to one data point from the three inputs. The first column of $\mathbf{A}$ should correspond to the x inputs that get multiplied by a, so the first number in each pair. The second column of $\mathbf{A}$ will contain all 1's since all terms must add the b term to be a line. The $\mathbf{b}$ will contain the expected output that we hope to match for each data point.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad \mathbf{b} \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

Using the same process as in Problem 2 we can find the values for a and b by solving this equation

$$\begin{bmatrix} a \\ b \end{bmatrix} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$$

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix} \qquad (\mathbf{A}^T\mathbf{A})^{-1} = \frac{1}{6} \begin{bmatrix} 3 & -3 \\ -3 & 5 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

The equation of the best fit line is

$$y = -3x + 5$$

Problem 4

Find the minimum polynomial for the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$

We can first start with the characteristic polynomial

$$(\lambda - 1) * (\lambda(\lambda - 3) + 2) = (\lambda - 1)(\lambda^2 - 3\lambda + 2); = (\lambda - 1)(\lambda - 2)(\lambda - 1);$$

We have 3 roots and we want the least common multiple of the roots we get that the polynomial is

$$\mu(X) = (X - 1)(X - 2) = X^2 - 3X + 2$$

Problem 5

Find the parabola that best approximates the data points,

$$(-1, \frac{1}{2}), \quad (1, -1), \quad (2, -\frac{1}{2}), \quad (3, 2)$$

What is the quantity being minimized?

Create a system of equations to find a solution for our line which has a form $y = ax^2 + bx + c$

$$\mathbf{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \mathbf{b}$$

Each row in **A** and **b** will correspond to one data point from the three inputs.
The first column of **A** should correspond to the x^2 inputs that get multiplied by a, so the square of the first number in each pair.
The second column of **A** should correspond to the x inputs that get multiplied by b, so the first number in each pair. The second column of **A** will contain all 1's since all terms must add the c term to be a line. The **b** will contain the expected output that we hope to match for each data point.

$$\mathbf{A} = \begin{bmatrix} (-1)^2 & -1 & 1 \\ 1^2 & 1 & 1 \\ 2^2 & 2 & 1 \\ 3^2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix} \qquad \mathbf{b} \begin{bmatrix} \frac{1}{2} \\ -1 \\ -\frac{1}{2} \\ 2 \end{bmatrix}$$

Using the same process as in Problem 2 we can find the values for a, b, and c by solving this equation

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$$

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 99 & 35 & 15 \\ 35 & 15 & 5 \\ 15 & 5 & 4 \end{bmatrix} \qquad (\mathbf{A}^T\mathbf{A})^{-1} = \frac{1}{440}\begin{bmatrix} 35 & -65 & -50 \\ -65 & 171 & 30 \\ -50 & 30 & 260 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{440}\begin{bmatrix} 265 \\ -379 \\ -410 \end{bmatrix}$$

The equation of the best fit line is

$$y = \frac{53}{88}x^2 + \frac{-379}{440}x + \frac{-41}{44}$$