Homework 5
ECES 511

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For the Following problems, **manually** calculate the expression for the response then **validate** your answer with MATLAB Simulation. Ex. If you get the response of the system $y(t) = e^{-t}$, please build your system in MATLAB and compare the output obtained from the system with $y(t) = e^{-t}$ in one plot for verification.

Problem 1

Find the response of the output variable

$$y = 2x_1 + x_2$$

in the system described by state equations

$$\dot{x}_1 = -2x_1 + u$$

$$\dot{x}_2 = x_1 - x_2$$

for a constant input $u(t) = 5$ for $t > 0$, i $x_1(0) = 0$, and $x_2 = 0$.

First we rewrite the equation to be in the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\dot{\mathbf{y}}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

$$\mathbf{A} = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 2 & 1 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\mathbf{x}(t) = e^{\mathbf{At}}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

$$e^{\mathbf{At}} = \begin{bmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{bmatrix}$$

$$e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau) = \begin{bmatrix} 5e^{-2(t-\tau)} \\ 5e^{-(t-\tau)} - 5e^{-2(t-\tau)} \end{bmatrix}$$

Since $x(0) = 0$ we only need to evaluate the integral.

$$x(t) = \int_0^t \begin{bmatrix} 5e^{-2(t-\tau)} \\ 5e^{-(t-\tau)} - 5e^{-2(t-\tau)} \end{bmatrix} d\tau = \frac{5}{2}\begin{bmatrix} 1 - e^{-2t} \\ e^{-2t}(e^t - 1)^2 \end{bmatrix}$$

Since $\mathbf{D}$ is zero we can find $\mathbf{y}$ with $\mathbf{y} = \mathbf{C}x(t)$

$$\mathbf{y}(t) = \mathbf{C}x(t) = \frac{5}{2}(3 - e^{-2t} - 2e^{-t})$$

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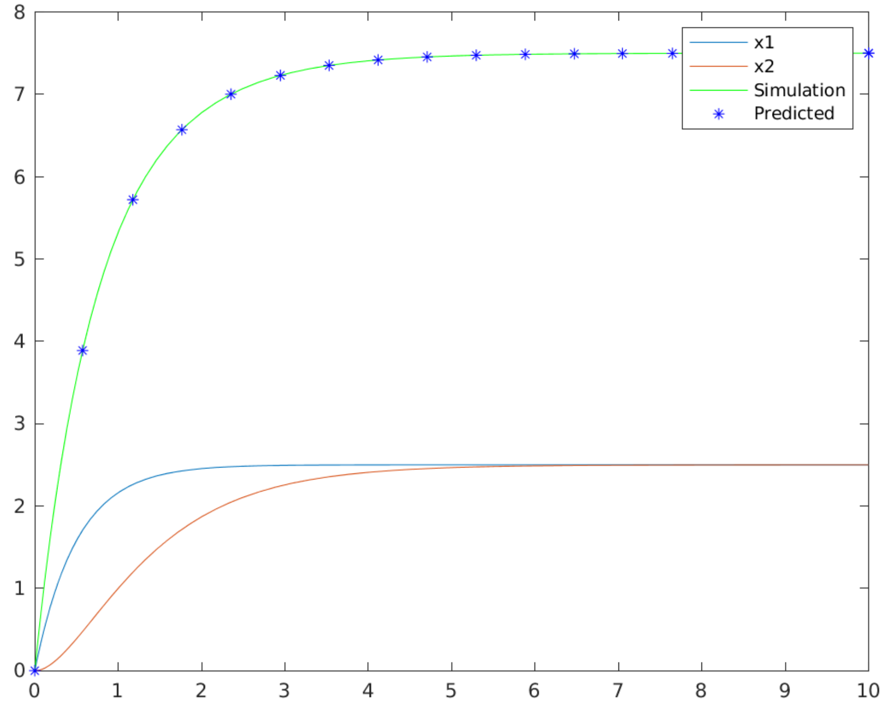
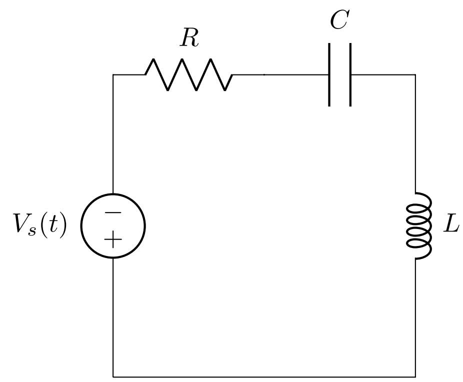

x1
x2
Simulation
Predicted
0
1
2
3
4
5
6
7
8
9
10

Problem 2



Given the standard RLC circuit (R = 1.2, C = 1, L = 0.2). To build the state space equations, we choose capacitor voltage $v_c(t) = x_1(t)$ and inductor current $i_l(t) = x_2(t)$ as state variables and $v_c(t)$ as output.

1) Find the state space equation for the RLC system
   We can start by expressing our state variables using Kirchoff's Voltage law.

$$V_s(t) = V_r(t) + V_c(t) + V_l(t)$$

$$V_s(t) = iR + V_c(t) + L\frac{di}{dt}$$

$$\frac{dV_c}{dt} = \frac{i}{C}$$

   Rewriting these in terms of our state variables we get the state equation.

$$\dot{x}_1 = \frac{x_2}{C}$$

$$\dot{x}_2 = \frac{1}{L}(V_s(t) - x_2R - x_1)$$

$$y = x_1$$

$$\mathbf{A} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

   Plugging in our values

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

2) If $v_c(0) = 0$,$i_l(0) = 5$,$v_s(t) = 0$, find the $\mathbf{x}(t)$ for the system

$$\mathbf{x}(t) = e^{\mathbf{A}\mathbf{t}}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

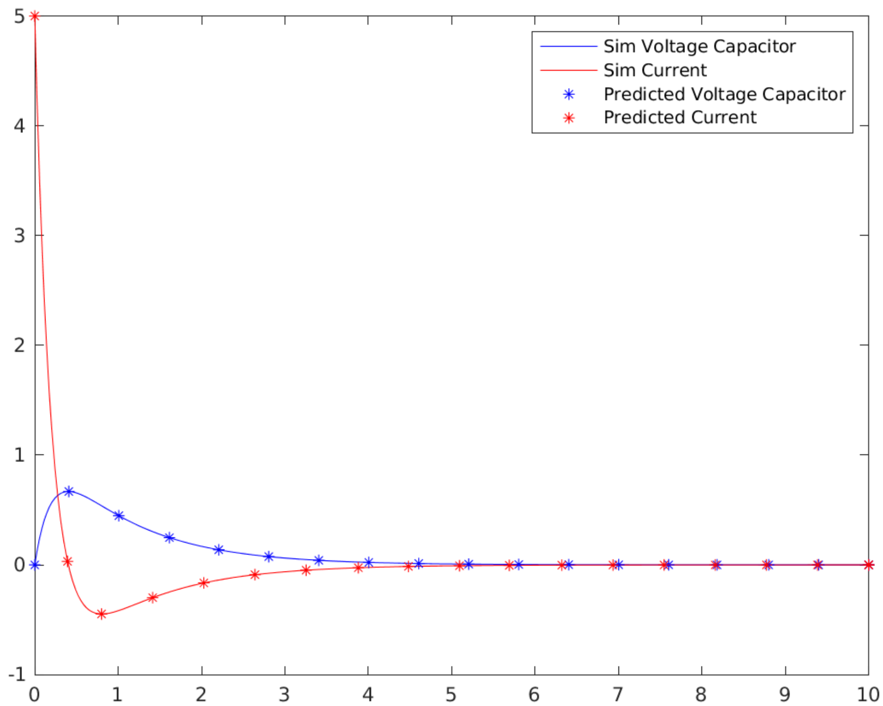
$$x(0) = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} .25e^{-1t} - 0.25e^{-5t} & 0.25e^{-1t} - 0.25e^{-5t} \\ 1.25e^{-5t} - 1.25e^{-1t} & 1.25e^{-5t} - 0.25e^{-1t} \end{bmatrix}$$

$$e^{\mathbf{A}t}x(0) = \begin{bmatrix} 1.25e^{-1t} - 1.25e^{-5t} \\ 6.25e^{-5t} - 1.25e^{-1t} \end{bmatrix}$$

$$\mathbf{u} = 0 \implies e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u} = 0$$

$$\implies \mathbf{x}(t) = \begin{bmatrix} 1.25e^{-1t} - 1.25e^{-5t} \\ 6.25e^{-5t} - 1.25e^{-1t} \end{bmatrix}$$



3) If $v_c(0) = 0$,$i_l(0) = 0$,$v_s(t) = 5$, find the $y(t)$ for the system
We have mostly the same situation just a few parameters have changed.

$$x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad v_s \neq 0$$

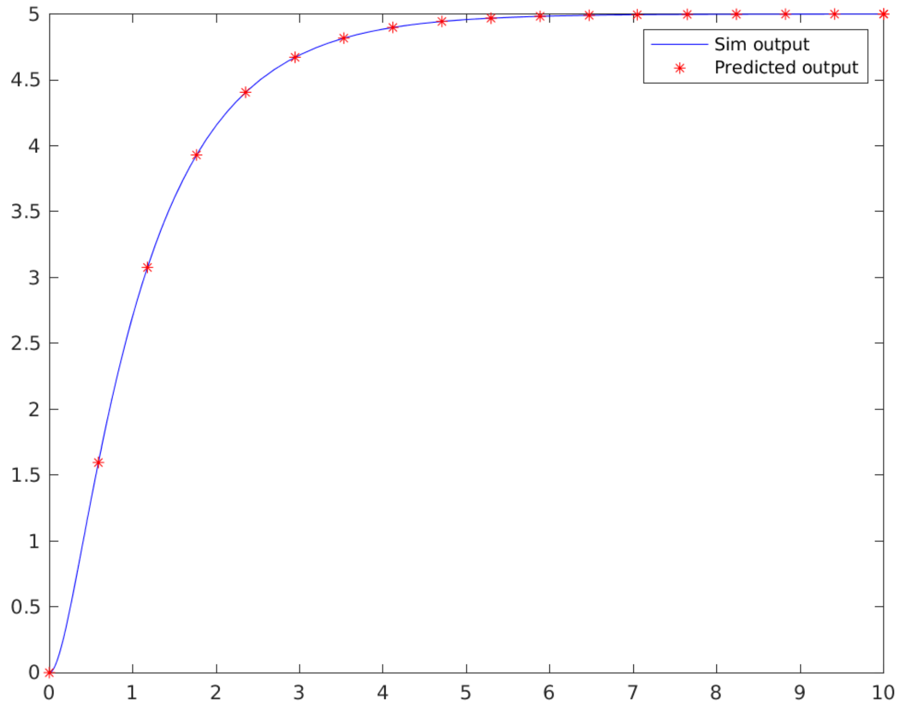
So we must evaluate the convolution.

$$\int_0^t e^{\mathbf{A}(t-\tau)} \mathbf{Bu}(\tau) d\tau = \begin{bmatrix} 5 + 1.25e^{-5t} - 6.25e^{-1t} \\ -1.7763 * 10^{-15} - 6.25e^{-5t} + 6.25e^{-1t} \end{bmatrix} = x(t)$$

And since the initial condition is zero the state variables are just the convolution.

$$y(t) = \mathbf{C}x(t)$$

$$y(t) = 5 + 1.25e^{-5t} - 6.25e^{-1t}$$


Sim output
Predicted output
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0
1
2
3
4
5
6
7
8
9
10

Problem 3

Consider the system

$$\ddot{x} + 4\dot{x} + 4x = u$$

1) Find the state space model for the system
Let $x_1 = x$ and $x_2 = \dot{x}$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -4x_2 - 4x_1 + u \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

2) Find the transfer function for the system. (Hint: you can use $\mathbf{C}(s\mathbf{I} - \mathbf{A}^{-1})\mathbf{B} + D$)

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & -1 \\ 4 & s+4 \end{bmatrix} \implies (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^2 + 4s + 4} \begin{bmatrix} s+4 & 1 \\ -4 & s \end{bmatrix}$$

$$\hat{\mathbf{x}}(s) = \frac{1}{s^2 + 4s + 4} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+4 & 1 \\ -4 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{s^2 + 4s + 4}$$

$$\hat{\mathbf{x}}(s) = \frac{1}{s^2 + 4s + 4}$$

3) Suppose $u(t) = e^{-2t} \sin(t)$ for $t \geq 0$, and $\ddot{x}(0) = \dot{x}(0) = 0$. Find the solution $x(t)$
We want to use $\hat{\mathbf{y}}(s) = \hat{\mathbf{G}}(s)\hat{\mathbf{u}}(s)$ where $\hat{\mathbf{G}}(s)$ is what we computed in part 2. First we must find the input's representation in the laplace space.

$$\hat{\mathbf{u}}(s) = \mathcal{L}\{e^{-2t} \sin(t)\} = \frac{1}{s^2 + 4s + 5} \qquad t > 0$$

$$\hat{\mathbf{y}}(s) = \frac{1}{s^2 + 4s + 5} \frac{1}{s^2 + 4s + 4} = \frac{1}{(2+s)^2(5 + 4s + s^2)}$$

$$y(t) = \mathcal{L}^{-1}(\hat{\mathbf{y}}) = e^{-2t}(t - \sin(t))$$