Homework 7
ECES 511

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Problem 1

Use two different methods to find the unit-step response of

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u$$

$$y = \begin{bmatrix} 2 & 3 \end{bmatrix}\mathbf{x}$$

1. Convolution

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

$$\mathbf{x}(t) = e^{\mathbf{At}}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

$$e^{\mathbf{At}} = \begin{bmatrix} e^{-t}(\sin(t)+\cos(t)) & e^{-t}\sin(t) \\ -2e^{-t}\sin(t) & e^{-t}(\cos(t)-\sin(t)) \end{bmatrix}$$

Since the initial conditon is not specified I will be assuming $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The first part of the equation will go to zero.

$$\int_0^t \begin{bmatrix} e^{\tau-t}(\sin(t-\tau)+\cos(t-\tau)) & e^{\tau-t}\sin(t-\tau) \\ -2e^{\tau-t}\sin(t-\tau) & e^{\tau-t}(\cos(t-\tau)-\sin(t-\tau)) \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}d\tau$$

$$= \int_0^t \begin{bmatrix} e^{\tau-t}(2\sin(t-\tau)+\cos(t-\tau)) \\ e^{\tau-t}(\cos(t-\tau)-3\sin(t-\tau)) \end{bmatrix}d\tau$$

$$= \begin{bmatrix} \frac{3}{2} - \frac{1}{2}e^{-t}(\sin(t)+3\cos(t)) \\ e^{-t}(2\sin(t)+\cos(t)) - 1 \end{bmatrix}$$

$$\implies y = \begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} \frac{3}{2} - \frac{1}{2}e^{-t}(\sin(t)+3\cos(t)) \\ e^{-t}(2\sin(t)+\cos(t)) - 1 \end{bmatrix}$$

$$y(t) = 5e^{-t}\sin(t)$$

2. Laplace Find the solution in the laplace space by computing $\hat{\mathbf{y}}(s) = \hat{\mathbf{G}}(s)\hat{\mathbf{u}}(s)$. Where $\hat{\mathbf{G}} = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} + D$

$$\hat{\mathbf{G}} = \begin{bmatrix} 2 & 3 \end{bmatrix}\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}\right)^{-1} = \begin{bmatrix} \frac{s+2}{s(s+2)+2} & \frac{1}{s(s+2)+2} \\ -\frac{2}{s(s+2)+2} & \frac{s}{s(s+2)+2} \end{bmatrix}$$

$$\hat{\mathbf{G}} = \frac{5s}{s(s+2)+2}$$

$$\hat{\mathbf{u}} = \mathcal{L}\{u\} = \frac{1}{s}$$

$$\hat{\mathbf{y}} = \hat{\mathbf{G}}\hat{\mathbf{u}} = \frac{5s}{s(s(s+2)+2)} = \frac{5}{(s+1)^2+1}$$

$$y(t) = \mathcal{L}^{-1}\{\hat{\mathbf{y}}\} = 5e^{t}\sin(t)$$

$$y(t) = 5e^{-t}\sin(t)$$

Problem 2

Discretize the state equation from Problem 1 for $T = 1$ and $T = \pi$

$$\mathbf{A}_d = e^{\mathbf{A}T} \qquad \mathbf{B}_d = \int_0^T e^{\mathbf{A}\alpha} d\alpha \mathbf{B} \qquad \mathbf{C}_d = \mathbf{C} \qquad \mathbf{D}_d = \mathbf{D}$$

$$\mathbf{A}_d = \begin{bmatrix} e^{-T}(\sin(T) + \cos(T)) & e^{-T}\sin(T) \\ -2e^{-T}\sin(T) & e^{-T}(\cos(T) - \sin(T)) \end{bmatrix}$$

$$\mathbf{B}_d = \begin{bmatrix} 1 - e^{-T}\cos(T) & \frac{1}{2} - \frac{1}{2}e^{-T}(\sin(T) + \cos(T)) \\ e^{-T}(\sin(T) + \cos(T)) - 1 & e^{-T}\sin(T) \end{bmatrix}$$

$$\mathbf{C}_d = \begin{bmatrix} 2 & 3 \end{bmatrix}$$

a) Let $T = 1$

$$\mathbf{A}_d = \begin{bmatrix} 0.508326 & 0.30956 \\ -0.61912 & -0.110794 \end{bmatrix}$$

$$\mathbf{B}_d = \begin{bmatrix} 1.04707 \\ -0.182114 \end{bmatrix}$$

$$\mathbf{C}_d = \begin{bmatrix} 2 & 3 \end{bmatrix}$$

b) Let $T = \pi$

$$\mathbf{A}_d = \begin{bmatrix} -0.0432139 & 0. \\ 0. & -0.0432139 \end{bmatrix}$$

$$\mathbf{B}_d = \begin{bmatrix} 1.56482 \\ -1.04321 \end{bmatrix}$$

$$\mathbf{C}_d = \begin{bmatrix} 2 & 3 \end{bmatrix}$$

$$x[k+1] = A_d x[k] + B_d[k]$$

$$y[k] = C_d x[k]$$

Problem 3

Given the scalar system $\dot{x} = ax + br$ where $x(0) = 10$ Let $a = -2 \quad b = 5$ and $r(t)$ is the unit step, $u(t)$
Part I

- Using the state transition equation compute the closed solution for $x(t)$ with the given initial condition and input $u(t)$. Use the Convolution integal form (not Laplace).

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}br(\tau)d\tau$$
$$x(t) = 10e^{-2t} + \int_0^t 5e^{-2(t-\tau)}d\tau$$
$$x(t) = 10e^{-2t} + \frac{5}{2} - \frac{5e^{-2t}}{2}$$
$$x(t) = \frac{5}{2}(3e^{-2t} + 1)$$

- Validate your solution using the Laplace transform from the State transition equation and invert.

$$\mathcal{L}(\dot{x}) = \mathcal{L}(ax + br)$$
$$s\hat{x}(s) - x(0) = a\hat{x}(s) + b\frac{1}{s}$$
$$\hat{x}(s) = \frac{1}{s}\frac{b}{s-a} + \frac{x(0)}{s-a}$$
$$\mathcal{L}^{-1}(\hat{x}(s)) = x(t) = \frac{5}{2}\left(3e^{-2t} + 1\right)$$

Part II $\quad$ Let $T = \frac{1}{8}$ sec

a) Find A_d

$$A_d = e^{aT}$$
$$A_d = e^{-2T} = e^{-\frac{1}{4}} \approx 0.7788$$

b) Find B_d (using both formulas given in the notes/text)

$$B_d = b\int_0^T e^{a\alpha}d\alpha = 5\int_0^{\frac{1}{8}} e^{-2\alpha}d\alpha = \frac{5}{2}(1 - e^{\frac{1}{4}}) \approx 0.553$$
$$B_d = A^{-1}[A_d - I]B = \frac{b(A_d - 1)}{a} = \frac{5(e^{-\frac{1}{4}} - 1)}{-2} = \frac{5}{2}(1 - e^{\frac{1}{4}}) \approx 0.553$$

c) Write the discrete time state equations $x[k+1] = f(x[k], u[k])$

$$x[k+1] = A_d x[k] + B_d u[k]$$

Since $u[k] = 1 \quad \forall k$

$$x[k+1] = e^{-\frac{1}{4}}x[k] + \frac{5}{2}(1 - e^{\frac{1}{4}})$$

d) Look at the Matlab function "c2d" use the "zoh" option - do you get the same results? See also CT DT Example Week7 file in WK6 handouts

$$A = 0.7788 \quad B = 0.553$$

Which matches what I found.

e) Using Matlab, Excel or something else recursively compute $x[1]$ through $x[10]$ note that $u[k] = 1 \quad \forall k$

1	0
2	0.552998
3	0.983673
4	1.31908
5	1.5803
6	1.78374
7	1.94217
8	2.06557
9	2.16166
10	2.2365