Analytic Solution

ECES 511

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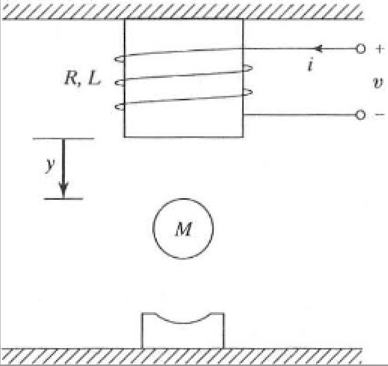
Electromechanical Magnetic-Ball Suspension

- Make an object levitate by controlling the current



The System

$$R = 1\Omega$$
$$L = 0.01\text{H}$$
$$m = 0.05\text{kg}$$
$$K = 0.0001\frac{\text{kg}\cdot\text{m}^2}{\text{A}^2\cdot\text{s}^2}$$
$$g = 9.81\frac{\text{m}}{\text{s}^2}$$

$$v(t) = Ri(t) + L\frac{di(t)}{dt}$$

$$\frac{d^2y(t)}{dt^2} = mg - K\frac{i^2(t)}{y(t)}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \frac{u(t)-Rx_1(t)}{L} \\ x_3(t) \\ -\frac{Kx_1^2(t)}{mx_2(t)} + g \end{bmatrix}$$

$$y(t) = x_2(t)$$

Equilibrium

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{u(t)-Rx_1(t)}{L} \\ x_3(t) \\ -\frac{Kx_1^2(t)}{mx_2(t)} + g \end{bmatrix}$$

Which gives us

$$x_1(t) = \frac{u(t)}{R}, \qquad x_3(t) = 0, \qquad x_2(t) = \frac{Ku^2(t)}{mgR^2}$$

For a constant input voltage we get

$$y_0 = \frac{Kv_0^2}{mgR^2}$$

If we want to hold the ball at a height y_0

$$v_0 = \sqrt{\frac{y_0 mgR^2}{K}}$$

Linearization about the Equilibrium

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \frac{u(t)-Rx_1(t)}{L} \\ x_3(t) \\ -\frac{Kx_1^2(t)}{mx_2(t)} + g \end{bmatrix}$$

$$A = \frac{\partial h}{\partial x} = \begin{bmatrix} -\frac{R}{L} & 0 & 0 \\ 0 & 0 & 1 \\ -\frac{2Kx_1(t)}{mx_2(t)} & \frac{K}{m}(\frac{x_1(t)}{x_2(t)})^2 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & 0 & 0 \\ 0 & 0 & 1 \\ -\frac{2Kx_{01}}{mx_{02}} & \frac{K}{m}(\frac{x_{01}}{x_{02}})^2 & 0 \end{bmatrix}$$

$$B = \frac{\partial h}{\partial u} = \begin{bmatrix} \frac{1}{L} \\ 0 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \end{bmatrix}$$

Sample Equilibria

$$v_0 = 7 \qquad x_0 = \begin{bmatrix} 7 \\ .00998 \\ 0 \end{bmatrix} \qquad A = \begin{bmatrix} -100 & 0 & 0 \\ 0 & 0 & 1 \\ -2.803 & 982 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Eigenvalues

$$(\lambda I - A)x = 0 \quad \begin{bmatrix} \lambda + 100 & 0 & 0 \\ 0 & \lambda & 1 \\ 2.803 & -982 & \lambda \end{bmatrix} x = 0$$

$$98200 + 982\lambda - 100\lambda^2 - \lambda^3$$

$$\implies \lambda_1 = -100 \lambda_2 = 31.3369 \quad \lambda_3 = -31.3369$$

Jordan Decomposition

$$(A - \lambda_1 I)q_1 = 0, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 100 & 1 \\ -2.803 & 982 & 100 \end{bmatrix} q_1 = 0 \implies q_1 = \begin{bmatrix} 3217.267 \\ 1 \\ -100 \end{bmatrix}$$

$$\begin{bmatrix} -131.3369 & 0 & 0 \\ 0 & -31.3369 & 1 \\ -2.803 & 982 & -31.3369 \end{bmatrix} q_2 = 0 \implies q_2 = \begin{bmatrix} 0 \\ 1 \\ 31.3369 \end{bmatrix}$$

$$\begin{bmatrix} -68.6631 & 0 & 0 \\ 0 & 31.3369 & 1 \\ -2.803 & 982 & 31.3369 \end{bmatrix} q_3 = 0 \implies q_3 = \begin{bmatrix} 0 \\ 1 \\ -31.3369 \end{bmatrix}$$

Jordan Form

Normalizing eigenvectors

$$q_1 = \begin{bmatrix} .999517 \\ -3.1067 \times 10^{-4} \\ .031067 \end{bmatrix} \quad q_2 = \begin{bmatrix} 0 \\ .031895 \\ .999491 \end{bmatrix} \, q_3 = \begin{bmatrix} 0 \\ .031895 \\ -.999491 \end{bmatrix}$$

$$Q = \begin{bmatrix} .999517 & 0 & 0 \\ -3.1067 \times 10^{-4} & .031895 & .031895 \\ .031067 & .999491 & -.999491 \end{bmatrix}$$

$$J = \begin{bmatrix} -100 & 0 & 0 \\ 0 & 31.3369 & 0 \\ 0 & 0 & -31.3369 \end{bmatrix}$$

$$Q^{-1} = \begin{bmatrix} 1.00048 & 0. & 0. \\ -0.0106764 & 15.6764 & 0.500255 \\ 0.0204215 & 15.6764 & -0.500255 \end{bmatrix}$$

Matrix Exponentiation

$$e^{At} = Qe^{J}Q^{-1} = Q\begin{bmatrix} e^{-100t} & 0 & 0 \\ 0 & e^{31.3369t} & 0 \\ 0 & 0 & e^{-31.3369t} \end{bmatrix} Q^{-1}$$

This Expression becomes very large so each column of the matrix will be broken up with the first column followed by the next two.

$$\begin{bmatrix} e^{\lambda_1 t} \\ -3.10823 \times 10^{-4} e^{\lambda_1 t} + 6.51349 \times 10^{-4} e^{\lambda_3 t} - 3.40526 \times 10^{-4} e^{\lambda_2 t} \\ 0.0310823 e^{\lambda_1 t} - 0.0204112 e^{\lambda_3 t} - 0.010671 e^{\lambda_2 t} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0.5e^{\lambda_3 t} + 0.5e^{\lambda_2 t} & 0.0159556e^{\lambda_2 t} - 0.0159556e^{\lambda_3 t} \\ 15.6684e^{\lambda_2 t} - 15.6684e^{\lambda_3 t} & 0.5e^{\lambda_3 t} + 0.5e^{\lambda_2 t} \end{bmatrix}$$

Step Response

$$\mathbf{u}(t) = \begin{bmatrix} 1 \end{bmatrix}$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) +$$

$$\begin{bmatrix} 1 - e^{-100.t} \\ 0.002854 + 3.1082 \times 10^{-4} e^{\lambda_1 t} - 0.002079 e^{\lambda_3 t} - 0.00109 e^{\lambda_2 t} \\ -0.03108 e^{\lambda_1 t} + 0.0651 e^{\lambda_3 t} - 0.03405 e^{\lambda_2 t} \end{bmatrix}$$

For a given initial condition

Step Response Analysis

For a general valid starting condition for the system the following will hold.

- Current term will approach a stable value of 1 amp
- Height will drop exponentially as the λ_1 and λ_2 terms go to zero while the λ_3 term corresponds to the constant gravity term which will accelerate the ball downwards until it hits the ground.
- The vertical velocity term will also continually increase due to gravity.

The system is unstable given the point we linearized around with an input $u(t) = 1$

Closed Form Solution

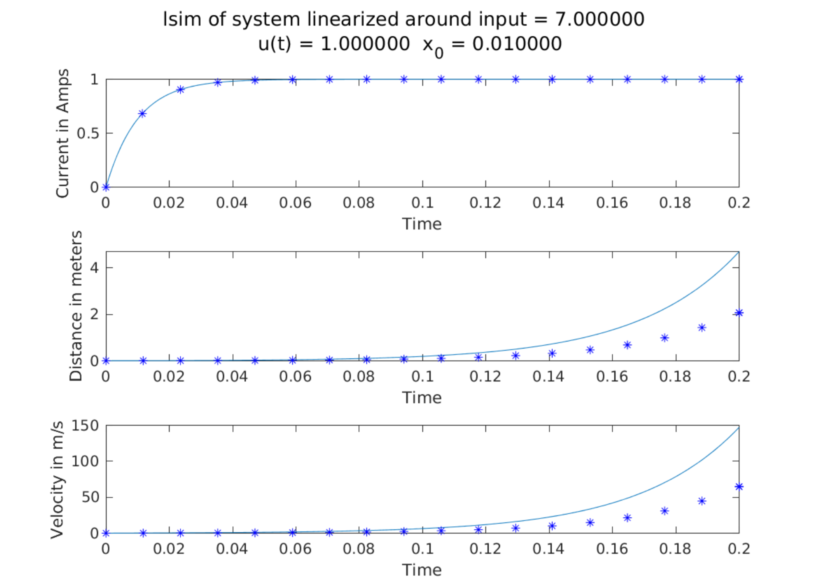
$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ .01 \\ 0 \end{bmatrix}$$

$$\mathbf{x}_1(t) = 1 - e^{-100t}$$

$$\mathbf{x}_2(t) = .002854 + 3.1082 \times 10^{-4} e^{-100t} + .0029215 e^{-31.3369t} + .0039133 e^{31.3}$$

$$\mathbf{x}_3(t) = -.091549 e^{-31.3369t} + .12263 e^{31.3369t}$$

Step Response Graph



Step Response Graph

- The Current appears to be the correct equation which make sense due to the fact that that part of the system can be described by a linear system.
- The Position and velocity graphs appear to be close but not accurate to the simulation. I would attribute this to lack of numerical precision since most of the numbers are quite small and being multiplied making them even smaller. Also while computing my close form solution I removed terms that were very close to zero so that I didn't have to carry them through the equations.

Poles and Zeros

$$\mathcal{L}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A}^{-1})\mathbf{B}$$

$$\hat{\mathbf{x}}(s) = \frac{-280.3}{s^3 + 100s^2 - 982s - 98200}$$

$$= \frac{-280.3}{(s - 31.3369)(s + 31.3369)(s + 100)}$$

$$\mathcal{L}(u) = \frac{1}{s}$$

$$\hat{\mathbf{y}}(s) = \frac{-280.3}{s(s - 31.3369)(s + 31.3369)(s + 100)}$$

We can see from the factored form that we have no zeros and 4 poles.

$$s = \lambda_1 \quad s = \lambda_2 \quad s = \lambda_3 \quad s = 0$$