Homework 5
ECES 511

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Problem 1

For the matrix $\mathbf{A}$ given as

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

Compute the state transition matrix $\varphi(t) = e^{\mathbf{A}t}$ by

1. Using the Laplace transform

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}(sI - A)^{-1}$$

$$(sI - A) = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} \implies (sI - A)^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$\mathcal{L}^{-1}(sI - A)^{-1} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-2t} - e^{-t} \end{bmatrix}$$

2. Using Cayley Hamilton's theorem
   Let $f(\lambda) = e^{\lambda t}$

$$\Delta(A) = \lambda^2 + 3\lambda + 2 = (\lambda + 2)(\lambda + 1)$$

$$\lambda_1 = -1 \qquad \lambda_2 = -2$$

   Let $h(\lambda) = \beta_0 + \beta_1 \lambda$
   With λ_1 and knowing it's multiplicity is 1 we have one equation

$$h(\lambda_1) = f(\lambda_1) \implies \beta_0 + \beta_1(-1) = e^{-t}$$

   With λ_2 and knowing it's multiplicity is 1 we have another equation

$$h(\lambda_2) = f(\lambda_2) \implies \beta_0 + \beta_1(-2) = e^{-2t}$$

$$\beta_0 = e^{-2t}(2e^t - 1) \qquad \beta_1 = e^{-2t}(e^t - 1)$$

$$e^{\mathbf{A}t} = f(\mathbf{A}) = h(\mathbf{A}) = \beta_1 \mathbf{A} + \beta_0 = (e^{-2t}(e^t - 1))\mathbf{A} + e^{-2t}(2e^t - 1)\mathbf{I}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-2t} - e^{-t} \end{bmatrix}$$

3. Determining the Jordan form $\mathbf{J}$ of $\mathbf{A}$ and computing $e^{\mathbf{A}t} = \mathbf{Q}e^{\mathbf{J}t}\mathbf{Q}^{-1}$
   We already found $\lambda_1 = -1 \quad \lambda_2 = -2$ so we just need to find the associated eigenvectors.

$$(-1I - A)q_1 = 0 = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} q_1 = 0 \implies q_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$(-2I - A)q_2 = 0 = \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} q_2 = 0 \implies q_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

   We can now generate our diagonalized matricies

$$\mathbf{Q} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \qquad \mathbf{J} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad Q^{-1} = \frac{1}{det(\mathbf{Q})} adj(\mathbf{Q})$$

$$\mathbf{Q} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \qquad \mathbf{J} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \qquad Q^{-1} = -\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$e^{\mathbf{A}t} = \mathbf{Q}e^{\mathbf{J}t}\mathbf{Q}^{-1}$$

$$\mathbf{Q} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2e^{-t} & e^{-t} \\ -e^{-2t} & -e^{-2t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-2t} - e^{-t} \end{bmatrix}$$

Problem 2

For the matrix $\mathbf{J}$ given as

$$\mathbf{J} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

1. Using the relationships in the notes, find $e^{\mathbf{J}t}$;

$$e^{\mathbf{J}t} = \begin{bmatrix} e^{2t} & te^{2t} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{2t} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-3t} & te^{-3t} & \frac{1}{2}t^2e^{-3t} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-3t} & te^{-3t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-3t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-3t} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{4t} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{4t} \end{bmatrix}$$

2. Identify each Jordan block and its order.
   There are 5 Jordan blocks:

   1. J_1 has order 2
   $$\begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}$$
   2. J_2 has order 3
   $$\begin{bmatrix} e^{-3t} & te^{-3t} & \frac{1}{2}t^2e^{-3t} \\ 0 & e^{-3t} & te^{-3t} \\ 0 & 0 & e^{-3t} \end{bmatrix}$$
   3. J_3 has order 1
   $$\begin{bmatrix} e^{-3t} \end{bmatrix}$$
   4. J_4 has order 1
   $$\begin{bmatrix} e^{4t} \end{bmatrix}$$
   5. J_5 has order 1
   $$\begin{bmatrix} e^{4t} \end{bmatrix}$$

Problem 3

Supose we are given a set of points $\{(1,1),(2,2),(3,1.7),(3.5,2.5),(4,3.6),(5,3.6)\}$. Between a linear model and a parabolic model which one fits these points better? Why?

- Linear Model

$$y = ax + b \implies y = \begin{bmatrix} x & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

Solve with least squares we find

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 3.5 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 1.7 \\ 2.5 \\ 3.6 \\ 3.6 \end{bmatrix} = \begin{bmatrix} .671 \\ .331 \end{bmatrix}$$

To see how good we did we can look at the error by looking at how close our equation is for the given points.

$$\text{Error} = \sum (\hat{y} - y)^2 = \Delta y^T \Delta y$$

$$\Delta y = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 3.5 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} .671 \\ .331 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 1.7 \\ 2.5 \\ 3.6 \\ 3.6 \end{bmatrix} = \begin{bmatrix} .002 \\ -.3269 \\ .6441 \\ .1796 \\ -.5849 \\ .0861 \end{bmatrix}$$

$$\text{Error} = \Delta y^T \Delta y = .9035$$

- Parabolic Model

$$y = ax^2 + bx + c \implies y = \begin{bmatrix} x^2 & x & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Solve with least squares we find

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \\ 3.5^2 & 3.5 & 1 \\ 16 & 4 & 1 \\ 25 & 5 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 1.7 \\ 2.5 \\ 3.6 \\ 3.6 \end{bmatrix} = \begin{bmatrix} 0.312 \\ .4862 \\ .5514 \end{bmatrix}$$

To see how good we did we can look at the error by looking at how close our equation is for the given points.

$$\text{Error} = \sum (\hat{y} - y)^2 = \Delta y^T \Delta y$$

$$\Delta y = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \\ 3.5^2 & 3.5 & 1 \\ 16 & 4 & 1 \\ 25 & 5 & 1 \end{bmatrix} \begin{bmatrix} 0.312 \\ .4862 \\ .5514 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 1.7 \\ 2.5 \\ 3.6 \\ 3.6 \end{bmatrix} = \begin{bmatrix} .0688 \\ -.3514 \\ .5906 \\ .1351 \\ -.6049 \\ .1618 \end{bmatrix}$$

$$\text{Error} = \Delta y^T \Delta y = .8875$$

Since our error is lower with the parabolic model it is better described within the region of points by it as well.