# Homework 6
# ECES 512

Damien Prieur

## Problem 1

Consider a plant described by

$$\dot{x} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

a. Find out if the system is stable.
We can look at the eigenvalues of the $A$ matrix as we are only looking at internal stability since the states are the output variables.

$$det(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}) = \begin{vmatrix} \lambda - 1 & -3 \\ -3 & \lambda - 1 \end{vmatrix} = 0$$

$$(\lambda - 1)^2 - 9 = 0$$

$$(\lambda - 1)^2 = 9$$

$$\lambda - 1 = \pm 3$$

$$\lambda = 1 \pm 3$$

$$\lambda_1 = -2 \quad \lambda_2 = 4$$

Since we have eigenvalues on the right half of the plane our system is unstable.

b. Find out if the system is controllable.
Since the plant is LTI we can use the rank of the controllability matrix $\begin{bmatrix} B & AB \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$C = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$$

We can see that the rank of the controllability matrix is 2 which is full rank. The system is controllable.

c. Use direct coefficients matching method to find the K that shifts the poles to $-1 \pm j2$.

$$u = -Kx \qquad \dot{x} = (\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix})x$$

$$\dot{x} = (\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ 0 & 0 \end{bmatrix})x = \begin{bmatrix} 1 - k_1 & 3 - k_2 \\ 3 & 1 \end{bmatrix} x$$

We want to find the $k$'s that satisfy this equation

$$|sI - (A - BK)| = (s - (-1 + j2))(s - (-1 - j2))$$

$$\begin{vmatrix} s + k_1 - 1 & k_2 - 3 \\ -3 & s - 1 \end{vmatrix} = s^2 + 2s + 5$$

$$(s + k_1 - 1)(s - 1) + 3(k_2 - 3) = s^2 + 2s + 5$$

$$s^2 + (k_1 - 2)s - k_1 + 3k_2 - 8 = s^2 + 2s + 5$$

Matching Coefficients we get two equations

$$2 = k_1 - 2 \qquad 5 = 3k_2 - k_1 - 8$$

Solving for each we get

$$k_1 = 4 \qquad k_2 = \frac{17}{3}$$

d. Apply Ackermann's formula to find the same K in part c.
We have the formula

$$k^T = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathcal{C}^{-1} \Delta(A)$$

We have the desired poles as $\Delta(x) = x^2 + 2x + 5$

$$k^T = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}^{-1} \left( \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}^2 + 2 \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \right)$$

$$k^T = \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix} \left( \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 6 \\ 6 & 7 \end{bmatrix} \right)$$

$$k^T = \frac{1}{3} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 17 & 12 \\ 12 & 17 \end{bmatrix}$$

$$k^T = \begin{bmatrix} 4 & \frac{17}{4} \end{bmatrix}$$

$$k_1 = 4 \qquad k_2 = \frac{17}{4}$$

e. Repeat part c and part d in Matlab by using the function *place* and *acker*.

```
%% Question 1
% System and pole definition

A = [ 1 3 ;
      3 1 ];
B = [ 1 ;
      0 ];
poles = [-1+2i -1-2i];

% Using place
gains_place = place(A,B,poles);
gains_place
% gains_place =
% 4.0000    5.6667


% Using Acker
gains_acker = acker(A,B,poles);
gains_acker

% gains_acker =
% 4.0000    5.6667

% lsim the systems
sys_orig = ss(A,B,[1 0],0);
sys_modi = ss(A-B*gains_place,B,[1 0],0);
t = 0:.01:10;
u = t >= 0;

[~,~,xs] = lsim(sys_orig, u,t);
```

```
plot(t,xs);
legend("x1", "x2");
saveas(gcf, "images/p1_original_system.png");

[~,~,xs] = lsim(sys_modi, u,t);
plot(t,xs);
legend("x1", "x2");
saveas(gcf, "images/p1_modified_system.png");
```

f. Simulate the time domain solution of the system before and after shifting the poles and explain. The input is the unit step function.

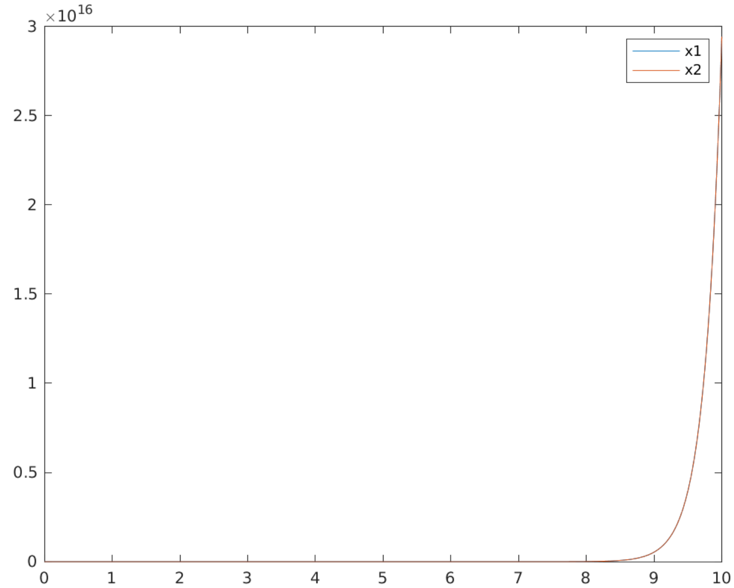


Figure 1: Original System

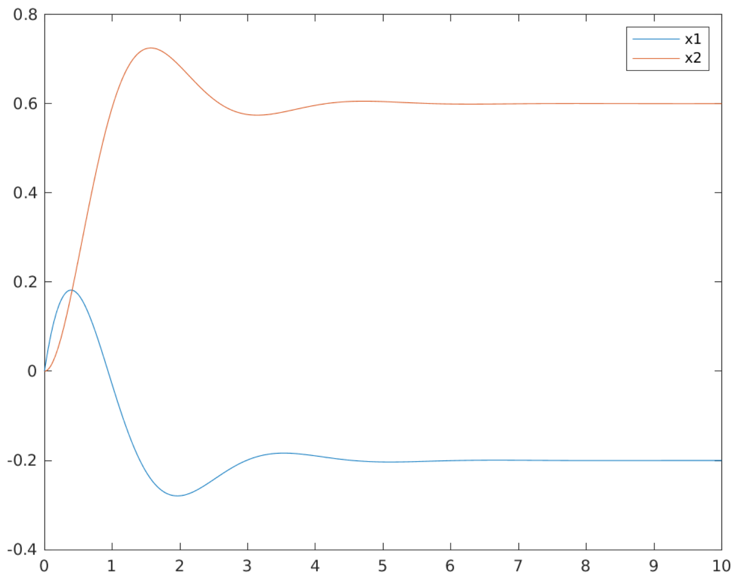


Figure 2: Modified System

In the original system the system is unstable due to the poles in the right right half plane. This causes the states to blow up towards infinity, but once we move the poles the system is no longer unstable. We can see that the system stabilized now that the poles are in the left half plane. The oscillation in the transient can be explained by the complex components of the poles which eventually get dampened out.

## Problem 2

Consider a plant described by

$$\dot{x} = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} x$$

a. Find out if the system is stable.
We can check for internal stability by looking at the eigenvalues and check for BIBO stability by looking at the transfer function.
Since the $A$ matrix is upperdiagonal the eigenvalues are already on the diagonal. We can see that we have 3 repeated eigenvalues of 1

$$\lambda_1 = \lambda_2 = \lambda_3 = 1$$

For BIBO stability we look at

$$C|sI - A|^{-1}B = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} \left( \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} s-1 & -1 & 2 \\ 0 & s-1 & -1 \\ 0 & 0 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

We can find the inverse of the Matrix more easily since it is upper triangular using the formula

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{a} & \frac{-b}{ad} & \frac{be-cd}{adf} \\ 0 & \frac{1}{d} & \frac{-e}{df} \\ 0 & 0 & \frac{1}{f} \end{bmatrix}$$

Plugging in we get

$$\begin{bmatrix} 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s-1)^2} & \frac{3-2s}{(s-1)^3} \\ 0 & \frac{1}{s-1} & \frac{1}{(s-1)^2} \\ 0 & 0 & \frac{1}{s-1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{2}{s-1} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\frac{2}{s-1}$$

Which only has a single pole in the right half plane. Therefore the system is not BIBO stable.

b. Find out if the system is controllable.
Since the plant is LTI we can use the rank of the controllability matrix $\begin{bmatrix} B & AB & A^2B \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

By performing the two row operations of subtracting the first two rows from the last row we get

$$\begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

We can see that the rank of the controllability matrix is 3 which is full rank. The system is controllable.

c. Use direct coefficients matching method to find the K so that the resulting system has eigenvalues -2 and $-1 \pm j1$.

$$u = -Kx \qquad \dot{x} = (\begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix})x$$

$$\dot{x} = (\begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 & k_3 \\ 0 & 0 & 0 \\ k_1 & k_2 & k_3 \end{bmatrix})x = \begin{bmatrix} 1-k_1 & 1-k_2 & -2-k_3 \\ 0 & 1 & 1 \\ -k_1 & -k_2 & 1-k_3 \end{bmatrix} x$$

We want to find the $k$'s that satisfy this equation

$$|sI - (A - BK)| = (s+2)(s-(-1+j))(s-(-1-j))$$

$$\begin{vmatrix} s+k_1-1 & k_2-1 & k_3+2 \\ 0 & s-1 & -1 \\ k_1 & k_2 & s+k_3-1 \end{vmatrix} = s^3 + 4s^2 + 6s + 4$$

$$(s+k_1-1)(s-1)(s+k_3-1)k_2 + k_1(1-k_2)(k_3+2)(1-s) = s^3 + 4s^2 + 6s + 4$$

$$s^3 + (k_1+k_3-3)s^2 + (-4k_1+k_2-2k_3+3)s + (4k_1-k_2+k_3-1) = s^3 + 4s^2 + 6s + 4$$

Matching Coefficients we get three equations

$$k_1 + k_3 - 3 = 4 \qquad -4k_1 + k_2 - 2k_3 + 3 = 6 \qquad 4k_1 - k_2 + k_3 - 1 = 4$$

We can rewrite this into matrix form for easier solving

$$\begin{bmatrix} 1 & 0 & 1 \\ -4 & 1 & -2 \\ 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 5 \end{bmatrix}$$

Solving for each we get

$$k_1 = 15 \qquad k_2 = 47 \qquad k_3 = -8$$

d. Apply Ackermann's formula to find the same K in part c.
We have the formula

$$k^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathcal{C}^{-1} \Delta(A)$$

We have the desired poles as $\Delta(x) = x^3 + 4x^2 + 6x + 4$

$$k^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}^{-1} (\begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^3 + 4\begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^2 + 6\begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix})$$

From part A we have a closed form solution to the matrix inverse since we are upper diagonal.

$$k^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -2 & -3 & -2 \\ 1 & 2 & -1 \end{bmatrix} (\begin{bmatrix} 1 & 3 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} + 4\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 10 & 6 & -12 \\ 0 & 10 & 6 \\ 0 & 0 & 10 \end{bmatrix})$$

$$k^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -2 & -3 & -2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 15 & 17 & -27 \\ 0 & 15 & 17 \\ 0 & 0 & 15 \end{bmatrix}$$

$$k^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 15 & 32 & -10 \\ -30 & -79 & 47 \\ 15 & 47 & -8 \end{bmatrix}$$

$$k^T = \begin{bmatrix} 15 & 47 & -8 \end{bmatrix}$$

$$k_1 = 15 \qquad k_2 = 47 \qquad k_3 = -8$$

e. Repeat part c and part d in Matlab by using the function *place* and *acker*.

```
%% Question 2
% System and pole definition

A = [ 1 1 -2 ;
      0 1  1 ;
      0 0  1];
B = [ 1 ;
      0 ;
      1];
poles = [-2 -1+i -1-i];

% Using place
gains_place = place(A,B,poles);
gains_place
% gains_place =
% 15.0000   47.0000   -8.0000



% Using Acker
gains_acker = acker(A,B,poles);
gains_acker

% gains_acker =
% 15    47    -8


% lsim the systems
sys_orig = ss(A,B,[1 0 0],0);
sys_modi = ss(A-B*gains_place,B,[1 0 0],0);
t = 0:.01:10;
u = t >= 0;

[~,~,xs] = lsim(sys_orig, u,t);
plot(t,xs);
legend("x1","x2","x3");
saveas(gcf, "images/p2_original_system.png");

[~,~,xs] = lsim(sys_modi, u,t);
plot(t,xs);
legend("x1","x2","x3");
saveas(gcf, "images/p2_modified_system.png");
```

f. Simulate the time domain solution of the system before and after shifting the poles and explain. The input is the unit step function.

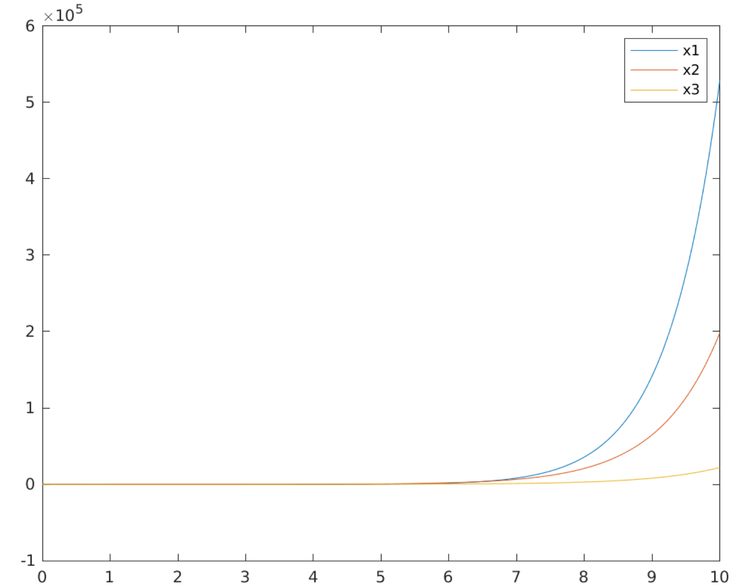


Figure 3: Original System

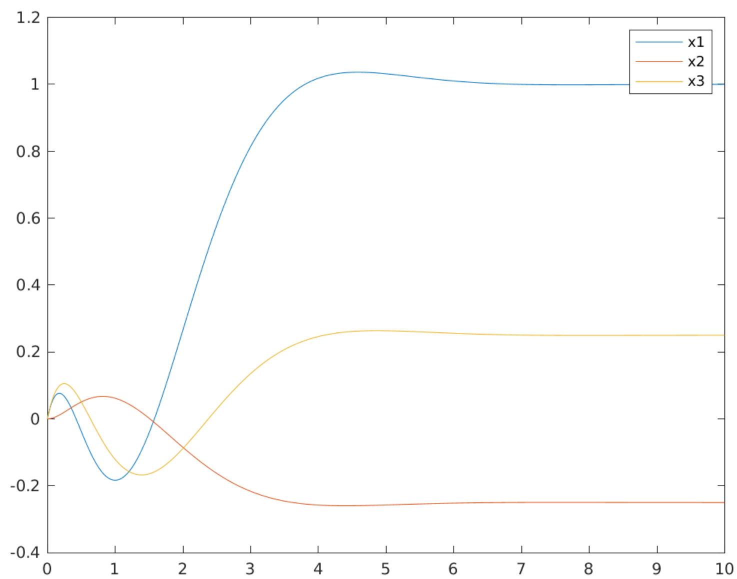


Figure 4: Modified System

In the original system the system is unstable due to the poles in the right right half plane. This causes the states to blow up towards infinity, but once we move the poles the system is no longer unstable. We can see that the system stabilized now that the poles are in the left half plane. The oscillation in the transient can be explained by the complex components of the poles which eventually get dampened out.