Homework 4
MEM 633
Group 1

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Problem 1

Consider the system described by the following state equaiton,

$$\dot{x}(t) = Ax(t) + Bu(t) = \begin{bmatrix} -18 & -19 & -15 \\ 20 & 21 & 16 \\ -5 & -5 & -4 \end{bmatrix} x(t) + \begin{bmatrix} -3 \\ 5 \\ -2 \end{bmatrix} u(t)$$

(a) Use the PBH Test to check if the system is controllable.

A system is controllable if the $rank\left(\begin{bmatrix} sI - A & b \end{bmatrix}\right) = n \quad \forall s \in \mathcal{C}$

$$\begin{bmatrix} s+18 & 19 & 15 & -3 \\ -20 & s-21 & -16 & 5 \\ 5 & 5 & s+4 & -2 \end{bmatrix}$$

Since we only need to check s equal to the eigenvalues of A since A is nonsingular.

$$\begin{vmatrix} \lambda+18 & 19 & 15 \\ -20 & \lambda-21 & -16 \\ 5 & 5 & \lambda+4 \end{vmatrix} = -\lambda^3 - \lambda^2 + 5\lambda - 3$$

$$\lambda_1 = -3 \quad \lambda_2 = 1 \quad \lambda_3 = 1$$

Plugging in $s = \lambda_i$ for the PBH test we get

$$rank(s=-3) = rank\left(\begin{bmatrix} 15 & 19 & 15 & -3 \\ -20 & -24 & -16 & 5 \\ 5 & 5 & 1 & -2 \end{bmatrix}\right) = 2$$

$$rank(s=1) = rank\left(\begin{bmatrix} 19 & 19 & 15 & -3 \\ -20 & -20 & -16 & 5 \\ 5 & 5 & 5 & -2 \end{bmatrix}\right) = 3$$

The system is uncontrollable, and the only uncontrollable eigenvalue is $\lambda = -3$.

(b) Characterize the controllable subpace using the controllability decomposition approach.

Let $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$, where $T_1 = \begin{bmatrix} b & Ab & \dots & A^{r-1}b \end{bmatrix}$ and T_2 be chosen such that T is full rank. Since we know that only 2 eigenvalues are controllable we only need the first two columns of T_1

$$T_1 = \begin{bmatrix} b & Ab \end{bmatrix} = \begin{bmatrix} -3 & -11 \\ 5 & 13 \\ -2 & -2 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T = \begin{bmatrix} -3 & -11 & 0 \\ 5 & 13 & 0 \\ -2 & -2 & 1 \end{bmatrix} \implies T^{-1} = \frac{1}{16}\begin{bmatrix} 13 & 11 & 0 \\ -5 & -3 & 0 \\ 16 & 16 & 16 \end{bmatrix}$$

Let $\hat{A} = T^{-1}AT, \hat{b} = T^{-1}b$ then we get

$$\hat{A} = \begin{bmatrix} 0 & -1 & -\frac{19}{16} \\ 1 & 2 & \frac{27}{16} \\ 0 & 0 & -3 \end{bmatrix} \quad \hat{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(c) Is the system stabilizable? Explain.

Yes, since the only uncontrollable eigenvalue is already stable since it's in the left half plane.

(d) Design a state feedback controller using the controllability decomposition approach so that the closed-loop system is internally stable.

Looking at the controllable subsystem given by the first $2x2$ subsystem we have

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The A matrix has eigenvalues $\lambda_1 = 1, \lambda_2 = 1$ as we expect. To control this system we can use pole placement to move the eigenvalues somewhere in the left half plane. Let those poles be $\{-5, -7\}$

$$det(\lambda I - (A - bF)) = (\lambda + 5)(\lambda + 7)$$

$$\begin{vmatrix} \lambda + f_1 & \lambda + 1 + 2f_2 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda + 5)(\lambda + 7)$$

$$\lambda^2 + \lambda(f_1 - 2) + 1 - 2f_1 + f_2 = \lambda^2 + 12\lambda + 35$$

$$f_1 = 14 \quad f_2 = 62$$

Putting this into the larger system we get $u = -Fx$

$$\begin{bmatrix} 0 & -1 & -\frac{19}{16} \\ 1 & 2 & \frac{27}{16} \\ 0 & 0 & -3 \end{bmatrix} \hat{x}(t) - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 14 & 62 & 0 \end{bmatrix} = \begin{bmatrix} -14 & -63 & -\frac{19}{16} \\ 1 & 2 & \frac{27}{16} \\ 0 & 0 & -3 \end{bmatrix}$$

Which has eigenvalues $-7, -5, -3$ as we expect. So the internal system is stable.

(e) Assume the initial state of the system is $x(0) = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}^T$, plot the state response $x(t)$ of the closed-loop system.

We can undo the similarity transform with

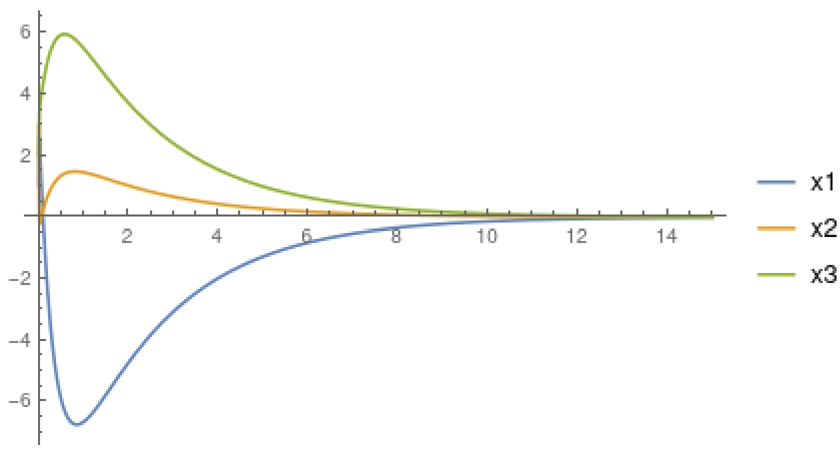
$$A = T\hat{A}T^{-1} \quad b = T\hat{b}$$

We can see that the feedback control values are unaffected by the similarity transformation so we get

$$\dot{x}(t) = \left(\begin{bmatrix} -18 & -19 & -15 \\ 20 & 21 & 16 \\ -5 & -5 & -4 \end{bmatrix} - \begin{bmatrix} -3 \\ 5 \\ -2 \end{bmatrix} \begin{bmatrix} 14 & 62 & 0 \end{bmatrix} \right) x(t) = \begin{bmatrix} 24 & 167 & -15 \\ -50 & -289 & 16 \\ 23 & 119 & -4 \end{bmatrix} x(t)$$

Which has a solution

$$x(t) = e^{At}x(0)$$



Problem 2

Consider the same system described by the state equation shown in problem 1.

$$\dot{x}(t) = Ax(t) + Bu(t) = \begin{bmatrix} -18 & -19 & -15 \\ 20 & 21 & 16 \\ -5 & -5 & -4 \end{bmatrix} x(t) + \begin{bmatrix} -3 \\ 5 \\ -2 \end{bmatrix} u(t)$$

(a) Find a similarity transformation to transform the state equation to one with a diagonal A matrix.

We know that the eigenvalues of this system are

$$\lambda_1 = -3 \quad \lambda_2 = 1 \quad \lambda_3 = 1$$

Since we have a repeated eigenvalue we will need to find a generalized eigenvector for the third eigenvector. So we are really finding the Jordan decomposition. Our normal eigenvalues are

$$\lambda = -3 \rightarrow \begin{bmatrix} 14 \\ -15 \\ 5 \end{bmatrix} \quad \lambda = 1 \rightarrow \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

To find the generalized eigenvalue for the repeated eigenvalue we need to solve

$$(A - \lambda I)v_3 = v_2$$

Where v_3 is the generalized eigenvalue and v_2 is the standard eigenvector for the associated eigenvalue. Solving this we get

$$\lambda = 1 \rightarrow \begin{bmatrix} \frac{1}{4} \\ 0 \\ -\frac{1}{4} \end{bmatrix}$$

So we have our generalized eigenvectors as

$$S = \begin{bmatrix} 14 & -1 & \frac{1}{4} \\ -15 & 1 & 0 \\ 5 & 0 & -\frac{1}{4} \end{bmatrix}$$

The diagonal matrix will be block diagonal since we have a repeated eigenvalue

$$J = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

So we have

$$A = SJS^{-1}$$

So our similarity transform is

$$T = S^{-1} \implies T^{-1} = S$$

(b) Characterize the controllable subspace using the eigenvectors obtained in problem 2(a).

An eigenvector is uncontrollable if and only if $qA = \lambda q$ and $qb = 0$ We can check them all at once by looking at TA and Tb

$$T = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{15}{4} & \frac{19}{4} & \frac{15}{4} \\ 5 & 5 & 1 \end{bmatrix} \quad TA = \begin{bmatrix} -\frac{3}{4} & -\frac{3}{4} & -\frac{3}{4} \\ \frac{35}{4} & \frac{39}{4} & \frac{19}{4} \\ 5 & 5 & 1 \end{bmatrix} \quad Tb = \begin{bmatrix} 0 \\ 5 \\ 8 \end{bmatrix}$$

We can see that the first row is uncontrollable and it corresponds to the eigenvalue $\lambda = -3$ Looking at our system in the diagonal case we taking just the controllable eigenvalues

$$\hat{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \hat{B} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

This is our controllable subspace in the transformed domain.

(c) Check the stabilizability of the system based on the result of problem 2(b).

Since the only uncontrollable eigenvector is already stable since $\lambda = -3$ we can stabilize the system as the other two eigenvectors are controllable.

(d) Design a state feedback controller using the eigen structure obtained in problem 2(b) so that the closed-loop system is internally stable.

Using our controllable subsystem we have

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) - \begin{bmatrix} 5 \\ 8 \end{bmatrix} \begin{bmatrix} f1 & f2 \end{bmatrix} x(t)$$

Using pole placement with the same poles as before, $\{-5, -7\}$.

$$det(\lambda I - (A - bF)) = (\lambda + 5)(\lambda + 7)$$

This gives us

$$F = \begin{bmatrix} 6 & -2 \end{bmatrix}$$

With a closed loop system of

$$\begin{bmatrix} -29 & 11 \\ -48 & 17 \end{bmatrix}$$

Which has eigenvalues $\lambda = -5$ and $\lambda = -7$ as we expect. Putting this back into the full system we have

$$\hat{A} - \hat{b} \begin{bmatrix} 0 \\ 6 \\ -2 \end{bmatrix}$$

And we know that converting back to our original domain doesn't effect the control. Our closed controller gives us

$$\dot{x}(t) = \begin{bmatrix} -18 & -1 & -21 \\ 20 & -9 & 26 \\ -5 & 7 & -8 \end{bmatrix} x(t)$$

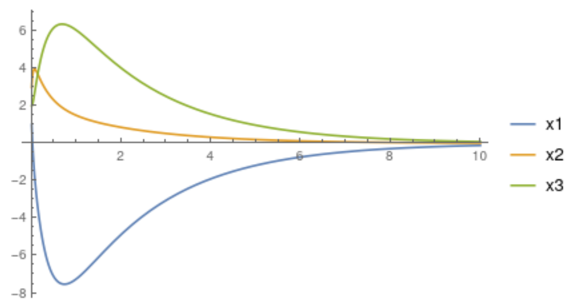
This has different eigenvalues which is due to the fact that we didn't have a fully diagonal matrix due to the repeated root requiring a jordan block. The final eigenvalues of our system are

$$\lambda = -3 \quad \lambda = -16 - \sqrt{241} \quad \lambda = -16 + \sqrt{241}$$

(e) Assume the initial state of the system is $x(0) = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}^T$, plot the state response $x(t)$ of the closed-loop system.

Again the system has the solution

$$x(t) = e^{At} x(0)$$



Problem 3

(a) If $\{A, b, c, d\}, d \neq 0$, is a realization with $H(s) = c(sI - A)^{-1}b + d$, show that

$$\left\{A - \frac{bc}{d}, \frac{b}{d}, \frac{-c}{d}, \frac{1}{d}\right\}$$

is a realization for a system with a transfer function $\frac{1}{H(s)}$.

Let the system $G(s) = \frac{1}{H(s)}$ be described by $\{\hat{A}, \hat{b}, \hat{c}, \hat{d}\}$. Its transfer function is given by

$$G(s) = \hat{c}(sI - \hat{A})^{-1}\hat{b} + \hat{d}$$

If we expand this and show that $G(s) = \frac{1}{H(s)}$ then the statement is true.

$$\frac{-c}{d}(sI - (A - \frac{bc}{d}))\frac{b}{d} + \frac{1}{d}$$

(b) If we are given $\{A, b, c, d\}, d \neq 0$, show that the zeros of $c(sI - A)^{-1}b + d$ are the eigenvalues of the matrix $A - \frac{bc}{d}$.

The eigenvalues of $A - \frac{bc}{d}$ can be expanded to

$$A - \frac{1}{d}\begin{bmatrix} b_1c_1 & b_1c_2 & \dots & b_1c_n \\ b_2c_1 & b_2c_2 & \dots & b_2c_n \\ \dots & \dots & \dots & \dots \\ b_nc_1 & b_nc_2 & \dots & b_nc_n \end{bmatrix}$$

Let $q_{i,j} = \frac{b_ic_j}{d}$ and Q be the matrix of $q_{i,j}$.

$$|\lambda I - (A - Q)| = 0$$

A matrix where

$$J(i,j) = \begin{cases} \lambda + q_{i,i} - a_{i,i} & i = j \\ q_{i,j} - a_{i,j} & i \neq j \end{cases}$$

Which we can find the determinant of if we take the cofactor expansion of each element.

$$c\frac{cofactor(sI - A)^T}{det(sI - A)}b + d$$

Let $\mathcal{A}$ be the adjoint of the matrix $(sI - A)$ and $\mathcal{C}_{i,j}$ be the cofactor of the element $\{i, j\}$

$$c\frac{\mathcal{A}}{det(sI - A)}b + d$$

Since we only care about zeros we can remove the denominator

$$c\mathcal{A}b = -d$$

Performing the matrix multiplication we get

$$c_1(b_1\mathcal{C}_{1,1} + b_2\mathcal{C}_{2,1} + \dots + b_n\mathcal{C}_{n,1}) + c_2(b_1\mathcal{C}_{1,2} + b_2\mathcal{C}_{2,2} + \dots + b_n\mathcal{C}_{n,2}) + \dots c_n(b_1\mathcal{C}_{1,n} + b_2\mathcal{C}_{2,n} + \dots + b_n\mathcal{C}_{n,n}) = -d$$

Which can be compact written as

$$\sum_j^n c_j(\sum_i^n b_i\mathcal{C}_{i,j}) = -d$$

Dividing both sides by d and the same definition for q we can see that we get

$$\sum_j^n \sum_i^n q_{i,j}\mathcal{C}_{i,j} = -1$$

This is the same sum we are searching for when we are solving for the determinant and therefore the solutions will be the same.
Q.E.D