Homework 5
MEM 633
Group 1

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Problem 1

Find a minimal controller-form (NOT a block controller form) realization of

$$H(s) = \begin{bmatrix} \frac{1}{(s-1)^2} & \frac{1}{(s-1)(s+3)} \\ \frac{-6}{(s-1)(s+3)^2} & \frac{s-2}{(s+3)^2} \end{bmatrix}$$

First we must find a Matrix Fraction Description (MFD) of $H(s) = N(s)D(s)^1$ Using the Smith-McMillan form we start by writing $H(s) = \frac{N(s)}{d(s)}$ where $d(s)$ is the monic least common multiple of the denominators of the entries of $H(s)$.

$$d(s) = (s-1)^2(s+3)^2 \implies N(s) = \begin{bmatrix} (s+3)^2 & (s-1)(s+3) \\ -6(s-1) & (s-1)^2(s-2) \end{bmatrix}$$

Now we need to find the Smith form of $N(s)$ by elementary row and column operations. First get the first entry to be a constant then we can easily remove all off diagonal entries.

$$\begin{bmatrix} 1 & \frac{s+7}{6} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (s+3)^2 & (s-1)(s+3) \\ -6(s-1) & (s-1)^2(s-2) \end{bmatrix} \begin{bmatrix} 1 & \frac{s^2-3s}{6} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 16 & \frac{1}{6}(s^4+3s^3-s^2-3s-32) \\ -6(s-1) & 2(s-1) \end{bmatrix}$$

We can rescale to make it a bit easier to see

$$\begin{bmatrix} 1 & 0 \\ \frac{3}{8}(s-1) & 1 \end{bmatrix} \begin{bmatrix} 16 & \frac{1}{6}(s^4+3s^3-s^2-3s-32) \\ -6(s-1) & 2(s-1) \end{bmatrix} \begin{bmatrix} 1 & \frac{-1}{16}(\frac{1}{6}s^4+\frac{1}{2}s^3-\frac{1}{6}s^2-\frac{1}{2}s-\frac{16}{3}) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & \frac{1}{16}(s^5+2s^4-4s^3-2s^2+ \end{bmatrix}$$

We can combine the elementary operations to find U and V

$$U = \begin{bmatrix} 1 & \frac{s+7}{6} \\ \frac{3(s-1)}{8} & \frac{1}{16}(s+3)^2 \end{bmatrix} \quad V = \begin{bmatrix} 1 & \frac{1}{96}\left(-s^4-3s^3+17s^2-45s+32\right) \\ 0 & 1 \end{bmatrix}$$

Which we can verify by checking $U(s)N(s)V(s) = \Lambda(s)$. We can also verify that U and V are unimodular as both their determinants are ALWAYS 1. We can get U_1 and U_2 by noting that $H(s) = U_1(s)\frac{\Lambda(s)}{d(s)}U_2(s)$ So $U_1(s) = U^{-1}$ and $U_2 = V^{-1}$

$$U_1(s) = \begin{bmatrix} \frac{1}{16}(s+3)^2 & \frac{1}{6}(-s-7) \\ -\frac{3}{8}(s-1) & 1 \end{bmatrix} \quad U_2(s) = \begin{bmatrix} 1 & \frac{1}{96}\left(s^4+3s^3-17s^2+45s-32\right) \\ 0 & 1 \end{bmatrix}$$

We can also verify these are unimodular as well by taking the determinant and seeing that they always equal 1 as well. We can get the Smith-McMillan Form of $H(s)$ with $H(s) = U_1(s)M(s)U_2(s)$ Which we can easily get our ϵ's and ψ's from

$$M(s) = \begin{bmatrix} \frac{16}{(s-1)^2(s+3)^2} & 0 \\ 0 & \frac{s(s+1)}{s+3} \end{bmatrix} \implies \epsilon_1 = 16, \quad \epsilon_2 = s^2+6s+15 \quad \psi_1 = (s-1)^2(s+3)^2, \quad \psi_2 = (s+3)$$

With the ψ's we can determine that the minimal realization of $H(s)$ is $n_{min} = 5$. Expanding $M(s) = E(s)\Psi_R(s)^{-1}$ we get $H(s) = U_1(s)E(s)[U_2(s)^{-1}\Psi_R(s)]^{-1}$ which gives us our MFD form We can also note that $U_2(s)^{-1} = V(s)$ which we already have $H(s) = N_0(s)D_0(s)^{-1}$ which will have minimal degree.

$$H(s) = \begin{bmatrix} \frac{1}{16}(s+3)^2 & \frac{1}{6}(-s-7) \\ -\frac{3}{8}(s-1) & 1 \end{bmatrix} \begin{bmatrix} 16 & 0 \\ 0 & s(s+1) \end{bmatrix} \left(\begin{bmatrix} 1 & \frac{1}{96}\left(-s^4 - 3s^3 + 17s^2 - 45s + 32\right) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (s-1)^2(s+3)^2 & 0 \\ 0 & 16(s+3) \end{bmatrix} \right)^{-1}$$

So we now have a MFD of $H(s)$.

$$H(s) = \begin{bmatrix} s^2 + 6s + 9 & -\frac{s^3}{6} - \frac{4s^2}{3} - \frac{7s}{6} \\ 6 - 6s & s^2 + s \end{bmatrix} \begin{bmatrix} s^4 + 4s^3 - 2s^2 - 12s + 9 & -\frac{s^5}{6} - s^4 + \frac{4s^3}{3} + s^2 - \frac{103s}{6} + 16 \\ 0 & 16s + 48 \end{bmatrix}^{-1}$$

We can also verify that it's true by computing the product. With this we can now create the state space model. We write $D(s) = D_{hc}S(s) + D_{IC}\Psi(s)$ where $S(s)$ is a diagonal matrix of s^k for each column where k is the largest degree of that column and $\Psi(s)$ recreates the remaining terms.

$$D(s) = \begin{bmatrix} s^4 & \frac{-s^5}{6} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 4s^3 - 2s^2 - 12s + 9 & -s^4 + \frac{4s^3}{3} + s^2 - \frac{103s}{6} + 16 \\ 0 & 16s + 48 \end{bmatrix}$$

Not sure why D_{hc} is singluar here, did I mess up on the Smith McMillian process? We can column reduce $D(s)$ with some matrix $Z(s)$ to get $N_1(s)$ and $D_1(s)$ where $H(s) = N(s)Z(s)(D(s)Z(s))^{-1} = N_1(s)D_1(s)^{-1}$

$$Z(s) = \begin{bmatrix} \frac{1}{98}(-7s^2 - 47s + 32) & \frac{s+2}{6} \\ \frac{-3}{49}(7s + 33) & 1 \end{bmatrix} \implies H(s) = \begin{bmatrix} \frac{144}{49} & \frac{7s}{3} + 3 \\ -\frac{48}{49}(7s - 2) & 2 \end{bmatrix} \begin{bmatrix} \frac{480}{49}\left(s^2 + 2s - 3\right) & \frac{1}{3}\left(7s^3 - 5s^2 - 59s + 57\right) \\ -\frac{48}{49}(s+3)(7s+33) & 16(s+3) \end{bmatrix}^{-1}$$

Also note that $Z(s)$ is unimodular as its determinant is always 1. With $D_1(s)$ in a better state we can now break it up properly.

$$D_1(s) = \begin{bmatrix} \frac{480}{49}s^2 & \frac{7}{3}s^3 \\ \frac{-48}{7}s^2 & 0 \end{bmatrix} + \begin{bmatrix} \frac{960}{49}s - \frac{1440}{49} & -\frac{5}{3}s^2 - \frac{59}{3}s + 19 \\ -\frac{2592}{49}s - \frac{4752}{49} & 16s + 48 \end{bmatrix}$$

Using the $S(s)$ and $\Psi(s)$ matrices we get

$$D_1(s) = \begin{bmatrix} \frac{480}{49} & \frac{7}{3} \\ \frac{-48}{7} & 0 \end{bmatrix} \begin{bmatrix} s^2 & 0 \\ 0 & s^3 \end{bmatrix} + \begin{bmatrix} \frac{960}{49} & -\frac{1440}{49} & -\frac{5}{3} & -\frac{59}{3} & 19 \\ -\frac{2592}{49} & -\frac{4752}{49} & 0 & 16 & 48 \end{bmatrix} \begin{bmatrix} s & 0 \\ 1 & 0 \\ 0 & s^2 \\ 0 & s \\ 0 & 1 \end{bmatrix}$$

We can also write the $N(s)$ matrix in terms of $\Psi(s)$

$$N_1(s) = \begin{bmatrix} 0 & \frac{144}{49} & 0 & \frac{7}{3} & 3 \\ -\frac{48}{7} & \frac{96}{49} & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} s & 0 \\ 1 & 0 \\ 0 & s^2 \\ 0 & s \\ 0 & 1 \end{bmatrix}$$

We can now find the controller form realization $\{A_c, B_c, C_c\}$ First we have A_c^o and B_c^o

$$A_c^o = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad B_c^o = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

With this and the relationships

$$A_c = A_c^o - B_c^o D_{hc}^{-1} D_{Ic} \quad B_c = B_c^o D_{hc}^{-1} \quad C_c = N_{ic}$$

We can find the state space representation

$$A_c = \begin{bmatrix} -\frac{54}{7} & -\frac{99}{7} & 0 & \frac{7}{3} & 7 \\ 1 & 0 & 0 & 0 & 0 \\ \frac{57600}{2401} & \frac{172800}{2401} & \frac{5}{7} & -\frac{67}{49} & -\frac{1839}{49} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad B_c = \begin{bmatrix} 0 & -\frac{7}{48} \\ 0 & 0 \\ \frac{3}{7} & \frac{30}{49} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad C_c = \begin{bmatrix} 0 & \frac{144}{49} & 0 & \frac{7}{3} & 3 \\ -\frac{48}{7} & \frac{96}{49} & 0 & 0 & 2 \end{bmatrix}$$

Putting it all together we have

$$\dot{x}(t) = \begin{bmatrix} -\frac{54}{7} & -\frac{99}{7} & 0 & \frac{7}{3} & 7 \\ 1 & 0 & 0 & 0 & 0 \\ \frac{57600}{2401} & \frac{172800}{2401} & \frac{5}{7} & -\frac{67}{49} & -\frac{1839}{49} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & -\frac{7}{48} \\ 0 & 0 \\ \frac{3}{7} & \frac{30}{49} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & \frac{144}{49} & 0 & \frac{7}{3} & 3 \\ -\frac{48}{7} & \frac{96}{49} & 0 & 0 & 2 \end{bmatrix} x(t)$$

As our state space represenation. We know this is minimal since the degree matches the McMillian degree we found earlier 5.

Problem 2

Let

$$A_c = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Show that the unimodular matrices

$$U(s) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & s \\ 1 & s+a_1 & s^2+a_1s+a_2 \end{bmatrix}, \quad V(s) = \begin{bmatrix} 0 & -1 & s^2 \\ -1 & 0 & s \\ 0 & 0 & 1 \end{bmatrix}$$

will convert $sI - A_c$ to its Smith form. Generalize the above result to its matrix case; i.e., each element in A_c is replaced by a corresponding nxn matix.

We can check the first case by just computing $U(s)A_cV(s)$

$$U(s)A_cV(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s^3+a_1s^2+a_2s+a_3 \end{bmatrix}$$

Which we can see is clearly in Simth form. If we rewrite the matricies with matrix indexes for a_i and note that I is the identity matrix that is nxn

$$A_c = \begin{bmatrix} -A_1 & -A_2 & -A_3 \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \quad U(s) = \begin{bmatrix} 0 & 0 & I \\ 0 & I & sI \\ I & sI+A_1 & s^2I+sA_1+A_2 \end{bmatrix} \quad V(s) = \begin{bmatrix} 0 & -I & s^2I \\ -I & 0 & sI \\ 0 & 0 & I \end{bmatrix}$$

Computing this we get

$$\begin{bmatrix} 0 & 0 & I \\ 0 & I & sI \\ I & sI+A_1 & s^2I+sA_1+A_2 \end{bmatrix} \begin{bmatrix} sI+A_1 & A_2 & A_3 \\ -I & sI & 0 \\ 0 & -I & sI \end{bmatrix} \begin{bmatrix} 0 & -I & s^2I \\ -I & 0 & sI \\ 0 & 0 & I \end{bmatrix}$$

Performing the right multiplication first we get

$$\begin{bmatrix} 0 & 0 & I \\ 0 & I & sI \\ I & sI+A_1 & s^2I+sA_1+A_2 \end{bmatrix} \begin{bmatrix} -A_2 & sI+A_1 & (sI+A_1)s^2+A_2s+A_3 \\ -sI & I & 0 \\ I & 0 & 0 \end{bmatrix}$$

Then we get

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -A_2-Is^2-A_1s+s^2I+sA_1+A_2 & sI+A_1-sI-A_1 & (sI+A_1)s^2+A_2s+A_3 \end{bmatrix}$$

Which cancels as we expect to

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & s^3I+A_1s^2+A_2s+A_3 \end{bmatrix}$$

Which is again in Smith form. This shows that the generalized matricies work for the matrix case of A.