Homework 2
MEM 633
Group 1

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Problem 1

$$A = T\Lambda T^{-1} \qquad \Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_n)$$

Where λ_i are distinct. Show that

(a) $e^{At} = Te^{\Lambda t}T^{-1}$
It is given that

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

Plugging in At we have

$$e^{At} = \sum_{i=0}^{\infty} \frac{(At)^i}{i!}$$

Since t is a scalar and A is not it can be helpful to group like terms together

$$e^{At} = \sum_{i=0}^{\infty} A^i \frac{t^i}{i!}$$

Now we plug in our definition of $A = T\Lambda T^{-1}$

$$e^{At} = \sum_{i=0}^{\infty} (T\Lambda T^{-1})^i \frac{t^i}{i!}$$

TODO: Show that the matrix power reduces to $(T\Lambda^i T^{-1}$ and factor the T's The powers of A can be expanded as such

$$A^n = T\Lambda T^{-1}T\Lambda T^{-1}...T\Lambda T^{-1} = T\Lambda^n T^{-1}$$

All of the T's and T^{-1} are paired off expect the first T and the last T^{-1}. Plugging this into our formula we get

$$e^{At} = \sum_{i=0}^{\infty} T\Lambda^i T^{-1} \frac{t^i}{i!}$$

Each term of the sum as a T on the left and a T^{-1} on the right so we can factor them out.

$$e^{At} = T(\sum_{i=0}^{\infty} \Lambda^i \frac{t^i}{i!})T^{-1}$$

By definition the sum $\sum_{i=0}^{\infty} \Lambda^i \frac{t^i}{i!}$ is equal to the matrix exponential $e^{\Lambda t}$. Substituting that in we get

$$e^{At} = Te^{\Lambda t}T^{-1}$$

(b) $e^{\Lambda t} = diag(e^{\lambda_1 t}, e^{\lambda_2 t}, e^{\lambda_n t})$

A the multiplication of two diagonal matricies $diag(a_1, a_2, ..., a_n)diag(b_1, b_2, ..., b_n)$ is another diagonal matrix with each entry multiplied by the corresponding entry $diag(a_1 b_1 a_2 b_2, ..., a_n b_n)$ So we have

$$e^{\Lambda t} = \sum_{i=0}^{\infty} \frac{(\Lambda t)^i}{i!}$$

Where each matrix will be a diagonal matrix of with ith powers of the eigenvalues. Looking at the sum we can look at each diagonal entry as such.

$$e_k^{\Lambda t} = \sum_{i=0}^{\infty} \frac{(\lambda_k t)^i}{i!}$$

Where $e_k^{\Lambda t}$ represents the value of the kth eigenvalue or diagonal entry. We can see that this is by defenintion equal to

$$e^{\lambda_k t} = \sum_{i=0}^{\infty} \frac{(\lambda_k t)^i}{i!}$$

So the kth entry on the diagonal of $e^{\Lambda t}$ will be $e^{\lambda t}$ Putting it all together we get

$$e^{\Lambda t} = diag(e^{\lambda_1 t}, e^{\lambda_2 t}, ..., e^{\lambda_n t})$$

Problem 2

Consider the time-invariant system $\dot{x}(t) = Ax(t)$ where the nxn matrix A has distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. The corresponding eigenvectors are $e_1, e_2, ..., e_n$. Let $T = [e_1, e_2, ..., e_n]$ and $v_1', v_2', ..., v_n'$ be the row vectors of T^{-1}. Show that the solution of $\dot{x}(t) = Ax(t)$ can be written as

$$x(t) = \sum_{i=1}^{n} v_i' x(0) e^{\lambda_i t} e_i$$

Problem 3

The A matrix in Problem 2 is given as

$$A = \begin{bmatrix} 0 & 1 \\ 6 & -5 \end{bmatrix}$$

Write down the solution of $\dot{x}(t) = Ax(t)$ by using the result of Problem 2. You will see the system is unstable. However, $x(t)$ will be bounded if the initial state vector is in the stable subspace. Describe the stable subspace of the system.

Problem 4

Find the realizations in controller and observability forms of the transfer function

$$H(s) = \frac{2s^3 + 13s^2 + 31s + 32}{s^3 + 6s^2 + 11s + 6}$$

Give both block diagrams and state-space equations.