Homework 3
MEM 633
Group 1

Anthony Siddique, Damien Prieur, Jachin Philip, Obinna Ekeh

Problem 1

$$A = T\Lambda T^{-1} \qquad \Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_n)$$

Where λ_i are distinct. Show that

(a) $e^{At} = Te^{\Lambda t}T^{-1}$
It is given that

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

Plugging in At we have

$$e^{At} = \sum_{i=0}^{\infty} \frac{(At)^i}{i!}$$

Since t is a scalar and A is not it can be helpful to group like terms together

$$e^{At} = \sum_{i=0}^{\infty} A^i \frac{t^i}{i!}$$

Now we plug in our definition of $A = T\Lambda T^{-1}$

$$e^{At} = \sum_{i=0}^{\infty} (T\Lambda T^{-1})^i \frac{t^i}{i!}$$

The powers of A can be expanded as such

$$A^n = T\Lambda T^{-1} T\Lambda T^{-1} ... T\Lambda T^{-1} = T\Lambda^n T^{-1}$$

All of the T's and T^{-1} are paired off expect the first T and the last T^{-1}. Plugging this into our formula we get

$$e^{At} = \sum_{i=0}^{\infty} T\Lambda^i T^{-1} \frac{t^i}{i!}$$

Each term of the sum as a T on the left and a T^{-1} on the right so we can factor them out.

$$e^{At} = T(\sum_{i=0}^{\infty} \Lambda^i \frac{t^i}{i!})T^{-1}$$

By definition the sum $\sum_{i=0}^{\infty} \Lambda^i \frac{t^i}{i!}$ is equal to the matrix exponential $e^{\Lambda t}$. Substituting that in we get

$$e^{At} = Te^{\Lambda t}T^{-1}$$

(b) $e^{\Lambda t} = diag(e^{\lambda_1 t}, e^{\lambda_2 t}, e^{\lambda_n t})$
The multiplication of two diagonal matrices $diag(a_1, a_2, ..., a_n)diag(b_1, b_2, ..., b_n)$ is another diagonal matrix with each entry multiplied by the corresponding entry $diag(a_1 b_1 a_2 b_2, ..., a_n b_n)$ So we have

$$e^{\Lambda t} = \sum_{i=0}^{\infty} \frac{(\Lambda t)^i}{i!}$$

Where each matrix will be a diagonal matrix of with ith powers of the eigenvalues. Looking at the sum we can look at each diagonal entry as such.

$$e_k^{\Lambda t} = \sum_{i=0}^{\infty} \frac{(\lambda_k t)^i}{i!}$$

Where $e_k^{\Lambda t}$ represents the value of the kth eigenvalue or diagonal entry. We can see that this is by definition equal to

$$e^{\lambda_k t} = \sum_{i=0}^{\infty} \frac{(\lambda_k t)^i}{i!}$$

So the kth entry on the diagonal of $e^{\Lambda t}$ will be $e^{\lambda t}$ Putting it all together we get

$$e^{\Lambda t} = diag(e^{\lambda_1 t}, e^{\lambda_2 t}, ..., e^{\lambda_n t})$$

Problem 2

Consider the time-invariant system $\dot{x}(t) = Ax(t)$ where the nxn matrix A has distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. The corresponding eigenvectors are $e_1, e_2, ..., e_n$. Let $T = [e_1, e_2, ..., e_n]$ and $v'_1, v'_2, ..., v'_n$ be the row vectors of T^{-1}. Show that the solution of $\dot{x}(t) = Ax(t)$ can be written as

$$x(t) = \sum_{i=1}^{n} v'_i x(0) e^{\lambda_i t} e_i$$

If we take the differential equation and take the Laplace transform $\mathbb{L}$ we get

$$\begin{aligned}
\mathbb{L}(\dot{x}(t)) &= \mathbb{L}(Ax(t)) \\
sX(s) - x(0) &= AX(s) \\
(s - A)X(s) &= x(0) \\
X(s) &= (s - A)^{-1} x(0)
\end{aligned}$$

Taking the inverse Laplace transform $\mathbb{L}^{-1}$ we can find the solution of $x(t)$

$$\begin{aligned}
\mathbb{L}^{-1}(X(s)) &= \mathbb{L}^{-1}((s - A)^{-1} x(0)) \\
x(t) &= e^{At} x(0) \\
x(t) &= Te^{\Lambda t} T^{-1} x(0)
\end{aligned}$$

Substituting T and T^{-1} we have

$$x(t) = \begin{bmatrix} e_1 & e_2 & ... & e_n \end{bmatrix} diag(e^{\lambda_1 t}, e^{\lambda_2 t}, ..., e^{\lambda_n t}) \begin{bmatrix} v'_1 \\ v'_2 \\ ... \\ v'_n \end{bmatrix} x(0)$$

Performing the multiplication on the left we get each λ_i multiplied by it's corresponding e_i. And performing the multiplication on the right we get $x(0)v'_i$

$$x(t) = \begin{bmatrix} e^{\lambda_1 t} e_1 & e^{\lambda_2 t} e_2 & ... & e^{\lambda_n t} e_n \end{bmatrix} \begin{bmatrix} v'_1 x(0) \\ v'_2 x(0) \\ ... \\ v'_n x(0) \end{bmatrix}$$

Writing this product as a general sum we get the form we are expecting

$$x(t) = \sum_{i=1}^{n} v'_i x(0) e^{\lambda_i t} e_i$$

Problem 3

The A matrix in Problem 2 is given as

$$A = \begin{bmatrix} 0 & 1 \\ 6 & -5 \end{bmatrix}$$

Write down the solution of $\dot{x}(t) = Ax(t)$ by using the result of Problem 2. You will see the system is unstable. However, $x(t)$ will be bounded if the initial state vector is in the stable subspace. Describe the stable subspace of the system.
We must first diagonalize the A matrix. Using Mathematica we find

$$\begin{bmatrix} 0 & 1 \\ 6 & -5 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} -6 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{7} & \frac{1}{7} \\ \frac{6}{7} & \frac{1}{7} \end{bmatrix}$$

We showed in the last problem that the sum is equivalent to the matrix product $Te^{\Lambda t}T^{-1}x(0)$ With this we get

$$x(t) = \begin{bmatrix} -1 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} e^{-6t} & 0 \\ 0 & e^{t} \end{bmatrix} \begin{bmatrix} -\frac{1}{7} & \frac{1}{7} \\ \frac{6}{7} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} \left(\frac{6et}{7} + \frac{t}{7e^6}\right) x_1(0) + \left(\frac{et}{7} - \frac{t}{7e^6}\right) x_2(0) \\ \left(\frac{6et}{7} - \frac{6t}{7e^6}\right) x_1(0) + \left(\frac{et}{7} + \frac{6t}{7e^6}\right) x_2(0) \end{bmatrix}$$

The only way this remains bounded is if $x(t) = 0 \quad \forall t \in \mathbb{R}$. So we want to find the nullspace of the matrix

$$t \begin{bmatrix} \left(\frac{6e}{7} + \frac{1}{7e^6}\right) & \left(\frac{e}{7} - \frac{1}{7e^6}\right) \\ \left(\frac{6e}{7} - \frac{6}{7e^6}\right) & \left(\frac{e}{7} + \frac{6}{7e^6}\right) \end{bmatrix} \begin{bmatrix} x_1(0) x_2(0) \end{bmatrix}$$

If this is invertible then there are two solutions and the null space only contains the zero vector.

$$\begin{vmatrix} \left(\frac{6e}{7} + \frac{1}{7e^6}\right) & \left(\frac{e}{7} - \frac{1}{7e^6}\right) \\ \left(\frac{6e}{7} - \frac{6}{7e^6}\right) & \left(\frac{e}{7} + \frac{6}{7e^6}\right) \end{vmatrix} = e^{-5}$$

So the null space only contains the zero vectors.
The stable subspace of the system occurs when $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Problem 4

Find the realizations in controller and observability forms of the transfer function

$$H(s) = \frac{2s^3 + 13s^2 + 31s + 32}{s^3 + 6s^2 + 11s + 6}$$

Give both block diagrams and state-space equations.

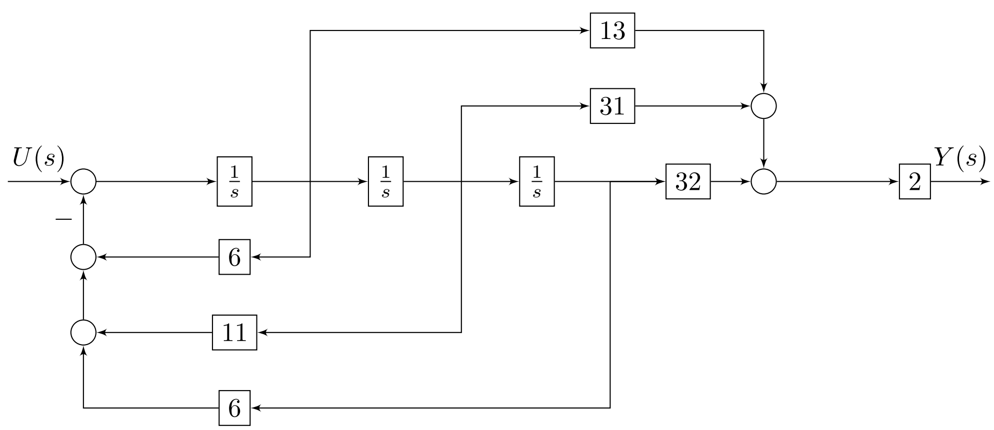
The controllable form is given by the state equations

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 32 - 12 & 31 - 22 & 13 - 12 \end{bmatrix} \quad D = \begin{bmatrix} 2 \end{bmatrix}$$

Reducing we get the following state space equation

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 20 & 9 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 2 \end{bmatrix}$$

The block diagram is given by



Controllable Canonical form

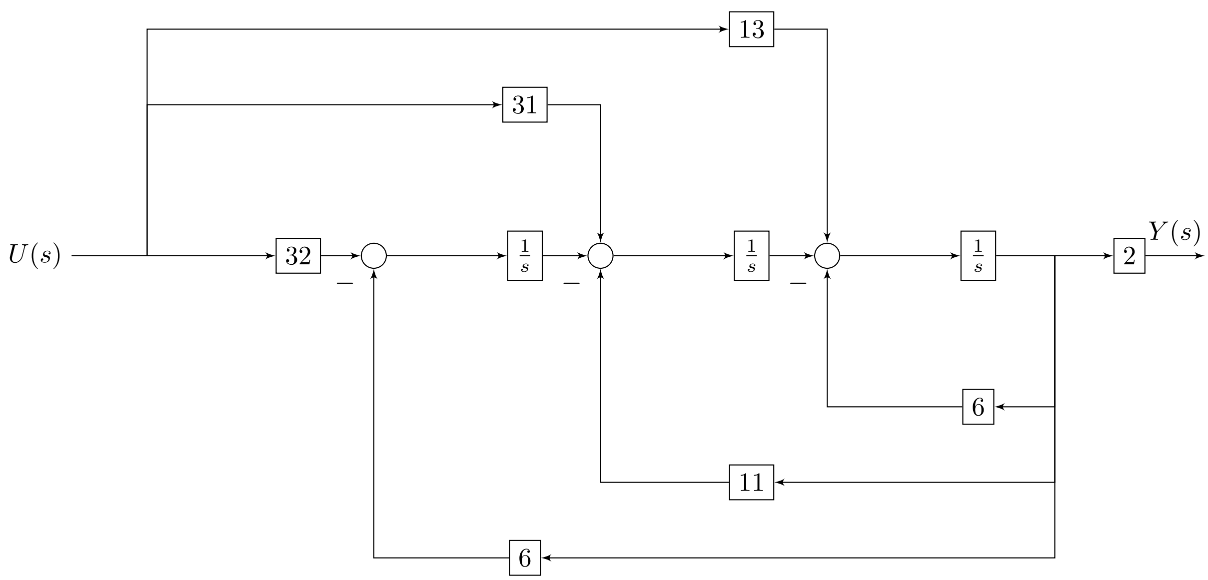
The observable form is given by the state equations

$$A = \begin{bmatrix} -6 & 1 & 0 \\ -11 & 0 & 1 \\ -6 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 13 - 12 \\ 31 - 22 \\ 32 - 12 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 2 \end{bmatrix}$$

Reducing we get the following state space equation

$$A = \begin{bmatrix} -6 & 1 & 0 \\ -11 & 0 & 1 \\ -6 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 9 \\ 20 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 2 \end{bmatrix}$$

The block diagram is given by



Observable Canonical form