# Homework 5
# MEM 633
# Group 1

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## Problem 1

Find a minimal controller-form (NOT a block controller form) realization of

$$H(s) = \begin{bmatrix} \frac{1}{(s-1)^2} & \frac{1}{(s-1)(s+3)} \\ \frac{-6}{(s-1)(s+3)^2} & \frac{s-2}{(s+3)^2} \end{bmatrix}$$

First we must find a Matrix Fraction Description (MFD) of $H(s) = N(s)D(s)^1$ Using the Smith-McMillan form we start by writing $H(s) = \frac{N(s)}{d(s)}$ where $d(s)$ is the monic least common multiple of the denominators of the entries of $H(s)$.

$$d(s) = (s-1)^2(s+3)^2 \implies N(s) = \begin{bmatrix} (s+3)^2 & (s-1)(s+3) \\ -6(s-1) & (s-1)^2(s-2) \end{bmatrix}$$

Now we need to find the Smith form of $H(s)$ by elementary row and column operations. First get the first entry to be a constant then we can easily remove all off diagonal entries.

$$\begin{bmatrix} 1 & \frac{s+7}{6} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (s+3)^2 & (s-1)(s+3) \\ -6(s-1) & (s-1)^2(s-2) \end{bmatrix} \begin{bmatrix} 1 & \frac{s^2-3s}{6} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 16 & \frac{1}{6}(s^4+3s^3-s^2-3s-32) \\ -6(s-1) & 2(s-1) \end{bmatrix}$$

We can rescale to make it a bit easier to see

$$\begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 16 & \frac{1}{6}(s^4+3s^3-s^2-3s-32) \\ -6(s-1) & 2(s-1) \end{bmatrix} \begin{bmatrix} \frac{1}{6} & \frac{1}{3} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 16 & s^4+3s^3-s^2-3s \\ (s-1) & 0 \end{bmatrix}$$

Now we can remove the off diagonal entries with

$$\begin{bmatrix} 1 & 0 \\ s-1 & 16 \end{bmatrix} \begin{bmatrix} 16 & s^4+3s^3-s^2-3s \\ (s-1) & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{-1}{16}(s^4+3s^3-s^2-3s) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & s^5+2s^4-4s^3-2s^2+3s \end{bmatrix}$$

We can combine the elementary operations to find $U$ and $V$

$$U = \begin{bmatrix} 6 & s+7 \\ 7(s-1) & (s+3)^2 \end{bmatrix} \quad V = \begin{bmatrix} \frac{1}{6} & \frac{-1}{96}(s^4+3s^3-17s^2+45s-32) \\ 0 & 1 \end{bmatrix}$$

Which we can verify by checking $U(s)d(s)H(s)V(s) = \Lambda(s)$. We can get $U_1$ and $U_2$ by noting that $H(s) = U_1(s)\frac{\Lambda(s)}{d(s)}U_2(s)$ So $U_1(s) = U^{-1}$ and $U_2 = V^{-1}$

$$U_1(s) = \frac{1}{16}\begin{bmatrix} \frac{1}{6}(s+3)^2 & \frac{-1}{6}(s+7) \\ s-1 & 1 \end{bmatrix} \quad U_2(s) = \begin{bmatrix} 6 & \frac{1}{16}(s^4+3s^3-17s^2+45s-32) \\ 0 & 1 \end{bmatrix}$$

We can get the Smith-McMillan Form of $H(s)$ with $H(s) = U_1(s)M(s)U_2(s)$ Which we can easily get our $\epsilon$'s and $\psi$'s from

$$M(s) = \begin{bmatrix} \frac{16}{(s-1)^2(s+3)^2} & 0 \\ 0 & \frac{s(s+1)}{s+3} \end{bmatrix} \implies \epsilon_1 = 16, \quad \epsilon_2 = s^2+6s+15 \quad \psi_1 = (s-1)^2(s+3)^2, \quad \psi_2 = (s+3)$$

With the $\psi$'s we can determine that the minimal realization of $H(s)$ is $n_{min} = 5$. Expanding $M(s) = E(s)\Psi_R(s)^{-1}$ we get $H(s) = U_1(s)E(s)[U_2(s)^{-1}\Psi_R(s)]^{-1}$ which gives us our MFD form $H(s) = N_0(s)D_0(s)^{-1}$ which will have minimal degree.

$$H(s) = \begin{bmatrix} \frac{1}{6}(s+3)^2 & \frac{-1}{6}(s+7) \\ s-1 & 1 \end{bmatrix} \begin{bmatrix} 16 & 0 \\ 0 & s^2+6s+15 \end{bmatrix} \left( \begin{bmatrix} 6 & \frac{1}{16}(s^4+3s^3-17s^2+45s-32) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (s-1)^2(s+3)^2 & 0 \\ 0 & (s+3) \end{bmatrix} \right)^{-1}$$

So we now have a MFD of $H(s)$.

$$H(s) = \begin{bmatrix} \frac{1}{6}(s-1)(s+3)^2 & -\frac{1}{96}(s-1)s(s+1)(s+7) \\ -(s-1)^2 & \frac{1}{16}(s-1)s(s+1) \end{bmatrix} \begin{bmatrix} \frac{1}{6}(s-1)^3(s+3)^2 & -\frac{1}{96}(s-1)(s+3)\left(s^4+3s^3-17s^2+45s-32\right) \\ 0 & (s-1)(s+3) \end{bmatrix}^{-1}$$

With this we can now create the state space model. We write $D(s) = D_{hc}S(s) + D_{IC}\Psi(s)$ where $S(s)$ is a diagonal matrix of $s^k$ for each column where $k$ is the largest degree of that column and $\Psi(s)$ recreates the remaining terms.

$$D(s) = \begin{bmatrix} \frac{s^5}{6} & \frac{-s^6}{96} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{s^4}{2} - s^3 - \frac{5s^2}{3} + \frac{7s}{2} - \frac{3}{2} & -\frac{5s^5}{96} + \frac{7s^4}{48} - \frac{s^3}{48} - \frac{109s^2}{96} + \frac{199s}{96} - 1 \\ 0 & s^2+2s-3 \end{bmatrix}$$

Not sure why $D_{hc}$ is singluar here, did I mess up on the Smith McMillian process?
Using the $S(s)$ and $\Psi(s)$ matrices we get

$$D(s) = \begin{bmatrix} \frac{1}{6} & \frac{-1}{96} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s^5 & 0 \\ 0 & s^6 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & -1 & \frac{-5}{3} & \frac{7}{2} & \frac{-3}{2} & \frac{-5}{96} & \frac{7}{48} & \frac{-1}{48} & \frac{-109}{96} & \frac{199}{96} & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} s^4 & 0 \\ s^3 & 0 \\ s^2 & 0 \\ s & 0 \\ 1 & 0 \\ 0 & s^5 \\ 0 & s^4 \\ 0 & s^3 \\ 0 & s^2 \\ 0 & s \\ 0 & 1 \end{bmatrix}$$

We can also write the $N(s)$ matrix in terms of $\Psi(s)$

$$N(s) = \begin{bmatrix} 0 & \frac{1}{6} & \frac{5}{6} & \frac{1}{2} & \frac{-3}{2} & 0 & \frac{-1}{96} & \frac{-7}{96} & \frac{1}{96} & \frac{7}{96} & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & \frac{1}{16} & 0 & \frac{-1}{16} & 0 \end{bmatrix} \begin{bmatrix} s^4 & 0 \\ s^3 & 0 \\ s^2 & 0 \\ s & 0 \\ 1 & 0 \\ 0 & s^5 \\ 0 & s^4 \\ 0 & s^3 \\ 0 & s^2 \\ 0 & s \\ 0 & 1 \end{bmatrix}$$

We can now find the controller form realization $\{A_c, B_c, C_c\}$ First we have $A_c^o$ and $B_c^o$

$$A_c^o = \begin{bmatrix} 0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&1 \end{bmatrix} \qquad B_c^o = \begin{bmatrix} 1&0 \\ 0&0 \\ 0&0 \\ 0&0 \\ 0&0 \\ 0&1 \\ 0&0 \\ 0&0 \\ 0&0 \\ 0&0 \\ 0&0 \end{bmatrix}$$

With this and the relationships

$$A_c = A_c^o - B_c^o D_{hc}^{-1} D_{Ic} \quad B_c = B_c^o D_{hc}^{-1} \quad C_c = N_{ic}$$

Due the the inverse of $D_{hc}$ not existing since the one I have is singular It can't be computed. We can find the state space representation

$$A_c = \begin{bmatrix} 9 \end{bmatrix}$$

# Problem 2

Let

$$A_c = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Show that the unimodular matrices

$$U(s) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & s \\ 1 & s+a_1 & s^2+a_1s+a_2 \end{bmatrix}, \quad V(s) = \begin{bmatrix} 0 & -1 & s^2 \\ -1 & 0 & s \\ 0 & 0 & 1 \end{bmatrix}$$

will convert $sI - A_c$ to its Smith form. Generalize the above result to its matrix case; i.e., each element in $A_c$ is replaced by a corresponding $nxn$ matix.

We can check the first case by just computing $U(s)A_cV(s)$

$$U(s)A_cV(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s^3+a_1s^2+a_2s+a_3 \end{bmatrix}$$

Which we can see is clearly in Simth form. If we rewrite the matricies with matrix indexes for $a_i$ and note that $I$ is the identity matrix that is $n$x$n$

$$A_c = \begin{bmatrix} -A_1 & -A_2 & -A_3 \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \quad U(s) = \begin{bmatrix} 0 & 0 & I \\ 0 & I & sI \\ I & sI+A_1 & s^2I+sA_1+A_2 \end{bmatrix} \quad V(s) = \begin{bmatrix} 0 & -I & s^2I \\ -I & 0 & sI \\ 0 & 0 & I \end{bmatrix}$$

Computing this we get

$$\begin{bmatrix} 0 & 0 & I \\ 0 & I & sI \\ I & sI+A_1 & s^2I+sA_1+A_2 \end{bmatrix} \begin{bmatrix} sI+A_1 & A_2 & A_3 \\ -I & sI & 0 \\ 0 & -I & sI \end{bmatrix} \begin{bmatrix} 0 & -I & s^2I \\ -I & 0 & sI \\ 0 & 0 & I \end{bmatrix}$$

Performing the right multiplication first we get

$$\begin{bmatrix} 0 & 0 & I \\ 0 & I & sI \\ I & sI+A_1 & s^2I+sA_1+A_2 \end{bmatrix} \begin{bmatrix} -A_2 & sI+A_1 & (sI+A_1)s^2+A_2s+A_3 \\ -sI & I & 0 \\ I & 0 & 0 \end{bmatrix}$$

Then we get

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -A_2-Is^2-A_1s+s^2I+sA_1+A_2 & sI+A_1-sI-A_1 & (sI+A_1)s^2+A_2s+A_3 \end{bmatrix}$$

Which cancels as we expect to

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & s^3I+A_1s^2+A_2s+A_3 \end{bmatrix}$$

Which is again in Smith form. This shows that the generalized matricies work for the matrix case of $A$.