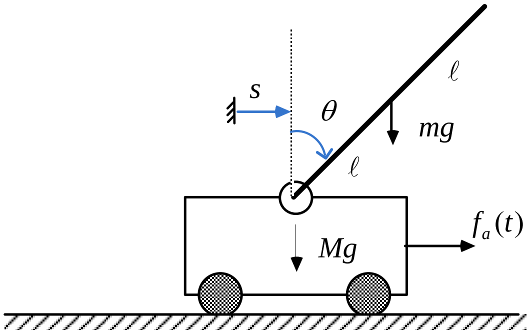
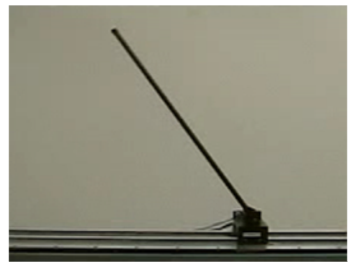
MEM 633 (Nov. 2021) HW 6

Consider the the following cart-inverted pendulum system:
(Although you may find more detailed information of the system in Kwatny and Chang's book, you do not need to read the book to complete this HW assignment.)





The equation of motion of the system has been derived in Section (4.5) of the Kwatny & Chang book,

$$\begin{aligned}(4/3)m\ell^2\ddot{\theta} + B_\theta\dot{\theta} - mg\ell\sin\theta + m\ell\cos\theta\ddot{s} = 0\\ (M+m)\ddot{s} + B_s\dot{s} - m\ell\sin\theta\dot{\theta}^2 + m\ell\cos\theta\ddot{\theta} = u\end{aligned} \qquad (4.59)$$

Assume both of the rotational and translational frictions are negligible, i.e., $B_s = 0$ Ns/m, and $B_\theta = 0$ Ns. Hence, the above equation can be simplified as follows,

$$\begin{aligned}(4/3)m\ell^2\ddot{\theta} - mg\ell\sin\theta + m\ell\cos\theta\ddot{s} = 0\\ (M+m)\ddot{s} - m\ell\sin\theta\dot{\theta}^2 + m\ell\cos\theta\ddot{\theta} = u\end{aligned} \qquad (4\text{-}59A)$$

(a) Plug the following experimental data $M = 1.79$ kg, $m = 0.104$ kg, $\ell = 0.3048$ m, g = 9.8 m/s^2 into Eq. (4.59A) to obtain the equation of motion of the system without the friction coefficients.

$$1.288\ddot{\theta} + 3.17\cos\theta\ddot{s} = 31.07\sin\theta$$

$$3.17\cos\theta\cdot\ddot{\theta} + 189.4\ddot{s} = 3.17\sin\theta\cdot\dot{\theta}^2 + 100u$$

(b) Define the state variables as $x_1(t) = \theta(t),\ x_2(t) = \dot{\theta}(t),\ x_3(t) = s(t),\ x_4(t) = \dot{s}(t)$. Find the state equations,

$$\dot{x}_i(t) = f_i(x,u), \quad i = 1,2,3,4 \qquad (7.87A)$$

Note that by inspection from the definition of state variables, we already have the following two state equations:

$$\dot{x}_1(t) = x_2(t) = f_1(x,u), \qquad \dot{x}_3(t) = x_4(t) = f_3(x,u)$$

To find the other two state equations, $\dot{x}_2(t) = f_2(x,u)$, and $\dot{x}_4(t) = f_4(x,u)$, we would need to replace $\ddot{\theta}$, $\dot{\theta}$, θ, $\ddot{s}$, $\dot{s}$, and s in Eq. (4.59A) by $\dot{x}_2$, x_2, x_1, $\dot{x}_4$, x_4, and x_3, respectively, and then solve for $\dot{x}_2$ and $\dot{x}_4$ in terms of the state variables x_1, x_2, x_3, x_4 and the input u.

$$\begin{bmatrix} 1.288 & 3.17\cos x_1 \\ 3.17\cos x_1 & 189.4 \end{bmatrix}\begin{bmatrix} \dot{x}_2 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 31.07\sin x_1 \\ 3.17\sin x_1\cdot x_2^2 + 100u \end{bmatrix} \qquad (7.84A)$$

$$\begin{bmatrix}\dot{x}_2\\ \dot{x}_4\end{bmatrix}=\begin{bmatrix}1.288 & 3.17\cos x_1\\ 3.17\cos x_1 & 189.4\end{bmatrix}^{-1}\begin{bmatrix}31.07\sin x_1\\ 3.17\sin x_1\cdot x_2^2+100u\end{bmatrix}$$
$$=\frac{1}{\Delta}\begin{bmatrix}189.4 & -3.17\cos x_1\\ -3.17\cos x_1 & 1.288\end{bmatrix}\begin{bmatrix}31.07\sin x_1\\ 3.17\sin x_1\cdot x_2^2+100u\end{bmatrix}$$
$$=\frac{1}{\Delta}\begin{bmatrix}5885\sin x_1-10\cos x_1\sin x_1\cdot x_2^2-317\cos x_1 u\\ -98.5\sin x_1\cos x_1+4.083\sin x_1\cdot x_2^2+128.8u\end{bmatrix},\qquad \text{where }\ \Delta=244-10\cos^2 x_1$$

$$\begin{aligned}\dot{x}_1 &= f_1(x,u)=x_2\\ \dot{x}_2 &= f_2(x,u)=\frac{1}{\Delta}\left(5885\sin x_1-10\cos x_1\sin x_1\cdot x_2^2-317\cos x_1 u\right)\\ \dot{x}_3 &= f_3(x,u)=x_4\\ \dot{x}_4 &= f_4(x,u)=\frac{1}{\Delta}\left(-98.5\sin x_1\cos x_1+4.083\sin x_1\cdot x_2^2+128.8u\right)\end{aligned}\qquad (7.87)$$

where $\Delta=244-10\cos^2 x_1$

(c) Find the equilibriums of the system. You will find two groups of equilibriums, one stable and the other unstable.

$$\begin{aligned}\dot{x}_1 &= f_1(x,u)=x_2=0 \quad\rightarrow\quad x_2=0\\ \dot{x}_2 &= f_2(x,u)=\frac{1}{\Delta}\left(5885\sin x_1-10\cos x_1\sin x_1\cdot x_2^2-317\cos x_1 u\right)=0 \quad\rightarrow\quad x_1=0 \ \text{ or } \ x_1=\pi\\ \dot{x}_3 &= f_3(x,u)=x_4=0 \quad\rightarrow\quad x_4=0\\ \dot{x}_4 &= f_4(x,u)=\frac{1}{\Delta}\left(-95.8\sin x_1\cos x_1+4.083\sin x_1\cdot x_2^2+128.8u\right) \quad\rightarrow\quad x_1=0 \ \text{ or } \ x_1=\pi\end{aligned}$$

There are two groups of equilibriums:

The unstable equilibriums: $x_{unstable}=\begin{bmatrix}0\\0\\x_3\\0\end{bmatrix}$, and the stable equilibriums: $x_{stable}=\begin{bmatrix}\pi\\0\\x_3\\0\end{bmatrix}$, .

where x_3 can be any displacement of the cart as long as the cart is inside the range of the rail. We will choose $x_3=0$ to represent the center of the rail.

(d) Find a linearized state-space model, $\dot{x}(t)=A_U x(t)+B_U u(t)$, at the equilibrium $x=\bar{x}_U=\begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix}^T$. Verify that the linearized system at $\bar{x}_U$ is unstable.

$$\dot{x}(t) = A_U x(t) + B_U u(t)$$

$$A_U = \begin{bmatrix} \frac{\delta f_1}{\delta x_1} & \frac{\delta f_1}{\delta x_2} & \frac{\delta f_1}{\delta x_3} & \frac{\delta f_1}{\delta x_4} \\ \frac{\delta f_2}{\delta x_1} & \frac{\delta f_2}{\delta x_2} & \frac{\delta f_2}{\delta x_3} & \frac{\delta f_2}{\delta x_4} \\ \frac{\delta f_3}{\delta x_1} & \frac{\delta f_3}{\delta x_2} & \frac{\delta f_3}{\delta x_3} & \frac{\delta f_3}{\delta x_4} \\ \frac{\delta f_4}{\delta x_1} & \frac{\delta f_4}{\delta x_2} & \frac{\delta f_4}{\delta x_3} & \frac{\delta f_4}{\delta x_4} \end{bmatrix}_{\bar{x}_U,\bar{u}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{\delta f_2}{\delta x_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\delta f_4}{\delta x_1} & 0 & 0 & 0 \end{bmatrix}_{\bar{x}_U,\bar{u}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 25.15 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.42094 & 0 & 0 & 0 \end{bmatrix}$$

$$B_U = \begin{bmatrix} \frac{\delta f_1}{\delta u} \\ \frac{\delta f_2}{\delta u} \\ \frac{\delta f_3}{\delta u} \\ \frac{\delta f_4}{\delta u} \end{bmatrix}_{\bar{x}_U,\bar{u}} = \begin{bmatrix} 0 \\ \frac{\delta f_2}{\delta u} \\ 0 \\ \frac{\delta f_4}{\delta u} \end{bmatrix}_{\bar{x}_U,\bar{u}} = \begin{bmatrix} 0 \\ -1.3547 \\ 0 \\ 0.5504 \end{bmatrix}$$

(e) Find a linearized state-space model, $\dot{x}(t) = A_S x(t) + B_S u(t)$, at the equilibrium $x = \bar{x}_S = \begin{bmatrix} \pi & 0 & 0 & 0 \end{bmatrix}^T$. Verify that the linearized system at $\bar{x}_S$ is stable.

$$\dot{x}(t) = A_S x(t) + B_S u(t)$$

$$A_S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{\delta f_2}{\delta x_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\delta f_4}{\delta x_1} & 0 & 0 & 0 \end{bmatrix}_{\bar{x}_S,\bar{u}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -25.15 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.42094 & 0 & 0 & 0 \end{bmatrix}$$

$$B_S = \begin{bmatrix} 0 \\ \frac{\delta f_2}{\delta u} \\ 0 \\ \frac{\delta f_4}{\delta u} \end{bmatrix}_{\bar{x}_S,\bar{u}} = \begin{bmatrix} 0 \\ 1.3547 \\ 0 \\ 0.5504 \end{bmatrix}$$

(f) Design a state feedback controller,

$$u(t) = Kx(t) = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix} \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) & x_4(t) \end{bmatrix}^T$$

so that the eigenvalues of $A_u + B_u K$ are in the right half of the complex plane. In other words, the closed-loop system is stable at the equilibrium $x = \bar{x}_u = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$ subject to the control input constraints u(t) is between -80N and +80N.

In this computer assignment, you will use the following three computer programs to help you answer some questions regarding the cart-inverted pendulum control system.

1. **`HW6_InvP_cntrl_1_wFriction.m`**
   This program will automatically call the Simulink simulation program, **`HW6_Inv_sim_2_wFriction.slx,`** and the MATLAB program, **`HW6_Inv_plot_3.m`**, which will automatically plot the simulation results for you.
   You are free to make changes on the initial conditions and feedback controller, and run the program for each case you want to explore.

2. **`HW6_Inv_sim_2_wFriction.slx`**
   `This Simulink program was built based on the nonlinear state-space model Eq.(7.87).`

3. **`HW6_Inv_plot_3.m`**
   `This program is called after the Simulink program to plot the simulation results. You are free to make changes of this program.`

(g) Run the program several times with random initial conditions to verify that $x = \bar{x}_U = \begin{bmatrix} 0 & 0 & x_3 & 0 \end{bmatrix}^T, u = \bar{u} = 0$ and $x = \bar{x}_S = \begin{bmatrix} \pi & 0 & x_3 & 0 \end{bmatrix}^T, u = \bar{u} = 0$ are equilibriums of the system, where x_3 can be any constant. **(20%)**

(h) Let initial conditions be $x_{10} = 15^\circ$ or 0.26 rad, $x_{30} = 0.2\text{m}$, and run the program with at least the following seven state feedback controllers: $F_1 = \begin{bmatrix} 120 & 22 & 20 & 28 \end{bmatrix}$, $F_2 = \begin{bmatrix} 120 & 22 & 20 & 28 \end{bmatrix} * 0.5$, $F_3 = \begin{bmatrix} 120 & 22 & 0 & 28 \end{bmatrix}$, $F_4 = \begin{bmatrix} 90 & 22 & 20 & 28 \end{bmatrix}$, $F_5 = \begin{bmatrix} 60 & 22 & 20 & 28 \end{bmatrix}$, $F_6 = \begin{bmatrix} 0 & 22 & 20 & 28 \end{bmatrix}$, $F_7 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$. Analyze the simulation result graphs, and have your comments on each case. **(60%)**

(i) The linearized state equation at the equilibrium $x = \bar{x}_u = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$ is given by

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 25.15 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.42094 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -1.3547 \\ 0 \\ 0.5504 \end{bmatrix} \quad \text{if the frictions are negligible.}$$

and

$$A_f = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 25.15 & -1.6188 & 0 & 0.33868 \\ 0 & 0 & 0 & 1 \\ -0.42094 & 0.027094 & 0 & -0.1376 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -1.3547 \\ 0 \\ 0.5504 \end{bmatrix}$$ if the frictions are considered.

Compute the eigenvalues of $A_0 + BF$ and $A_f + BF$ for each of the seven cases in Problem **(h)** and explain their physical meanings and relevance to the simulation results from Problem **(h)**. **(20%)**