Abbotsleigh 1999 MX2 Trial Q4(b)

Q4 (b)(i)

Show that the tangent to the rectangular hyperbola $xy = c^2$ at the point $T(ct, \frac{c}{t})$ has equation $x^2 + t^2y = 2ct$

Q4 (b)(ii)

The tangents to the rectangular hyperbola $xy = c^2$ at the points $P(cp, \frac{c}{p})$ and $Q(cq, \frac{c}{q})$, where $pq = 1$, intersect at R.

Find the equation of the locus of R and state any restrictions on the value of x for this locus.

Worked Solution:

Q4 (b)(i)

Firstly to find the derivative of the rectangular hyperbola (as a graph)

$xy = c^2 \Rightarrow y = \dfrac{c^2}{x}$
$\Rightarrow \dfrac{dy}{dx} = -\dfrac{c^2}{x^2}$

Therefore at $T\left(c, \dfrac{c}{t}\right)$, $\dfrac{dy}{dx} = -\dfrac{c^2}{(ct)^2}$
$= -\dfrac{1}{t^2}$

Now, using the gradient-point form of a line:

$y - \dfrac{c}{t} = -\dfrac{1}{t^2}(x - ct)$
$t^2y - ct = -x + ct$

Therefore the tangent has equation

$$x + t^2y = 2ct$$

Q4 (b)(ii)

Similar the equations of tangents at P and Q are:

$$x + p^2 y = 2cp \tag{1}$$

and

$$x + q^2 y = 2cq \tag{2}$$

Multiple equation (1) by q and (2) by p to give:

$$qx + p^2 qy = 2cpq$$

$$px + pq^2 y = 2cpq$$

and since it is given that $pq = 1$, one has equations (3) and (4) respectively:

$$qx = 2c - py \tag{3}$$

$$px = 2c - qy \tag{4}$$

Intersect equations (3) and (4) to find R, that is (4) $-$ (3):

$$(p - q)x = (p - q)y$$

Divide through by $p - q$ to give the locus of R as the line $y = x$.

But $x = 0$ given $c \neq 0$, when substituted into (1) and (2) imply that $p = 0$ and $q = 0$, which is a contradiction. Therefore, the locus of R is the line $y = x$ where $x \neq 0$