

## James Ruse 2006 MX2 Trial Q8(c)

Given  $a_1, a_2, a_3, \dots, a_n$  and  $b_1, b_2, b_3, \dots, b_n$  are positive real numbers, where  $A_n = a_1 + a_2 + a_3 + \dots + a_n$  and  $B_n = b_1 + b_2 + b_3 + \dots + b_n$  are such that  $a_1, a_2, a_3, \dots, a_n > 0$ ,  $b_1, b_2, b_3, \dots, b_n > 0$  and  $A_r \leq B_r$ , for  $r = 1, 2, 3, \dots, n$

Q8 (c)(i)

Prove by mathematical induction for  $n = 1, 2, 3, \dots$  that:

$$\begin{aligned} \frac{1}{\sqrt{b_n}} B_n + \left( \frac{1}{\sqrt{b_{n-1}}} - \frac{1}{\sqrt{b_n}} \right) B_{n-1} + \left( \frac{1}{\sqrt{b_{n-2}}} - \frac{1}{\sqrt{b_{n-1}}} \right) B_{n-2} + \dots + \left( \frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}} \right) B_1 \\ = \sqrt{b_1} + \sqrt{b_2} + \sqrt{b_3} + \dots + \sqrt{b_n} \end{aligned}$$

Q8 (c)(ii)

Hence, given:

$$\begin{aligned} \frac{a_1}{\sqrt{b_1}} + \frac{a_2}{\sqrt{b_2}} + \frac{a_3}{\sqrt{b_3}} + \dots + \frac{a_n}{\sqrt{b_n}} \\ = \frac{1}{\sqrt{b_n}} A_n + \left( \frac{1}{\sqrt{b_{n-1}}} - \frac{1}{\sqrt{b_n}} \right) A_{n-1} + \left( \frac{1}{\sqrt{b_{n-2}}} - \frac{1}{\sqrt{b_{n-1}}} \right) A_{n-2} + \dots + \left( \frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}} \right) A_1 \end{aligned}$$

Q8 (c)(iii)

Deduce that:  $\sum_{r=1}^n \sqrt{a_r} \leq \sum_{r=1}^n \sqrt{b_r}$

### Worked Solutions:

Q8 (c)(i)

For  $n = 1$ : This case is trivial.

For  $n = 2$ :

$$\begin{aligned} \text{LHS} &= \frac{1}{\sqrt{b_2}} B_2 + \left( \frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}} \right) B_1 \\ &= \frac{1}{\sqrt{b_2}} (b_1 + b_2) + \left( \frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}} \right) b_1 \\ &= \frac{b_1}{\sqrt{b_2}} + \sqrt{b_2} + \sqrt{b_1} - \frac{b_1}{\sqrt{b_2}} \\ &= \sqrt{b_2} \quad \text{as required. Therefore, true for } n = 2 \end{aligned}$$

Assume result holds up to some  $n = k$  (strong induction), that is

$$\frac{1}{\sqrt{b_k}} B_k + \left( \frac{1}{\sqrt{b_{k-1}}} - \frac{1}{\sqrt{b_k}} \right) B_{k-1} + \dots + \left( \frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}} \right) B_1 = \sum_{r=1}^k \sqrt{b_r}$$

For  $n = k + 1$

$$\begin{aligned}
\text{LHS} &= \frac{1}{\sqrt{b_{k+1}}} (B_{k+1} + \left( \frac{1}{\sqrt{b_k}} - \frac{1}{\sqrt{b_{k+1}}} \right) B_k + \left( \frac{1}{\sqrt{b_{k-1}}} - \frac{1}{\sqrt{b_k}} \right) B_{k-1} + \cdots + \left( \frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}} \right) B_1) \\
&= \frac{1}{\sqrt{b_{k+1}}} (B_{k+1} - B_k) + \left( \frac{1}{\sqrt{b_k}} B_k + \left( \frac{1}{\sqrt{b_{k-1}}} - \frac{1}{\sqrt{b_k}} \right) B_{k-1} + \cdots + \left( \frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}} \right) B_1 \right) \\
&= \frac{1}{\sqrt{b_{k+1}}} (b_{k+1}) + \sum_{r=1}^k \sqrt{b_r} \\
&= \sqrt{b_{k+1}} + \sum_{r=1}^k \sqrt{b_r} \\
&= \sum_{r=1}^{k+1} \sqrt{b_r}
\end{aligned}$$

Therefore, since the initial case and two consecutive cases hold, by the principle of Mathematical Induction, the proposition is true  $\square$ .

*Q8 (c)(ii)*

$$\begin{aligned}
\sum_{r=1}^n \frac{a_r}{\sqrt{b_r}} &= \frac{1}{\sqrt{b_k}} A_n + \left( \frac{1}{\sqrt{b_{n-1}}} - \frac{1}{\sqrt{b_n}} \right) A_{n-1} + \cdots + \left( \frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}} \right) A_1 \\
&\leq \frac{1}{\sqrt{b_k}} B_n + \left( \frac{1}{\sqrt{b_{n-1}}} - \frac{1}{\sqrt{b_n}} \right) B_{n-1} + \cdots + \left( \frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}} \right) B_1 \quad \text{since } A_i \leq B_i \text{ for all } 0 < i \leq n \\
&= \sum_{r=1}^n \sqrt{b_r}
\end{aligned}$$

Q8 (c)(iii)

Since all  $a_r > 0$  and  $b_r > 0$  for all  $0 < r \leq n$ , therefore from Q8(b)(ii)

$$\sqrt{a_r} < \frac{1}{2} \left( \frac{a_r}{\sqrt{b_r}} + \sqrt{b_r} \right), \quad \text{for all } r \text{ where } 0 < r \leq n$$

Summing over  $n$  terms gives:

$$\begin{aligned} \sum_{r=1}^n \sqrt{a_r} &\leq \sum_{r=1}^n \frac{1}{2} \left( \frac{a_r}{\sqrt{b_r}} + \sqrt{b_r} \right) \\ &= \frac{1}{2} \left( \sum_{r=1}^n \frac{a_r}{\sqrt{b_r}} \right) + \frac{1}{2} \left( \sum_{r=1}^n \sqrt{b_r} \right) \quad (\text{rearranging the sum}) \\ &\leq \frac{1}{2} \left( \sum_{r=1}^n \sqrt{b_r} \right) + \frac{1}{2} \left( \sum_{r=1}^n \sqrt{b_r} \right) \quad (\text{using the result from Q8(c)(ii)}) \\ &= \sum_{r=1}^n \sqrt{b_r} \quad \text{as required} \end{aligned}$$