James Ruse 2006 MX2 Trial Q8(c)

Given $a_1, a_2, a_3, \ldots, a_n$ and $b_1, b_2, b_3, \ldots, b_n$ are positive real numbers,
where $A_n = a_1 + a_2 + a_3 + \cdots + a_n$ and $B_n = b_1 + b_2 + b_3 + \cdots + b_n$ are such that
$a_1, a_2, a_3, \ldots, a_n$ ¿ 0, $b_1, b_2, b_3, \ldots, b_n$ ¿ 0 and $A_r \leq B_r$, for $r = 1, 2, 3, \ldots, n$

Q8 (c)(i)

Prove by mathematical induction for $n = 1, 2, 3, \ldots$ that:

$$\frac{1}{\sqrt{b_n}}\,B_n + \left(\frac{1}{\sqrt{b_{n-1}}} - \frac{1}{\sqrt{b_n}}\right) B_{n-1} + \left(\frac{1}{\sqrt{b_{n-2}}} - \frac{1}{\sqrt{b_{n-1}}}\right) B_{n-2} + \cdots + \left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}\right) B_1 = \sqrt{b_1} + \sqrt{b_2} + \sqrt{b_3} + \cdots +$$

Q8 (c)(ii)

Hence, given:

$$\frac{a_1}{\sqrt{b_1}} + \frac{a_2}{\sqrt{b_2}} + \frac{a_3}{\sqrt{b_3}} + \cdots + \frac{a_n}{\sqrt{b_n}} = \frac{1}{\sqrt{b_n}}\,A_n + \left(\frac{1}{\sqrt{b_{n-1}}} - \frac{1}{\sqrt{b_n}}\right) A_{n-1} + + \left(\frac{1}{\sqrt{b_{n-2}}} - \frac{1}{\sqrt{b_{n-1}}}\right) A_{n-2} + \cdots + \left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}\right.$$

Q8 (c)(iii)

Deduce that: $\displaystyle\sum_{r=1}^{n} \sqrt{a_r} \leq \sum_{r=1}^{n} \sqrt{b_r}$

Worked Solutions:

Q8 (c)(i)

<u>For $n = 1$:</u> This case is trivial.

<u>For $n = 2$:</u>

$$\begin{aligned}
\text{LHS} &= \frac{1}{\sqrt{b_2}}\,B_2 + \left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}\right) B_1 \\
&= \frac{1}{\sqrt{b_2}}(b_1 + b_2) + \left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}\right) b_1 \\
&= \frac{b_1}{\sqrt{b_2}} + \sqrt{b_2} + \sqrt{b_1} - \frac{b_1}{\sqrt{b_2}} \\
&= \sqrt{b_2} \quad \text{as required. Therefore, true for } n = 2
\end{aligned}$$

Assume result holds up to some $n = k$ (strong induction), that is

$$\frac{1}{\sqrt{b_k}}\,B_k + \left(\frac{1}{\sqrt{b_{k-1}}} - \frac{1}{\sqrt{b_k}}\right) B_{k-1} + \cdots + \left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}\right) B_1 = \sum_{r=1}^{k} \sqrt{b_r}$$

<u>For $n = k + 1$</u>

$$\begin{aligned}
\text{LHS} &= \frac{1}{\sqrt{b_{k+1}}}(B_{k+1} + \left(\frac{1}{\sqrt{b_k}} - \frac{1}{\sqrt{b_{k+1}}}\right) B_k + \left(\frac{1}{\sqrt{b_{k-1}}} - \frac{1}{\sqrt{b_k}}\right) B_{k-1} + \cdots + \left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}\right) B_1 \\
&= \frac{1}{\sqrt{b_{k+1}}}(B_{k+1} - B_k) + \left(\frac{1}{\sqrt{b_k}} B_k + \left(\frac{1}{\sqrt{b_{k-1}}} - \frac{1}{\sqrt{b_k}}\right) B_{k-1} + \cdots + \left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}\right) B_1\right) \\
&= \frac{1}{\sqrt{b_{k+1}}}(b_{k+1}) + \sum_{r=1}^{k} \sqrt{b_r} \\
&= \sqrt{b_{k+1}} + \sum_{r=1}^{k} \sqrt{b_r} \\
&= \sum_{r=1}^{k+1} \sqrt{b_r}
\end{aligned}$$

Therefore, since the initial case and two consecutive cases hold, by the principle of Mathematical Induction, the proposition is true $\square$.

Q8 (c)(ii)

$$\begin{aligned}
\sum_{r=1}^{n} \frac{a_r}{\sqrt{b_r}} &= \frac{1}{\sqrt{b_k}} A_n + \left(\frac{1}{\sqrt{b_{n-1}}} - \frac{1}{\sqrt{b_n}}\right) A_{n-1} + \cdots + \left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}\right) A_1 \\
&\leq \frac{1}{\sqrt{b_k}} B_n + \left(\frac{1}{\sqrt{b_{n-1}}} - \frac{1}{\sqrt{b_n}}\right) B_{n-1} + \cdots + \left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}\right) B_1 \quad \text{since } A_i \leq B_i \text{ for all } 0 < i \leq n \\
&= \sum_{r=1}^{n} \sqrt{b_r}
\end{aligned}$$

Q8 (c)(iii)

Since all $a_r > 0$ and $b_r > 0$ for all $0 < r \le n$, therefore from Q8(b)(ii)

$$\sqrt{a_r} < \frac{1}{2}\left(\frac{a_r}{\sqrt{b_r}} + \sqrt{b_r}\right), \quad \text{for all } r \text{ where } 0 < r \le n$$

Summing over n terms gives:

$$\begin{aligned}
\sum_{r=1}^{n} \sqrt{a_r} &\le \sum_{r=1}^{n} \frac{1}{2}\left(\frac{a_r}{\sqrt{b_r}} + \sqrt{b_r}\right) \\
&= \frac{1}{2}\left(\sum_{r=1}^{n} \frac{a_r}{\sqrt{b_r}}\right) + \frac{1}{2}\left(\sum_{r=1}^{n} \sqrt{b_r}\right) \quad \text{(rearranging the sum)} \\
&\le \frac{1}{2}\left(\sum_{r=1}^{n} \sqrt{b_r}\right) + \frac{1}{2}\left(\sum_{r=1}^{n} \sqrt{b_r}\right) \quad \text{(using the result from Q8(c)(ii))} \\
&= \sum_{r=1}^{n} \sqrt{b_r} \quad \text{as required}
\end{aligned}$$