WEIBULL: W_{pdf} maximum likelihood estimation, analysis

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For Weibull - We apply the MLE calculation:

We start by assuming Weibull pdf that is "Independent Identically Distributed" which is usually designated as i.i.d. (or perhaps Markov depending on how you define or scope the analysis):

$$X = \{x_1, \ldots, x_n\}, \quad W_{pdf}(k, \lambda)$$

$$f(x; \lambda, k) = \begin{cases} \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$L(\Theta \mid X) = L(x_1, \cdots x_n; \gamma, \theta) = \prod_{i=1}^{n} (\gamma/\theta) x_i^{\gamma-1} \exp\left(-x_i^{\gamma}/\theta\right)$$

Consider the maximum of $\log\left(L(\Theta \mid X)\right)$:

$$\frac{\partial(\log L(\Theta \mid X))}{\partial\gamma} = 0 \quad and \quad \frac{\partial(\log L(\Theta \mid X))}{\partial\theta} = 0 \quad and$$

Solving the first order differentials:

$$\frac{\partial \ln L}{\partial \gamma} = \frac{n}{\gamma} + \sum_{1}^{n} \ln x_i - \frac{1}{\theta} \sum_{1}^{n} x_i^{\gamma} \ln x_i = 0$$

$$\frac{\partial \ln L}{\partial \theta} = -\frac{n}{\theta} + \frac{1}{\theta^2} \sum_{1}^{n} x_i^{\gamma} \quad = 0$$

On eliminating θ between these two equations and simplifying, we have

$$\left[\frac{\sum_1^n x_i^{\gamma} \ln x_i}{\sum_1^n x_i^{\gamma}} - \frac{1}{\gamma}\right] = \frac{1}{n} \sum_{1}^{n} \ln x_i,$$

We have to now solve the MLE for the best estimate of $\hat{\gamma}$. There is no analytic closed solution here for $\hat{\gamma}$, unfortunately. However, this can be solved by iteration, or trial and error. Once two values γ_1 and γ_2 have been found in an interval such that $\gamma_1 < \gamma < \gamma_2$, end with a linear interpolation for $\hat{\gamma}$.

With $\hat{\gamma}$ thus determined, θ is estimated as

$$\hat{\theta} = \sum_{1}^{n} x_i^{\hat{\gamma}}/n$$

INterestingly enough - this simply looks like the average of the x_i at some power $\hat{\gamma}$.

Discussion

This data is what I like to refere to a ceiling (or floor) data - in other words the data ranges to 100 (or 0) percent. In this example it is a cell culture viability distribution. Therefore, it won't be normally distributed. This type of distribution is often seen in the Biotech area - where we have a top 100 percent value as a ceiling.

We have a few choices: Transform it, do it non-parametrically, or use a suitable pdf. In this case the Weibull pdf is used since the shape is reasonable. Here we set the LSL (spec limit) to 80 percent. And it shows a calculated Ppk of 3.9 ... showing the spec limit may be able to be raised based on the process ANOVA for the viability for the expample vial thaw.

VIAL THAW Viability Distribution ANOVA: For the purposes of a NON-Normal distibution in Cpk type capability analysis. This follows from applying a Weibull pdf approach and is supported by MLE theory (roughly) derived above.

Fitted Weibull Distribution

Parameter		Estimate	Std Error	Lower 95%	Upper 95%
Scale	α	99.221547	0.0578749	99.106403	99.334698
Shape	β	124.59903	7.3481352	110.67423	139.48078

Measures	
-2*LogLikelihood	584.73057
AICc	588.78998
BIC	595.37659

Goodness-of-Fit Test

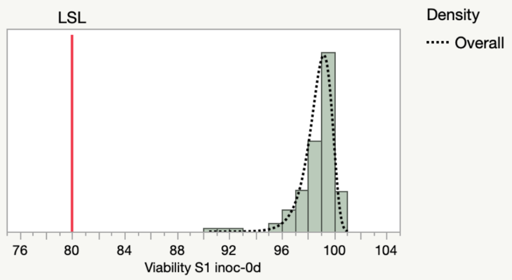
	A2	Simulated p-Value
Anderson-Darling	3.2079181	<.0001*

Process Capability

Viability S1 inoc-0d(Weibull) Capability

Nonnormal capability indices calculated with the Percentiles method.

Histogram



Process Summary

LSL	80
N	205
Sample Mean	98.70732
Sample Std Dev	1.344085

Overall Sigma Capability

Index	Estimate
Ppk	3.917
Ppl	3.917

Parameter Estimates

Parameter		Estimate
Scale	α	99.221547
Shape	β	124.59903