WEIBULL: W_{pdf} maximum likelihood estimation, analysis

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October 23, 2024

version 1.1: for repo release

For Weibull - We apply the MLE calculation:

We start by assuming Weibull pdf that is “Independent Identically Distributed” which is usually designated as i.i.d. (or perhaps Markov depending on how you define or scope the analysis):

$$X = \{x_1, \ldots, x_n\}, \quad W_{pdf}(k, \lambda)$$

$$f(x; \lambda, k) = \begin{cases} \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$L(\Theta \mid X) = L(x_1, \cdots x_n; \gamma, \theta) = \prod_{i=1}^{n} (\gamma/\theta) x_i^{\gamma-1} \exp\left(-x_i^\gamma/\theta\right)$$

Consider the maximum of $\log\left(L(\Theta \mid X)\right)$:

$$\frac{\partial(\log L(\Theta \mid X))}{\partial\gamma} = 0 \quad and \quad \frac{\partial(\log L(\Theta \mid X))}{\partial\theta} = 0 \quad and$$

Solving the first order differentials:

$$\frac{\partial \ln L}{\partial \gamma} = \frac{n}{\gamma} + \sum_1^n \ln x_i - \frac{1}{\theta} \sum_1^n x_i^\gamma \ln x_i = 0$$

$$\frac{\partial \ln L}{\partial \theta} = -\frac{n}{\theta} + \frac{1}{\theta^2} \sum_1^n x_i^\gamma \quad = 0$$

On eliminating θ between these two equations and simplifying, we have

$$\left[\frac{\sum_1^n x_i^\gamma \ln x_i}{\sum_1^n x_i^\gamma} - \frac{1}{\gamma}\right] = \frac{1}{n}\sum_1^n \ln x_i,$$

We have to now solve the MLE for the best estimate of $\hat{\gamma}$. There is no analytic closed solution here for $\hat{\gamma}$, unfortunately. However, this can be solved by iteration, or trial and error. Once two values γ_1 and γ_2 have been found in an interval such that $\gamma_1 < \gamma < \gamma_2$, end with a linear interpolation for $\hat{\gamma}$.

With $\hat{\gamma}$ thus determined, θ is estimated as

$$\hat{\theta} = \sum_1^n x_i^{\hat{\gamma}}/n$$

Interestingly enough - this simply looks like the average of the x_i at some power $\hat{\gamma}$.

Example of WEIBULL PDF and Cpk in JMP

This data is what I like to refer to as ceiling (or floor) data - the data ranges up to a limit of 100 (or 0) percent. In this example it is a cell culture vial thaw viability distribution. Therefore, typically it won't be normally distributed. This type of distribution is often seen in the Biotech area - where we have a top 100 (or bottom 0) percent value as a ceiling.

We have a few choices: Transform it, do it non-parametrically, or use a suitable pdf. In this case the Weibull pdf is used since the shape is reasonable. Here we set the LSL (spec limit) to 80 percent, giving a calculated Ppk of 3.9. Therefore, the spec limit can be raised based on the process ANOVA for the viability in the vial thaw.

VIAL THAW Viability Distribution ANOVA: For the purposes of a non-Normal pdf in Cpk type capability analysis, here is the JMP restult. This follows from applying a Weibull pdf approach and is supported by MLE theory (roughly) derived above.

Fitted Weibull Distribution

Parameter		Estimate	Std Error	Lower 95%	Upper 95%
Scale	α	99.221547	0.0578749	99.106403	99.334698
Shape	β	124.59903	7.3481352	110.67423	139.48078

Measures	
-2*LogLikelihood	584.73057
AICc	588.78998
BIC	595.37659

Goodness-of-Fit Test

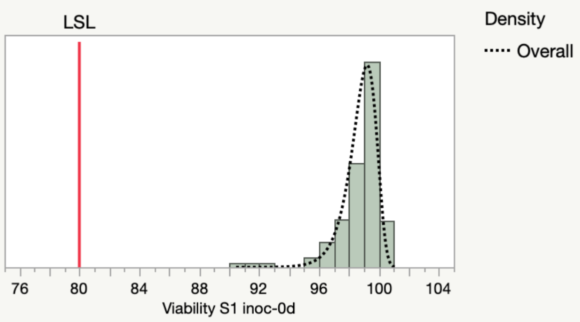
	A2	Simulated p-Value
Anderson-Darling	3.2079181	<.0001*

Process Capability

Viability S1 inoc-0d(Weibull) Capability

Nonnormal capability indices calculated with the Percentiles method.

Histogram



Process Summary

LSL	80
N	205
Sample Mean	98.70732
Sample Std Dev	1.344085

Overall Sigma Capability

Index	Estimate
Ppk	3.917
Ppl	3.917

Parameter Estimates

Parameter		Estimate
Scale	α	99.221547
Shape	β	124.59903