Università degli Studi di Padova

Dipartimento di Ingegneria dell'Informazione

A Branch-and-Cut based Pricer for the Capacitated Vehicle Routing Problem

Graduate:
Davide Paro
ID: 1207154

Supervisor:
Prof. Domenico Salvagnin

Co-Supervisor:
Prof. Roberto Roberti

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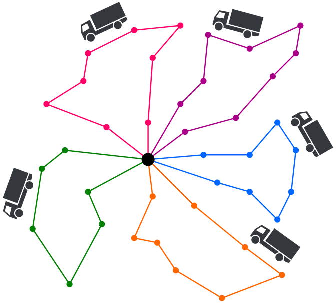
Introduction

The Capacitated Vehicle Routing Problem (**CVRP**) [1] is an NP-hard combinatorial optimization routing problem with applications in logistics (transportation, distribution, delivery).



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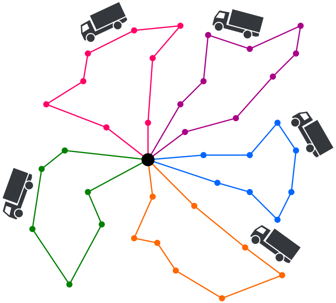

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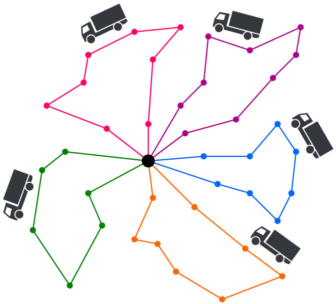

- CVRP defined on a complete graph. We are given:
  - The amount of available vehicles with their capacity.
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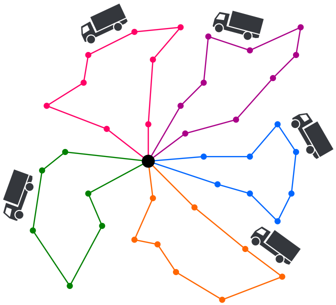

- CVRP defined on a complete graph. We are given:
  - The amount of available vehicles with their capacity.
  - A central depot where vehicles are stationed.
  - Customers' locations.
  - The demands of the individual customers.
- Objective:
  - Minimize overall routing costs while meeting the needs of **all** customers.

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\min_{\lambda} \quad & z_{\mathrm{SC}}(\lambda) = \sum_{p \in P} c_p \lambda_p \\
& \sum_{p \in P} \lambda_p = K \\
& \sum_{p \in P} a_{ip} \lambda_p \geq 1 \quad \forall i \in V_0 \\
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- How?
  - Column Generation (CG): decision variables are generated lazily.
    - In VRP, decision variables represent single vehicle feasible routes.
  - **Pricing** in CVRP, is the art of feeding the BPC approach with good reduced cost routes to bring it towards fast optimality convergence.

Contemporary Approaches for Pricing

- Pricing Problem (PP) in its "natural" form is an **NP-hard** combinatorial optimization problem [12].
  - Relaxation of the PP to make it solvable in pseudo-polynomial time [13, 14].

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- **Two major issues with contemporary approaches**:
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     - Improved in last decade thanks to "smart" relaxations [17].
  2. (strong) **Labeling** algorithm's performance degrades as the vehicle capacity increases (longer routes), limiting its applicability to modern distribution problems.

Contributions

1. An exact branch-and-cut approach for the **non**-relaxed pricing problem.
   - Almost no works on this domain, except [18].
2. Verify its competitiveness at solving the PP as the vehicle capacity increases.

Implementation

- Branch-and-cut implementation based on the IBM ILOG CPLEX optimizer.
- The **Capacitated Profitable Tour Problem** (CPTP) [18] was used as the base IP formulation to model the PP.
- Implemented in `C` and available online[a] under a permissive license.

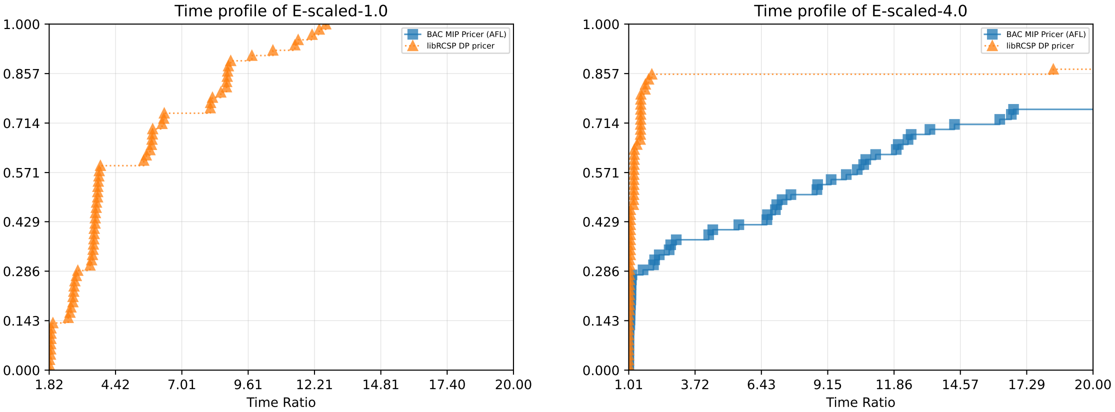
[a] `https://github.com/dparo/master-thesis`

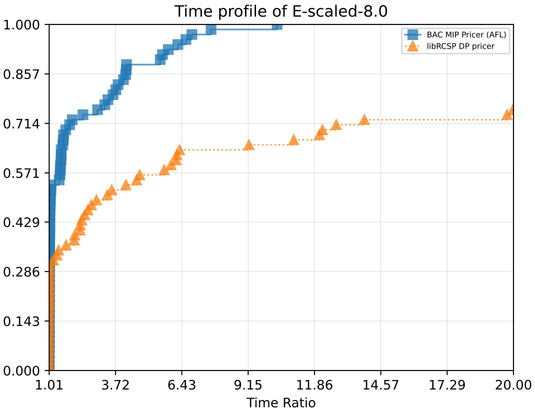
- Some implementation details:
  - Heuristics for warm-starting (constructive + local-search).
  - Separation of cutting-planes both for integral and fractional solutions:
    - GSEC, RCC [19], GLM [20].
  - Fractional separation through push relabel max-flow [21] and Gomory-Hu trees [22].

Empirical evaluation

- We've modified the commonly employed traditional instances proposed in [1, 23, 24, 25, 26, 27, 28].
- *BaPCod*: State-of-the-art BPC of [29] developed in France at the Bordeaux University and Bordeaux Research Center.
  - Parametrized accordingly (disabled non-robust inequalities).
- Time the *BaPCod*'s labeling algorithm [11] and our BAC in solving the harder PPs.
- Running time depicted through **performance profiles** [30].

Results (1/2)


Time profile of E-scaled-1.0
BAC MIP Pricer (AFL)
libRCSP DP pricer
1.000
0.857
0.714
0.571
0.429
0.286
0.143
0.000
1.82
4.42
7.01
9.61
12.21
14.81
17.40
20.00
Time Ratio
Time profile of E-scaled-4.0
BAC MIP Pricer (AFL)
libRCSP DP pricer
1.000
0.857
0.714
0.571
0.429
0.286
0.143
0.000
1.01
3.72
6.43
9.15
11.86
14.57
17.29
20.00
Time Ratio



Time profile of E-scaled-8.0
BAC MIP Pricer (AFL)
libRCSP DP pricer
1.000
0.857
0.714
0.571
0.429
0.286
0.143
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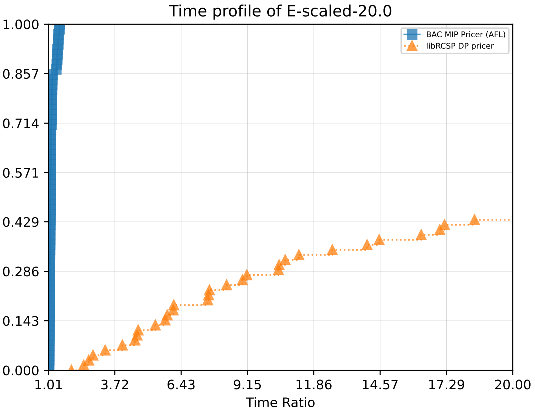

Time profile of E-scaled-20.0
BAC MIP Pricer (AFL)
libRCSP DP pricer
1.000
0.857
0.714
0.571
0.429
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1.01
3.72
6.43
9.15
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14.57
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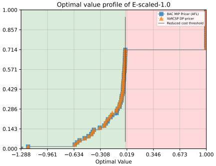
Results (2/2)


Optimal value profile of E-scaled-1.0
BAC MIP Pricer (AFL)
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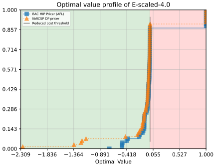
Reduced cost threshold
Optimal Value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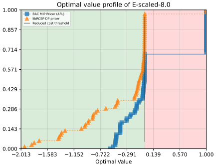
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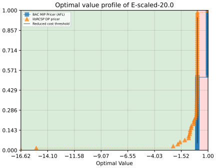
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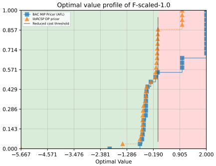
Optimal value profile of F-scaled-1.0
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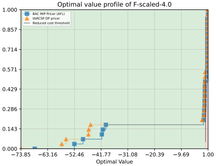
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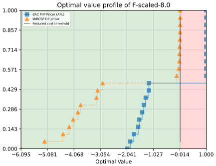
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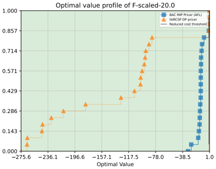
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Conclusions and Future Work

The proposed branch-and-cut pricer proved competitive at solving some PP:

- Branch-and-cut approaches may supplement the traditional labeling algorithm.
- Suggesting future research on branch-and-cut approaches in the context of **pricing for the CVRP**. Benefits:
  - Improve performance of contemporary CVRP solvers.

The end

Thank, you.

Additional material.

Integer Programming

MIP solvers are rather general and can be used to solve a wide range of problems from various fields [31]. MIP models are, in spirit, a way to mathematically program a solver to achieve the desired solution. A MIP solver can solve a mixed-integer linear programming formulation expressed as [32]:

$$\max_{x,y} \quad c^T x + d^T y \tag{1}$$

$$\text{s.t.} \quad Ax + By \leq b \tag{2}$$

$$x \in \mathbb{R}^n \tag{3}$$

$$y \in \mathbb{Z}_+^k, \tag{4}$$

where $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{m \times k}$ are matrices and $c \in \mathbb{R}^n, d \in R^k, b \in \mathbb{R}^m$ are vector coefficients. The bound in eq. (2) can also be rewritten in equality and/or greater form.

History on CVRP exact solvers

- 1961: First branch-and-price applied to the Cutting Stock Problem in [33].
- Up to late 80s: tree search algorithms employing branch-and-bound schemes: Lagrangian duality relaxation, ...
- 1984, 1989: First Column Generation attempts [34, 35].
- From late 80s to first 2000s: branch-and-cut approaches.
- 1992: labeling algorithm for pricing in [15].
- 1999: infant BPC framework in [36].
- 2006: Turning point in modern BPC frameworks thanks to [2].
- 2011: smart ng-routes relaxation for the pricing problem [17].
- 2014-now: State-of-the-art BPC algorithms [7, 10, 9, 37, 38, 11].

Set Covering formulation

Let $P = \{p \mid p \text{ is a single-truck elementary feasible route}\}$ be the set of all feasible routes.

$$\min_{\lambda} \quad z_{\mathrm{SC}}(\lambda) = \sum_{p \in P} c_p \lambda_p \tag{5}$$

$$\sum_{p \in P} \lambda_p = K \tag{6}$$

$$\sum_{p \in P} a_{ip} \lambda_p \geq 1 \qquad \forall i \in V_0 \tag{7}$$

$$\lambda_p \in \{0, 1\} \qquad \forall p \in P. \tag{8}$$

Master Problem (Primal)

$$\min_{\lambda} \quad z_{\mathrm{MP}}(\lambda) = \sum_{p \in P} c_p \lambda_p \tag{9}$$

$$\sum_{p \in P} \lambda_p = K \tag{10}$$

$$\sum_{p \in P} a_{ip} \lambda_p = 1 \qquad \forall i \in V_0 \tag{11}$$

$$0 \leq \lambda_p \leq 1 \qquad \forall p \in P. \tag{12}$$

Master Problem (Dual)

$$\max_{\pi} \quad z_{\mathrm{DMP}}(\pi) = K\pi_0 + \sum_{i \in V_0} \pi_i \tag{13}$$

$$\pi_0 + \sum_{i \in V_0} a_{ip}\pi_i \leq c_p \qquad \forall p \in P \tag{14}$$

$$\pi_0 \in \mathbb{R} \tag{15}$$

$$\pi_i \in \mathbb{R} \qquad \forall i \in V_0, \tag{16}$$

where $\pi_0 \in \mathbb{R}, \pi_i \in \mathbb{R} \quad \forall i \in V_0$ represents the dual variables associated respectively with constraints (10) and (11).

Restricted Master Problem

Due to the enormous size of the set of routes P, evaluating the dual variables $\pi \in \mathbb{R}^N$ is computationally intractable. As a result, in BAP frameworks we consider only a small subset of columns $\mathscr{P} \subseteq P$:

$$\min_{\lambda} \quad z_{\mathrm{RMP}}(\lambda) = \sum_{p \in \mathscr{P}} c_p \lambda_p \tag{17}$$

$$\sum_{p \in \mathscr{P}} \lambda_p = K \tag{18}$$

$$\sum_{p \in \mathscr{P}} a_{ip} \lambda_p = 1 \qquad \forall i \in V_0 \tag{19}$$

$$\lambda_p \geq 0 \qquad \forall p \in \mathscr{P}. \tag{20}$$

Pricing sub-problem

We look for a column to enter the basis of the RMP, which in turn necessitates the resolution of the following sub-problem:

$$c_p^\star = \min_{p \in P} \left\{ \bar{c}_p = \sum_{e=\{i,j\} \in E} \left(c_e - \frac{\pi_i + \pi_j}{2} \right) a_{ep} \right\}, \tag{21}$$

which takes the name of *Pricing Problem* (PP). Any $p \in P$ which satisfies $c_p < 0$ is a valid column which can enter the basis of the RMP. The column generation procedure stops mainly under two scenarios: (i) when no more negative reduced cost routes exist, i.e. the PP outputs a $p^\star \in P$ achieving $c_p^\star \geq 0$ or (ii) the CG procedure tails off.

The Pricing Sub-problem

To advance the column generation, the pricer, a critical component in BPC frameworks, needs to solve the pricing sub-problem (PP):

- An Elementary Shortest Path Problem with Capacity Constraints (**ESPPCC**) in a reduced cost network with negative cycles.
  - NP-hard problem [12].
- **Relax elementarity condition** to make it solvable in pseudo-polynomial time:
  - q-routes with 2-cycles elimination [39].
  - q-routes with arbitrary k-cycles elimination [16, 40].
  - ng-routes [17].
- State-of-the-art solutions for the relaxed PP are based on dynamic programming:
  - **labeling algorithm** [15, 16].

$$\min_{x,y} \quad z_{\mathrm{CPTP}}(x,y) = \sum_{i \in E} c_e x_e - \sum_{i \in V} p_i y_i \tag{22}$$

$$y_0 = 1 \tag{23}$$

$$\sum_{i \in V} q_i y_i \le Q \tag{24}$$

$$\sum_{e \in \delta(i)} x_e = 2 y_i \qquad \forall i \in V \tag{25}$$

$$\sum_{e \in \delta(S)} x_e \ge 2 y_i \qquad \forall i \in S,\ \forall S \subseteq V_0,\ |S| \ge 2 \tag{26}$$

$$x_e \in \{0,1\} \qquad \forall e \in E \tag{27}$$

$$y_i \in \{0,1\} \qquad \forall i \in V. \tag{28}$$

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Dipartimento di Ingegneria dell'Informazione

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Graduate:
Davide Paro
ID: 1207154

Supervisor:
Prof. Domenico Salvagnin

Co-Supervisor:
Prof. Roberto Roberti

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