Università degli Studi di Padova

Dipartimento di Ingegneria dell'Informazione

A Branch-and-Cut based Pricer for the Capacitated Vehicle Routing Problem

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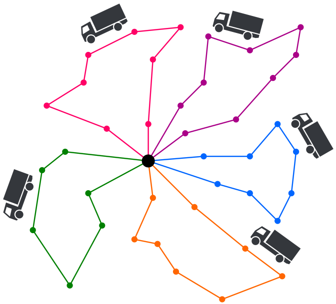
Introduction

The Capacitated Vehicle Routing Problem (**CVRP**) [1] is an NP-hard combinatorial optimization routing problem with applications in logistics (transportation, distribution, delivery).



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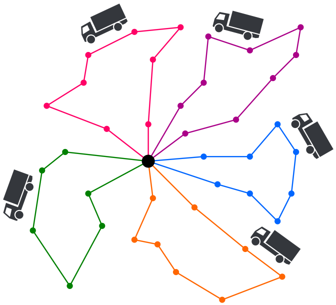

- CVRP defined on a complete graph. We are given:
  - The amount of available vehicles with their capacity.
  - A central depot where vehicles are stationed.
  - Customers' locations.
  - The demands (in units) of the individual customers.

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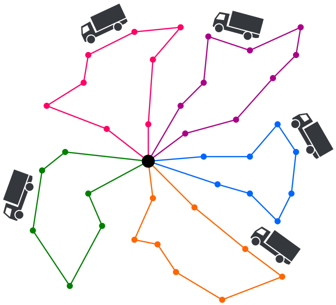

- CVRP defined on a complete graph. We are given:
  - The amount of available vehicles with their capacity.
  - A central depot where vehicles are stationed.
  - Customers' locations.
  - The demands (in units) of the individual customers.
- Objective:
  - Minimize overall routing costs while meeting the needs of **all** customers.

Branch-price-and-cut

- In the last two decades, the most efficient VRP solvers are all based on branch-price-and-cut approaches [2, 3, 4, 5, 6, 7, 8, 9, 10, 11].
- Branch-price-and-cut (BPC) is an **exact approach** for solving combinatorial optimization problems.
  - Extension of traditional branch-and-cut (BAC).
  - Can tackle **extensive** integer programming models.

$$\begin{aligned}
\min_{\lambda} \quad & z_{\mathrm{SC}}(\lambda) = \sum_{p \in P} c_p \lambda_p \\
& \sum_{p \in P} \lambda_p = K \\
& \sum_{p \in P} a_{ip} \lambda_p \geq 1 \quad \forall i \in V_0 \\
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$$

- How?
  - Column Generation (CG): decision variables are generated lazily.
    - In VRP, decision variables represent single vehicle feasible routes.
  - **Pricing**: determines the optimal reduced-cost decision variables for their inclusion.

Contemporary Approaches for Pricing

- Pricing Problem (PP) in its "natural" form is an induced **NP-hard** combinatorial optimization problem [12].
  - An Elementary Shortest Path Problem with Resource Constraints (**ESPPRC**) in a reduced cost network with negative cycles:

$$c_p^\star = \min_{p \in P} \left\{ \bar{c}_p = \sum_{e=\{i,j\} \in E} \left(c_e - \frac{\pi_i + \pi_j}{2} \right) a_{ep} \right\}. \tag{1}$$

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- PP hard $\rightarrow$ Relaxation of the PP.
  - Relaxed PP is solvable in pseudo-polynomial time [13, 14] via efficient dynamic programming algorithms.
    - **Labeling algorithm** proposed in 1992 in [15].
    - Labeling algorithm extended to handle elementarity condition in 2004 in [16].

Issues with Contemporary Approaches

Two major issues with contemporary approaches:

1. (weak) relaxing the PP weakens the dual bounds fed to the BPC (increasing column generation time).
   - Recently alleviated thanks to recent developments of efficacious relaxations [17].
2. (strong) **Labeling** algorithm's speed decreases as the vehicle capacity increases $O(N^2 Q)$ (longer routes), limiting its applicability to modern distribution problems.

Contributions

We propose a different approach to pricing:

1. An exact branch-and-cut algorithm for solving the **non**-relaxed pricing problem.
   - Almost no works on this domain, except [18].
2. We compare its competitiveness against state-of-the-art solutions as the vehicle capacity increases.

Implementation

- Implemented in `C` using the commercial IBM ILOG CPLEX optimizer.
- The PP is modeled as a **Capacitated Profitable Tour Problem** (CPTP) [18].
- Available online[a] under a permissive license.

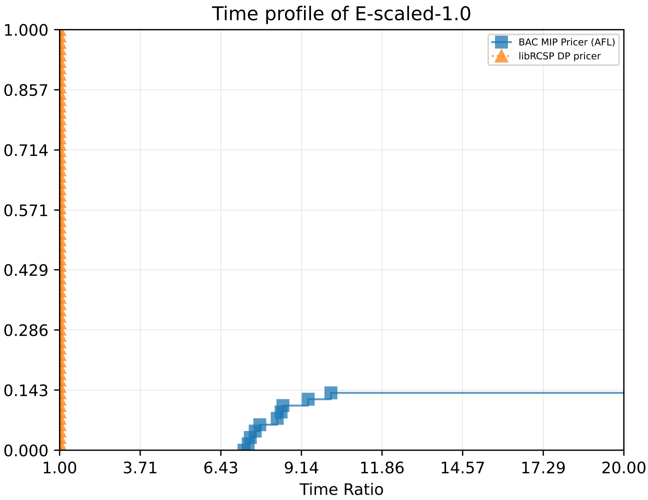
[a] https://github.com/dparo/master-thesis

- Some implementation details:
  - Heuristics for warm-starting (constructive + local-search).
  - Inclusion of several cutting-planes:
    - GSEC, RCC [19], GLM [20].
    - Integral separation by tracking major connected components.
    - Fractional separation through push relabel max-flow [21] and Gomory-Hu trees [22].

Empirical evaluation

- We've modified the commonly employed traditional instances proposed in [1, 23, 24, 25, 26, 27, 28].
- Measure the running time of the current state-of-the-art labeling algorithm [11] and compare it against our BAC pricer.
- Running time depicted through **performance profiles** [29].

Results (1/2)



Time profile of E-scaled-1.0
1.000
0.857
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0.143
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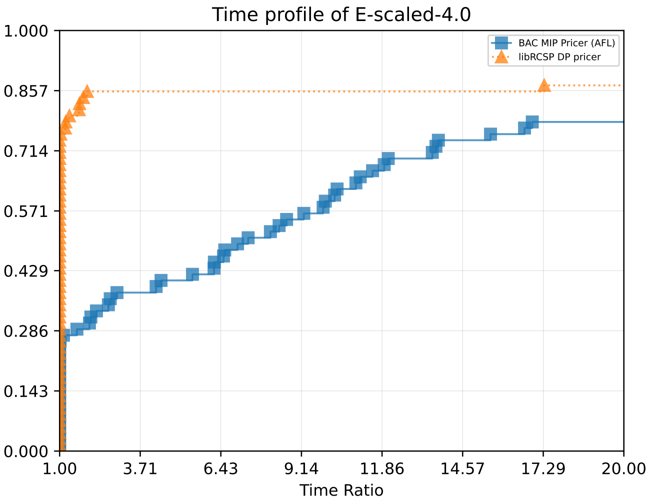
9.14
11.86
14.57
17.29
20.00
Time Ratio
BAC MIP Pricer (AFL)
libRCSP DP pricer


Time profile of E-scaled-4.0
BAC MIP Pricer (AFL)
libRCSP DP pricer
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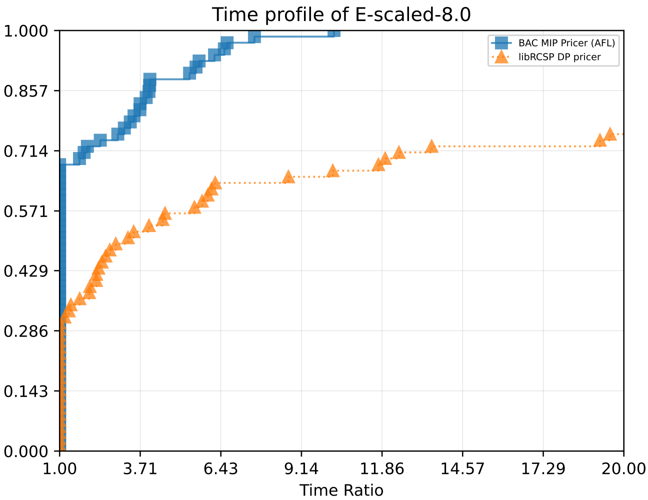
Time profile of E-scaled-8.0
BAC MIP Pricer (AFL)
libRCSP DP pricer
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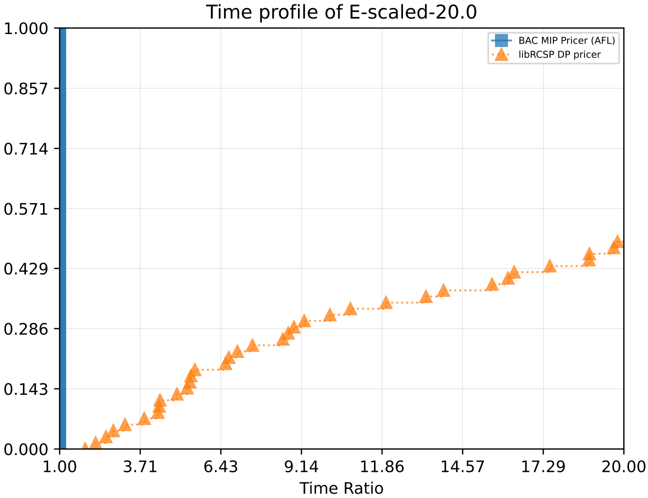

Time profile of E-scaled-20.0
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Time Ratio
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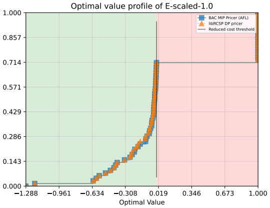
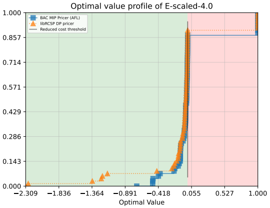
Results (2/2)


Optimal value profile of E-scaled-1.0
BAC MIP Pricer (AFL)
libRCSP DP pricer
Reduced cost threshold
Optimal Value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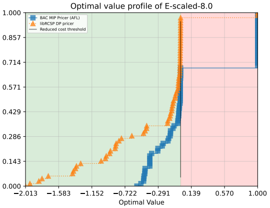
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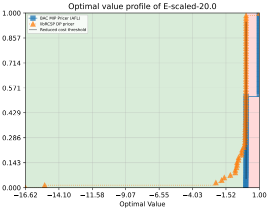
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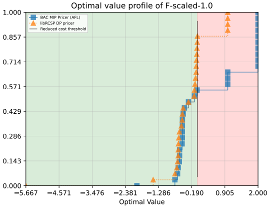
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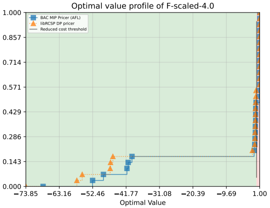
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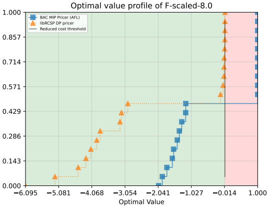
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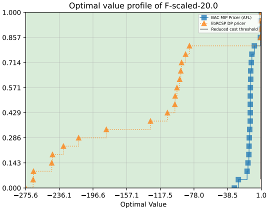
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Conclusions and Future Work

Although our study is to be considered an indication on the genuine efficiency of the two approaches:

- We proved that branch-and-cut may supplement the traditional labeling algorithm, especially in solving PP with non-stringent vehicle capacities.
- Suggesting future research on branch-and-cut approaches in the context of **pricing for the CVRP**, may bring substantial benefits to contemporary CVRP solvers.

The end

Thank you all!

Additional material.

MIP solvers are rather general and can be used to solve a wide range of problems from various fields [30]. MIP models are, in spirit, a way to mathematically program a solver to achieve the desired solution. A MIP solver can solve a mixed-integer linear programming formulation expressed as [31]:

$$\max_{x,y} \quad c^T x + d^T y \tag{2}$$
$$\text{s.t.} \quad Ax + By \leq b \tag{3}$$
$$x \in \mathbb{R}^n \tag{4}$$
$$y \in \mathbb{Z}^k_+, \tag{5}$$

where $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{m \times k}$ are matrices and $c \in \mathbb{R}^n, d \in R^k, b \in \mathbb{R}^m$ are vector coefficients. The bound in eq. (3) can also be rewritten in equality and/or greater form.

History on CVRP exact solvers

- 1959: The CVRP was first introduced in [1].
- 1961: First branch-and-price applied to the Cutting Stock Problem in [32].
- Up to late 80s: tree search algorithms employing branch-and-bound schemes: Lagrangian duality relaxation, ...
- Mid 80s: First Column Generation attempts [33, 34].
- From late 80s to first 2000s: branch-and-cut approaches [19, 35, 36, 37, 28, 38, 39, 40, 41, 42].
- 1992: labeling algorithm for pricing in [15].
- 1999: basic implementation of a BPC framework in [43].
- 2006: Turning point in modern BPC frameworks thanks to [2].
- 2011: smart ng-routes relaxation for the pricing problem [17].
- 2014-now: State-of-the-art BPC algorithms [7, 10, 9, 44, 45, 11].

Set Covering formulation

Let $P = \{p \mid p \text{ is a single-truck elementary feasible route}\}$ be the set of all feasible routes.

$$\min_{\lambda} \quad z_{\mathrm{SC}}(\lambda) = \sum_{p \in P} c_p \lambda_p \tag{6}$$

$$\sum_{p \in P} \lambda_p = K \tag{7}$$

$$\sum_{p \in P} a_{ip} \lambda_p \geq 1 \qquad \forall i \in V_0 \tag{8}$$

$$\lambda_p \in \{0, 1\} \qquad \forall p \in P. \tag{9}$$

Master Problem (Primal)

$$\min_{\lambda} \quad z_{\mathrm{MP}}(\lambda) = \sum_{p \in P} c_p \lambda_p \tag{10}$$

$$\sum_{p \in P} \lambda_p = K \tag{11}$$

$$\sum_{p \in P} a_{ip} \lambda_p = 1 \qquad \forall i \in V_0 \tag{12}$$

$$0 \leq \lambda_p \leq 1 \qquad \forall p \in P. \tag{13}$$

Master Problem (Dual)

$$\max_{\pi} \quad z_{\text{DMP}}(\pi) = K\pi_0 + \sum_{i \in V_0} \pi_i \tag{14}$$

$$\pi_0 + \sum_{i \in V_0} a_{ip}\pi_i \leq c_p \qquad \forall p \in P \tag{15}$$

$$\pi_0 \in \mathbb{R} \tag{16}$$

$$\pi_i \in \mathbb{R} \qquad \forall i \in V_0, \tag{17}$$

where $\pi_0 \in \mathbb{R}, \pi_i \in \mathbb{R} \quad \forall i \in V_0$ represents the dual variables associated respectively with constraints (11) and (12).

Restricted Master Problem

Due to the enormous size of the set of routes P, evaluating the dual variables $\pi \in \mathbb{R}^N$ is computationally intractable. As a result, in BAP frameworks we consider only a small subset of columns $\mathscr{P} \subseteq P$:

$$\min_{\lambda} \quad z_{\text{RMP}}(\lambda) = \sum_{p \in \mathscr{P}} c_p \lambda_p \tag{18}$$

$$\sum_{p \in \mathscr{P}} \lambda_p = K \tag{19}$$

$$\sum_{p \in \mathscr{P}} a_{ip} \lambda_p = 1 \qquad \forall i \in V_0 \tag{20}$$

$$\lambda_p \geq 0 \qquad \forall p \in \mathscr{P}. \tag{21}$$

Pricing sub-problem

We look for a column to enter the basis of the RMP, which in turn necessitates the resolution of the following sub-problem:

$$c_p^\star = \min_{p \in P} \left\{ \bar{c}_p = \sum_{e=\{i,j\} \in E} \left(c_e - \frac{\pi_i + \pi_j}{2} \right) a_{ep} \right\}, \qquad (22)$$

which takes the name of *Pricing Problem* (PP). Any $p \in P$ which satisfies $c_p < 0$ is a valid column which can enter the basis of the RMP. The column generation procedure stops mainly under two scenarios: (i) when no more negative reduced cost routes exist, i.e. the PP outputs a $p^\star \in P$ achieving $c_p^\star \geq 0$ or (ii) the CG procedure tails off.

Pricing Sub-problem Relaxations

- **Relax elementarity condition** to make the PP solvable in pseudo-polynomial time:
  - *q*-routes with 2-cycles elimination [46].
  - *q*-routes with arbitrary *k*-cycles elimination [16, 47].
  - ng-routes [17].

$$\min_{x,y} \quad z_{\text{CPTP}}(x,y) = \sum_{i \in E} c_e x_e - \sum_{i \in V} p_i y_i \tag{23}$$

$$y_0 = 1 \tag{24}$$

$$\sum_{i \in V} q_i y_i \leq Q \tag{25}$$

$$\sum_{e \in \delta(i)} x_e = 2y_i \qquad \forall i \in V \tag{26}$$

$$\sum_{e \in \delta(S)} x_e \geq 2y_i \qquad \forall i \in S,\ \forall S \subseteq V_0,\ |S| \geq 2 \tag{27}$$

$$x_e \in \{0,1\} \qquad \forall e \in E \tag{28}$$

$$y_i \in \{0,1\} \qquad \forall i \in V. \tag{29}$$

BaPCod

BaPCod [45] is a software package developed in France at the Bordeaux University and Bordeaux Research Center that embeds a sophisticated column generation approach embedded in a generic and modern branch-price-and-cut (BPC) algorithm.

- Currently state-of-the-art in solving routing problems.
- Takes as input a compact midex-integer programming model and solves it using a Dantzig-Wolfe reformulation [48].
- Sses an automatic dual price smoothing stabilization [49].

VRPSolver

The *VRPSolver* extension [11], is a *BaPCod* extension distributed by the same authors. This extension includes an advanced implementation of a bidirectional dynamic programming labeling algorithm [50] for solving the pricing problem. The included labeling algorithm can be used as an exact or heuristic pricer. The labeling algorithm contains two successively lighter heuristic implementations; for more information, see Sadykov et al. [50].
The *VRPSolver* extension, also, includes the implementation of some specific cutting planes and branching decisions aimed at efficiently solving routing-like problems (or problems that exhibit similar structures) such as the CVRP, VRPTW, and also others (see [11]).

BaPCod parametrization

- Disabled non-robust inequalities.
- Disabled multiple columns per pricing iteration.
- Increased ng-sets size and disable tailing off condition.
- Disabled any form of branching and cut-generation to stop the resolution process at the root node.

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