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A Branch-and-Cut based Pricer for the Capacitated Vehicle Routing Problem

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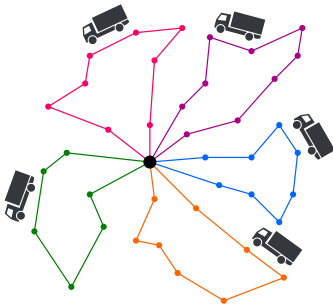
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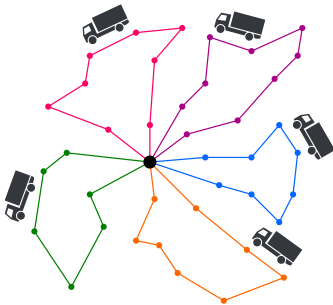
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The Capacitated Vehicle Routing Problem (**CVRP**) [1] is an **NP-hard combinatorial optimization routing problem** with applications in **logistics** (transportation, distribution, delivery).

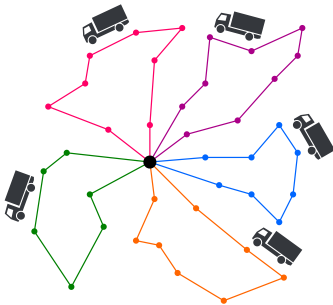


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- CVRP defined on a complete graph. We are given:
 - The amount of **available vehicles** with their **capacity**.
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 - Customers' **locations**.
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- CVRP defined on a complete graph. We are given:
 - The amount of **available vehicles** with their **capacity**.
 - A **central depot** where vehicles are stationed.
 - Customers' **locations**.
 - The **demands** of the individual customers.
- Objective:
 - **Minimize overall routing costs while meeting the needs of **all** customers.**

- In the last two decades, the most efficient VRP solvers are all based on branch-price-and-cut approaches [2, 3, 4, 5, 6, 7, 8, 9, 10, 11].
- **Branch-price-and-cut** (BPC) is an **exact approach** for solving combinatorial optimization problems.
 - Extension of traditional **branch-and-cut** (BAC).
 - Can tackle **extensive integer programming models**.

$$\begin{aligned} \min_{\lambda} \quad & z_{SC}(\lambda) = \sum_{p \in P} c_p \lambda_p \\ & \sum_{p \in P} \lambda_p = K \\ & \sum_{p \in P} a_{ip} \lambda_p \geq 1 \quad \forall i \in V_0 \\ & \lambda_p \in \{0, 1\} \quad \forall p \in P. \end{aligned}$$

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■ How?

- **Column Generation** (CG): decision variables are generated lazily.
 - In VRP, decision variables represent **single vehicle feasible routes**.
- **Pricing** in CVRP, is the art of feeding the BPC approach with good reduced cost routes to bring it towards fast optimality convergence.

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- Pricing Problem (PP) in its "natural" form is an **NP-hard** combinatorial optimization problem [12].
 - An **Elementary Shortest Path Problem with Resource Constraints (ESPPRC)** in a reduced cost network with negative cycles:

$$c_p^* = \min_{p \in P} \left\{ \bar{c}_p = \sum_{e=\{i,j\} \in E} \left(c_e - \frac{\pi_i + \pi_j}{2} \right) a_{ep} \right\}. \quad (1)$$

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- **Relaxation** of the PP to make it solvable in **pseudo-polynomial time** [13, 14].
- **Dynamic programming** algorithms (exact and heuristics) to solve the **relaxed** PP:
 - **labeling algorithm** proposed in 1992 by Desrochers et al. [15].
 - Extended to handle elementarity condition in 2004 by Feillet et al. [16].

Two major issues with contemporary approaches:

- 1 (weak) **relaxing** the PP worsens the dual bounds fed to the BPC (increasing column generation time).
 - Improved in last decade thanks to "smart" **relaxations** [17].
- 2 (strong) **Labeling** algorithm's performance degrades as the **vehicle capacity increases** (longer routes), limiting its applicability to modern distribution problems.



- 1 An exact **branch-and-cut** approach for the **non-relaxed** pricing problem.
 - Almost no works on this domain, except [18].
- 2 Verify its competitiveness at solving the PP as the **vehicle capacity increases**.

- Implemented in C and available online^a under a permissive license.
- Branch-and-cut implementation based on the **IBM ILOG CPLEX optimizer**.
- The **Capacitated Profitable Tour Problem** (CPTP) [18] was used as the base IP formulation to model the PP.



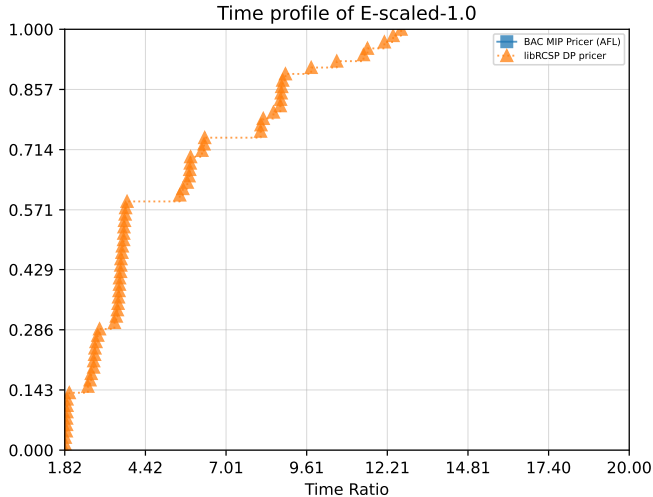
^a<https://github.com/dparo/master-thesis>

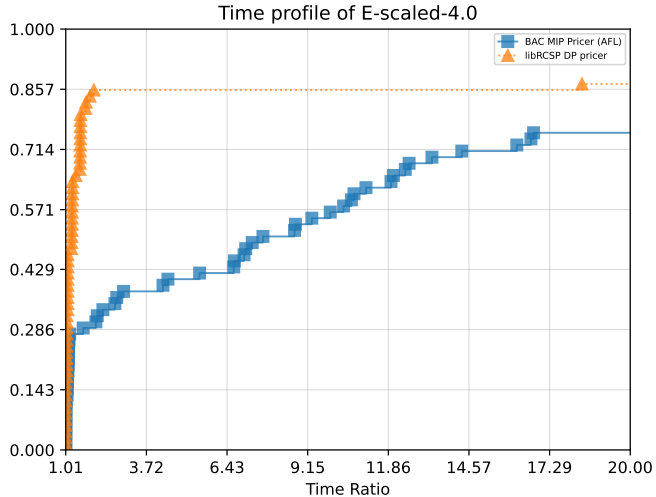
- Some implementation details:
 - Heuristics for warm-starting (constructive + local-search).
 - Inclusion of several **cutting-planes**:
 - GSEC, RCC [19], GLM [20].
 - Integral separation by tracking major connected components.
 - Fractional separation through push relabel max-flow [21] and Gomory-Hu trees [22].

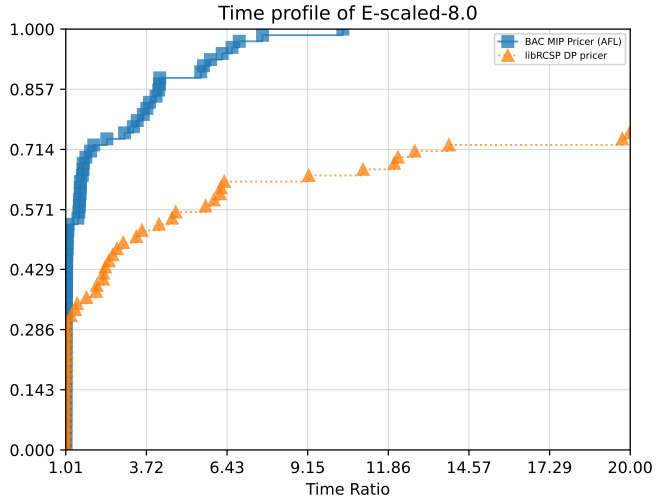


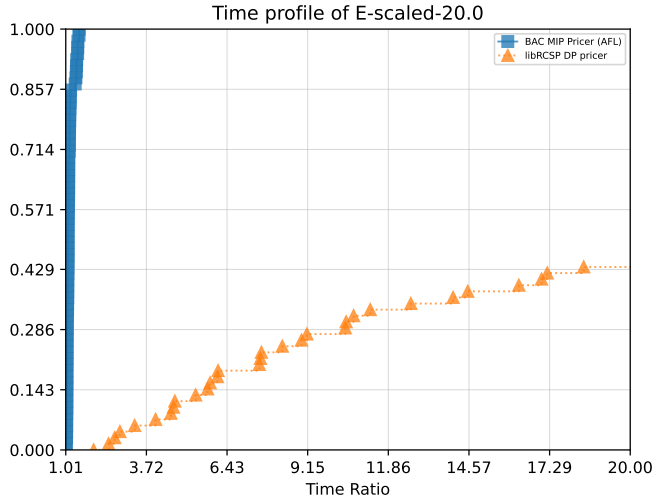
- We've modified the commonly employed traditional instances proposed in [1, 23, 24, 25, 26, 27, 28].
- *BaPCod*: State-of-the-art BPC of [29] developed in France at the Bordeaux University and Bordeaux Research Center.
 - Parametrized accordingly (disabled non-robust inequalities).
- Time the *BaPCod*'s labeling algorithm [11] and our BAC in solving the harder PPs.
- Running time depicted through **performance profiles** [30].

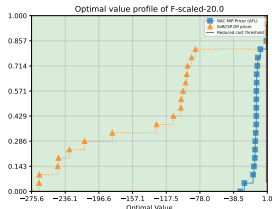
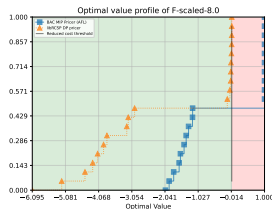
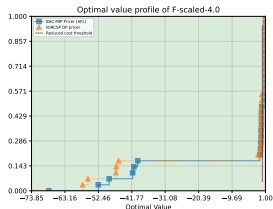
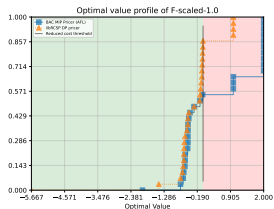
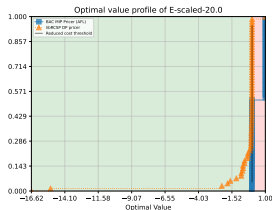
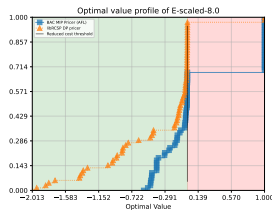
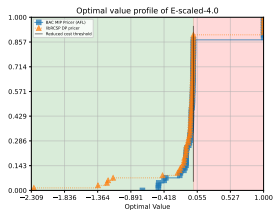
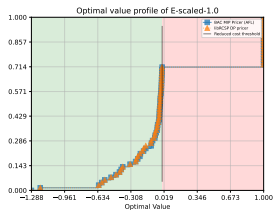
Results (1/2)











The proposed branch-and-cut pricer proved competitive at solving some PP:

- **Branch-and-cut** approaches may supplement the traditional **labeling algorithm**.
- Suggesting future research on **branch-and-cut** approaches in the context of **pricing for the CVRP**. Benefits:
 - Improve performance of contemporary CVRP solvers.



The end



Thank you all!

Additional material.

MIP solvers are rather general and can be used to solve a wide range of problems from various fields [31]. MIP models are, in spirit, a way to mathematically program a solver to achieve the desired solution. A MIP solver can solve a mixed-integer linear programming formulation expressed as [32]:

$$\max_{x,y} \quad c^T x + d^T y \quad (2)$$

$$\text{s.t.} \quad Ax + By \leq b \quad (3)$$

$$x \in \mathbb{R}^n \quad (4)$$

$$y \in \mathbb{Z}_+^k, \quad (5)$$

where $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times k}$ are matrices and $c \in \mathbb{R}^n$, $d \in \mathbb{R}^k$, $b \in \mathbb{R}^m$ are vector coefficients. The bound in eq. (3) can also be rewritten in equality and/or greater form.

- 1961: First branch-and-price applied to the Cutting Stock Problem in [33].
- Up to late 80s: tree search algorithms employing branch-and-bound schemes: Lagrangian duality relaxation,
- Mid 80s: First Column Generation attempts [34, 35].
- From late 80s to first 2000s: branch-and-cut approaches [19, 36, 37, 38, 28, 39, 40, 41, 42, 43].
- 1992: labeling algorithm for pricing in [15].
- 1999: basic implementation of a BPC framework in [44].
- 2006: Turning point in modern BPC frameworks thanks to [2].
- 2011: smart ng-routes relaxation for the pricing problem [17].
- 2014-now: State-of-the-art BPC algorithms [7, 10, 9, 45, 46, 11].

Set Covering formulation

Let $P = \{p \mid p \text{ is a single-truck elementary feasible route}\}$ be the set of all feasible routes.

$$\min_{\lambda} \quad z_{SC}(\lambda) = \sum_{p \in P} c_p \lambda_p \quad (6)$$

$$\sum_{p \in P} \lambda_p = K \quad (7)$$

$$\sum_{p \in P} a_{ip} \lambda_p \geq 1 \quad \forall i \in V_0 \quad (8)$$

$$\lambda_p \in \{0, 1\} \quad \forall p \in P. \quad (9)$$

Master Problem (Primal)

$$\min_{\lambda} \quad z_{\text{MP}}(\lambda) = \sum_{p \in P} c_p \lambda_p \quad (10)$$

$$\sum_{p \in P} \lambda_p = K \quad (11)$$

$$\sum_{p \in P} a_{ip} \lambda_p = 1 \quad \forall i \in V_0 \quad (12)$$

$$0 \leq \lambda_p \leq 1 \quad \forall p \in P. \quad (13)$$

$$\max_{\pi} \quad z_{\text{DMP}}(\pi) = K\pi_0 + \sum_{i \in V_0} \pi_i \quad (14)$$

$$\pi_0 + \sum_{i \in V_0} a_{ip} \pi_i \leq c_p \quad \forall p \in P \quad (15)$$

$$\pi_0 \in \mathbb{R} \quad (16)$$

$$\pi_i \in \mathbb{R} \quad \forall i \in V_0, \quad (17)$$

where $\pi_0 \in \mathbb{R}, \pi_i \in \mathbb{R} \quad \forall i \in V_0$ represents the dual variables associated respectively with constraints (11) and (12).

Due to the enormous size of the set of routes P , evaluating the dual variables $\pi \in \mathbb{R}^N$ is computationally intractable. As a result, in BAP frameworks we consider only a small subset of columns $\mathcal{P} \subseteq P$:

$$\min_{\lambda} \quad z_{\text{RMP}}(\lambda) = \sum_{p \in \mathcal{P}} c_p \lambda_p \quad (18)$$

$$\sum_{p \in \mathcal{P}} \lambda_p = K \quad (19)$$

$$\sum_{p \in \mathcal{P}} a_{ip} \lambda_p = 1 \quad \forall i \in V_0 \quad (20)$$

$$\lambda_p \geq 0 \quad \forall p \in \mathcal{P}. \quad (21)$$

We look for a column to enter the basis of the RMP, which in turn necessitates the resolution of the following sub-problem:

$$c_p^* = \min_{p \in P} \left\{ \bar{c}_p = \sum_{e=\{i,j\} \in E} \left(c_e - \frac{\pi_i + \pi_j}{2} \right) a_{ep} \right\}, \quad (22)$$

which takes the name of *Pricing Problem* (PP). Any $p \in P$ which satisfies $c_p < 0$ is a valid column which can enter the basis of the RMP. The column generation procedure stops mainly under two scenarios: (i) when no more negative reduced cost routes exist, i.e. the PP outputs a $p^* \in P$ achieving $c_p^* \geq 0$ or (ii) the CG procedure tails off.

- **Relax elementarity condition** to make the PP solvable in pseudo-polynomial time:
 - q -routes with 2-cycles elimination [47].
 - q -routes with arbitrary k -cycles elimination [16, 48].
 - ng-routes [17].

$$\min_{x,y} \quad z_{\text{CPTP}}(x,y) = \sum_{e \in E} c_e x_e - \sum_{i \in V} p_i y_i \quad (23)$$

$$y_0 = 1 \quad (24)$$

$$\sum_{i \in V} q_i y_i \leq Q \quad (25)$$

$$\sum_{e \in \delta(i)} x_e = 2y_i \quad \forall i \in V \quad (26)$$

$$\sum_{e \in \delta(S)} x_e \geq 2y_i \quad \forall i \in S, \forall S \subseteq V_0, |S| \geq 2 \quad (27)$$

$$x_e \in \{0,1\} \quad \forall e \in E \quad (28)$$

$$y_i \in \{0,1\} \quad \forall i \in V. \quad (29)$$

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