Università degli Studi di Padova

Dipartimento di Ingegneria dell'Informazione

A Branch-and-Cut based Pricer for the Capacitated Vehicle Routing Problem

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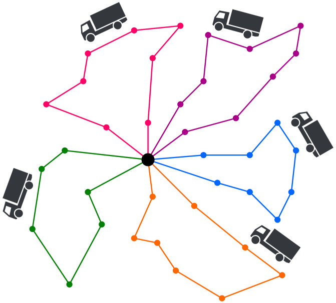
The Capacitated Vehicle Routing Problem (**CVRP**) [1] is an NP-hard combinatorial optimization routing problem with applications in logistics (transportation, distribution, delivery).



Introduction

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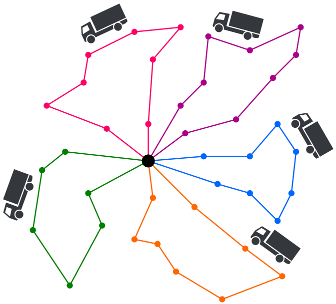

- CVRP defined on a complete graph. We are given:
  - The amount of available vehicles with their capacity.
  - A central depot where vehicles are stationed.
  - Customers' locations.
  - The demands (in units) of the individual customers.

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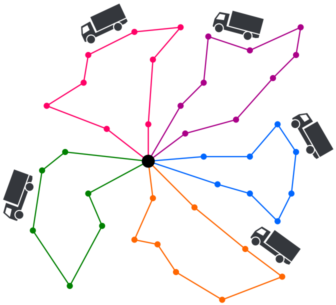

- CVRP defined on a complete graph. We are given:
  - The amount of available vehicles with their capacity.
  - A central depot where vehicles are stationed.
  - Customers' locations.
  - The demands (in units) of the individual customers.
- Objective:
  - Minimize overall routing costs while meeting the needs of **all** customers.

Branch-price-and-cut

- In the last two decades, the most efficient VRP solvers are all based on branch-price-and-cut approaches [2, 3, 4, 5, 6, 7, 8, 9, 10, 11].
- Branch-price-and-cut (BPC) is an **exact approach** for solving combinatorial optimization problems.
  - Extension of traditional branch-and-cut (BAC).
  - Can tackle **extensive** integer programming models.

$$
\begin{aligned}
\min_{\lambda} \quad & z_{\mathrm{SC}}(\lambda) = \sum_{p \in P} c_p \lambda_p \\
& \sum_{p \in P} \lambda_p = K \\
& \sum_{p \in P} a_{ip} \lambda_p \geq 1 \quad \forall i \in V_0 \\
& \lambda_p \in \{0, 1\} \quad \forall p \in P.
\end{aligned}
$$

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$$\min_{\lambda} \quad z_{SC}(\lambda) = \sum_{p \in P} c_p \lambda_p$$
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$$\lambda_p \in \{0, 1\} \quad \forall p \in P.$$

- How?
  - Column Generation (CG): decision variables are generated lazily.
    - In VRP, decision variables represent single vehicle feasible routes.
  - **Pricing**: determines the optimal reduced-cost decision variables for their inclusion.

- Pricing Problem (PP) in its "natural" form is an induced **NP-hard** combinatorial optimization problem [12].
  - An Elementary Shortest Path Problem with Resource Constraints (**ESPPRC**) in a reduced cost network with negative cycles:

$$c_p^\star = \min_{p \in P} \left\{ \bar{c}_p = \sum_{e=\{i,j\} \in E} \left(c_e - \frac{\pi_i + \pi_j}{2} \right) a_{ep} \right\}. \tag{1}$$

Contemporary Approaches for Pricing

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- PP hard $\rightarrow$ Relaxation of the PP.
  - Relaxed PP is solvable in pseudo-polynomial time [13, 14] via efficient dynamic programming algorithms.
    - **Labeling algorithm** proposed in 1992 in [15].
    - Labeling algorithm extended to handle elementarity condition in 2004 in [16].

Two major issues with contemporary approaches:

1. (weak) relaxing the PP weakens the dual bounds fed to the BPC (increasing column generation time).
   - Recently alleviated thanks to recent developments of efficacious relaxations [17].
2. (strong) **Labeling** algorithm's speed decreases as the vehicle capacity increases $O(N^2Q)$ (longer routes), limiting its applicability to modern distribution problems.

Contributions

We propose a different approach to pricing:

1. An exact branch-and-cut algorithm for solving the **non**-relaxed pricing problem.
   - Almost no works on this domain, except [18].
2. We compare its competitiveness against the current state-of-the-art solution [11] as the vehicle capacity increases.

Implementation

- Implemented in `C` using the commercial IBM ILOG CPLEX optimizer.
- The PP is modeled as a **Capacitated Profitable Tour Problem** (CPTP) [18].
- Available online[a] under a permissive license.

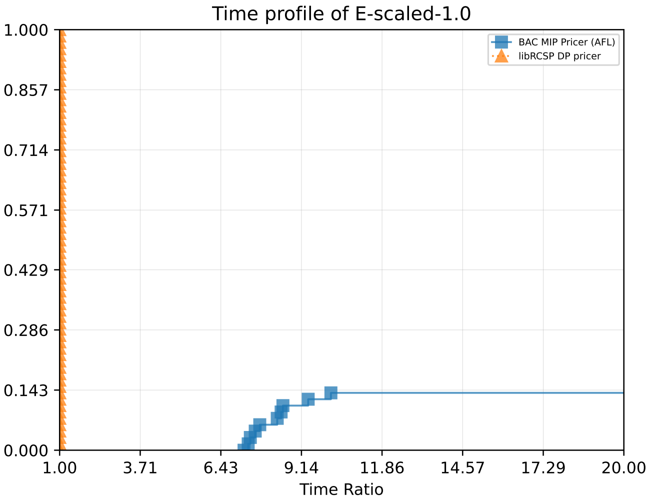
[a] https://github.com/dparo/master-thesis

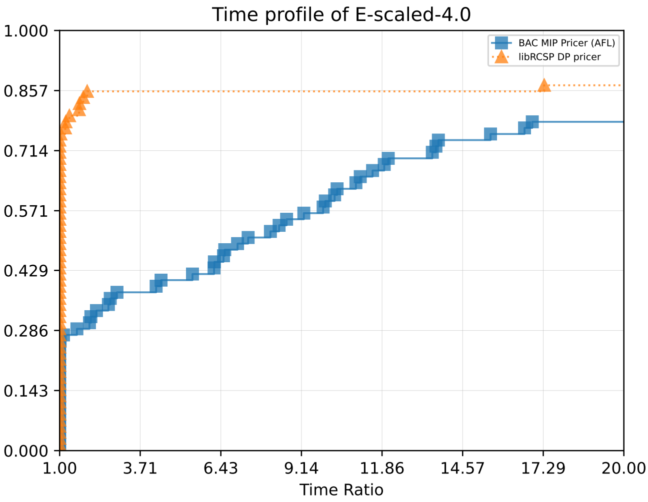
- Some implementation details:
  - Heuristics for warm-starting (constructive + local-search).
  - Inclusion of several cutting-planes:
    - GSEC, RCC [19], GLM [20].
    - Integral separation by tracking major connected components.
    - Fractional separation through push relabel max-flow [21] and Gomory-Hu trees [22].

Empirical evaluation

- We've modified the commonly employed traditional instances proposed in [1, 23, 24, 25, 26, 27, 28].
  - Generated new instances by scaling the vehicle capacities.
- The running times of the two pricing approaches are depicted through **performance profiles** [29].

Results (1/2)


Time profile of E-scaled-1.0
BAC MIP Pricer (AFL)
libRCSP DP pricer
1.000
0.857
0.714
0.571
0.429
0.286
0.143
0.000
1.00
3.71
6.43
9.14
11.86
14.57
17.29
20.00
Time Ratio


Time profile of E-scaled-4.0
BAC MIP Pricer (AFL)
libRCSP DP pricer
1.000
0.857
0.714
0.571
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0.286
0.143
0.000
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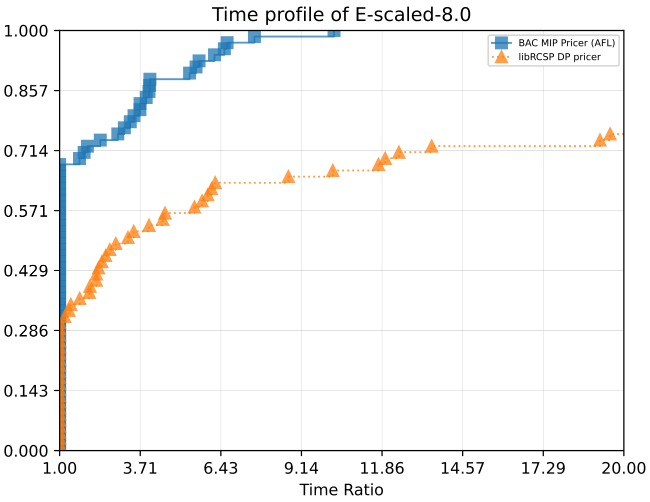
6.43
9.14
11.86
14.57
17.29
20.00
Time Ratio



Time profile of E-scaled-8.0
BAC MIP Pricer (AFL)
libRCSP DP pricer
1.000
0.857
0.714
0.571
0.429
0.286
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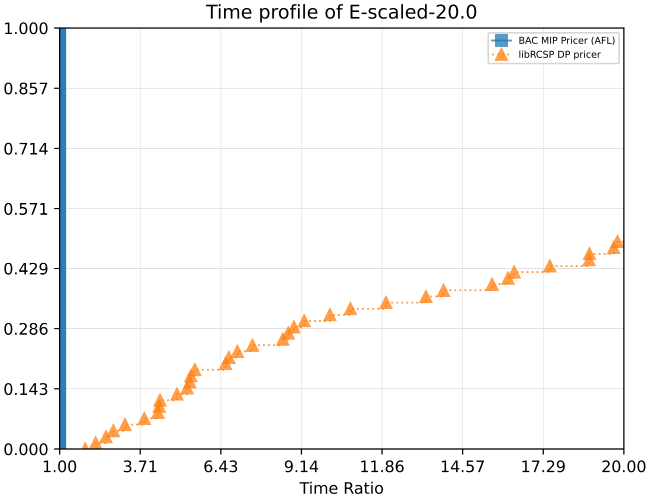
11.86
14.57
17.29
20.00
Time Ratio


Time profile of E-scaled-20.0
1.000
0.857
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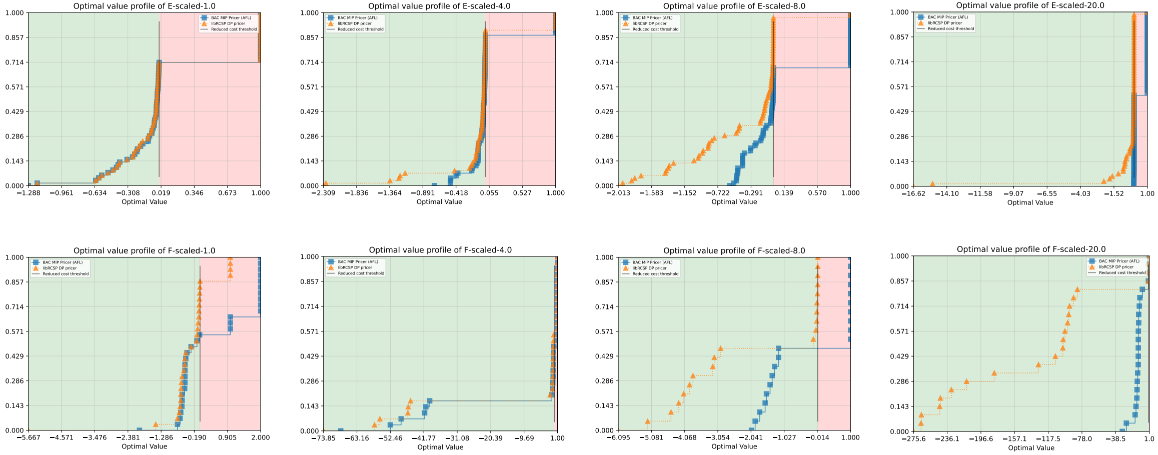
0.571
0.429
0.286
0.143
0.000
1.00
3.71
6.43
9.14
11.86
14.57
17.29
20.00
Time Ratio
BAC MIP Pricer (AFL)
libRCSP DP pricer

Results (2/2)


Optimal value profile of E-scaled-1.0
Optimal value profile of E-scaled-4.0
Optimal value profile of E-scaled-8.0
Optimal value profile of E-scaled-20.0
Optimal value profile of F-scaled-1.0
Optimal value profile of F-scaled-4.0
Optimal value profile of F-scaled-8.0
Optimal value profile of F-scaled-20.0
BAC MIP Pricer (AFL)
libRCSP DP pricer
Reduced cost threshold
Optimal Value

Conclusions and Future Work

Although our study is to be considered an indication on the genuine efficiency of the two approaches:

- We proved that branch-and-cut may supplement the traditional labeling algorithm, especially in solving PP with non-stringent vehicle capacities.
- Suggesting future research on branch-and-cut approaches in the context of **pricing for the CVRP**, may bring substantial benefits to contemporary CVRP solvers.

The end

Thank you for your time!

Additional material.

Integer Programming

MIP solvers are rather general and can be used to solve a wide range of problems from various fields [30]. MIP models are, in spirit, a way to mathematically program a solver to achieve the desired solution. A MIP solver can solve a mixed-integer linear programming formulation expressed as [31]:

$$\max_{x,y} \quad c^T x + d^T y \tag{2}$$

$$\text{s.t.} \quad Ax + By \leq b \tag{3}$$

$$x \in \mathbb{R}^n \tag{4}$$

$$y \in \mathbb{Z}_+^k, \tag{5}$$

where $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{m \times k}$ are matrices and $c \in \mathbb{R}^n, d \in R^k, b \in \mathbb{R}^m$ are vector coefficients. The bound in eq. (3) can also be rewritten in equality and/or greater form.

History on CVRP exact solvers

- 1959: The CVRP was first introduced in [1].
- 1961: First branch-and-price applied to the Cutting Stock Problem in [32].
- Up to late 80s: tree search algorithms employing branch-and-bound schemes: Lagrangian duality relaxation, ...
- Mid 80s: First Column Generation attempts [33, 34].
- From late 80s to first 2000s: branch-and-cut approaches [19, 35, 36, 37, 28, 38, 39, 40, 41, 42].
- 1992: labeling algorithm for pricing in [15].
- 1999: basic implementation of a BPC framework in [43].
- 2006: Turning point in modern BPC frameworks thanks to [2].
- 2011: smart ng-routes relaxation for the pricing problem [17].
- 2014-now: State-of-the-art BPC algorithms [7, 10, 9, 44, 45, 11].

Set Covering formulation

Let $P = \{p \mid p$ is a single-truck elementary feasible route$\}$ be the set of all feasible routes.

$$\min_{\lambda} \quad z_{\mathrm{SC}}(\lambda) = \sum_{p \in P} c_p \lambda_p \tag{6}$$

$$\sum_{p \in P} \lambda_p = K \tag{7}$$

$$\sum_{p \in P} a_{ip} \lambda_p \geq 1 \qquad \forall i \in V_0 \tag{8}$$

$$\lambda_p \in \{0,1\} \qquad \forall p \in P. \tag{9}$$

Master Problem (Primal)

$$\min_{\lambda} \quad z_{\mathrm{MP}}(\lambda) = \sum_{p \in P} c_p \lambda_p \tag{10}$$

$$\sum_{p \in P} \lambda_p = K \tag{11}$$

$$\sum_{p \in P} a_{ip} \lambda_p = 1 \qquad \forall i \in V_0 \tag{12}$$

$$0 \leq \lambda_p \leq 1 \qquad \forall p \in P. \tag{13}$$

Master Problem (Dual)

$$\max_{\pi} \quad z_{\mathrm{DMP}}(\pi) = K\pi_0 + \sum_{i \in V_0} \pi_i \tag{14}$$

$$\pi_0 + \sum_{i \in V_0} a_{ip}\pi_i \leq c_p \qquad \forall p \in P \tag{15}$$

$$\pi_0 \in \mathbb{R} \tag{16}$$

$$\pi_i \in \mathbb{R} \qquad \forall i \in V_0, \tag{17}$$

where $\pi_0 \in \mathbb{R}, \pi_i \in \mathbb{R} \quad \forall i \in V_0$ represents the dual variables associated respectively with constraints (11) and (12).

Restricted Master Problem

Due to the enormous size of the set of routes P, evaluating the dual variables $\pi \in \mathbb{R}^N$ is computationally intractable. As a result, in BAP frameworks we consider only a small subset of columns $\mathscr{P} \subseteq P$:

$$\min_{\lambda} \quad z_{\mathrm{RMP}}(\lambda) = \sum_{p \in \mathscr{P}} c_p \lambda_p \tag{18}$$

$$\sum_{p \in \mathscr{P}} \lambda_p = K \tag{19}$$

$$\sum_{p \in \mathscr{P}} a_{ip} \lambda_p = 1 \qquad \forall i \in V_0 \tag{20}$$

$$\lambda_p \geq 0 \qquad \forall p \in \mathscr{P}. \tag{21}$$

Pricing sub-problem

We look for a column to enter the basis of the RMP, which in turn necessitates the resolution of the following sub-problem:

$$c_p^\star = \min_{p \in P} \left\{ \bar{c}_p = \sum_{e=\{i,j\} \in E} \left(c_e - \frac{\pi_i + \pi_j}{2} \right) a_{ep} \right\}, \tag{22}$$

which takes the name of *Pricing Problem* (PP). Any $p \in P$ which satisfies $c_p < 0$ is a valid column which can enter the basis of the RMP. The column generation procedure stops mainly under two scenarios: (i) when no more negative reduced cost routes exist, i.e. the PP outputs a $p^\star \in P$ achieving $c_p^\star \geq 0$ or (ii) the CG procedure tails off.

Pricing Sub-problem Relaxations

- **Relax elementarity condition** to make the PP solvable in pseudo-polynomial time:
  - q-routes with 2-cycles elimination [46].
  - q-routes with arbitrary k-cycles elimination [16, 47].
  - ng-routes [17].

$$\min_{x,y} \quad z_{\text{CPTP}}(x,y) = \sum_{i \in E} c_e x_e - \sum_{i \in V} p_i y_i \tag{23}$$

$$y_0 = 1 \tag{24}$$

$$\sum_{i \in V} q_i y_i \leq Q \tag{25}$$

$$\sum_{e \in \delta(i)} x_e = 2y_i \qquad \forall i \in V \tag{26}$$

$$\sum_{e \in \delta(S)} x_e \geq 2y_i \qquad \forall i \in S,\ \forall S \subseteq V_0,\ |S| \geq 2 \tag{27}$$

$$x_e \in \{0,1\} \qquad \forall e \in E \tag{28}$$

$$y_i \in \{0,1\} \qquad \forall i \in V. \tag{29}$$

BaPCod

BaPCod [45] is a software package developed in France at the Bordeaux University and Bordeaux Research Center that embeds a sophisticated column generation approach embedded in a generic and modern branch-price-and-cut (BPC) algorithm.

- Currently, state-of-the-art in solving routing problems.
- Takes as input a compact mixed-integer programming model and solves it using a Dantzig-Wolfe reformulation [48].
- Uses an automatic dual price smoothing stabilization [49].

VRPSolver

The *VRPSolver* extension [11], is a *BaPCod* extension distributed by the same authors. This extension includes an advanced implementation of a bidirectional dynamic programming labeling algorithm [50] for solving the pricing problem. The included labeling algorithm can be used as an exact or heuristic pricer. The labeling algorithm contains two successively lighter heuristic implementations; for more information, see Sadykov et al. [50].

The *VRPSolver* extension, also, includes the implementation of some specific cutting planes and branching decisions aimed at efficiently solving routing-like problems (or problems that exhibit similar structures) such as the CVRP, VRPTW, and also others (see [11]).

BaPCod parametrization

- Disabled non-robust inequalities.
- Disabled multiple columns per pricing iteration.
- Increased ng-sets size and disable tailing off condition.
- Disabled any form of branching and cut-generation to stop the resolution process at the root node.

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