



UNIVERSITÀ
DEGLI STUDI
DI PADOVA



DIPARTIMENTO
DI INGEGNERIA
DELL'INFORMAZIONE

A Branch-and-Cut based Pricer for the Capacitated Vehicle Routing Problem

Graduate:

Davide PARO

ID: 1207154

Supervisor:

Prof. Domenico SALVAGNIN

Co-Supervisor:

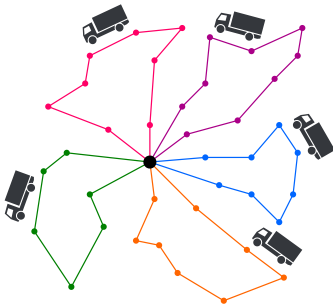
Prof. Roberto ROBERTI

Master's Degree in Computer Engineering

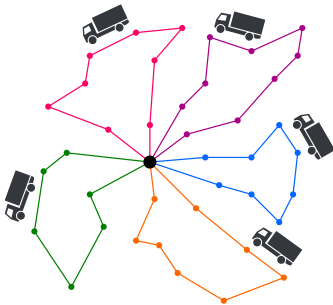
July 14th, 2022

Academic Year: 2021–2022

The Capacitated Vehicle Routing Problem (**CVRP**) [1] is an **NP-hard combinatorial optimization routing problem** with applications in **logistics** (transportation, distribution, delivery).

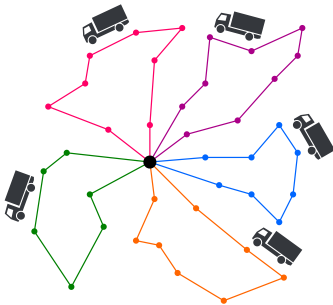


The Capacitated Vehicle Routing Problem (**CVRP**) [1] is an **NP-hard combinatorial optimization routing problem** with applications in **logistics** (transportation, distribution, delivery).



- CVRP defined on a complete graph. We are given:
 - The amount of **available vehicles** with their **capacity**.
 - A **central depot** where vehicles are stationed.
 - Customers' **locations**.
 - The **demands** (in units) of the individual customers.

The Capacitated Vehicle Routing Problem (**CVRP**) [1] is an **NP-hard combinatorial optimization routing problem** with applications in **logistics** (transportation, distribution, delivery).



- CVRP defined on a complete graph. We are given:
 - The amount of **available vehicles** with their **capacity**.
 - A **central depot** where vehicles are stationed.
 - Customers' **locations**.
 - The **demands** (in units) of the individual customers.
- Objective:
 - **Minimize overall routing costs while meeting the needs of **all** customers.**

- In the last two decades, the most efficient VRP solvers are all based on branch-price-and-cut approaches [2, 3, 4, 5, 6, 7, 8, 9, 10, 11].
- **Branch-price-and-cut** (BPC) is an **exact approach** for solving combinatorial optimization problems.
 - Extension of traditional **branch-and-cut** (BAC).
 - Can tackle **extensive integer programming models**.

$$\begin{aligned} \min_{\lambda} \quad & z_{SC}(\lambda) = \sum_{p \in P} c_p \lambda_p \\ & \sum_{p \in P} \lambda_p = K \\ & \sum_{p \in P} a_{ip} \lambda_p \geq 1 \quad \forall i \in V_0 \\ & \lambda_p \in \{0, 1\} \quad \forall p \in P. \end{aligned}$$

- In the last two decades, the most efficient VRP solvers are all based on branch-price-and-cut approaches [2, 3, 4, 5, 6, 7, 8, 9, 10, 11].
- **Branch-price-and-cut** (BPC) is an **exact approach** for solving combinatorial optimization problems.
 - Extension of traditional **branch-and-cut** (BAC).
 - Can tackle **extensive integer programming models**.

■ How?

- **Column Generation** (CG): decision variables are generated lazily.
 - In VRP, decision variables represent **single vehicle feasible routes**.
- **Pricing**: determines the optimal **reduced-cost decision variables** for their inclusion.

$$\begin{aligned} \min_{\lambda} \quad & z_{SC}(\lambda) = \sum_{p \in P} c_p \lambda_p \\ & \sum_{p \in P} \lambda_p = K \\ & \sum_{p \in P} a_{ip} \lambda_p \geq 1 \quad \forall i \in V_0 \\ & \lambda_p \in \{0, 1\} \quad \forall p \in P. \end{aligned}$$

- Pricing Problem (PP) in its "natural" form is an induced **NP-hard** combinatorial optimization problem [12].
 - An **Elementary Shortest Path Problem with Resource Constraints (ESPPRC)** in a reduced cost network with negative cycles:

$$c_p^* = \min_{p \in P} \left\{ \bar{c}_p = \sum_{e \in \{i,j\} \in E} \left(c_e - \frac{\pi_i + \pi_j}{2} \right) a_{ep} \right\}. \quad (1)$$

- Pricing Problem (PP) in its "natural" form is an induced **NP-hard** combinatorial optimization problem [12].
 - An **Elementary Shortest Path Problem with Resource Constraints (ESPPRC)** in a reduced cost network with negative cycles:

$$c_p^* = \min_{p \in P} \left\{ \bar{c}_p = \sum_{e=\{i,j\} \in E} \left(c_e - \frac{\pi_i + \pi_j}{2} \right) a_{ep} \right\}. \quad (1)$$

- PP hard \rightarrow **Relaxation** of the PP.
 - **Relaxed** PP is solvable in **pseudo-polynomial time** [13, 14] via efficient **dynamic programming** algorithms.
 - **Labeling algorithm** proposed in 1992 in [15].
 - Labeling algorithm extended to handle elementarity condition in 2004 in [16].

Two major issues with contemporary approaches:

- 1 (weak) **relaxing** the PP weakens the dual bounds fed to the BPC (increasing column generation time).
 - **Recently alleviated** thanks to recent developments of **efficacious relaxations** [17].
- 2 (strong) **Labeling** algorithm's **speed decreases** as the **vehicle capacity increases** $O(N^2Q)$ (longer routes), limiting its applicability to modern distribution problems.



We propose a different approach to pricing:

- 1 An exact **branch-and-cut** algorithm for solving the **non-relaxed** pricing problem.
 - Almost no works on this domain, except [18].
- 2 We study its competitiveness as the **vehicle capacity increases** by comparing it with **the current state-of-the-art solution** [11].

- Implemented in C using the commercial **IBM ILOG CPLEX optimizer**.
- The PP is modeled as a **Capacitated Profitable Tour Problem (CPTP)** [18].
- Available online^a under a permissive license.



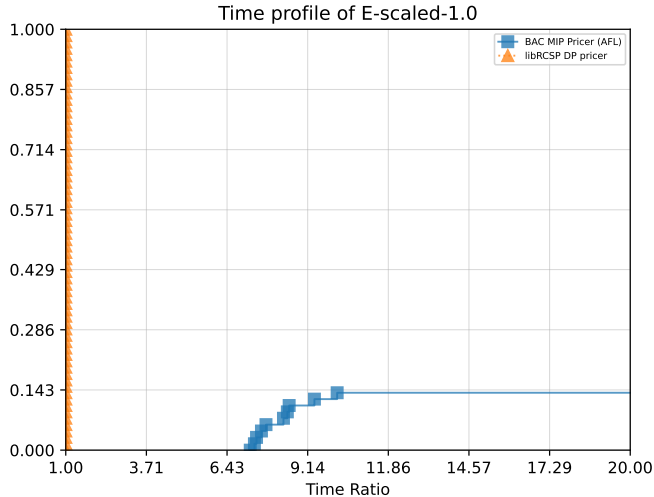
^a<https://github.com/dparo/master-thesis>

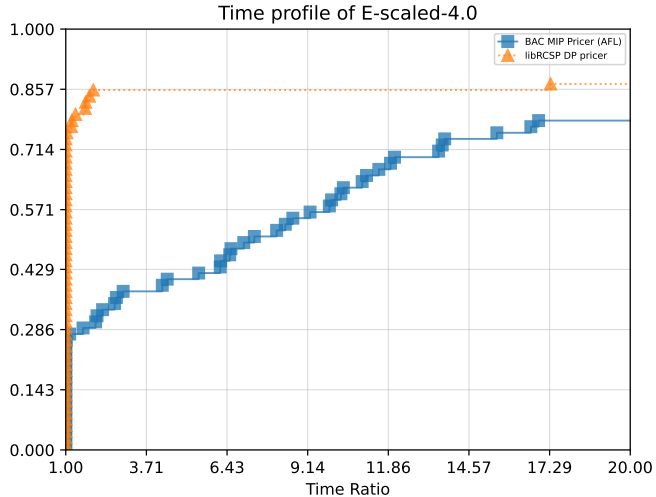
- Some implementation details:
 - Heuristics for warm-starting (constructive + local-search).
 - Inclusion of several **cutting-planes**:
 - GSEC, RCC [19], GLM [20].
 - Integral separation by tracking major connected components.
 - Fractional separation through push relabel max-flow [21] and Gomory-Hu trees [22].

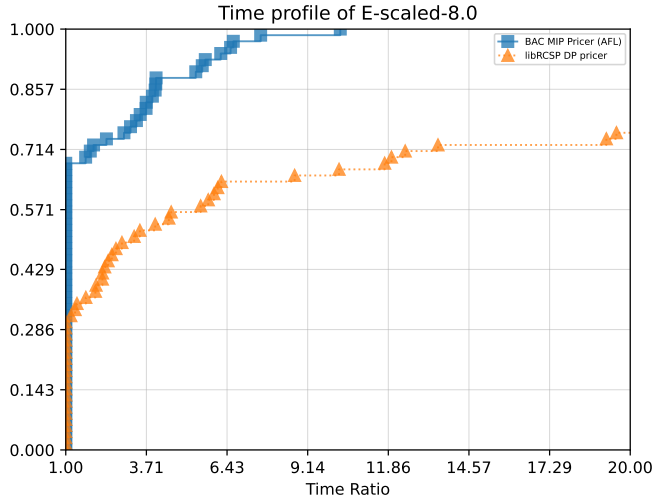


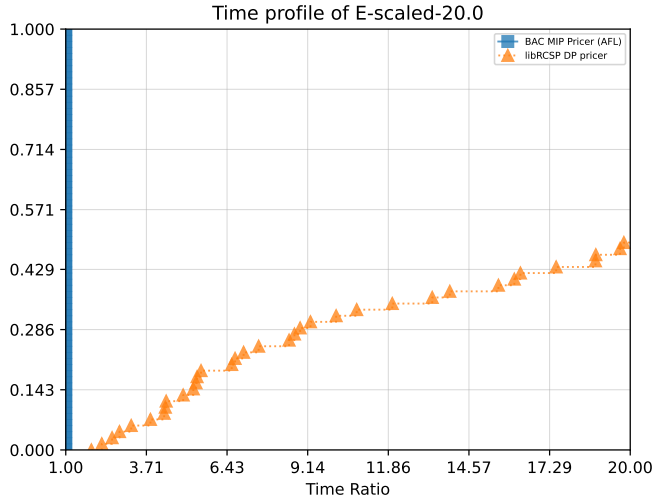
- We've modified the commonly employed traditional instances proposed in [1, 23, 24, 25, 26, 27, 28].
 - Generated new instances by scaling the **vehicle capacities**.
- The running times of the two pricing approaches are depicted through **performance profiles** [29]:
 - **Slim representation** to compare optimization algorithms.

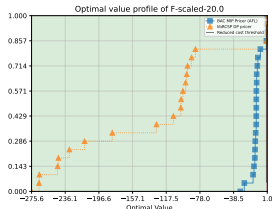
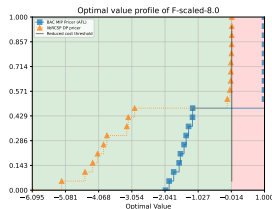
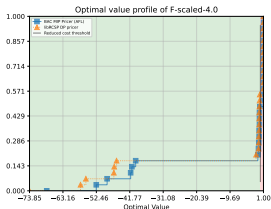
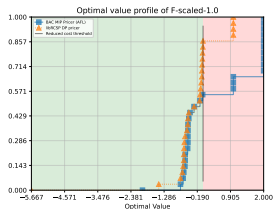
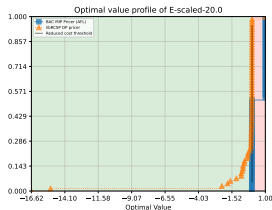
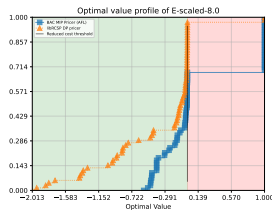
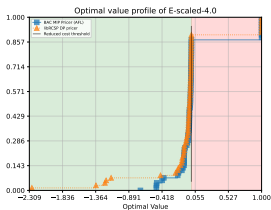
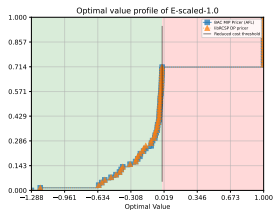
Results (1/2)











Although our study is an indication on the genuine efficiency of the two approaches:

- We proved that **branch-and-cut** may supplement the traditional **labeling algorithm**, especially in solving PP with **non-stringent vehicle capacities**.
- Suggesting future research on **branch-and-cut** approaches in the context of **pricing for the CVRP**, may bring substantial benefits to contemporary CVRP solvers.





The end



Thank you for your time!

■ Technological level:

- Extend the approach to other routing problems, such as the VRPTW.
- Integrate newer cutting-plane strategies.
- Hybridize the BAC approach with the labeling algorithm.
- Add support for non-robust cuts.
- Implement the efficient ng-routes relaxation [17] within a BAC scheme.
- Implement Decremental State Space Relaxation (DSSR) [30, 31].

■ Implementation level:

- Use heuristic separation procedures whenever possible to separate the cutting-planes.
- Implement an effective branching strategy.
- Try out compact IP formulations and their cutting-planes.

■ Setup level:

- Integrate the BAC algorithm inside a BPC framework to assess more metrics for the running time comparison.

Additional material.

MIP solvers are rather general and can be used to solve a wide range of problems from various fields [32]. MIP models are, in spirit, a way to mathematically program a solver to achieve the desired solution. A MIP solver can solve a mixed-integer linear programming formulation expressed as [33]:

$$\max_{x,y} \quad c^T x + d^T y \quad (2)$$

$$\text{s.t.} \quad Ax + By \leq b \quad (3)$$

$$x \in \mathbb{R}^n \quad (4)$$

$$y \in \mathbb{Z}_+^k, \quad (5)$$

where $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times k}$ are matrices and $c \in \mathbb{R}^n$, $d \in \mathbb{R}^k$, $b \in \mathbb{R}^m$ are vector coefficients. The bound in eq. (3) can also be rewritten in equality and/or greater form.

- 1959: The CVRP was first introduced in [1].
- 1961: First branch-and-price applied to the Cutting Stock Problem in [34].
- Up to late 80s: tree search algorithms employing branch-and-bound schemes: Lagrangian duality relaxation,
- Mid 80s: First Column Generation attempts [35, 36].
- From late 80s to first 2000s: branch-and-cut approaches [19, 37, 38, 39, 28, 40, 41, 42, 43, 44].
- 1992: labeling algorithm for pricing in [15].
- 1999: basic implementation of a BPC framework in [45].
- 2006: Turning point in modern BPC frameworks thanks to [2].
- 2011: smart ng-routes relaxation for the pricing problem [17].
- 2014-now: State-of-the-art BPC algorithms [7, 10, 9, 46, 47, 11].

Set Covering formulation

Let $P = \{p \mid p \text{ is a single-truck elementary feasible route}\}$ be the set of all feasible routes.

$$\min_{\lambda} \quad z_{SC}(\lambda) = \sum_{p \in P} c_p \lambda_p \quad (6)$$

$$\sum_{p \in P} \lambda_p = K \quad (7)$$

$$\sum_{p \in P} a_{ip} \lambda_p \geq 1 \quad \forall i \in V_0 \quad (8)$$

$$\lambda_p \in \{0, 1\} \quad \forall p \in P. \quad (9)$$

Master Problem (Primal)

$$\min_{\lambda} \quad z_{\text{MP}}(\lambda) = \sum_{p \in P} c_p \lambda_p \quad (10)$$

$$\sum_{p \in P} \lambda_p = K \quad (11)$$

$$\sum_{p \in P} a_{ip} \lambda_p = 1 \quad \forall i \in V_0 \quad (12)$$

$$0 \leq \lambda_p \leq 1 \quad \forall p \in P. \quad (13)$$

$$\max_{\pi} \quad z_{\text{DMP}}(\pi) = K\pi_0 + \sum_{i \in V_0} \pi_i \quad (14)$$

$$\pi_0 + \sum_{i \in V_0} a_{ip} \pi_i \leq c_p \quad \forall p \in P \quad (15)$$

$$\pi_0 \in \mathbb{R} \quad (16)$$

$$\pi_i \in \mathbb{R} \quad \forall i \in V_0, \quad (17)$$

where $\pi_0 \in \mathbb{R}, \pi_i \in \mathbb{R} \quad \forall i \in V_0$ represents the dual variables associated respectively with constraints (11) and (12).

Due to the enormous size of the set of routes P , evaluating the dual variables $\pi \in \mathbb{R}^N$ is computationally intractable. As a result, in BAP frameworks we consider only a small subset of columns $\mathcal{P} \subseteq P$:

$$\min_{\lambda} \quad z_{\text{RMP}}(\lambda) = \sum_{p \in \mathcal{P}} c_p \lambda_p \quad (18)$$

$$\sum_{p \in \mathcal{P}} \lambda_p = K \quad (19)$$

$$\sum_{p \in \mathcal{P}} a_{ip} \lambda_p = 1 \quad \forall i \in V_0 \quad (20)$$

$$\lambda_p \geq 0 \quad \forall p \in \mathcal{P}. \quad (21)$$

We look for a column to enter the basis of the RMP, which in turn necessitates the resolution of the following sub-problem:

$$c_p^* = \min_{p \in P} \left\{ \bar{c}_p = \sum_{e=\{i,j\} \in E} \left(c_e - \frac{\pi_i + \pi_j}{2} \right) a_{ep} \right\}, \quad (22)$$

which takes the name of *Pricing Problem* (PP). Any $p \in P$ which satisfies $c_p < 0$ is a valid column which can enter the basis of the RMP. The column generation procedure stops mainly under two scenarios: (i) when no more negative reduced cost routes exist, i.e. the PP outputs a $p^* \in P$ achieving $c_p^* \geq 0$ or (ii) the CG procedure tails off.

- **Relax elementarity condition** to make the PP solvable in pseudo-polynomial time:
 - q -routes with 2-cycles elimination [48].
 - q -routes with arbitrary k -cycles elimination [16, 49].
 - ng-routes [17].

$$\min_{x,y} \quad z_{\text{CPTP}}(x,y) = \sum_{e \in E} c_e x_e - \sum_{i \in V} p_i y_i \quad (23)$$

$$y_0 = 1 \quad (24)$$

$$\sum_{i \in V} q_i y_i \leq Q \quad (25)$$

$$\sum_{e \in \delta(i)} x_e = 2y_i \quad \forall i \in V \quad (26)$$

$$\sum_{e \in \delta(S)} x_e \geq 2y_i \quad \forall i \in S, \forall S \subseteq V_0, |S| \geq 2 \quad (27)$$

$$x_e \in \{0,1\} \quad \forall e \in E \quad (28)$$

$$y_i \in \{0,1\} \quad \forall i \in V. \quad (29)$$

BaPCod [47] is a software package developed in France at the Bordeaux University and Bordeaux Research Center that embeds a sophisticated column generation approach embedded in a generic and modern branch-price-and-cut (BPC) algorithm.

- Currently, state-of-the-art in solving routing problems.
- Takes as input a compact mixed-integer programming model and solves it using a Dantzig-Wolfe reformulation [50].
- Uses an automatic dual price smoothing stabilization [51].

The *VRPSolver* extension [11], is a *BaPCod* extension distributed by the same authors. This extension includes an advanced implementation of a bidirectional dynamic programming labeling algorithm [52] for solving the pricing problem. The included labeling algorithm can be used as an exact or heuristic pricer. The labeling algorithm contains two successively lighter heuristic implementations; for more information, see Sadykov et al. [52].

The *VRPSolver* extension, also, includes the implementation of some specific cutting planes and branching decisions aimed at efficiently solving routing-like problems (or problems that exhibit similar structures) such as the CVRP, VRPTW, and also others (see [11]).

- Disabled non-robust inequalities.
- Disabled multiple columns per pricing iteration.
- Increased ng-sets size and disable tailing off condition.
- Disabled any form of branching and cut-generation to stop the resolution process at the root node.

References.



George B Dantzig and John H Ramser. “The Truck Dispatching Problem”. In: *Management science* 6.1 (1959), pp. 80–91 (cit. on pp. 2–4, 12, 24).



Ricardo Fukasawa, Humberto Longo, Jens Lysgaard, Marcus Poggi de Aragão, Marcelo Reis, Eduardo Uchoa, and Renato F. Werneck. “Robust Branch-and-Cut-and-Price for the Capacitated Vehicle Routing Problem”. In: *Mathematical Programming* 106.3 (May 2006), pp. 491–511. ISSN: 0025-5610, 1436-4646. DOI: 10.1007/s10107-005-0644-x. URL: <http://link.springer.com/10.1007/s10107-005-0644-x> (visited on 03/24/2022) (cit. on pp. 5, 6, 24).



Artur Pessoa, Marcus Poggi De Aragao, and Eduardo Uchoa. “Robust Branch-Cut-and-Price Algorithms for Vehicle Routing Problems”. In: *The Vehicle Routing Problem: Latest Advances and New Challenges*. Springer, 2008, pp. 297–325 (cit. on pp. 5, 6).



Gabriel Gutiérrez-Jarpa, Guy Desaulniers, Gilbert Laporte, and Vladimir Marianov. “A Branch-and-Price Algorithm for the Vehicle Routing Problem with Deliveries, Selective Pickups and Time Windows”. In: *European Journal of Operational Research* 206.2 (2010), pp. 341–349 (cit. on pp. 5, 6).



Claudia Archetti, Mathieu Bouchard, and Guy Desaulniers. “Enhanced Branch and Price and Cut for Vehicle Routing with Split Deliveries and Time Windows”. In: *Transportation Science* 45.3 (2011), pp. 285–298 (cit. on pp. 5, 6).



Andrea Bettinelli, Alberto Ceselli, and Giovanni Righini. “A Branch-and-Cut-and-Price Algorithm for the Multi-Depot Heterogeneous Vehicle Routing Problem with Time Windows”. In: *Transportation Research Part C: Emerging Technologies* 19.5 (2011), pp. 723–740 (cit. on pp. 5, 6).

-  Claudio Contardo and Rafael Martinelli. “A New Exact Algorithm for the Multi-Depot Vehicle Routing Problem under Capacity and Route Length Constraints”. In: *Discrete Optimization* 12 (2014), pp. 129–146 (cit. on pp. 5, 6, 24).
-  Claudio Contardo, Guy Desaulniers, and François Lessard. “Reaching the Elementary Lower Bound in the Vehicle Routing Problem with Time Windows”. In: *Networks. An International Journal* 65.1 (2015), pp. 88–99 (cit. on pp. 5, 6).
-  Diego Pecin, Claudio Contardo, Guy Desaulniers, and Eduardo Uchoa. “New Enhancements for the Exact Solution of the Vehicle Routing Problem with Time Windows”. In: *INFORMS Journal on Computing* 29.3 (2017), pp. 489–502 (cit. on pp. 5, 6, 24).






Diego Pecin, Artur Pessoa, Marcus Poggi, and Eduardo Uchoa. “Improved Branch-Cut-and-Price for Capacitated Vehicle Routing”. In: *Mathematical Programming Computation* 9.1 (2017), pp. 61–100. ISSN: 18672957. DOI: 10.1007/s12532-016-0108-8 (cit. on pp. 5, 6, 24).



Artur Pessoa, Ruslan Sadykov, Eduardo Uchoa, and François Vanderbeck. “A Generic Exact Solver for Vehicle Routing and Related Problems”. In: *Mathematical Programming* 183.1-2 (2020), pp. 483–523. ISSN: 14364646. DOI: 10.1007/s10107-020-01523-z (cit. on pp. 5, 6, 10, 24, 33).



Moshe Dror. “Note on the Complexity of the Shortest Path Models for Column Generation in VRPTW”. In: *Operations Research* 42.5 (Oct. 1994), pp. 977–978. ISSN: 0030-364X, 1526-5463. DOI: 10.1287/opre.42.5.977. URL: <http://pubsonline.informs.org/doi/abs/10.1287/opre.42.5.977> (visited on 03/24/2022) (cit. on pp. 7, 8).

-  Martin Desrochers and François Soumis. “A Generalized Permanent Labelling Algorithm for the Shortest Path Problem with Time Windows”. In: *INFOR: Information Systems and Operational Research* 26.3 (1988), pp. 191–212 (cit. on pp. 7, 8).
-  Stefan Irnich and Guy Desaulniers. “Shortest Path Problems with Resource Constraints”. In: *Column Generation*. Springer, 2005, pp. 33–65 (cit. on pp. 7, 8).
-  Martin Desrochers, Jacques Desrosiers, and Marius Solomon. “A New Optimization Algorithm for the Vehicle Routing Problem with Time Windows”. In: *Operations research* 40.2 (1992), pp. 342–354 (cit. on pp. 7, 8, 24).



Dominique Feillet, Pierre Dejax, Michel Gendreau, and Cyrille Gueguen. “An Exact Algorithm for the Elementary Shortest Path Problem with Resource Constraints: Application to Some Vehicle Routing Problems”. In: *Networks* 44.3 (2004), pp. 216–229. ISSN: 00283045. DOI: 10.1002/net.20033 (cit. on pp. 7, 8, 30).



Roberto Baldacci, Aristide Mingozzi, and Roberto Roberti. “New Route Relaxation and Pricing Strategies for the Vehicle Routing Problem”. In: *Operations Research* 59.5 (2011), pp. 1269–1283. ISSN: 0030364X. DOI: 10.1287/opre.1110.0975 (cit. on pp. 9, 20, 24, 30).







Mads Kehlet Jepsen, Bjørn Petersen, Simon Spoorendonk, and David Pisinger. “A Branch-and-Cut Algorithm for the Capacitated Profitable Tour Problem”. In: *Discrete Optimization* 14 (2014), pp. 78–96. ISSN: 15725286. DOI: 10.1016/j.disopt.2014.08.001. URL: <http://dx.doi.org/10.1016/j.disopt.2014.08.001> (cit. on pp. 10, 11).











G. Laporte and Y. Nobert. “A Branch and Bound Algorithm for the Capacitated Vehicle Routing Problem”. In: *OR Spektrum* 5.2 (1983), pp. 77–85. ISSN: 01716468. DOI: 10.1007/BF01720015 (cit. on pp. 11, 24).







Luis Gouveia. “A Result on Projection for the Vehicle Routing Problem”. In: *European Journal of Operational Research* 85.3 (Sept. 1995), pp. 610–624. ISSN: 03772217. DOI: 10.1016/0377-2217(94)00025-8. URL: <https://linkinghub.elsevier.com/retrieve/pii/0377221794000258> (visited on 02/01/2022) (cit. on p. 11).

-  Andrew V Goldberg. “An Efficient Implementation of a Scaling Minimum-Cost Flow Algorithm”. In: *Journal of algorithms* 22.1 (1997), pp. 1–29 (cit. on p. 11).
-  Ralph E Gomory and Tien Chung Hu. “Multi-Terminal Network Flows”. In: *Journal of the Society for Industrial and Applied Mathematics* 9.4 (1961), pp. 551–570 (cit. on p. 11).
-  Nicos Christofides and Samuel Eilon. “An Algorithm for the Vehicle-Dispatching Problem”. In: *Journal of the Operational Research Society* 20.3 (1969), pp. 309–318 (cit. on p. 12).
-  TJ Gaskell. “Bases for Vehicle Fleet Scheduling”. In: *Journal of the Operational Research Society* 18.3 (1967), pp. 281–295 (cit. on p. 12).

-  Billy E Gillett and Leland R Miller. “A Heuristic Algorithm for the Vehicle-Dispatch Problem”. In: *Operations research* 22.2 (1974), pp. 340–349 (cit. on p. 12).
-  Nicos Christofides. “The Vehicle Routing Problem”. In: *Combinatorial optimization* (1979) (cit. on p. 12).
-  Marshall L. Fisher. “Optimal Solution of Vehicle Routing Problems Using Minimum K-Trees”. In: *Operations Research* 42.4 (1994), pp. 626–642. JSTOR: 171617 (cit. on p. 12).
-  P Augerat, J M Bealeguer, E Benavent, A Corberan, D Naddef, and G Rinaldi. “Computational Results with a Branch and Cut Code for the Capacitated Vehicle Routing Problem”. In: (1995), p. 33 (cit. on pp. 12, 24).

-  E. D. Dolan and J.J. Moré. “Benchmarking Optimization Software with Performance Profiles”. In: *Mathematical Programming* 91.2 (2002), pp. 201–213 (cit. on p. 12).
-  Natasha Boland, John Dethridge, and Irina Dumitrescu. “Accelerated Label Setting Algorithms for the Elementary Resource Constrained Shortest Path Problem”. In: *Operations Research Letters* 34.1 (2006), pp. 58–68 (cit. on p. 20).
-  Giovanni Righini and Matteo Salani. “New Dynamic Programming Algorithms for the Resource Constrained Elementary Shortest Path Problem”. In: *Networks* 51.3 (2008), pp. 155–170. ISSN: 00283045. DOI: 10.1002/net.20212 (cit. on p. 20).
-  Robert Bixby and Edward Rothberg. “Progress in Computational Mixed Integer Programming—a Look Back from the Other Side of the Tipping Point”. In: *Annals of Operations Research* 149.1 (2007), p. 37 (cit. on p. 23).

-  Laurence A Wolsey and George L Nemhauser. *Integer and Combinatorial Optimization*. Vol. 55. John Wiley & Sons, 1999 (cit. on p. 23).
-  P. C. Gilmore and R. E. Gomory. “A Linear Programming Approach to the Cutting-Stock Problem”. In: *Operations Research* 9.6 (1961), pp. 849–859. JSTOR: 167051 (cit. on p. 24).
-  Jacques Desrosiers, François Soumis, and Martin Desrochers. “Routing with Time Windows by Column Generation”. In: *Networks. An International Journal* 14.4 (1984), pp. 545–565 (cit. on p. 24).
-  Yogesh Agarwal, Kamlesh Mathur, and Harvey M Salkin. “A Set-Partitioning-Based Exact Algorithm for the Vehicle Routing Problem”. In: *Networks. An International Journal* 19.7 (1989), pp. 731–749 (cit. on p. 24).






Gilbert Laporte, Yves Nobert, and Martin Desrochers. “Optimal Routing under Capacity and Distance Restrictions”. In: *Operations research* 33.5 (1985), pp. 1050–1073 (cit. on p. 24).





Philippe Augerat. “Approche Polyédrale Du Problème de Tournées de Véhicules”. Institut National Polytechnique de Grenoble-INPG, 1995 (cit. on p. 24).







J. R. Araque G, G. Kudva, T. L. Morin, and J. F. Pekny. “A Branch-and-Cut Algorithm for Vehicle Routing Problems”. In: *Annals of Operations Research* 50.1 (Dec. 1994), pp. 37–59. ISSN: 0254-5330, 1572-9338. DOI: 10.1007/BF02085634. URL: <http://link.springer.com/10.1007/BF02085634> (visited on 03/25/2022) (cit. on p. 24).

-  NR Achuthan, L Caccetta, and SP Hill. “A New Subtour Elimination Constraint for the Vehicle Routing Problem”. In: *European Journal of Operational Research* 91.3 (1996), pp. 573–586 (cit. on p. 24).
-  Ulrich Blasum and Winfried Hochstattler. “Application of the Branch and Cut Method to the Vehicle Routing Problem”. In: (2000), p. 22 (cit. on p. 24).
-  T.K. Ralphs, L. Kopman, W.R. Pulleyblank, and L.E. Trotter. “On the Capacitated Vehicle Routing Problem”. In: *Mathematical Programming* 94.2-3 (Jan. 1, 2003), pp. 343–359. ISSN: 0025-5610, 1436-4646. DOI: 10.1007/s10107-002-0323-0. URL: <http://link.springer.com/10.1007/s10107-002-0323-0> (visited on 03/25/2022) (cit. on p. 24).

 N. R. Achuthan, L. Caccetta, and S. P. Hill. “An Improved Branch-and-Cut Algorithm for the Capacitated Vehicle Routing Problem”. In: *Transportation Science* 37.2 (May 2003), pp. 153–169. ISSN: 0041-1655, 1526-5447. DOI: 10.1287/trsc.37.2.153.15243. URL: <http://pubsonline.informs.org/doi/abs/10.1287/trsc.37.2.153.15243> (visited on 03/25/2022) (cit. on p. 24).

 R. Baldacci, E. Hadjiconstantinou, and A. Mingozzi. “An Exact Algorithm for the Capacitated Vehicle Routing Problem Based on a Two-Commodity Network Flow Formulation”. In: *Operations Research* 52.5 (2004), pp. 723–738. ISSN: 0030364X. DOI: 10.1287/opre.1040.0111 (cit. on p. 24).

 Niklas Kohl, Jacques Desrosiers, Oli BG Madsen, Marius M Solomon, and Francois Soumis. “2-Path Cuts for the Vehicle Routing Problem with Time Windows”. In: *Transportation Science* 33.1 (1999), pp. 101–116 (cit. on p. 24).

-  Artur Pessoa and Eduardo Uchoa. “Solving Bin Packing Problems Using VRPSolver Models”. In: *Operations Research Forum*. Vol. 2. Springer, 2020, pp. 1–25 (cit. on p. 24).
-  Ruslan Sadykov and François Vanderbeck. “BaPCod-a Generic Branch-and-Price Code”. PhD thesis. Inria Bordeaux Sud-Ouest, 2021 (cit. on pp. 24, 32).
-  N. Christofides, A. Mingozzi, and P. Toth. “Exact Algorithms for the Vehicle Routing Problem, Based on Spanning Tree and Shortest Path Relaxations”. In: *Mathematical Programming* 20.1 (1981), pp. 255–282. ISSN: 00255610. DOI: 10.1007/BF01589353 (cit. on p. 30).
-  Alain Chabrier. “Vehicle Routing Problem with Elementary Shortest Path Based Column Generation”. In: *Computers and Operations Research* 33.10 (2006), pp. 2972–2990. ISSN: 03050548. DOI: 10.1016/j.cor.2005.02.029 (cit. on p. 30).



George B Dantzig and Philip Wolfe. “Decomposition Principle for Linear Programs”. In: *Operations research* 8.1 (1960), pp. 101–111 (cit. on p. 32).



Artur Pessoa, Ruslan Sadykov, Eduardo Uchoa, and François Vanderbeck. “Automation and Combination of Linear-Programming Based Stabilization Techniques in Column Generation”. In: *INFORMS Journal on Computing* 30.2 (2018), pp. 339–360 (cit. on p. 32).



Ruslan Sadykov, Eduardo Uchoa, and Artur Pessoa. “A Bucket Graph–Based Labeling Algorithm with Application to Vehicle Routing”. In: *Transportation Science* 55.1 (2021), pp. 4–28. ISSN: 15265447. DOI: 10.1287/TRSC.2020.0985 (cit. on p. 33).



UNIVERSITÀ
DEGLI STUDI
DI PADOVA



DIPARTIMENTO
DI INGEGNERIA
DELL'INFORMAZIONE

A Branch-and-Cut based Pricer for the Capacitated Vehicle Routing Problem

Graduate:

Davide PARO

ID: 1207154

Supervisor:

Prof. Domenico SALVAGNIN

Co-Supervisor:

Prof. Roberto ROBERTI

Master's Degree in Computer Engineering

July 14th, 2022

Academic Year: 2021–2022