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# A Branch-and-Cut based Pricer for the Capacitated Vehicle Routing Problem

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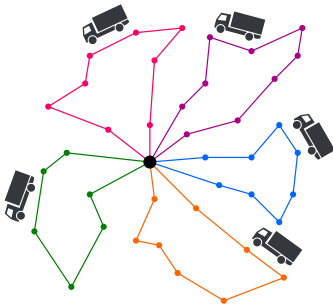
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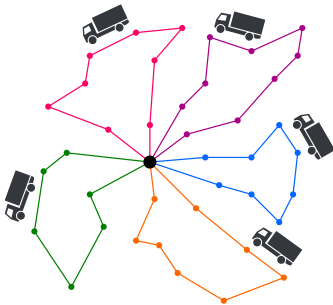
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The Capacitated Vehicle Routing Problem (**CVRP**) [1] is an **NP-hard combinatorial optimization routing problem** with applications in **logistics** (transportation, distribution, delivery).

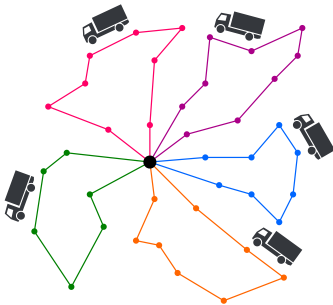


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- CVRP defined on a complete graph. We are given:
  - The amount of **available vehicles** with their **capacity**.
  - A **central depot** where vehicles are stationed.
  - Customers' **locations**.
  - The **demands** (in units) of the individual customers.

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  - Customers' **locations**.
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- Objective:
  - **Minimize overall routing costs while meeting the needs of **all** customers.**

- In the last two decades, the most efficient VRP solvers are all based on branch-price-and-cut approaches [2, 3, 4, 5, 6, 7, 8, 9, 10, 11].
- **Branch-price-and-cut** (BPC) is an **exact approach** for solving combinatorial optimization problems.
  - Extension of traditional **branch-and-cut** (BAC).
  - Can tackle **extensive integer programming models**.

$$\begin{aligned} \min_{\lambda} \quad & z_{SC}(\lambda) = \sum_{p \in P} c_p \lambda_p \\ & \sum_{p \in P} \lambda_p = K \\ & \sum_{p \in P} a_{ip} \lambda_p \geq 1 \quad \forall i \in V_0 \\ & \lambda_p \in \{0, 1\} \quad \forall p \in P. \end{aligned}$$

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## ■ How?

- **Column Generation** (CG): decision variables are generated lazily.
  - In VRP, decision variables represent **single vehicle feasible routes**.
- **Pricing**: determines the optimal **reduced-cost decision variables** for their inclusion.

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- Pricing Problem (PP) in its "natural" form is an induced **NP-hard** combinatorial optimization problem [12].
  - An **Elementary Shortest Path Problem with Resource Constraints (ESPPRC)** in a reduced cost network with negative cycles:

$$c_p^* = \min_{p \in P} \left\{ \bar{c}_p = \sum_{e \in \{i,j\} \in E} \left( c_e - \frac{\pi_i + \pi_j}{2} \right) a_{ep} \right\}. \quad (1)$$

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- PP hard  $\rightarrow$  **Relaxation** of the PP.
  - **Relaxed** PP is solvable in **pseudo-polynomial time** [13, 14] via efficient **dynamic programming** algorithms.
    - **Labeling algorithm** proposed in 1992 in [15].
    - Labeling algorithm extended to handle elementarity condition in 2004 in [16].



## Two major issues with contemporary approaches:

- 1 (weak) **relaxing** the PP weakens the dual bounds fed to the BPC (increasing column generation time).
  - **Recently alleviated** thanks to recent developments of **efficacious relaxations** [17].
- 2 (strong) **Labeling** algorithm's **speed decreases** as the **vehicle capacity increases**  $O(N^2Q)$  (longer routes), limiting its applicability to modern distribution problems.



We propose a different approach to pricing:

- 1 An exact **branch-and-cut** algorithm for solving the **non-relaxed** pricing problem.
  - Almost no works on this domain, except [18].
- 2 We study its competitiveness as the **vehicle capacity increases** by comparing it with **the current state-of-the-art solution** [11].

- Implemented in C using the commercial **IBM ILOG CPLEX optimizer**.
- The PP is modeled as a **Capacitated Profitable Tour Problem (CPTP)** [18].
- Available online<sup>a</sup> under a permissive license.



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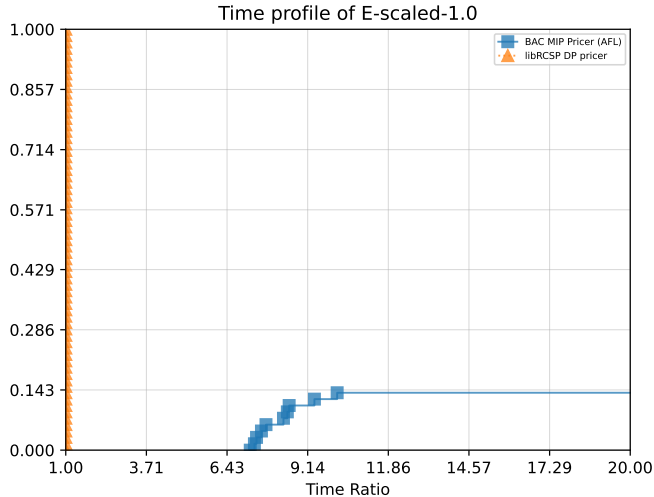
<sup>a</sup><https://github.com/dparo/master-thesis>

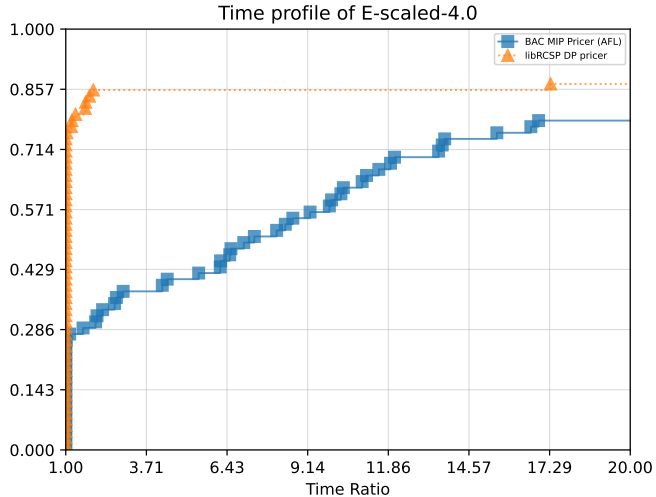
- Some implementation details:
  - Heuristics for warm-starting (constructive + local-search).
  - Inclusion of several **cutting-planes**:
    - GSEC, RCC [19], GLM [20].
    - Integral separation by tracking major connected components.
    - Fractional separation through push relabel max-flow [21] and Gomory-Hu trees [22].

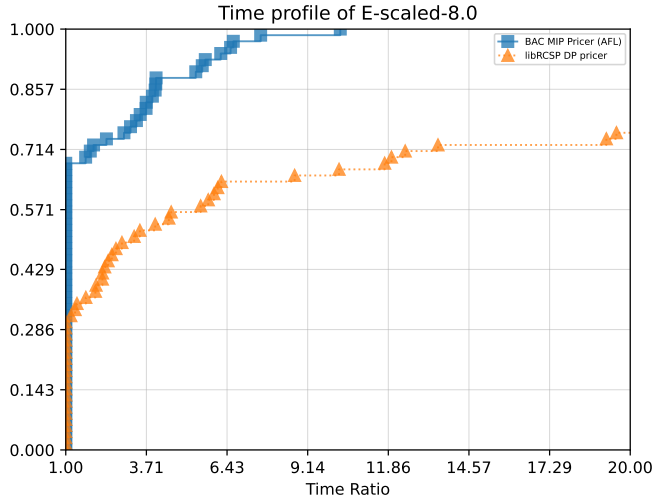


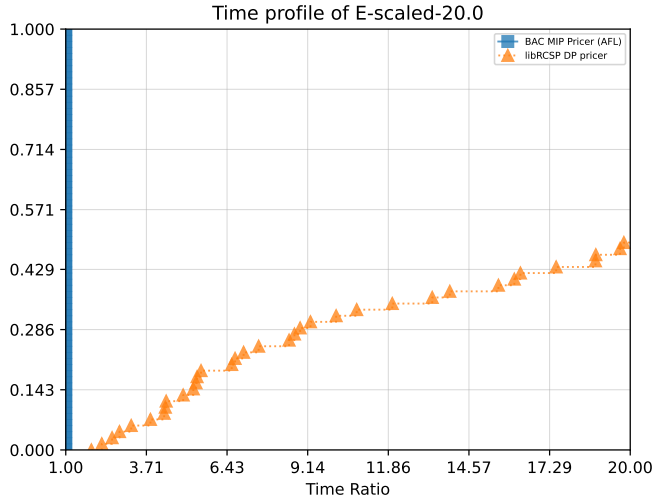
- We've modified the commonly employed traditional instances proposed in [1, 23, 24, 25, 26, 27, 28].
  - Generated new instances by scaling the **vehicle capacities**.
- The running times of the two pricing approaches are depicted through **performance profiles** [29]:
  - **Slim representation** to compare optimization algorithms.

# Results (1/2)

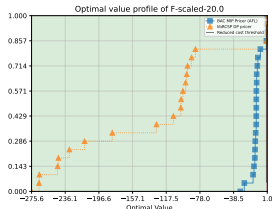
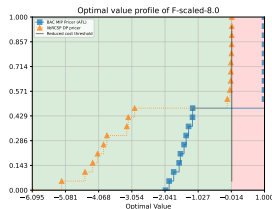
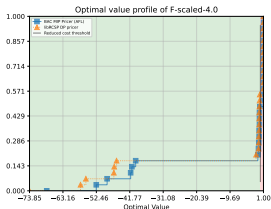
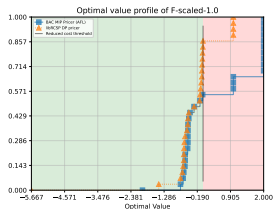
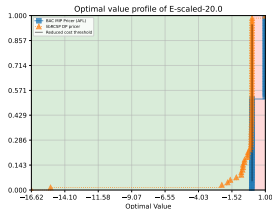
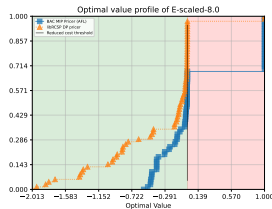
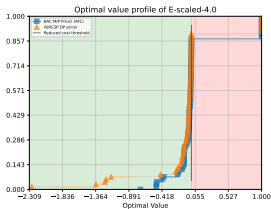
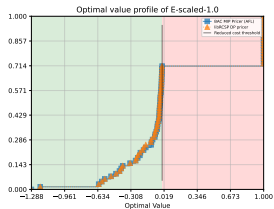












Although our study is an indication on the genuine efficiency of the two approaches:

- We proved that **branch-and-cut** may supplement the traditional **labeling algorithm**, especially in solving PP with **non-stringent vehicle capacities**.
- Suggesting future research on **branch-and-cut** approaches in the context of **pricing for the CVRP**, may bring substantial benefits to contemporary CVRP solvers.





# The end



Thank you for your time!



Additional material.

MIP solvers are rather general and can be used to solve a wide range of problems from various fields [30]. MIP models are, in spirit, a way to mathematically program a solver to achieve the desired solution. A MIP solver can solve a mixed-integer linear programming formulation expressed as [31]:

$$\max_{x,y} \quad c^T x + d^T y \quad (2)$$

$$\text{s.t.} \quad Ax + By \leq b \quad (3)$$

$$x \in \mathbb{R}^n \quad (4)$$

$$y \in \mathbb{Z}_+^k, \quad (5)$$

where  $A \in \mathbb{R}^{m \times n}$ ,  $B \in \mathbb{R}^{m \times k}$  are matrices and  $c \in \mathbb{R}^n$ ,  $d \in \mathbb{R}^k$ ,  $b \in \mathbb{R}^m$  are vector coefficients. The bound in eq. (3) can also be rewritten in equality and/or greater form.

- 1959: The CVRP was first introduced in [1].
- 1961: First branch-and-price applied to the Cutting Stock Problem in [32].
- Up to late 80s: tree search algorithms employing branch-and-bound schemes: Lagrangian duality relaxation, ....
- Mid 80s: First Column Generation attempts [33, 34].
- From late 80s to first 2000s: branch-and-cut approaches [19, 35, 36, 37, 28, 38, 39, 40, 41, 42].
- 1992: labeling algorithm for pricing in [15].
- 1999: basic implementation of a BPC framework in [43].
- 2006: Turning point in modern BPC frameworks thanks to [2].
- 2011: smart ng-routes relaxation for the pricing problem [17].
- 2014-now: State-of-the-art BPC algorithms [7, 10, 9, 44, 45, 11].

# Set Covering formulation

Let  $P = \{p \mid p \text{ is a single-truck elementary feasible route}\}$  be the set of all feasible routes.

$$\min_{\lambda} \quad z_{SC}(\lambda) = \sum_{p \in P} c_p \lambda_p \quad (6)$$

$$\sum_{p \in P} \lambda_p = K \quad (7)$$

$$\sum_{p \in P} a_{ip} \lambda_p \geq 1 \quad \forall i \in V_0 \quad (8)$$

$$\lambda_p \in \{0, 1\} \quad \forall p \in P. \quad (9)$$



# Master Problem (Primal)

$$\min_{\lambda} \quad z_{\text{MP}}(\lambda) = \sum_{p \in P} c_p \lambda_p \quad (10)$$

$$\sum_{p \in P} \lambda_p = K \quad (11)$$

$$\sum_{p \in P} a_{ip} \lambda_p = 1 \quad \forall i \in V_0 \quad (12)$$

$$0 \leq \lambda_p \leq 1 \quad \forall p \in P. \quad (13)$$

$$\max_{\pi} \quad z_{\text{DMP}}(\pi) = K\pi_0 + \sum_{i \in V_0} \pi_i \quad (14)$$

$$\pi_0 + \sum_{i \in V_0} a_{ip} \pi_i \leq c_p \quad \forall p \in P \quad (15)$$

$$\pi_0 \in \mathbb{R} \quad (16)$$

$$\pi_i \in \mathbb{R} \quad \forall i \in V_0, \quad (17)$$

where  $\pi_0 \in \mathbb{R}, \pi_i \in \mathbb{R} \quad \forall i \in V_0$  represents the dual variables associated respectively with constraints (11) and (12).

Due to the enormous size of the set of routes  $P$ , evaluating the dual variables  $\pi \in \mathbb{R}^N$  is computationally intractable. As a result, in BAP frameworks we consider only a small subset of columns  $\mathcal{P} \subseteq P$ :

$$\min_{\lambda} \quad z_{\text{RMP}}(\lambda) = \sum_{p \in \mathcal{P}} c_p \lambda_p \quad (18)$$

$$\sum_{p \in \mathcal{P}} \lambda_p = K \quad (19)$$

$$\sum_{p \in \mathcal{P}} a_{ip} \lambda_p = 1 \quad \forall i \in V_0 \quad (20)$$

$$\lambda_p \geq 0 \quad \forall p \in \mathcal{P}. \quad (21)$$

We look for a column to enter the basis of the RMP, which in turn necessitates the resolution of the following sub-problem:

$$c_p^* = \min_{p \in P} \left\{ \bar{c}_p = \sum_{e=\{i,j\} \in E} \left( c_e - \frac{\pi_i + \pi_j}{2} \right) a_{ep} \right\}, \quad (22)$$

which takes the name of *Pricing Problem* (PP). Any  $p \in P$  which satisfies  $c_p < 0$  is a valid column which can enter the basis of the RMP. The column generation procedure stops mainly under two scenarios: (i) when no more negative reduced cost routes exist, i.e. the PP outputs a  $p^* \in P$  achieving  $c_p^* \geq 0$  or (ii) the CG procedure tails off.

- **Relax elementarity condition** to make the PP solvable in pseudo-polynomial time:
  - $q$ -routes with 2-cycles elimination [46].
  - $q$ -routes with arbitrary  $k$ -cycles elimination [16, 47].
  - ng-routes [17].

$$\min_{x,y} \quad z_{\text{CPTP}}(x,y) = \sum_{e \in E} c_e x_e - \sum_{i \in V} p_i y_i \quad (23)$$

$$y_0 = 1 \quad (24)$$

$$\sum_{i \in V} q_i y_i \leq Q \quad (25)$$

$$\sum_{e \in \delta(i)} x_e = 2y_i \quad \forall i \in V \quad (26)$$

$$\sum_{e \in \delta(S)} x_e \geq 2y_i \quad \forall i \in S, \forall S \subseteq V_0, |S| \geq 2 \quad (27)$$

$$x_e \in \{0,1\} \quad \forall e \in E \quad (28)$$

$$y_i \in \{0,1\} \quad \forall i \in V. \quad (29)$$

*BaPCod* [45] is a software package developed in France at the Bordeaux University and Bordeaux Research Center that embeds a sophisticated column generation approach embedded in a generic and modern branch-price-and-cut (BPC) algorithm.

- Currently, state-of-the-art in solving routing problems.
- Takes as input a compact mixed-integer programming model and solves it using a Dantzig-Wolfe reformulation [48].
- Uses an automatic dual price smoothing stabilization [49].

The *VRPSolver* extension [11], is a *BaPCod* extension distributed by the same authors. This extension includes an advanced implementation of a bidirectional dynamic programming labeling algorithm [50] for solving the pricing problem. The included labeling algorithm can be used as an exact or heuristic pricer. The labeling algorithm contains two successively lighter heuristic implementations; for more information, see Sadykov et al. [50].

The *VRPSolver* extension, also, includes the implementation of some specific cutting planes and branching decisions aimed at efficiently solving routing-like problems (or problems that exhibit similar structures) such as the CVRP, VRPTW, and also others (see [11]).



- Disabled non-robust inequalities.
- Disabled multiple columns per pricing iteration.
- Increased ng-sets size and disable tailing off condition.
- Disabled any form of branching and cut-generation to stop the resolution process at the root node.



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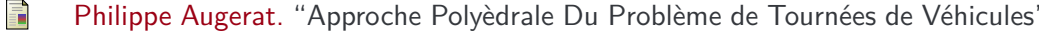


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