DS 5: Vector Spaces and Linear Independence

Philip Cherian

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1 Checking for Linear Independence

Recap: In order to show that some set of vectors (belonging to some vector space V) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ is linearly independent, one simply has to show that there is no linear combination that results in the additive identity (i.e. the "zero" element) of the vector space. In other words, that the only solution to

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \mathbf{0},$$

is when all the c_is are 0.

(a) Show that any set of vectors that contains the **0** element of the vector space is linearly independent.

(b) Determine whether the following set of vectors is linearly independent or linearly dependent. If the set is linearly dependent, express one of the vectors as a linear combination of the other.

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \\ 7 \\ 11 \end{pmatrix} \right\}$$

(c) Determine whether the following matrices are linearly dependent or linearly independent:

$$P = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \quad R = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \quad S = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$$

(d) For which value (or values) of a is the following set linearly independent?

$$S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ a \end{pmatrix}, \begin{pmatrix} a \\ 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ a^2 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 3 \\ a^3 \end{pmatrix} \right\}$$

(e) Determine whether the following functions (defined over all $x \in (-\infty, \infty)$) are linearly independent:

(i) $f_1(x) = e^x$, $f_2(x) = x^2 e^x$

(ii) $f_1(x) = x$, $f_2(x) = x + x^2$, $f_3(x) = 2x - x^2$,

(iii) $f_1(x) = \sin(x)$, $f_2(x) = \cos(x)$

(iv) $f_1(x) = x^2$, $f_2(x) = \begin{cases} 2x^2, & x \geq 0 \\ -x^2 & x < 0 \end{cases}$