DS 11: Motivating the Fourier Transform

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1 The complex Fourier Series representation

From our earlier classes, you should remember that if a function is periodic over some period T_0, i.e. $f(t+T_0) = f(t)$, then $f(t)$ could equivalently be represented by an infinite series of sines and cosines, called a **Fourier Series**:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(2\pi n \frac{t}{T_0}\right) + b_n \sin\left(2\pi n \frac{t}{T_0}\right)\right) \tag{1}$$

An equivalent, though somewhat neater, way to write this series is to use the fact that the sines and cosines can be expressed in terms of complex exponentials.

Exercise: Show that you can write

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{-in\omega_0 t}, \tag{2}$$

where $\omega_0 = 2\pi/T_0$, and c_n is some coefficient you need to find in terms of a_n and b_n.

Exercise: Show that $c_{-n} = c_n^\star$.

Exercise: Can you now see why in the original definition of the Fourier Series the constant term had an extra factor of $1/2$ that the other terms did not have?

Notice how all the information contained in the function $f(t)$, or rather, all the information required to reconstruct the function $f(t)$ is present in the infinite (but *countably* infinite) set of coefficients $\{c_n\}$. The components c_n tell us how much of the different frequencies (all of which are some integral multiple of ω_0) are present in the function $f(t)$. Thus, we can say that

$$\begin{aligned} f(t) \quad &\longrightarrow \quad \text{contains information in the “time” domain,} \\ c_n \quad &\longrightarrow \quad \text{contains information in the “frequency” domain,} \end{aligned}$$

The coefficients c_n can be obtained from the function $f(t)$ by performing the integral:

$$c_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t) e^{in\omega_0 t}\,\mathrm{d}t \tag{3}$$

These are obtained in more or less exactly the same way that the coefficients a_n and b_n are obtained in the usual definition of the Fourier Series.

Exercise: Show that the functions $e^{in\omega_0 t}$ are orthogonal over a period of $T_0 = 2\pi/\omega_0$. i.e., show that:

$$\int_{-T_0/2}^{T_0/2} e^{im\omega_0 t} e^{in\omega_0 t} \mathrm{d}t = T_0\, \delta_{mn} \tag{4}$$

2 Extending our definition to non-periodic functions

Notice how the above definition only works for periodic functions. Suppose, instead, that we don't have a periodic function, but we are only interested in the function between two points, say, $(-T_0/2, T_0/2)$. We can use a trick: the idea is that we imagine that the function is periodic, with period T_0. Notice how it's not the same function, but both functions overlap over the interval of interest. We have thus constructed a periodic function over all space, but we're only interested in a small part, from $-T_0/2$ to $T_0/2$. We can of course get the Fourier components for this function, provided we understand that they are only to be used to approximate the function $f(t)$ in the appropriate regime.

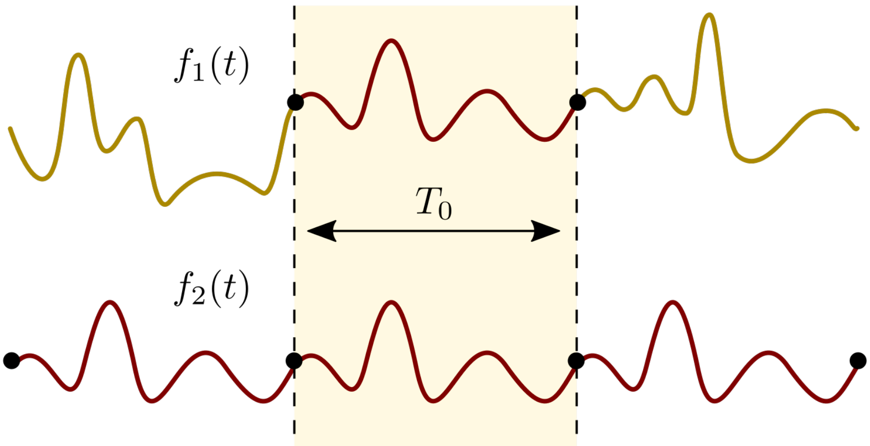

Figure 1: Two functions that are not equal in general, but are equal in an interval of width T_0.

Thus, over the interval T_0, we can express $f(t)$ as a sum over sinusoidal functions with frequencies that are integral multiples of a smallest frequency ω_0, when

$$\omega_0 = \frac{2\pi}{T_0}. \tag{5}$$

Of course, since the function is inherently non-periodic, there is nothing special about the value of T_0 that we chose. We could always have defined a different interval (say, $T_1 > T_0$). In this way, we could re-express $f(t)$ between $-T_1/2$ and $T_1/2$ as a combination of sinusoidal functions with a smaller fundamental frequency,

$$\omega_1 = \frac{2\pi}{T_1}. \tag{6}$$

But can we use this technique to completely describe an arbitrarily long signal, i.e. a purely non-periodic function. We can think of such a function as have a period of $T_0 \to \infty$. We will need to take this limit very carefully, to remain consistent. First, we notice that as T_0 increases, the fundamental frequency ω_0 decreases. Indeed, this is a reflection of the fact that the spacing between the different frequencies required to describe the function goes to zero. Therefore, the discrete variable $\omega_n = n\omega_0$ now goes to a *continuous* variable, ω, since the separation between subsequent values goes to zero. Furthermore, since the frequencies ω_n are now continuous, the discrete Fourier coefficients must *also* be continuous. Thus, to summarise, in this limit:

$$\begin{aligned}
T_0 &\to \infty && \text{(an aperiodic function has an infinite period)}\\
\Delta\omega \equiv \omega_0 &\to \mathrm{d}\omega && \text{(an infinitesimal quantity)}\\
n\Delta\omega &\to \omega && \text{(a continuous variable)}\\
c_n &\to c(\omega) && \text{(a function of a continuous variable)}
\end{aligned}$$

Given our discussion above, we would expect that the discrete Fourier sum must now be replaced by an integral. We can do this by writing $f(t)$ as a limit of a sum, and taking the limit $T_0 \to \infty$, i.e.

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{-in\omega_0 t} = \frac{1}{\Delta\omega}\sum_{n=-\infty}^{\infty} c_n e^{-in\omega_0 t}\Delta\omega.$$

Using the above scheme, we now have two equations for $f(t)$ and $c(\omega)$ that look more symmetric:

$$\begin{aligned}
f(t) &= \frac{1}{\Delta\omega}\sum_{n=-\infty}^{\infty} c_n e^{-in\omega_0 t}\Delta\omega &&\xrightarrow{T_0\to\infty} \frac{1}{\Delta\omega}\int_{-\infty}^{\infty} c(\omega)e^{-i\omega t}\mathrm{d}\omega,\\
c(\omega) &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)e^{in\omega_0 t}\mathrm{d}t &&\xrightarrow{T_0\to\infty} \frac{1}{T_0}\int_{-\infty}^{\infty} f(t)e^{i\omega t}\mathrm{d}t.
\end{aligned}\tag{7}$$

However, the astute among you would be right to be concerned: the above equations do look a little bogus; the biggest problems with them have been highlighted in red: the constant terms in front of the integrals both behave undesirably as $T_0 \to \infty$. In the first case, $\Delta\omega \to 0$, as so the constant term blows up, while in the second case, the $1/T_0$ term takes the right-hand-side to 0. So while this idea nearly seems to work, it doesn't *quite* work yet. The trick is to realise that while $\Delta\omega$ and T_0 go to 0 and ∞ respectively, they do so at the same rate, such that $T_0 \times \Delta\omega = 2\pi$. This seems to indicate that $c(\omega)$ might not be the ideal function to work with when $\Delta\omega \to 0$. Thus, the solution is to redefine things and work instead with

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}\mathrm{d}t \tag{8}$$

Exercise: Using the above definition, show that $f(t)$ can be written in terms of $F(\omega)$ as

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega t}\mathrm{d}\omega. \tag{9}$$

As you can see – using $F(\omega)$ – all the undesirable behaviour of both functions seem to have been dealt with. The function $F(\omega)$ is called the **Fourier Transform** of $f(t)$: it takes $f(t)$ – a function in the "time" domain – and produces a function in the *frequency* domain ω. You should, in a coarse sense, consider $F(\omega)$ to play the same role that the Fourier Coefficients did in the case of periodic functions: it is a measure of "how much" of the frequency ω the function is composed of. Similarly, $f(t)$ is called the *inverse* Fourier Transform of $F(\omega)$.

Now, for all of this to make any sense at all, we require that the inverse Fourier Transform of the Fourier Transform of $f(t)$ just give back $f(t)$ again, since this is just a basic consistency requirement. Thus, we require

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, e^{i\omega t} \underbrace{\int_{-\infty}^{\infty} \mathrm{d}t'\, f(t') e^{-i\omega t'}}_{F(\omega)}. \tag{10}$$

The above condition is known as the Fourier Integral Theorem. We can prove it as follows: we first rearrange the terms in this integral (since we're assuming everything converges, and so on[1]) and get

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}t'\, f(t') \int_{-\infty}^{\infty} \mathrm{d}\omega\; e^{i\omega(t-t')}. \tag{11}$$

Now, we perform the second integral on the right-hand side. In order to do this, we will need a result that you will prove in this week's assignment:

$$\int_{-\infty}^{\infty} \mathrm{d}\omega\; e^{i\omega(t-t')} = 2\pi\delta(t-t'), \tag{12}$$

where $\delta(\tau)$ is the Dirac Delta function. As a result, we can show that Equation (11) is indeed consistent.

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}t'\, f(t') \times 2\pi\delta(t-t') = f(t), \tag{13}$$

3 Problems:

(a) Prove the Fourier Shift Theorem, i.e. if we denote the Fourier Transform of a function $f(t)$ by $\mathscr{F}\Big[f(t)\Big](\omega)$, prove that

$$\mathscr{F}\Big[f(t-t_0)\Big](\omega) = e^{i\omega t_0}\mathscr{F}\Big[f(t-t_0)\Big](\omega) = e^{i\omega t_0}F(\omega). \tag{14}$$

(b) Prove the Convolution Theorem. Defining the convolution of two functions $f(t)$ and $g(t)$ as being

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(t) g(u-t)\mathrm{d}t, \tag{15}$$

show that:

$$\mathscr{F}\Big[f(t) * g(t)\Big] = F(\omega)G(\omega). \tag{16}$$

(c) Prove Parseval's Theorem for Fourier Transforms:

$$\int_{-\infty}^{\infty} \big|f(t)\big|^2 \mathrm{d}t = \int_{-\infty}^{\infty} |F(\omega)|^2 \mathrm{d}\omega \tag{17}$$

(d) Prove the Differentiation Theorem for the Fourier Transform

$$\mathscr{F}\Big[f'(t)\Big] = -i\omega F(\omega), \tag{18}$$

and generalise this to find $\mathscr{F}\Big[f^{(n)}(t)\Big]$ in terms of $F(\omega)$.

[1] The more mathematically minded of you would be right to be wary of the reckless abandon with which this was done. I'm sure there are many mathematical subtleties that I am sweeping under the rug, but they are not pertinent to our discussion here.