# Privacy in Statistics and Machine Learning — Spring 2021
# In-class Exercises for Lecture 15 (Multiplicative Weights)
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*Problems with marked with an asterisk* **(*)** *are more challenging or open-ended.*

1. **(Online Learning Requires Randomization)** Show that for every method $\mathcal{D}$ that plays deterministic actions (where $\mathbf{p}^t$ puts probability 1 on a single action) there is an adversary for which $\mathcal{D}$'s average regret is $\Omega(1)$.

2. **($\sqrt{\ln(k)/T}$ lower bound)** Show that for any method (randomized or not), if the adversary picks cost vectors unfiormly at random in $\{0,1\}^k$, the expected regret will be $\Theta\Big(\sqrt{\log(k)/T}\Big)$. (*Hint:* The expected cost paid by the algorithm is exactly $T/2$. Show that in hindsight, with probability at least $1/2$, one of the choices will have cost less than $\frac{T}{2} - \Omega(\sqrt{T\ln(k)})$.)

   This means that the guarantee obtained by multiplicative weights is tight, in general.

3. **(MW with a perfect action)** Suppose we know ahead of time that there is a perfect choice $a^*$ that always has cost 0. Show that we can set $\eta$ so that the algorithm achieves total cost at most $2\ln(k)$. (Be careful: our proof required $\eta$ to be at most $1/2$.)

4. **(*)** Show that if we know $OPT$ ahead of time, we can set $\eta$ to get an average cost of at most $\frac{OPT}{T} + \frac{2}{T}\sqrt{\ln(k)\cdot\max(OPT,\ln(k))}$.

5. Theorem 3.1 requires us to know the number of steps $T$ ahead of time. Show that one can modify the algorithm to adapt automatically to the length of the process. Specifically, there is a standard trick known as "repeated doubling": we start the algorithm assuming we will run for $T_0 = 4\ln(k)$ steps. If the number of steps exceeds $T_0$, we restart the algorithm assuming a length of $T_1 = 2T_0$. If the number of steps exceeds $T_0 + T_1$, we expand our time horizon to $T_2 = 2T_1$, and so on. Show that this variation achieves average regret $O\big(\sqrt{\ln(k)/T}\big)$ of $T$ (without knowing $T$).

6. How important is the fact that probabilities of higher-cost actions be selected with exponentially small probability? Consider an algorithm that, at each time $t$, selects action $a$ with probability that scales polynomially in its cost so far $c_a^{<t}$. Specifically, suppose $p_a^t = \frac{1}{1+c_a^{<t}}$. Show a sequence of cost vectors on which the algorithm has expected average regret at least $\Omega k/T$ (one can actually prove a stronger bound).