Final Exam Question 2(a) Devon DeBellis

Exercise 5.2

This page shows the derivation of an algorithm that, given a symmetric indefinite tridiagonal matrix A, computes $A = UDU^T$. The filled-out Figure 9 is on the next page

- Relative to the algorithm in Figure 8, which computes UDU^T factorization of a general symmetric matrix, we know if we further specify a tridiagonal A, we know we can exploit the banded structure.

- For tridiagonal A, at a typical point in the algorithm, we can partition:

$$A = \left(\begin{array}{ccc|c|ccc} \times & \times & & & & & \\ \times & \times & \times & & & 0 & \\ & \times & \times & \alpha_{MF} & & & \\ \hline & & \alpha_{MF} & \alpha_{MM} & \alpha_{LM} & & \\ \hline & & & \alpha_{LM} & \times & \times & \\ & 0 & & & \times & \times & \times \\ & & & & & \times & \times \end{array}\right) = \left(\begin{array}{c|c|c} A_{FF} & \alpha_{MF} e_L & 0 \\ \hline \alpha_{MF} e_L^T & \alpha_{MM} & \alpha_{LM} e_F^T \\ \hline 0 & \alpha_{LM} e_F & A_{LL} \end{array}\right)$$

- to expose parts needed for computation, re-partition again:

$$\left(\begin{array}{c|c|c} A_{FF} & \alpha_{FM} e_L & 0 \\ * & \alpha_{MM} & \alpha_{ML} e_F^T \\ * & * & A_{LL} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c|c} A_{00} & a_{01} e_L & 0 & 0 \\ * & \alpha_{11} & \alpha_{12} & 0 \\ \hline * & * & \alpha_{22} & \alpha_{23} e_F^T \\ \hline * & * & * & A_{33} \end{array}\right)$$

and then further re-partition A_{00}, knowing we need to build up from the lower-right corner of A and work towards the top-left:

$$A_{00} = \left(\begin{array}{c|c} * & * \\ \hline * & \alpha_{00} \end{array}\right)$$

- the above allows the insight that, referencing Figure 8, a_{01} for a tridiagonal A is given by $a_{01} = \left(\frac{0}{\alpha_{01}}\right)$ s.t., for example, $a_{01} := a_{01}/\alpha_{11}$ becomes

$$\left(\frac{0}{\alpha_{01}}\right) := \left(\frac{0}{\alpha_{01}}\right) / \alpha_{11} = \left(\frac{0}{\alpha_{01}/\alpha_{11}}\right)$$

then: $A_{00} := A_{00} - \alpha_{11} a_{01} a_{01}^T$ becomes

$$\left(\begin{array}{c|c} * & * \\ \hline * & \alpha_{00} \end{array}\right) := \left(\begin{array}{c|c} * & * \\ \hline * & \alpha_{00} \end{array}\right) - \alpha_{11} \left(\frac{0}{\alpha_{01}}\right) \left(\frac{0}{\alpha_{01}}\right)^T = \left(\begin{array}{c|c} * & * \\ \hline * & \alpha_{00} - \alpha_{11} \alpha_{01}^2 \end{array}\right)$$

$\rightarrow$ the rest of the options follow intuitively from the above. Reference Figure 9 for the filled-out algorithm on the next page

for a quick example, consider $A = \begin{pmatrix} 5 & 2 & 0 \\ 2 & 3 & 3 \\ 0 & 3 & 2 \end{pmatrix} \rightarrow \left(\begin{array}{cc|c} 5 & 2 & 0 \\ 2 & 3 & 3 \\ \hline 0 & 3 & 2 \end{array}\right) \Rightarrow A_{00} = \begin{pmatrix} 5 & 2 \\ 2 & 3 \end{pmatrix} \Rightarrow \alpha_{00} = 3$, $\alpha_{01} = 3$, $\alpha_{11} = 2$ $\longrightarrow$

$\rightarrow$ update $\alpha_{01} := 3/2$, $\alpha_{00} := 3 - (2)(3/2)^2 = -3/2 \Rightarrow A = \left(\begin{array}{c|c|c} 5 & 2 & 0 \\ * & -3/2 & 3/2 \\ \hline * & * & 2 \end{array}\right) \Rightarrow A_{00} = 5 \Rightarrow \alpha_{00} = 5$, $\alpha_{01} = 2$, $\alpha_{11} = -3/2$ $\longrightarrow$

$\rightarrow$ update $\alpha_{01} := 2 \cdot -\frac{2}{3} = -\frac{4}{3}$, $\alpha_{00} := 5 - (-\frac{3}{2})(-\frac{4}{3})^2 = \frac{23}{3} \Rightarrow A = \begin{pmatrix} 23/3 & -4/3 & 0 \\ * & -3/2 & 3/2 \\ * & * & 2 \end{pmatrix}$ $\longrightarrow$

$\rightarrow$ note that $\begin{pmatrix} 1 & -4/3 & 0 \\ * & 1 & 3/2 \\ * & * & 1 \end{pmatrix} \begin{pmatrix} 23/3 & 0 & 0 \\ 0 & -3/2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & -4/3 & 0 \\ * & 1 & 3/2 \\ * & * & 1 \end{pmatrix}^T = A \rightarrow$ done ✓

Figure 9 next page $\rightarrow$

question 2(a) continued

Algorithm: $A := \text{UDUT\_TRI}(A)$

Partition $A \rightarrow \left(\begin{array}{c|c|c} A_{FF} & \alpha_{FM} e_L^T & 0 \\ \hline \star & \alpha_{MM} & \alpha_{ML} e_F^T \\ \hline \star & \star & A_{LL} \end{array}\right)$

where A_{LL} is 0×0

while $m(A_{LL}) < m(A)$ **do**

$A_{00} = \left(\begin{array}{c|c} * & * \\ \hline * & \alpha_{00} \end{array}\right)$

Repartition

$$\left(\begin{array}{c|c|c} A_{FF} & \alpha_{FM} e_L & 0 \\ \hline \star & \alpha_{MM} & \alpha_{ML} e_F^T \\ \hline \star & \star & A_{LL} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c|c} A_{00} & \alpha_{01} e_L & 0 & 0 \\ \hline \star & \alpha_{11} & \alpha_{12} & 0 \\ \hline \star & \star & \alpha_{22} & \alpha_{23} e_F^T \\ \hline \star & \star & \star & A_{33} \end{array}\right) = \left(\begin{array}{cc|c|c|c} * & * & \alpha_{01} e_L & 0 & 0 \\ * & \alpha_{00} & & & \\ \hline * & & \alpha_{11} & \alpha_{12} & 0 \\ \hline * & & * & \alpha_{22} & \alpha_{23} e_F^T \\ \hline * & & * & * & A_{33} \end{array}\right)$$

where α_{00} is scalar

option 1	option 2	option 3
$\lambda_{01} := \alpha_{01} / \alpha_{11}$	$\alpha_{01} := \alpha_{01} / \alpha_{11}$	$\alpha_{00} := \alpha_{00} - \frac{1}{\alpha_{11}} \alpha_{01}^2$
$\alpha_{00} := \alpha_{00} - \lambda_{01} \alpha_{01}$ (updating upper triangle)	$\alpha_{00} := \alpha_{00} - \alpha_{11} \alpha_{01}^2$ (updating upper triangle)	(updating upper triangle)
$\alpha_{01} := \lambda_{01}$		$\alpha_{01} := \alpha_{01} / \alpha_{11}$

Continue with

$A_{00} = \left(\begin{array}{c|c} * & * \\ \hline * & \alpha_{00} \end{array}\right)$

$$\left(\begin{array}{c|c|c} A_{FF} & \alpha_{FM} e_L & 0 \\ \hline \star & \alpha_{MM} & \alpha_{ML} e_F^T \\ \hline \star & \star & A_{LL} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c|c} A_{00} & \alpha_{01} e_L & 0 & 0 \\ \hline \star & \alpha_{11} & \alpha_{12} & 0 \\ \hline \star & \star & \alpha_{22} & \alpha_{23} e_F^T \\ \hline \star & \star & \star & A_{33} \end{array}\right) = \left(\begin{array}{cc|c|c|c} * & * & \alpha_{01} e_L & 0 & 0 \\ * & \alpha_{00} & & & \\ \hline * & & \alpha_{11} & \alpha_{12} & 0 \\ \hline * & & * & \alpha_{22} & \alpha_{23} e_F^T \\ \hline * & & * & * & A_{33} \end{array}\right)$$

endwhile

Figure 9: Algorithm for computing the the UDU^T factorization of a tridiagonal matrix.