Final Exam Question 2(b) Devon DeBaisi

Exercise 6.1 WTS $\phi_1 = \delta_1 + \Sigma_1 - \alpha_{11}$ given $A, L, U, D,$ and E from the problem statement

(a)

Approach: from section 6.2, we assume we are given the $A = LDL^T$ and $A = UEU^T$ factorizations of A with the provided block formatting of $A, L, U, D,$ and E. We will multiply out $A = LDL^T$ and $A = UEU^T$, then compare with the blocked factored equation in the problem statement

i) $A = LDL^T = \left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline \lambda_{10}e_L^T & 1 & 0 \\ \hline 0 & \lambda_{21}e_F & L_{22} \end{array}\right)\left(\begin{array}{c|c|c} D_{00} & 0 & 0 \\ \hline 0 & \delta_1 & 0 \\ \hline 0 & 0 & D_{22} \end{array}\right) L^T = \left(\begin{array}{c|c|c} L_{00}D_{00} & 0 & 0 \\ \hline \lambda_{10}e_L^T D_{00} & \delta_1 & 0 \\ \hline 0 & \lambda_{21}e_F\delta_1 & L_{22}D_{22} \end{array}\right)\left(\begin{array}{c|c|c} L_{00} & \lambda_{10}e_L^T & 0 \\ \hline 0 & 1 & \lambda_{21}e_F \\ \hline 0 & 0 & L_{22} \end{array}\right)$

$= \left(\begin{array}{c|c|c} L_{00}D_{00}L_{00} & L_{00}D_{00}\lambda_{10}e_L^T & 0 \\ \hline \lambda_{10}e_L^T D_{00}L_{00} & \lambda_{10}e_L^T D_{00}\lambda_{10}e_L^T + \delta_1 & \delta_1\lambda_{21}e_F \\ \hline 0 & \lambda_{21}e_F\delta_1 & \lambda_{21}e_F\delta_1\lambda_{21}e_F + L_{22}D_{22}L_{22} \end{array}\right)$

ii) $A = UEU^T = \left(\begin{array}{c|c|c} U_{00} & v_{01}e_L & 0 \\ \hline 0 & 1 & v_{12}e_F^T \\ \hline 0 & 0 & U_{22} \end{array}\right)\left(\begin{array}{c|c|c} E_{00} & 0 & 0 \\ \hline 0 & \Sigma_1 & 0 \\ \hline 0 & 0 & E_{22} \end{array}\right) U^T = \left(\begin{array}{c|c|c} U_{00}E_{00} & v_{01}e_L\Sigma_1 & 0 \\ \hline 0 & \Sigma_1 & v_{12}e_F^T E_{22} \\ \hline 0 & 0 & U_{22}E_{22} \end{array}\right)\left(\begin{array}{c|c|c} U_{00} & 0 & 0 \\ \hline v_{01}e_L & 1 & 0 \\ \hline 0 & v_{12}e_F^T & U_{22} \end{array}\right)$

$= \left(\begin{array}{c|c|c} U_{00}E_{00}U_{00} + v_{01}e_L\Sigma_1 v_{01}e_L & v_{01}e_L\Sigma_1 & 0 \\ \hline \Sigma_1 v_{01}e_L & \Sigma_1 + v_{12}e_F^T E_{22}v_{12}e_F^T & v_{12}e_F^T E_{22}U_{22} \\ \hline 0 & U_{22}E_{22}v_{12}e_F^T & U_{22}E_{22}U_{22} \end{array}\right)$

iii) given $\underbrace{\left(\begin{array}{c|c|c} A_{00} & \alpha_{10}e_L & 0 \\ \hline \alpha_{10}e_L^T & \alpha_{11} & \alpha_{21}e_F^T \\ \hline 0 & \alpha_{21}e_F & A_{22} \end{array}\right)}_{A} = \left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline \lambda_{10}e_L^T & 1 & v_{12}e_F^T \\ \hline 0 & 0 & U_{22} \end{array}\right)\left(\begin{array}{c|c|c} D_{00} & 0 & 0 \\ \hline 0 & \Phi_1 & 0 \\ \hline 0 & 0 & E_{22} \end{array}\right)\left(\begin{array}{ccc} L_{00} & 0 & 0 \\ \lambda_{10}e_L^T & 1 & v_{12}e_F^T \\ 0 & 0 & U_{22} \end{array}\right)^T$

$= \left(\begin{array}{c|c|c} L_{00}D_{00} & 0 & 0 \\ \hline \lambda_{10}e_L^T D_{00} & \phi_1 & v_{12}e_F^T E_{22} \\ \hline 0 & 0 & U_{22}E_{22} \end{array}\right)\left(\begin{array}{c|c|c} L_{00} & \lambda_{10}e_L^T & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & v_{12}e_F^T & U_{22} \end{array}\right)$

$= \left(\begin{array}{c|c|c} L_{00}D_{00}L_{00} & L_{00}D_{00}\lambda_{10}e_L^T & 0 \\ \hline \lambda_{10}e_L^T D_{00}L_{00} & \lambda_{10}e_L^T D_{00}\lambda_{10}e_L^T + \phi + v_{12}e_F^T E_{22}v_{12}e_F^T & v_{12}e_F^T E_{22}U_{22} \\ \hline 0 & U_{22}E_{22}v_{12}e_F^T & U_{22}E_{22}U_{22} \end{array}\right)$

→ We observe:
- the leftmost vertical partition from iii) ≡ the leftmost vertical partition from i)
- the rightmost vertical partition from iii) ≡ the rightmost vertical partition from ii)
- $\left(\begin{array}{c|c|c} & * & \\ \hline & & \\ \hline & & \end{array}\right)$ from iii) ≡ $\left(\begin{array}{c|c|c} & * & \\ \hline & & \\ \hline & & \end{array}\right)$ from i)
- $\left(\begin{array}{c|c|c} & & \\ \hline & & \\ \hline & * & \end{array}\right)$ from iii) ≡ $\left(\begin{array}{c|c|c} & & \\ \hline & & \\ \hline & * & \end{array}\right)$ from ii)

→ this is all intuitive, as all 3 expressions = A. All that is left is $\left(\begin{array}{c|c|c} & & \\ \hline & * & \\ \hline & & \end{array}\right)$. Knowing i), ii), iii) all $= A$, we have:

(A)

$$\alpha_{11} = \underbrace{\lambda_{10}e_L^T D_{00}\lambda_{10}e_L^T + \delta_1}_{(i)_{11}} = \underbrace{\Sigma_1 + v_{12}e_F^T E_{22}v_{12}e_F^T}_{(ii)_{11}} = \underbrace{\lambda_{10}e_L^T D_{00}\lambda_{10}e_L^T + \phi + v_{12}e_F^T E_{22}v_{12}e_F^T}_{(iii)_{11}}$$

$\Rightarrow \alpha_{11} - \lambda_{10}e_L^T D_{00}\lambda_{10}e_L^T = \delta_1 \Rightarrow \delta_1 = \phi + v_{12}e_F^T E_{22}v_{12}e_F^T = \phi + (\alpha_{11} - \Sigma_1)$

$\therefore \Rightarrow \boxed{\phi = \delta_1 + \Sigma_1 - \alpha_{11}}$

(cost analysis next page) →

Final Exam Question 2(b)

[b] Cost analysis

- Cost of computing **one** twisted factorization given we have already computed the LDL^T and UEU^T factorizations, and know the full structure of A per piazza @1158

  - We know from the document and piazza @1098 that each twisted factorization starts w/ an assumption about **where** we twist the matrix, so we start knowing:

$$A = \left(\begin{array}{c|c|c} A_{00} & \alpha_{10} e_L & 0 \\ \hline \alpha_{10} e_L^T & \alpha_{11} & \alpha_{21} e_F^T \\ \hline 0 & \alpha_{21} e_F & A_{22} \end{array}\right), \quad D = \left(\begin{array}{c|c|c} D_{00} & 0 & 0 \\ \hline 0 & \delta_1 & 0 \\ \hline 0 & 0 & D_{22} \end{array}\right), \quad E = \left(\begin{array}{c|c|c} E_{00} & 0 & 0 \\ \hline 0 & \varepsilon_1 & 0 \\ \hline 0 & 0 & E_{22} \end{array}\right)$$

  and specifically know α_{11}, δ_1, and ε_1 b/c **we pick** the location of the twist

  - From the first part of this question, we know the only unknown for a given twisted factorization ϕ_1 can be computed as: $\phi_1 = \delta_1 + \varepsilon_1 - \alpha_{11}$

  - So, w/ the assumption we already know A, $A = LDL^T$, $A = UEU^T$, the cost of computing **one** twisted factorization is just the cost of computing the only unknown ϕ_1 which is 2 flops, so **O(1)**

- Now, we know for an $m \times m$ matrix A, the number of possible twisted factorizations is the number of diagonal elements: m. → confirmed w/ Piazza @1158
  So, w/ the same assumption as above that we already know A, $A = LDL^T$, and $A = UEU^T$, the cost of computing **all** twisted factorizations is given by the cost of computing m ϕ_1 values, so **O(m)**

note: per instructions, only counting floating point computations