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Exercise 7.1

i) want to compute x_0, χ_1, x_2 s.t.:

$$\left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline \lambda_{10} e_L^T & 1 & v_{12} e_F^T \\ \hline 0 & 0 & U_{22} \end{array}\right)\left(\begin{array}{c|c|c} D_{00} & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & E_{22} \end{array}\right)\left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline \lambda_{10} e_L^T & 1 & v_{12} e_F^T \\ \hline 0 & 0 & U_{22} \end{array}\right)^T \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right) = \left(\begin{array}{c} 0 \\ \hline 0 \\ \hline 0 \end{array}\right)$$

where $x = \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right) \neq 0$

$$\underbrace{\left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline \lambda_{10} e_L^T & 0 & v_{12} e_F^T \\ \hline 0 & 0 & U_{22} \end{array}\right)\left(\begin{array}{c|c|c} D_{00} & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & E_{22} \end{array}\right)}\underbrace{\left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline \lambda_{10} e_L^T & 1 & v_{12} e_F^T \\ \hline 0 & 0 & U_{22} \end{array}\right)^T \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right)}$$

$$= \left(\begin{array}{c|c|c} L_{00} D_{00} & 0 & 0 \\ \hline \lambda_{10} e_L^T D_{00} & 0 & v_{12} e_F^T E_{22} \\ \hline 0 & 0 & U_{22} E_{22} \end{array}\right) \left(\begin{array}{c} L_{00} x_0 + \lambda_{10} e_L^T \chi_1 \\ \hline \chi_1 \\ \hline v_{12} e_F^T \chi_1 + U_{22} x_2 \end{array}\right)$$

$$= \left(\begin{array}{c} (L_{00} D_{00})(L_{00} x_0 + \lambda_{10} e_L^T \chi_1) \\ \hline (\lambda_{10} e_L^T D_{00})(L_{00} x_0 + \lambda_{10} e_L^T \chi_1) + (v_{12} e_F^T E_{22})(v_{12} e_F^T \chi_1 + U_{22} x_2) \\ \hline (U_{22} E_{22})(v_{12} e_F^T \chi_1 + U_{22} x_2) \end{array}\right) \quad (*)$$

to get this to be $= \left(\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right)$ w/o having $x = 0$, we will choose (per given hint) s.t.:

$$\left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline \lambda_{10} e_L^T & 1 & v_{12} e_F^T \\ \hline 0 & 0 & U_{22} \end{array}\right)^T \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right) = \left(\begin{array}{c} L_{00} x_0 + \lambda_{10} e_L^T \chi_1 \\ \hline \chi_1 \\ \hline v_{12} e_F^T \chi_1 + U_{22} x_2 \end{array}\right) = \left(\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right)$$

$\Rightarrow$ we choose $\chi_1 = 1$, so:

a) $L_{00} x_0 + \lambda_{10} e_L^T \chi_1 = 0 \rightarrow L_{00} x_0 + \lambda_{10} e_L^T = 0 \Rightarrow x_0 = \text{BiSolve}(L_{00} x_0 = -\lambda_{10} e_L^T)$

b) $v_{12} e_F^T \chi_1 + U_{22} x_2 = 0 \rightarrow v_{12} e_F^T + U_{22} x_2 = 0 \Rightarrow x_2 = \text{BiSolve}(U_{22} x_2 = -v_{12} e_F^T)$

$$x = \left(\begin{array}{c} \text{BiSolve}(L_{00} x_0 = -\lambda_{10} e_L^T) \\ \hline 1 \\ \hline \text{BiSolve}(U_{22} x_2 = -v_{12} e_F^T) \end{array}\right) = \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right)$$

which clearly makes $(*) = 0$, where BiSolve is a scheme that exploits the bidiagonal structure of L_{00} and U_{22}, it explained on next page.

(cost analysis) next page

ignore below

ii) ~~want~~ to report cost of computing $x = \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right)$ given L_{00} and U_{22} have bidiagonal structure

We know both L_{00} and U_{22} are bidiagonal, therefore, to solve $x_0 = (L_{00}^{-1})(-\lambda_{10} e_L^T)$ and $x_2 = (U_{22}^{-1})(-v_{12} e_F^T)$: (no cost associated w/ $\chi_1 = 1$)

a) compute QR factorizations of L_{00} and U_{22} via implicitly shifted bidiagonal QR algorithm from section 11.2.4

b) Because $L_{00} = Q_L R_L$, $L_{00}^{-1} = R_L^{-1} Q_L^{-1} = R_L^{-1} Q^H$ and $U_{22} = Q_U R_U$, $U_{22}^{-1} = R_U^{-1} Q_U^{-1} = R_U^{-1} Q^H$ and we can ignore the cost of $Q \rightarrow Q^H$, so, the cost here comes from calculation of inverse of triangular matrix

c) multiply $R_L^{-1} Q_L^H$ and $R_U^{-1} Q_U^H$ to recover L_{00}^{-1} and U_{00}^{-1}

d) ~~multiply~~ L_{00}^{-1} and U_{00}^{-1} by scalars $\underbrace{(-\lambda_{10} e_L^T)}_{\text{flop, ignore}}$ and $\underbrace{(-v_{12} e_F^T)}_{\text{flop, ignore}}$ respectively.

Big O analysis of a)-d) follows on the next page

Final Exam Question 2(c)

Cost analysis

- to analyze the cost of computing X_i, we need to analyze the cost of solving 2 bidiagonal systems:

i) $L_{00} x_0 = -\lambda_{10} e_L^T$
ii) $U_{22} x_2 = -v_{12} e_F^T$

) we assume both L_{00} and U_{22} are $m \times m$

- we know i) looks like

$$\underbrace{\begin{pmatrix} 1 & & & & 0 \\ x & 1 & & & \\ & x & \ddots & & \\ & & \ddots & 1 & \\ 0 & & x & 1 & \\ & & & x & 1 \end{pmatrix}}_{\text{from section 4}} \begin{pmatrix} x_{0,0} \\ \vdots \\ x_{0,m-1} \end{pmatrix} = -\lambda_{10} e_L^T = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\lambda_{10} \end{pmatrix}$$

→ clearly, here $x_{0,m-1} = -\lambda_{10}$

→ Clearly, we can construct a simple looping scheme that repartitions L_{00} s.t. we solve for $x_{0,k}$ @ the k^{th} iteration, in a forward substitution scheme similar to that detailed in section 5.3.5. Each iteration only has 1 flop, to solve for $x_{0,k}$, but we need m iterations, making the cost of solving for x_0 $O(m)$

Figure 5.3.5.1 but we know only one multiplication needed here

- very similarly, we know ii) looks like

$$\begin{pmatrix} 1 & x & & 0 \\ & 1 & x & \\ & & \ddots & \ddots \\ & & & 1 & x \\ 0 & & & & 1 \end{pmatrix} \begin{pmatrix} x_{2,0} \\ \vdots \\ x_{0,m-1} \end{pmatrix} = -v_{12} e_F^T = \begin{pmatrix} -v_{12} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

→ clearly, here $x_{2,0} = -v_{12}$

→ as above, we can clearly construct a simple looping scheme that repartitions U_{22} s.t. we solve for $x_{2,k}$ @ the k^{th} iteration in a backward substitution scheme similar to that detailed in section 5.3.5. Each iteration has 1 flop to solve for $x_{2,k}$ but we need m iterations, making the cost of solving for x_2 $O(m)$

Homework 5.3.5.2 but we know only one multiplication is needed here

So, from the above, we determine the cost of solving for X_i, one approximate eigenvector, to be $O(m)$. To solve for all m approximate eigenvectors, the cost becomes $O(m^2)$. This is consistent with the summary given in section 1. of this document