**Powell, M.S. , Weerasekara, N.S. , Cole, S., LaRoche, R.D. , Favier, J. DEM modeling of liner evolution and its influence on grinding rate in ball mills. Minerals Engineering 24 (2011), 341-351**

This paper is from EDEM group. It describes with detail the EDEM model, and how the slippage condition is modeled as , so .

Also presents a wear model.

**Tuley, R., Danby, M., Shrimpton, J., Palmer, M. On the optimal numerical time integration for Lagrangian DEM within implicit flow solvers. Computers and Chemical Engineering 34 (2010), 886-899.**

For powder flows in which a fluid particle coupling exists, there can be defined micro-scale, meso-scale and macro-scale techniques for numerically predict its behavior.

It covers aspects of numerical integration methods and errors associated with them.

The test case is a binary collision between two particles, one initially stationary and the other with a certain velocity.

It mentions to Kruggel-Emden 2007, and 2008 as the basis for their model.

**Xu, B.H., Yu, A.B. Numerical simulation of the gas-solid flow in a fluidized bed by combining discrete particle method with computational fluid dynamics. Chemical Engineering Science 52 (1997), 2785-2809.**

Here as in the original Cundall and Strack paper, clearly states that the slipping condition is:

**Geng, Y., Che, D. An extended DEM-CFD model for char combustion in a bubbling fluidized bed combustor of inert sand. Chemical Engineering Science 66 (2011), 207-219.**

Also, they cite and uses the model as in Cundall and Strack, Tsuji et al 1993, and Xu and Yu 1997.

**Siiriä, S., Yliruusi, J. DEM simulation of influence of parameters on energy and degree of mixing. Particuology 9 (2011), 406-413.**

It refers to another paper by the same authors in 2007. Also refers to the review made by Zhu, Zhou, Yang & Yu 2007 to introduce different methods for simulating powder behavior including the DEM method.

**Zhu, H.P., Zhou, Z.Y., Yang, R.Y. Discrete particle simulation of particulate systems: Theoretical developments. Chemical Engineering Science 62 (2007), 3378-3396.**

A general review about DEM and CFDEM. It includes review about non-contact forces (van der Waals, and electrostatic forces), liquid bridge between particles, as well as different contact forces models. They include the Langston method as an alternative to a full Hertz-Mindlin model.

They use a Langston modified model, but as it appears in the table 1, the slipping is handled in an equivalent way as Cundall and Strack although in a very different form (it’s a hysteretic method).

**Siiriä, S., Yliruusi, J. Particle packing simulations based on Newtonian mechanics. Powder Technology 174 (2007), 82-92.**

They use some kind of a hysteretic model in normal direction. It is given by: . When particles approach, but as <0, then adopts a value less than unity. So the hysteretic behavior occurs instantaneously.

They introduce the energy balance equation, and consider the energy error. First the total energy inside the simulation is , gravitational potential, plus kinetic (linear and rotational), and spring potential energy terms. There is also the Energy lost (dissipated) in the collisions , and the error associated to numerical and model inaccuracies, so the energy is not conserved. Then is

**Maw, N., Barber, J.R., Fawcett, J.N. The oblique impact of elastic spheres. Wear 38 (1976), 101-114**

This paper handles non-collinear impact of spheres. Mindlin and Deresevicz stated the theory of non-linear solution for tangential forces, in the same way than Hertz did for the normal forces. This solution is based in normal and tangential displacements independently applied (first one, and then the other). Nevertheless it’s very important to state that always there is some kind of slipping. When particles are separating always some dissipation of energy due to micro slipping occurs. As the particles separate, the area of contact is diminishing, and then in an infinitesimal area the normal force is approaching to zero, so the slipping occurs. In this way, before the particles lose contact, the accumulated tangential strain is released with frictional slippage. Three regimes exist: i) initial stick, and then micro-slipping; ii) initial gross slip, then stick, and then micro-slipping; iii) gross slip during whole collision.

**Maw, N., Barber, J.R., Fawcett, J.N. The rebound of elastic bodies in oblique impact. Mech. Res. Comm. 4 (1977) , 17-22**

Normal frequency is different from Tangential frequency. Here the definition for is given

**Maw, N., Barber, J.R., Fawcett, J.N. The Role of Elastic Tangential compliance in Oblique Impact. Journal of Lubrication Technology 103 (1981), 101-114**

Elastic tangential compliance. The model has 5 hypothesis.

1. Area of contact << D
2. Static coefficient of friccion equal to the dynamic coefficient of friction
3. The material has a linear response
4. Hertz quasi-static regime
5. Tangential traction axysimmetric

It’s shown that the behavior of steel and rubber is mostly the same, but the curves are displaced and with different values.

**Cundall, P.A., Strack, O.D.L. A discrete numerical model for granular assemblies. Géotechnique 29 (1979), 47-65.**

Basic paper in which the DEM method is presented. The model is in normal direction: linear spring + viscosity force. In tangential direction is also linear spring + viscosity force. The slipping condition is handled as: .

**Walton, O.R., Braun, R.L. Viscosity, Granular-Temperature, and Stress Calculations for Shearing Assemblies of Inelastic, Frictional Disks. Journal of Rheology 30 (1986), 949-980.**

This is a basic paper in which the method of Walton and Braun is presented. Is an hysteretic model in normal direction and in tangential direction. The tangential direction force model is based in a simplified Mindlin-Deresewicz theory. The limit of the tangential force is , so it would be equivalent to .

**Zhou, Y.C., Wright, B.D., Yang, R.Y., Xu, B.H., Yu, A.B. Rolling friction in the dynamic simulation of sandpile formation. Physica A 269 (1999), 536-553.**

They use the Langston model, but in this paper they add two different rolling friction models. They get better agreement with the angular velocity independent model. The angular velocity dependent friction model has worse experimental agreement.

This phrase is very good:

The interaction between particle and fluid and the long-range forces, such as van der Waals and electrostatic forces, can be ignored in the present work which deals with large particles in static, low-viscosity fluid.

**Brillantov, N.V., Pöschel, T. Rolling friction of a viscous sphere on a hard plane. Europhysics Letters 42 (1998), 511-516.**

**Brillantov, N.V., Spahn F., Hertzsch J.M., Pöschel T. Model for collisions in granular gases. Physical Review E 53 (1996), 5382-5392.**

This second reference (1996) is the one in which all the ideas come from.

They give a quasi-static approximation for the visco-elastic sphere. For this approximation to be true the condition is that the time of collission dissipative relaxation times (I do not have an estimation for this time). For the elastic part to be modeled as quasi-static Hertz approximation, the characteristic velocity has to be much more smaller than the sound velocity in the solid (this is very easy).

One idea is that there is a quasi-stationary state sequence, where the elastic part rules the friction limit to slippage. There is a instantaneous dissipation, so the dissipation force is used to decelerate the solid (macroscopic solid), but immediately is dissipated so in the interface there is only the elastic part interaction. In the case of a restoring force, then only the elastic part should be used to accelerate/decelerate the macroscopic solid.

They cite to forester, bridges (nature) and Goldstein (Impact book).

This paper is an example of a viscoelastic model in normal direction. The dependence of the viscous term is , as opposed as the liggghts dependence of

Also from the equations, it can be seen that they take as friction limit the value given by , and always is applied the viscous force

**Langston P.A., Tüzün U., Continuous potential discrete particle simulations of stress and velocity fields in hoppers: transition from fluid to granular flow. Chemical Engineering Science 49 (1994), 1259-1275.**

Here is defined the normal interaction as a continuous potential, and the tangential interaction as the model of Langston . This expression will be optimized in next papers.

The way in which the friction limit is handled is seen better in the next paper.

**Langston P.A., Tüzün U., Heyes D.M., Discrete element simulation of granular flow in 2D and 3D hoppers: dependence of discharge rate and wall stress on particle interactions. Chemical Engineering Science 50 (1995), 967-987.**

Here it’s specified more in deep the normal and tangential models of interactions. For normal interaction they use linear model, continuous potential, and Hertz model. For the tangential force they improve their model, and also it is clearly stated that they use , and , being only the elastic part. So the limit is always , and always is applied the viscous part to both normal and tangential forces.

**Foerster S.F., Louge M.Y., Chang H., Allia K., Measurements of the collision properties of small spheres. Phys. Fluids 4 (1994), 1108-1115.**

Experimental values. It states the dependence of the normal restitution coefficient with velocity as: nearly constant at low velocities, but as fully plastic deformations occur at large velocities, it may decrease like .

It is also worth that in the graphs representing the non dimensional tangential velocity after collision, as function of the initial non dimensional tangential velocity, in the regions of small , the value of seems to be always negative. This is an obscure point of the Maw model, as in one of their papers they show this value also as negative.

**Schäfer J., Dippel S., Wolf D.E., Force Schemes in Simulations of Granular Materials. J. Phys. I France 6 (1996), 5-20**

It points out that there is a plastic deformation velocity. . It also relates that for Hertz normal interaction model, sometimes a viscous dissipation is introduced “Ad-hoc”, but if the coefficient is constant (dashpot), the restitution coefficient grows with impact velocity, and the experiments shows that this is not the case, but it goes as . Then the model of Kubara & Kono, as well as the proposed by Brilliantov gave similar results.

It also defines the total coefficient of restitution as

**Kuwara G., Kono K., Restitution Coeffcient in a Collision between Two Spheres. Japanese Journal of Applied Physics 26 (1987), 1230-1233.**

Here also it’s deduced the dependence . The variation of the restitution coefficient is very low for steel balls. Only for brass like materials has a big dependence.



**Johnson D.L., Norris A.N., Rough elastic spheres in contact: memory effects and the transverse force. J. Mech. Phys. Solids 45 (1997), 1025-1036.**

It analyzes the path dependent tangential force. Also points out that the hertz model should be numerically formulated as a incremental method, and not as a integral model. It also points out that there is a work done by tangential forces that is path dependent.

**Brendel L., Dippel S. Lasting Contacts in Molecular Dynamics Simulations. In: Herrmann, H.J., Hovi, J.-P., Luding, S. (Eds.), Physics of Dry Granular Media. Kluwer Academic Publishers. (1998), p. 313.**

They show that the model for tangential elongation calculation presented in Cundall & Strack, is not correct. It shall be cut up to the value in which .

**Zhang D., Whiten W.J., An efficient calculation method for particle motion in discrete element simulations. Powder Technology 98 (1998), 223-230.**

It’s almost no use. Only cite to Tsuji 1992, and express the dissipation force as

**Tsuji Y., Tanaka T., Ishida T. Lagrangian numerical simulation of plug flow of cohesionless particles in a horizontal pipe. Powder Technology 71 (1992) 239-250.**

The first one in propose a viscoelastic damping model with constant normal coefficient of restitution. It’s derived heuristically from the equations of motion, and looking for a restitution coefficient that only depends on a constant . Also is probably the first in which the stiffness in Hertz model is defined from material constants, and the tangential stiffness is deduced from Mindlin-Deresewicz theory.

**Vu-Quoc L., Zhang X., An accurate and efficient tangential force-displacement model for elastic frictional contact in particle flow simulations. Mechanics of Materials 31 (1999) 235-269.**

A very complex model trying to cover aspects of Mindlin-Deresewicz theory. Its application it’s limited to simple-loading.

**Matchett A.J., Yanagida T., Okudaira Y., Kobayashi S. Vibrating powder beds: a comparison of experimental and Distinct Element method simulated data. Powder Technology 107 (2000), 13-30.**

They measure the energy input to the system, and compare it with the calculated by the DEM code. With the DEM code they calculate the internal Energy as , and the dissipated in the time increment , is . The change in internal Energy is , so the change in total energy is . This is for each step. If summed for all time steps and dividing by t, they have the power loss. They represent the power loss for each acceleration value in stationary conditions. The experimental and numerical results do not agree. Only some qualitative observations can be made by DEM. They also represent as an attractor the energy dissipation rate as wall position. The Energy dissipation rate is given by .

It can be worth to explore how to get this value in our applications.

**Rajamani R.K., Mishra B.K., Venugopal R., Datta A., Discrete element analysis of tumbling mills. Powder Technology 109 (2000), 105-112.**

They calculate the dissipated energy and compare it with the experimental power needed for the tumbling mill to work. In stationary conditions, the kinetic and gravitational terms smear out, and the power needed is due only because of dissipation. Do not consider how the different energy terms are conserved, but only compares de dissipated power with the input power to the mill. Also they introduce the rotation tensor in 3D, and derives one approximation that needs iteration. Also is noticeable that they calculate the moments of the tangential forces only with the spring force, and not with the damping force.

It is seems that they use the condition

**Kharaz A.H., Gorham D.A., Salman A.D. An experimental study of the elastic rebound of spheres. Powder Technology 120 (2001) 281-291.**

Experimental study.

They state that:

*Another potentially important issue is the adhesive effects due to van der Waals forces, which become increasingly important as the particle size decreases and which dominate the impact behavior for particles in the micronsize range. However, the present work deals with particles of millimeter dimensions in which adhesive effects are negligible and so these will not be considered further in this paper.*

It’s a nice problem definition.

Here in this experiment it’s shown clearly the positive region of for , then the negative region of for , and the linear ascending shape of for .

Also they study the energy partition after the collision for each collision angle. It could be interesting to try to postprocess our results to obtain this, but their partition study the linear kinetic energy, the rotational, the associated with the normal coefficient of restitution (the viscous term in our case), and the slipping term.

**C. Thornton, Z. Ning, C.-Y. Wu, M. Nasrullah, L.-Y. Li, Contact mechanics and coefficients of restitution, in: T. Pöschel, S. Luding (Eds.), Granular Gases, Springer, Berlin, 2001, pp. 184–194**

It’s an interesting paper. They compare a viscoelastic model with a elastoplastic model. One important conclusion is that:

*“It is worth noting that the elastoplastic model of impact predicts that the unloading period is shorter than the loading period, in contrast to the predictions of viscoelastic impact models”*

This is probably related to the normal force artifact with the viscoelastic model. It would also to be shorter, so it’s interesting to try to cut the force as soon as it becomes attractive. There is also an expression to approximate the time of collision as: