להחזיר לתא מס' 06

עיבוד תמונות ואותות במחשב
236327

תרגיל: 3

הוגש ע"י:

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Question 1

Part 1:

a. (i) The standard function for N = 4 are:

$$\Psi_i{}^s(t) = \begin{cases} 1 & , t \in [\frac{i-1}{4}, \frac{i}{4}] \\ 0 & , \quad otherwise \end{cases}$$

The Walsh Hadamard matrix for N = 4 is:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

Using the definition $\Psi_i{}^{HW}(t) = \sum_{k=1}^{4} w_{k,i}\, \Psi_i{}^s(t)$ the functions are:

$$\Psi_1{}^{HW}(t) = 1$$

$$\Psi_2{}^{HW}(t) = \begin{cases} 1 \ \ , t \in [0, \frac{1}{2}) \\ -1\, , t \in [\frac{1}{2}, 1) \\ 0\, , otherwise \end{cases}$$

$$\Psi_3{}^{HW}(t) = \begin{cases} 1 \ \ , t \in \left[0, \frac{1}{4}\right) \ or\ t \in \left[\frac{3}{4}, 1\right) \\ -1 & , t \in [\frac{1}{4}, \frac{3}{4}) \\ 0 & , otherwise \end{cases}$$

$$\Psi_4{}^{HW}(t) = \begin{cases} 1 \ \ , t \in \left[0, \frac{1}{4}\right) \ or\ t \in \left[\frac{1}{2}, \frac{3}{4}\right) \\ -1 \ \ , t \in [\frac{1}{4}, \frac{1}{2}) or\ t \in \left[\frac{3}{4}, 1\right) \\ 0 & , otherwise \end{cases}$$

(ii) First we will calculate the coefficients using this formula:

$$\varphi_i^{HW} = \int_0^1 \varphi(t)\, \Psi_i{}^{HW}(t) dt = \int_0^1 (1-t)\, \Psi_i{}^{HW}(t) dt$$

$$\varphi_1^{HW} = \int_0^1 (1-t)\, dt = \left(1 - \frac{1}{2}\right) = \frac{1}{2}$$

$$\varphi_2^{HW} = \int_0^{1/2} (1-t)\, dt - \int_{\frac{1}{2}}^1 (1-t)\, dt = \frac{3}{8} - \frac{1}{8} = \frac{1}{4}$$

$$\varphi_3^{HW} = \int_0^{1/4} (1-t)\, dt + \int_{3/4}^1 (1-t)\, dt - \int_{\frac{1}{4}}^{\frac{3}{4}} (1-t)\, dt = \frac{7}{32} + \frac{1}{32} - \frac{1}{4} = 0$$

$$\varphi_4^{HW} = \int_0^{1/4} (1-t)\, dt + \int_{1/2}^{3/4} (1-t)\, dt - \int_{\frac{1}{4}}^{\frac{1}{2}} (1-t)\, dt - \int_{\frac{3}{4}}^1 (1-t)\, dt = \frac{7}{32} + \frac{3}{32} - \frac{5}{32} - \frac{1}{32} = \frac{1}{8}$$

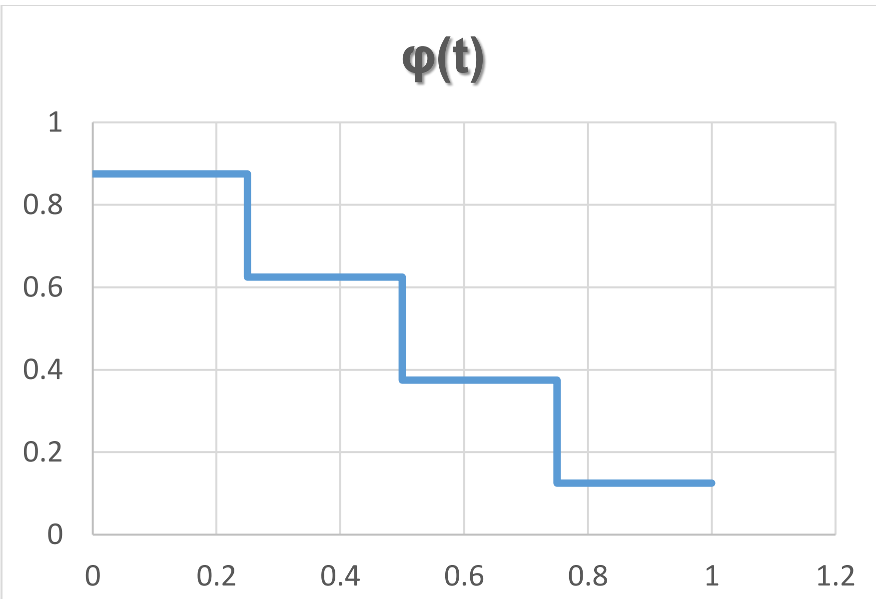
Using the coefficient and the ${\Psi_i}^{HW}(t)$ functions we are getting the following approximation:

$$\hat{\varphi}(t) = \sum_{i=1}^{4} \varphi_i^{HW} {\Psi_i}^{HW}(t) = \varphi_1^{HW} * {\Psi_1}^{HW}(t) + \varphi_2^{HW} * {\Psi_2}^{HW}(t) + \varphi_3^{HW} * {\Psi_3}^{HW}(t) + \varphi_4^{HW} * {\Psi_4}^{HW}(t)$$

$$= 1 * {\Psi_1}^{HW}(t) + \frac{1}{4} * {\Psi_2}^{HW}(t) + 0 * {\Psi_3}^{HW}(t) + \frac{1}{8} * {\Psi_4}^{HW}(t) =$$

$$= \frac{1}{2} + \frac{1}{4} * {\Psi_2}^{HW}(t) + \frac{1}{8} * {\Psi_4}^{HW}(t)$$

The function we got look like this:



b. (i) We got zero coefficient because the transformation between $\widehat{\varphi_3}(t)\ \ to\ \widehat{\varphi_4}(t)$ doesn't change the MSE. The best representation for this signal by Hadamard-Walsh basis for N = 4 is when used only 3 coefficients.
(ii) The function $\varphi(t) = (1-t)^2$ doesn't get zero coefficient for ${\Psi_3}^{HW}$.
Calculations:

$$\varphi_3^{HW} = \int_0^{\frac{1}{4}} (1-t)^2\, dt + \int_{\frac{3}{4}}^{1} (1-t)^2\, dt - \int_{\frac{1}{4}}^{\frac{3}{4}} (1-t)^2\, dt = \frac{37}{192} + \frac{1}{192} - \frac{13}{96} = \frac{1}{16} \neq 0$$

Part 2

a. The functions that the matrix defines are

$$\Psi_i^{Haar}(t) = \sum_{k=1}^{4} h_{k,i}\, \Psi_i^{s}(t)$$

$$\Psi_1^{Haar}(t) = 1$$

$$\Psi_2^{Haar}(t) = \begin{cases} 1 \ \ , t \in [0, \frac{1}{2}) \\ -1 \, , t \in [\frac{1}{2}, 1) \\ 0 \, , otherwise \end{cases}$$

$$\Psi_3^{Haar}(t) = \begin{cases} \sqrt{2} \ \ , t \in [0, \frac{1}{4}) \\ -\sqrt{2} \, , t \in [\frac{1}{4}, \frac{1}{2}) \\ 0 \, , otherwise \end{cases}$$

$$\Psi_3^{Haar}(t) = \begin{cases} \sqrt{2} \ \ , t \in [\frac{1}{2}, \frac{3}{4}) \\ -\sqrt{2} \, , t \in [\frac{3}{4}, 1) \\ 0 \, , otherwise \end{cases}$$

b. First we will calculate the coefficients:

$$\varphi_i^{Haar} = \int_0^1 \varphi(t)\, \Psi_i^{HW}(t) dt = \int_0^1 (1-t)\, \Psi_i^{Haar}(t) dt$$

$$\varphi_1^{Haar} = \int_0^1 (1-t)\, dt = \left(1 - \frac{1}{2}\right) = \frac{1}{2}$$

$$\varphi_2^{Haar} = \int_0^{1/2} (1-t)\, dt - \int_{\frac{1}{2}}^{1} (1-t)\, dt = \frac{3}{8} - \frac{1}{8} = \frac{1}{4}$$

$$\varphi_3^{Haar} = \int_0^{1/4} (1-t)\, dt - \int_{\frac{1}{4}}^{\frac{1}{2}} (1-t)\, dt = \sqrt{2}\left(\frac{7}{32} - \frac{5}{32}\right) = \frac{\sqrt{2}}{16}$$

$$\varphi_4^{Haar} = \int_{1/2}^{3/4} (1-t)\, dt - \int_{3/4}^{1} (1-t)\, dt = \sqrt{2}\left(\frac{3}{32} - \frac{2}{32}\right) = \frac{\sqrt{2}}{16}$$

Calculations of the $\hat{\varphi}_i(t) = \sum_{k=1}^{i} \varphi_i^{Haar}\, \Psi_i^{Haar}$ approximations:

$$\hat{\varphi}_1(t) = \frac{1}{2}$$

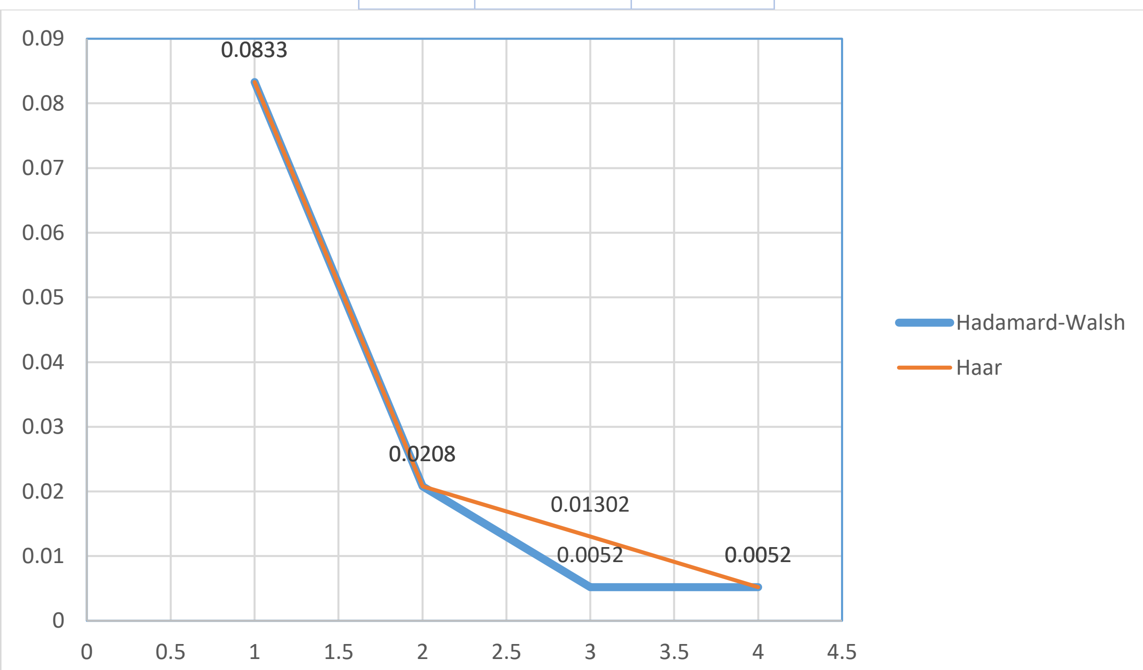
$$\hat{\varphi}_2(t) = \frac{1}{4} * \Psi_2^{Haar}$$

$$\hat{\varphi}_3(t) = \frac{\sqrt{2}}{16} * {\Psi_3}^{Haar}$$

$$\hat{\varphi}_4(t) = \frac{\sqrt{2}}{16} * {\Psi_4}^{Haar}$$

c. Comparison between the two representations of the linear signal:

	Walsh-Hadamard	**Haar**
MSE(1)	0.0833	0.0833
MSE(2)	0.0208	0.0208
MSE(3)	0.0052	0.013
MSE(4)	0.0052	0.0052



From the table and the graph we can see that for this signal both of the representations have the same MSE for N = 4 functions as expected. also , We can see that the Hadamard-Walsh representation is better for the 3-term approximations, not as expected.

Question 2

$$\langle \Psi_i, \Psi_i \rangle = \int_0^1 \left(\sum_{k=1}^{N} {u_i}^k \Psi_k^s(t) \right)^2 dt = \int_0^1 \sum_{k=1}^{N} {u_i}^{k^2} \Psi_k^s(t)^2 \, dt + 2 \int_0^1 \sum_{\substack{k=1 \\ k<l}}^{N-1} {u_i}^k \Psi_k^s(t) * {u_i}^l \Psi_l^s(t) \, dt =$$

From Unitary properties:

$$u^k \cdot {u^k}^T = {u_1}^{k^2} + {u_2}^{k^2} + \cdots + {u_N}^{k^2} = \sum_{k=1}^{N} {u_i}^{k^2} = 1$$

$$u^k \cdot {u^k}^T \cdot \sum_{k=1}^{N} \Psi_k^s(t)^2 = \left({u_i}^{1^2} \Psi_1^s(t) + {u_i}^{2^2} \Psi_2^s(t) + \cdots + \Psi_N^s(t) {u_i}^{N^2} \right) = \sum_{k=1}^{N} {u_i}^{k^2} \Psi_k^s(t)^2$$

$$= \sum_{k=1}^{N} 1 * \Psi_k^s(t)^2$$

$\Psi_k^s(t)$ *are ortho* $-$ *normal*: $\int_{\frac{k-1}{N}}^{\frac{k}{N}} \Psi_k^s(t)^2 = \langle \Psi_k^s(t), \Psi_k^s(t) \rangle = 1$

$$\int_0^1 \sum_{k=1}^{N} {u_i}^{k^2} \Psi_k^s(t)^2 \, dt = \int_0^1 \sum_{k=1}^{N} \Psi_k^s(t)^2 \, dt = \sum_{k=1}^{N} \int_{\frac{k-1}{N}}^{\frac{k}{N}} \Psi_k^s(t)^2 \, dt = \sum_{k=1}^{N} \int_{\frac{k-1}{N}}^{\frac{k}{N}} 1 \, dt = N * 1 * \frac{1}{N} = 1$$

$\Psi_k^s(t) \Psi_l^s(t) = 0$ for k≠l from their definition:

$$\int_0^1 \sum_{\substack{k=1 \\ k<l}}^{N-1} {u_i}^k \Psi_k^s(t) * {u_i}^l \Psi_l^s(t) \, dt = \int_0^1 \sum_{\substack{k=1 \\ k<l}}^{N-1} \Psi_k^s(t) \Psi_l^s(t) {u_i}^k * {u_i}^l \, dt = 0$$

$$\langle \Psi_i, \Psi_j \rangle = \int_0^1 \left(\sum_{k=1}^{N} {u_i}^k \Psi_k^s(t) \right) \left(\sum_{k=1}^{N} {u_i}^l \Psi_l^s(t) \right) dt$$

$$= 2 \int_0^1 \sum_{\substack{k=1 \\ k<l}}^{N-1} {u_i}^k \Psi_k^s(t) * {u_i}^l \Psi_l^s(t) \, dt + \int_0^1 \sum_{k=1}^{N} {u_i}^{k^2} {u_j}^{k^2} \Psi_k^s(t)^2 \, dt$$

From Unitary properties:

$$u^i \cdot {u^j}^T = {u_1}^i {u_1}^j + {u_2}^i {u_2}^j + \cdots + {u_N}^i {u_N}^j = \sum_{k=1}^{N} {u_i}^k {u_j}^k = 0$$

Therefore,

$$u^i \cdot {u^j}^T \cdot \sum_{k=1}^{N} \Psi_k^s(t)^2 = \left({u_i}^1 {u_j}^1 \Psi_1^s(t) + {u_i}^2 {u_j}^2 \Psi_2^s(t) + \cdots + {u_i}^N {u_j}^N \Psi_N^s(t)\right) = \sum_{k=1}^{N} {u_i}^k {u_j}^k \Psi_k^s(t)^2$$

$$= \sum_{k=1}^{N} 0 * \Psi_k^s(t)^2 = 0$$

$\Psi_k^s(t)\Psi_l^s(t) = 0$ for k≠l from their definition:

$$\int_0^1 \sum_{\substack{k=1 \\ k<l}}^{N-1} {u_i}^k \Psi_k^s(t) * {u_i}^l \Psi_l^s(t)\, dt = \int_0^1 \sum_{\substack{k=1 \\ k<l}}^{N-1} \Psi_k^s(t)\, \Psi_l^s(t) * {u_i}^k {u_i}^l\, dt$$

In conclusion, we got the wanted result:

$$\langle \Psi_i, \Psi_j \rangle = \begin{cases} 1\,, i = j \\ 0\,, i \neq j \end{cases}$$

Question 3:

If we express $\varphi_{|\Delta|}{}^{smooth}(t)$ as the convolution F$\{h * \varphi\}(t)$ then :

$$< \varphi_{|\Delta|}{}^{smooth}, e^{i2\pi kt} >= \int_0^1 \varphi_{|\Delta|}{}^{smooth}(t)e^{-i2\pi kt}dt = \int_0^1 (\int_{-\infty}^{\infty} g(\tau)\varphi_p(t-\tau)d\tau)\, e^{-i2\pi kt}dt =$$

From the linearity of the convolution:

$$\int_0^1 (\int_{-\infty}^{\infty} g(\tau)d\tau)\varphi(t)\, e^{-i2\pi kt}dt = \int_{-\infty}^{\infty} g(\tau)d\tau \int_0^1 e^{-i2\pi kt}\,\varphi(t)\,dt$$

And Given that $\int_0^1 \varphi(t)e^{-i2\pi kt} = 0 \quad for\ |k| > N_0$

We get that :

$$< \varphi_{|\Delta|}{}^{smooth}, e^{i2\pi kt} > = 0$$

meaning the smoothed signal is N_0 bandlimited as well

Question 4

a. The operator is linear:

$$H\{a_1\varphi_1(t)+a_2\varphi_2(t)\} = \int_{-\infty}^{\infty} w(\xi,\mathrm{t})(a_1\varphi_1(\xi)+a_2\varphi_1(\xi))\,d\xi =$$

$$= \int_{-\infty}^{\infty} w(\xi,\mathrm{t})a_1\varphi_1(\xi) + w(\xi,\mathrm{t})a_2\varphi_1(\xi)\,d\xi =$$

$$= a_1\int_{-\infty}^{\infty} w(\xi,\mathrm{t})\varphi_1(\xi)d\xi + a_2\int_{-\infty}^{\infty} w(\xi,\mathrm{t})a_2\varphi_1(\xi)\,d\xi =$$

$$= a_1H\{\varphi_1(t)\} + a_2H\{\varphi_2(t)\}$$

The operator is not shift invariant:

$$H\left\{T_{t_o}\{\varphi(t)\}\right\} = H\{\varphi(t-t_0)\} = \int_{-\infty}^{\infty} w(\xi,\mathrm{t})\varphi(\xi-t_0)d\xi$$

$$T_{t_0}\{H\{\varphi(t)\} = T_{t_0}\left\{\int_{-\infty}^{\infty} w(\xi,\mathrm{t})\varphi(\xi)d\xi\right\} = \int_{-\infty}^{\infty} w(\xi,\mathrm{t}-\mathrm{t}_0)\varphi(\xi)d\xi$$

b. The operator is linear:

$$H\{a_1\varphi_1(t)+a_2\varphi_2(t)\} = \int_{-\infty}^{\infty} h(t-\xi)(a_1\varphi_1(t)+a_2\varphi_2(t))\,d\xi =$$

$$= a_1\int_{-\infty}^{\infty} h(t-\xi)\varphi_1(t) + a_2\int_{-\infty}^{\infty} h(t-\xi)\varphi_1(t) = a_1H\{\varphi_1(t)\} + a_2H\{\varphi_2(t)\}$$

The operator is shift invariant:

$$H\left\{T_{t_o}\{\varphi(t)\}\right\} = H\{\varphi(t-t_0)\} = \int_{-\infty}^{\infty} h(t-t_0-\xi)\varphi(\xi-t_0)d\xi$$

$$T_{t_0}\{H\{\varphi(t)\} = T_{t_0}\left\{\int_{-\infty}^{\infty} h(t-\xi)\varphi(\xi)d\xi\right\} = \int_{-\infty}^{\infty} h(t-t_0-\xi)\varphi(\xi-t_0)d\xi$$

$$\xrightarrow{yields}\quad H\left\{T_{t_o}\{\varphi(t)\}\right\} = T_{t_0}\{H\{\varphi(t)\}$$

Question 5

$$x[n] = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$ ones at the 0, T, … (c-1)T entries.

Calculate the DFT by definition:

$$x[n]^F = \begin{bmatrix} w^{*0\cdot 0} & \cdots & w^{*0\cdot N-1} \\ \vdots & \ddots & \vdots \\ w^{*0\cdot N-1} & \cdots & w^{*N-1\cdot N-1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} w^{*0\cdot 0} + w^{*0\cdot T} + w^{*0\cdot 2T} + \cdots + w^{*0\cdot (c-1)T} \\ w^{*1\cdot 0} + w^{*1\cdot T} + w^{*1\cdot 2T} + \cdots + w^{*1\cdot (c-1)T} \\ w^{*2\cdot 0} + w^{*2\cdot T} + w^{*2\cdot 2T} + \cdots + w^{*2\cdot (c-1)T} \\ \vdots \\ \vdots \\ \vdots \\ w^{*N-1\cdot 0} + w^{*N-1\cdot T} + w^{*N-1\cdot 2T} + \cdots + w^{*N-1\cdot (c-1)T} \end{bmatrix} =$$

Each component is geometric progression with $q = w^{*kT} = e^{-\frac{2\pi i}{N}KT}$ when k is the index of the component: $k = 0,1 \dots, N-1$. For $\frac{kT}{N} \in \mathbb{Z} \rightarrow q = 1 \; and \; k \; is \; a \; divider \; of \frac{N}{T} = c$.

$$= \begin{bmatrix} 1 + 1 + \cdots 1 \\ w^{*1}\left(\dfrac{w^{*c\cdot 1\cdot T} - 1}{w^{*T} - 1}\right) \\ w^{*2}\left(\dfrac{w^{*c\cdot 2\cdot T} - 1}{w^{*2T} - 1}\right) \\ \vdots \\ 1 + 1 + \cdots 1 \\ w^{*2}\left(\dfrac{w^{*c\cdot (k+1)\cdot T} - 1}{w^{*(k+1)T} - 1}\right) \\ \vdots \\ \vdots \\ w^{*N-1}\left(\dfrac{w^{*c\cdot (N-1)\cdot T} - 1}{w^{*T} - 1}\right) \end{bmatrix} =$$

Use of $cT = N$ and $e^{\frac{-2\pi i k N}{N}} = w^{*N*k} = 1$

$$
= \begin{bmatrix} 1+1+\cdots 1 \\ w^{*1}\left(\frac{w^{*N*1}-1}{w^{*T}-1}\right) \\ w^{*2}\left(\frac{w^{*N*2}-1}{w^{*2T}-1}\right) \\ \vdots \\ 1+1+\cdots 1 \\ w^{*2}\left(\frac{w^{*(k+1)\cdot N}-1}{w^{*(k+1)T}-1}\right) \\ \vdots \\ \vdots \\ w^{*N-1}\left(\frac{w^{*(N-1)\cdot N}-1}{w^{*T}-1}\right) \end{bmatrix} = \begin{bmatrix} c \\ 0 \\ \vdots \\ c \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}
$$

All components are 0 except from components with their index is a divider of c:

$$
x^F[x] = \begin{cases} c & , c \bmod k = 0 \\ 0 & , othewise \end{cases}
$$

Matlab Part:

a.

Original Image

Deteriorated Image, Increasing of all pixels belonging to columns that their index is an integer multiplication of 16 by 40 gray levels:

Deteriorated Image

b. According to question 5 the transform of the interference is $cT = N \rightarrow c = \frac{N}{T} \rightarrow c = \frac{512}{16} = 32$:

$$40 * \begin{bmatrix} c \\ 0 \\ \vdots \\ c \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} = 40 * \begin{bmatrix} 32 \\ 0 \\ \vdots \\ 32 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1280 \\ 0 \\ \vdots \\ 1280 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

Matlab results:

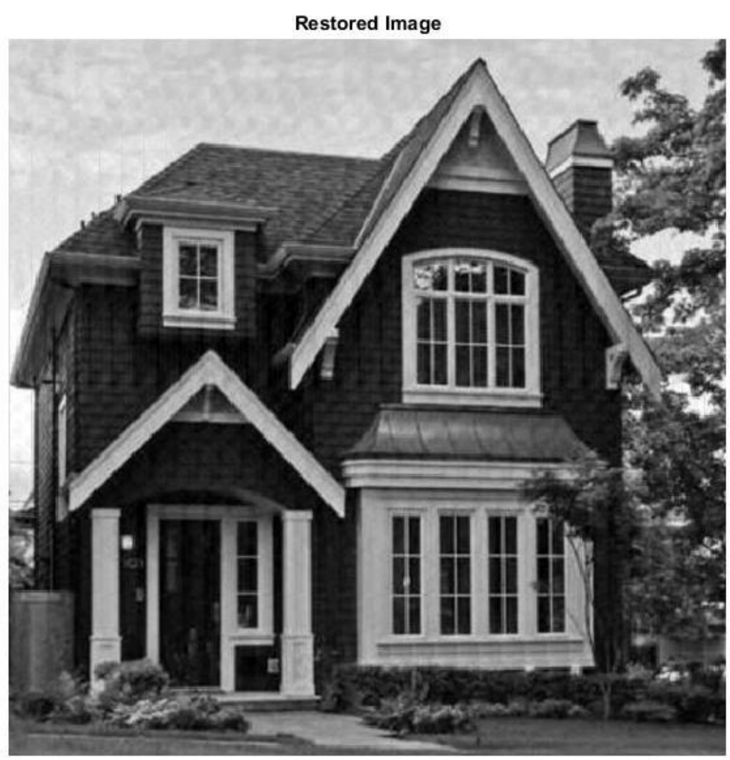
DFT components in which the iterference have strictly positive values: 0 32 64 96 128 160 192 224 256 288 320 352 384 416 448 480

The value of each positive components is: 1280

The value of all other components is: 0

We can see that the result of the calculations match the theoretical result.

c.



MSE of the deteriorated image: 97.8059

MSE of the restored image: 22.6476

As expected, the MSE of the restored image is much lower than the MSE of the deteriorated image. In the restored Image we barely see defects.