Class Two: General Physics 3114
Kigali Institute of Science and Technology

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Outline for Today:

1. Dimensional Analysis
2. Precision vs. Accuracy
3. Uncertainties and propagating uncertainties
4. Standard deviation

Dimensional Analysis: All terms in a physics equation must have the same units. This information can help us figure out lots of useful information. For example, the distance d an object of mass m travels when it undergoes an acceleration a for a time interval t. How might d depend on a and t? Let's try a formula of the following form:

$$d = ca^x m^y t^z \ , \tag{1}$$

where c is a numerical constant. The units for acceleration are meters/s^2. Written in terms of dimensions, $a = L/T^2$. Eq. (1) then becomes

$$L^1 = c\left(\frac{L^1}{T^2}\right)^x M^t T^z \ . \tag{2}$$

The dimensions must be the same on both sides of this equation. I.e., there need to be the same power of L, T, and M. So this is really three equations in one!

$$1 = x \tag{3}$$

$$0 = z - 2x \tag{4}$$

$$0 = y \tag{5}$$

Solving these, and plugging into d, we obtain

$$d = cat^2 \ . \tag{6}$$

We can't determine the value of c, as it is a dimensionless number. Nevertheless, we have essentially derived a formula by thinking *only* about units. Dimensional analysis can be a powerful and useful technique. You will practice it in tutorial and lab.

Uncertainty: Every measurement is an approximation to the true value. As scientists we need to know how to calculate, analyze, and report uncertainties. There are two types of uncertainty. These are illustrated in Fig. 1.

1. **Systematic** errors arise from error or bias in measurement apparatus or technique. Ex: calibration error. If there are few systematic errors, the measurement is **accurate**.
2. **Random** errors are due to unpredictable variations in conditions under which the measurement occurs. Ex. Measure something multiple times and you will get slightly different values each time. If there are few random errors, the measurement is **precise**.

How might we measure the uncertainty of a single measurement? Make a reasonable guess, based on the equipment you have. For example, in Fig. 2, we might say that our best estimate for the length of the object is 18.5 mm. The low estimate would be 18.2 and the high estimate 18.8. We would report this as $L = 18.5 \pm 0.3$ mm. In general, if we are measuring measuring a quantity x, we would report its value as

$$x = x_{\text{best guess}} \pm \delta x \ \text{ or } \ x = x_{\text{best guess}} \pm \sigma_x \ . \tag{7}$$

Try and chose an δx such that you are roughly 70% confident that the true value is contained in the range.

Combining Uncertainties: Suppose you measure the length of five different objects and obtain:

$$L_1 = 50\text{cm} \pm 1\text{cm} \ , \ L_2 = 50\text{cm} \pm 3\text{cm} \ , \ L_3 = 100\text{cm} \pm 1\text{cm} \ , \ L_4 = 50\text{cm} \pm 2\text{cm} \ , \ L_5 = 50\text{cm} \pm 1\text{cm} \ , \tag{8}$$

How would you report the uncertainty of the total length?

$$L_{\text{total}} = 300\text{cm} \pm \underline{\qquad}??? \ . \tag{9}$$

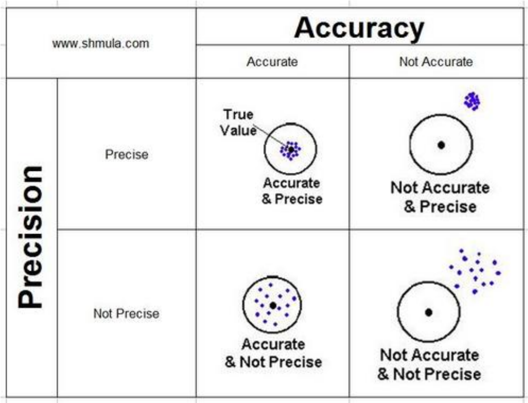

Figure 1: Accuracy vs. precision. Source: `scienceblogs.com/startswithabang/2011/11/a_new_challenge_a_new_job_and.php`

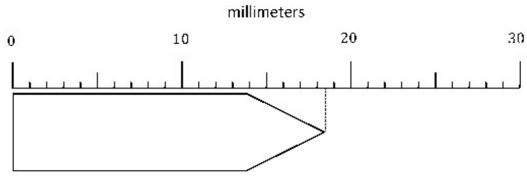

Figure 2: Measuring length. Figure made by Christian Kwisanga.

Hmmmm.... Perhaps the uncertainty should be 9 cm. But this is too large. Some of the uncertainties are likely to cancel. E.g., the measured value for L_1 might be too big, while the value for L_2 is too small. The way to account for this is as follows:

$$\text{uncertainty in } L_{\text{total}} = \sqrt{1^2 + 3^2 + 1^2 + 2^2 + 1^2} \tag{10}$$

This yields $L_{\text{total}} = 300\text{cm} \pm 4\text{cm}$.

The general results is, that if

$$\text{if } C = A + B \text{ , then } \sigma_c = \sqrt{\sigma_A^2 + \sigma_B^2} \text{ .} \tag{11}$$

We can think of the errors in A and B as existing in different directions, and σ_C is like a hypotenuse. The justification of this expression requires some math that is beyond the scope of this course. Nevertheless, I hope the above formula seems reasonable.

Let's do another example that will lead to a general method for propagating errors. Suppose we want to estimate the uncertainty of the area of a rectangle and we have measured $b = 15 \pm 2$cm and $h = 10 \pm 1$ cm, for the base and the height. What is the uncertainty of the area A? There are two sources of uncertainty:

$$\text{uncertainty in } A \text{ due to uncertainty in } h = (15\text{cm})(1\text{cm}) = 15\text{cm}^2 \text{ .} \tag{12}$$

$$\text{uncertainty in } A \text{ due to uncertainty in } b = (10\text{cm})(2\text{cm}) = 20\text{cm}^2 \text{ .} \tag{13}$$

As in the previous example, we don't just add these two uncertainties, but combine them using the Pythagorean theorem:

$$\sigma_A = \sqrt{(15\text{cm}^2)^2 + (20\text{cm}^2)^2} = 25\text{cm}^2 \text{ .} \tag{14}$$

Thus, $A = 150\text{cm}^2 \pm 25\text{cm}^2$.

Let's repeat this a little more generally. Let $A = BH$. Then

$$\sigma_A = \sqrt{(B\sigma_H)^2 + (H\sigma_B)^2} \text{ , or } \sigma_A^2 = B^2\sigma_H^2 + H^2\sigma_B^2 \text{ .} \tag{15}$$

Dividing the left side by A^2 and the right side by $(BH)^2$ (this is a legal thing to do since $A = BH$), we obtain

$$\frac{\sigma_A^2}{A^2} = \frac{\sigma_H^2}{H^2} + \frac{\sigma_B^2}{B^2} \ . \tag{16}$$

This equation is nice, since each term is a percent uncertainty.

Let me state the general result. We'll then do another example. Let f be some function of x, y, and z. Let x, y, and z be measured with uncertainties σ_x, σ_y and σ_z. Then,

$$\text{uncertainty in } f \text{ due to uncertainty in } x = \left|\frac{\partial f}{\partial x}\right| \sigma_x \ . \tag{17}$$

This result follows immediately from the definition of the derivative; $\partial f/\partial x$ tells us how changes in x are related to changes in f. A similar result holds for y and z. We then combine these uncertainties to obtain

$$\sigma_f = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2 + \left(\frac{\partial f}{\partial z}\right)^2 \sigma_z^2} \tag{18}$$

All error propagation results follow from this formula.

Another important topic is that of *significant digits.* Please see the handout I distributed with these notes. You will practice this in lab and tutorial. The basic ideas is that a statement like 5.2934 ± 0.1 does not make sense. The 3 and 4 in the number are not significant; they do not carry any real information, since the uncertainty is larger than them. Instead, we would write 5.3 ± 0.1 and say that the number 5.3 has two significant digits.

Example: You measure the radius of a sphere to be $r = 1 \pm 0.05$ m. What is the uncertainty of the volume? I will do this example in lecture.

Standard Deviation: Another (and often better way) to estimate the uncertainty on a single measurement is to repeat that measurement many times and perform a statistical analysis. For example, suppose we have made the following 5 time measurements:

$$t_1 = 5\text{s} \ , \ t_2 = 4\text{s} \ , \ t_3 = 6\text{s} \ , \ t_4 = 3\text{s} \ , \ t_5 = 7\text{s} \ . \tag{19}$$

The idea here is that we measured the *same* event five different times. Our best estimate for the time is just the average:

$$t_{\text{best estimate}} = t_{\text{average}} = \bar{t} = \frac{1}{5}(5\text{s} + 4\text{s} + 6\text{s} + 3\text{s} + 7\text{s}) = 5\text{s} \ . \tag{20}$$

What uncertainty should we list for our measurements? We might be tempted to compute the average difference from the average. I.e.,

$$\text{average difference from average} = \frac{1}{5}\left((5-5) + (4-5) + (6-5) + (3-5) + (7-5)\right) \tag{21}$$

But this equals zero! Positive and negative terms cancel out. To get around this problem, we square the differences. The formula is:

$$\text{square of average difference from average} = \frac{1}{4}\left((5-5)^2 + (4-5)^2 + (6-5)^2 + (3-5)^2 + (7-5)^2\right) \tag{22}$$

Then to get the uncertainty, we take the square root. The general result for N measurements $x_1, x_2, \cdots x_N$ is:

$$\sigma_x = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2} \tag{23}$$

Note that we divide by $N-1$ not N. The quantity σ_x defined in Eq. (23) is known as the *standard deviation.* It is the standard way to describe the spread or variance in a series of measurements. Under most conditions, around 70% of the measurements will be within $\pm\sigma_x$ of the average.