Differential Equations
Homework Four
Due Friday, November 21, 2014

In this set of exercises we will solve the one-dimensional heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \, . \tag{1}$$

Here $u(x,t)$ is a function of position x and time t, and describes the temperature of the rod at point x and time t. We will assume that the diffusion or heat conductivity is 1.

We will consider a metal rod of length L. The left and right hand sides of the rod are fixed at zero temperature:

$$u(0,t) = 0 = u(L,t) \, . \tag{2}$$

In order to solve this equation we need to specify an initial condition $u(x,t)$ in addition to the above boundary conditions. We will get to this in a bit. For now, let's see how far we can get with using only the boundary conditions, Eq. (2).

Note: This problem potentially involves lots of integrals. You should use wolframalpha to evaluate these integrals without guilt or hesitation.

To solve Eq. (1) we will use a technique called *separation of variables*. This method works for some, but not all, partial differential equations. The method begins with the guess that the solution can be written as the product of two functions, one of which depends only on x, and the other of which depends only on time t. That is, we write u as:

$$u(x,t) = X(x)T(t) \, . \tag{3}$$

The idea here is that $X(x)$ is an unknown function of x and $T(t)$ is an unknown function of t.

Question 1. *Plug Eq. (3) into Eq. (1), and show that one can obtain:*

$$\frac{T'}{T} = \frac{X''}{X} \, . \tag{4}$$

Think for a moment about why the derivatives are now ordinary derivatives and not partials.

If Eq. (4) is true for all x and t, then the only way this can occur is that if the left and right hand sides both equal a constant.[1] Experience suggests that it will be nice to call this constant $-k^2$. We have now converted our pde into two odes.

[1] It will likely require a few moments of quiet reflection to convince yourself that this is true.

Question 2. *Show that these two odes are:*

$$T' = -k^2 T \ , \tag{5}$$

and

$$X'' = -k^2 X \ . \tag{6}$$

Question 3. *Write down the general solution to Eq. (6). Note that since this is a second-order equation there will be two solutions, not one.*

Question 4. *Apply the boundary conditions, Eq. (2) and show that that*

$$k = \frac{n\pi}{L} \ , \ n = 1, 2, 3, \dots \ , \tag{7}$$

and

$$X(x) = a_n \sin(\frac{n\pi x}{L}) \ , \ n = 1, 2, 3, \dots \ , \tag{8}$$

where a_n is a constant that will be determined later on by the initial conditions.

Question 5. *Write down the solution to Eq. (5), using k from Eq. (7).*

Question 6. *Take your $X(x)$ and $T(t)$ solutions, plug in to Eq. (3) to obtain solutions to the heat equation, Eq. (1). You should find that:*

$$u_n(x,t) = a_n e^{\frac{-n^2\pi^2}{L^2}t} \sin(\frac{n\pi x}{L}) \ , \ n = 1, 2, 3, \dots \ . \tag{9}$$

Note that we have an infinite number of solutions, one for each value of n.

Question 7. *Describe qualitatively the x and t behavior of $u_n(x,t)$. Make a few sketches or write a few sentences. What happens as t gets very large? Why does this make sense physically?*

The heat equation is linear. This means that if $u(x,t)$ and $v(x,t)$ both solve Eq. (1) then a linear combination of these two solutions is also a solution to Eq. (1). That is,

$$w(x,t) = c_1 u(x,t) + c_2 v(x,t) \tag{10}$$

solves Eq. (1) provided that $u(x,t)$ and $v(x,t)$ solve the Eq. (1).

Question 8. Optional. *Show that the heat equation is linear.*

The linearity of the heat equation tells us that the most general solution is a sum of all the different solutions (one for each n) given in Eq. (9). (This is similar to what we did with two-dimensional linear odes, but now we have an infinite number of solutions, not just two.) We thus write the most general solution as

$$u(x,t) \;=\; \sum_{n=1}^{\infty} a_n e^{\frac{-n^2\pi^2}{L^2}t} \sin(\frac{n\pi x}{L}) \; . \tag{11}$$

In order to proceed further, we need to specify initial conditions $u(x,0)$.

Let's suppose that we have an initial condition $u(x,0) = \phi(x)$. This means that the function $\phi(x)$ specifies the initial temperature of every point in the rod. Our task now is to find the a_n's so that

$$u(x,0) \;=\; \phi(x) \; . \tag{12}$$

So we see what we're up against, let's plug in Eq. (11) to the above equation:

$$\sum_{n=1}^{\infty} a_n \sin(\frac{n\pi x}{L}) \;=\; \phi(x) \; . \tag{13}$$

Note that the exponential term disappears because we plug in $t = 0$. Oy. This looks like bad news. We have an infinite number unknown coefficients a_n. How can we ever solve for them all?

It turns out that there is a very elegant, fun, and super useful trick that makes it possible to solve for the a_n's. Here's how it works. The spatial part of the solutions,

$$X(x) \;=\; \sin(\frac{n\pi x}{L}) \; , \tag{14}$$

have some very nice properties. It turns out that

$$\int_0^L \sin(\frac{n\pi x}{L}) \sin(\frac{m\pi x}{L}) \, dx \;=\; 0 \; , \; m \neq n \; , \tag{15}$$

and

$$\int_0^L \sin(\frac{n\pi x}{L}) \sin(\frac{n\pi x}{L}) \, dx \;=\; \frac{L}{2} \; . \tag{16}$$

I'm now going to take Eq. (13), multiply both sides by $\sin(m\pi x/L)$ and integrate from 0 to L:

$$\int_0^L \sin(\frac{m\pi x}{L}) \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi x}{L}) \, dx \;=\; \int_0^L \sin(\frac{m\pi x}{L}) \phi(x) \, dx \; . \tag{17}$$

It looks like we just made things worse, but they are actually about to get a lot better.

Question 9. *Use Eq. (16) and (15) to simplify Eq. (17) to obtain*

$$a_n \;=\; \frac{2}{L} \int_0^L \phi(x) \sin(\frac{m\pi x}{L}) \, dx \; . \tag{18}$$

This is great progress! We now have a formula for a_n.

In order to proceed further, we need specify the initial condition $\phi(x)$. Let's choose a "triangle function." By this I mean a function that increases linearly, reaching a maximum at $x = L/2$, and then decreases linearly until $x = L$. This is shown in Fig. 1. The function $phi(x)$ can be defined as a piecewise function as follows:

$$\phi(x) = \begin{cases} \frac{2T}{L}x & : \ 0 < x < L/2 \\ 2T - \frac{2T}{L}x & : L/2 \le x < L \end{cases} \tag{19}$$

Now that we have specified the initial condition $\phi(x)$ we can use Eq. (18) to solve for the a_n's.

Question 10. *Use Eqs. (19) and (18) to derive the following formula for a_n:*

$$a_n \ = \ \frac{8T}{\pi^2 n^2} \sin(\frac{\pi n}{2}) \ . \tag{20}$$

You will surely want to use wolframalpha to evaluate the integrals. Note that you need to do the integral in two pieces: from 0 to $L/2$ and then from $L/2$ to L.

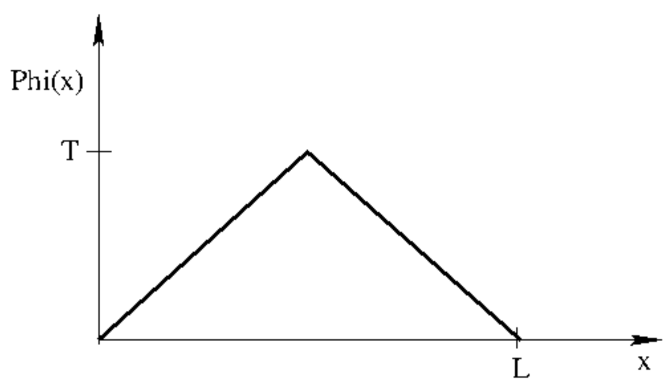

Figure 1: The initial temperature on the metal rod.

We have now solved the problem! The solution is:

$$u(x,t) \ = \ \sum_{n=1}^{\infty} \frac{8T}{\pi^2 n^2} \sin(\frac{\pi n}{2}) e^{\frac{-n^2\pi^2}{L^2}t} \sin(\frac{n\pi x}{L}) \ . \tag{21}$$

All we need to do now is get python to plot solutions for us. We can't, of course, plot all terms in the infinite sum. So we'll have to truncate—use only the first 10 or 20 terms.

Getting Eq. (21) into python takes a steady hand and an empty mind. For readability, I broke the terms on the right into three functions and wrote it as:

$$u(x,t) \ = \ \sum_{n=1}^{\infty} a(n) T(n,t) X(n,x) \ . \tag{22}$$

If nothing else, doing it like this may make it easier to debug and catch typos.

Question 11. Optional: *Make python plot Eq. (21) for* $t = 0$. *As you include more and more terms in the sum, you should see* $u(x, 0)$ *getting closer and closer to the triangle function of Fig. 1. Pause and reflect on the wonder of this: a sum of smooth and wiggly functions is converging to a straight and pointy function.*

Question 12. Optional: *Make python plot Eq. (21) for* $t = 0, 20, 40, 60, 80,$ *and* 100. *Is the behavior what you would expect?*

Question 13. Optional: *Have python solve the heat equation with these boundary and initial conditions using Euler's method and a discretized second derivative. Compare your numerical results with the analytic results from the previous question.*

If you have the time and energy, doing these last three questions would a great way to solidify and tie together some of the python work we've done this term. But I understand that you probably have quite a bit on your plate the next week.