Extensions to Shannon Entropy

- One of the requirements on the Shannon entropy H that is used to derive it is that H is independent of the way we group probabilities.
- Let's state this more precisely. We'll do so via an example.
- Consider the random variable X that can take on three outcomes, a, b, and c:
- $\Pr(a) = 1/2$, $\Pr(b) = 1/2$, and $\Pr(c) = 1/4$.
- It turns out that $H[X] = 3/2$.
- We can also view this as follows: Y can be a or Z, each with probability $1/2$. And Z can be b or c with probability $1/2$.
- $H[Y] = 1$, and $H[Z] = 1$.
- $H[X] = H[Y] + \frac{1}{2}H[Z]$.
- This last condition is an example of requiring H be independent of the way we group probabilities.

Rényi Entropy

- Let's relax the condition that H be independent of grouping.
- However, let's still require that the entropy of independent variables be additive:

$$\begin{aligned} \Pr(X,Y) = \Pr(X)\Pr(Y) & \quad \Longrightarrow \\ H[X,Y] = H[X] + H[Y] & \quad . \end{aligned}$$

- The result is a one-parameter family, the Rényi entropies:

$$H_q \equiv \frac{1}{q-1} \log_2 \sum_i p_i^q \ . \tag{1}$$

- This can be rewritten in the following, slightly less odd-looking way.

$$H_q = \frac{1}{q-1} \log_2 \sum_i p_i p_i^{q-1} \ . \tag{2}$$

$$H_q = \frac{1}{q-1} \log_2 \langle p_i^{q-1} \rangle \ . \tag{3}$$

Rényi Entropies: Properties and Comments

- H_1 is the Shannon entropy.
- H_0 is the topological entropy, the log of the number of states.
- There is a coding theorem for Rényi entropy. Campbell. *Information and Control.* 8:423. 1966.
- H_q is a non-increasing function of q.

Rényi and Thermodynamics

- The Rényi entropy allows one to apply the formalism of thermodynamics to any probability distribution. q plays a role similar to inverse temperature.

Escort Distributions

- Given a set of probabilities, we can always make a new set of probabilities as follows:

$$p_i \longrightarrow \frac{p_i^\beta}{Z} \ .$$

- β is a number that acts like $1/\mathrm{Temperature}$.
- $\beta = 1$: initial distribution
- $\beta = 0$: all states equally likely $\Rightarrow T = \infty$.
- $\beta = \infty$: only most probable state remains. This is the $T = 0$ “ground state.”
- $\beta = -\infty$: only least probable state remains. This is the $T = 0^-$ “anti-ground” state.
- Loosely speaking, the Rényi entropy can be thought of the average surprise of the escort distribution with $\beta = q - 1$.
- The parameter β allows one to probe different regions of the distribution.

Thermodynamic Formalism

- The ideas on the previous slide can be extended in an elegant way to develop thermodynamics to any probability distribution.
- This goes by many names; thermodynamic formalism, $S(u)$, $f(\alpha)$, multifractals, fluctuation spectrum, large deviation theory.
- This is a well developed, well understood approach. It is very enticing and very cool.
- In my experience, this approach doesn't speak directly to complexity or pattern, largely because thermodynamics doesn't have direct measures of complexity.
- For example, a biased coin (i.e. no correlations), has a "multifractal spectrum."
- There are many confusing things written about the thermodynamic formalism. Some clear references:
  - Young and Crutchfield. Chaos, Solitons, and Fractals. 4:5. 1993.
  - Beck and Schlögl, *Thermodynamics of Chaotic Systems*. Cambridge University Press. 1993.

Tsallis Entropy

- Define the following generalized entropy

$$S_q \equiv \frac{1 - \sum_i p^q}{q - 1} \; . \tag{4}$$

- This q is not the same as Rényi's q.
- S_q has the property that:

$$\begin{aligned} \Pr(X,Y) &= \Pr(X)\Pr(Y) \Longrightarrow \\ S_q[X,Y] &= S_q[X] + S_q[Y] + \\ & \quad (1-q)S_q[X]S_q[Y] \; . \end{aligned} \tag{5}$$

- I.e., S_q is not additive for independent events.
- One can generate a statistical mechanics and thermodynamics using Eq. (4) as a starting point.
- However, it is hard to see how a non-additive entropy can be physical.
- It has been claimed that S_q works well for systems with strong correlations. But it seems to me that the non-additivity creates spurious correlations rather than measuring correlations that are really there.

Tsallis Entropy, references

But, you should read the papers and decide for yourself.

Some reviews:

- Tsallis. *Physica D*, **193**:3. 2004. `http://arxiv.org/cond-mat/0403012`.
- Tsallis, et al. `arXiv.org/cond-mat/0309093`. 2003.
- Tsallis and Brigatti, *Continuum Mechanics & Thermodynamics*, **16**:223 `arXiv.org/cond-mat/0305606`. 2004.

Some critiques, and responses:

- Grassberger. *Physical Review Letters*, 95. 140601. 2005. `http://arxiv.org/cond-mat/0508110`
- Responses to Grassberger:
  - Robledo. `arxiv.org/cond-mat/0510293`
  - Tsallis. `arxiv.org/cond-mat/0511213`
- Nauenberg. *Physical Review E*. **67**:036114. 2003. `arxiv.org/cond-mat/0210561`

- Responses to Nauenberg and discussion:
  - Tsallis. *Physical Review E*. **69**:038102. 2004. `arxiv.org/cond-mat/0304696`
  - Nauenberg. *Physical Review E*. **69**:038102. 2004 `arxiv.org/cond-mat/0305365`

See also:

- `www.cbpf.br/GrupPesq/StatisticalPhys/biblio.htm` for an extensive biobiography on Tsallis entropy.