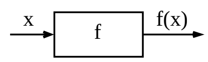
A Brief, Introductory Overview of Dynamical Systems and Chaos

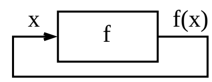
- A **Dynamical System** is any system that changes over time
  - A Differential Equation
  - A system of differential equations
  - Iterated functions
  - Cellular Automata
- The goal of this brief introduction is to define a handful of terms and introduce the phenomena associated with chaos.
- I will focus on iterated functions.
- Let's start with an example.

Example: Iterating the squaring rule, $f(x) = x^2$

- Consider the function $f(x) = x^2$. What happens if we start with a number and repeatedly apply this function to it?
- E.g., $3^2 = 9$, $9^2 = 81$, $81^2 = 6561$, etc.
- The iteration process can also be written $x_{n+1} = x_n^2$.
- In this is example, the initial value 3 is the **seed**. The seed is often denoted x_0.
- The sequence $3, 9, 81, 6561, \cdots$ is the **orbit** or the **itinerary** of 3.
- Picture the function as a "box" that takes x as an input and outputs $f(x)$:



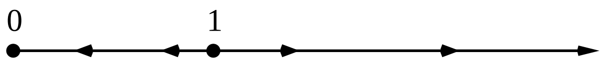
- Iterating the function is then achieved by feeding the output back to the function, making a feedback loop:



The squaring rule, continued

In dynamics, we are usually interested in the long-term behavior of the orbit, not in the particulars of the orbit.

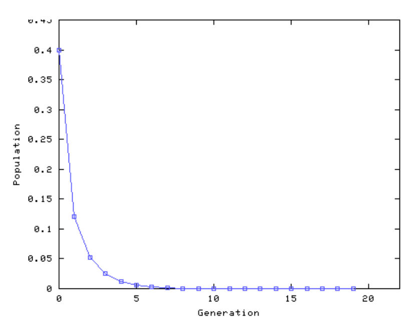
- The seed 3 tends toward infinity—it gets bigger and bigger.
- Any $x_0 > 1$ will tend toward infinity.
- If $x_0 = 1$ or $x_0 = 0$, then the point never changes. These are fixed points.
- If $0 \leq x_0 < 1$, then x_0 approaches 0.
- We can summarize this with the following diagram:



- 0 and 1 are both **fixed points**
- 0 is a **stable** or **attracting** fixed point
- 1 is an **unstable** or **repelling** fixed point

Logistic Equation

- Logistic equation: $f(x) = rx(1-x)$.
- A simple model of resource-limited population growth.
- The population x is expressed as a fraction of the carrying capacity. $0 \leq x \leq 1$.
- r is a parameter—the growth rate—that we will vary.
- Let's first see what happens if $r = 0.5$.



- This graph is known as a **time series plot**.
- 0 is an attracting fixed point.

Logistic Equation, $r = 2.5$

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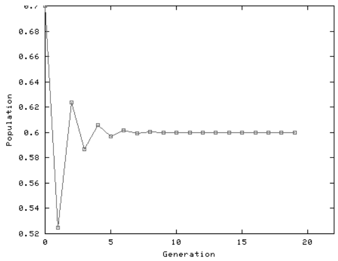

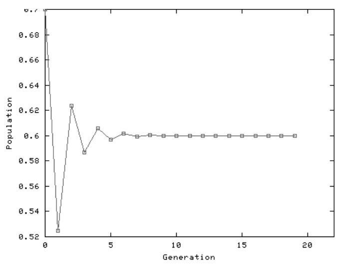

- All initial conditions are pulled toward 0.6.
- 0.6 is an attracting fixed point.

Logistic Equation, $r = 3.2$

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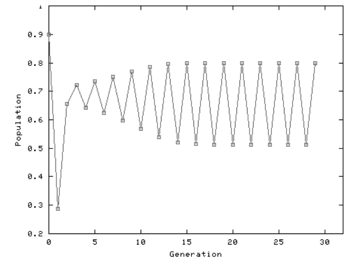

- Initial conditions are pulled toward a **cycle** of period 2.
- The orbit oscillates between 0.513045 and 0.799455.
- This cycle is an attractor. Many different initial conditions get pulled to it.

Logistic Equation, $r = 4.0$

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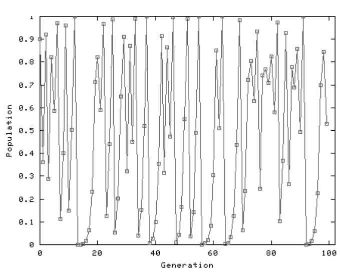
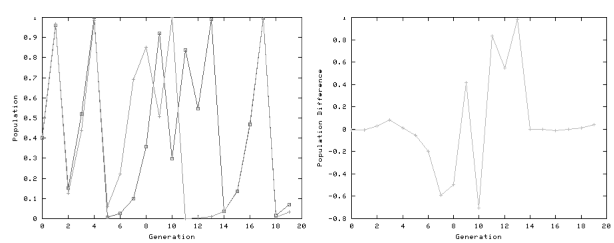


- What's going on here?!
- The orbit is not periodic. In fact, it never repeats.
- This is a rigorous result; it doesn't rely on computers.
- This cycle is an attractor. Many different initial conditions get pulled to it.
- What happens if we try different initial conditions?

Different Initial conditions

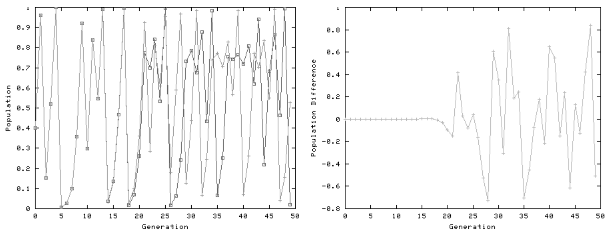
- Logistic equation, $r = 4.0$. Two different initial conditions, $x_0 = 0.4$ and $x_0 = 0.41$.



- The right graph plots the different between the two orbits on the left with slightly different initial conditions.
- Note that the difference between the two orbits grows.
- Can think of one initial condition as the true one, and the other as the measured one.
- The plot on the right then shows what happens to our prediction error over time.
- What happens if the two initial conditions are closer together?

Sensitive Dependence on Initial Conditions

- Logistic equation, $r = 4.0$. Two different initial conditions, $x_0 = 0.4$ and $x_0 = 0.4000001$.



- The two initial conditions differ by one part in one million
- The orbits differ significantly after around 20 iterations, whereas before they differed after around 4 iterations.
- Increasing the accuracy of the initial condition by a factor of 10^5 allow us to predict the outcome 5 times further.
- Thus, for all practical purposes, this system is unpredictable, even though it is deterministic.
- This phenomena is known as **Sensitive Dependence on Initial Conditions**, or, more colloquially, **The Butterfly Effect**.

Definition of Sensitive Dependence on Initial Conditions

- A dynamical system has sensitive dependence on initial conditions (SDIC) if arbitrarily small differences in initial conditions eventually lead to arbitrarily large differences in the orbits.

More formally

- Let X be a metric space, and let f be a function that maps X to itself: $f : X \mapsto X$.
- The function f has SDIC if there exists a $\delta > 0$ such that $\forall x_1 \in X$ and $\forall \epsilon > 0$, there is an $x_2 \in X$ and a natural number $n \in N$ such that $d[x_1, x_2] < \epsilon$ and $d[f^n(x_1), f_n(x_2)] > \delta$.
- In other words, two initial conditions that start ϵ apart will, after n iterations, be separated by a distance δ.

Definition of Chaos

There is not a 100% standard definition of chaos. But here is one of the most commonly used ones:

An iterated function is **chaotic** if:

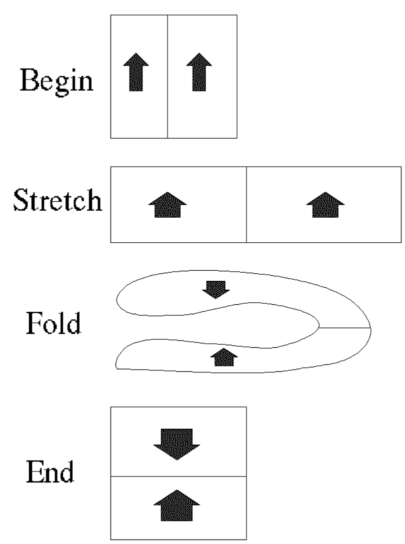
1. The function is **deterministic**.
2. The system's orbits are **bounded**.
3. The system's orbits are **aperiodic**; i.e., they never repeat.
4. The system has **sensitive dependence on initial conditions**.

Other properties of a chaotic dynamical system ($f : X \mapsto X$) that are sometimes taken as defining features:

1. **Dense periodic points:** The periodic points of f are dense in X.
2. **Topological transitivity:** For all open sets $U, V \in X$, there exists an $x \in U$ such that, for some $n < \infty$, $f_n(x) \in V$. I.e., in any set there exists a point that will get arbitrarily close to any other set of points.

Geometry of Chaos

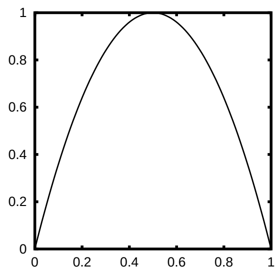
Geometrically, all chaotic systems involve stretching and folding:



- Stretching pulls nearby points apart, leading to **sensitive dependence on initial conditions.**
- Folding keeps the orbits **bounded**.

Geometry of Chaos, continued

The logistic equation may be viewed as stretching and folding the unit interval onto itself:



- Note that the amount of stretching is captured by the slope of the function.
- We shall see that the “average slope” is related to the degree of SDIC, which is in turn related to the unpredictability.
- Thus, SDIC is a geometric property of the system.

Chaos and Dynamical Systems: Selected References

There are many excellent references and textbooks on dynamical systems. Some of my favorites:

- Peitgen, et al. *Chaos and Fractals: New Frontiers of Science*. Springer-Verlag. 1992. *Huge (almost 1000 pages), and very clear. Excellent balance of rigor and intuition.*
- Cvitanović, *Universality in Chaos, second edition*, World Scientific. 1989. *Comprehensive collection of reprints. Very handy. Nice introduction by Cvitanović*.
- Gleick, *Chaos: Making a New Science*. Penguin Books. 1988. *Popular science book. But very good. Extremely well written and accurate.*
- Devaney. *An Introduction to Chaotic Dynamical Systems, second edition*. Perseus Publishing. 1989. *Advanced undergrad math textbook. Very clear.*