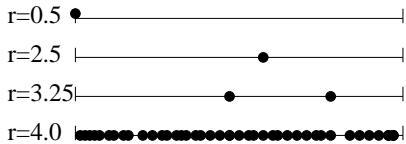


### What else can the logistic equation do?

Thus far, we have seen several possible long-term behaviors for the logistic equation:

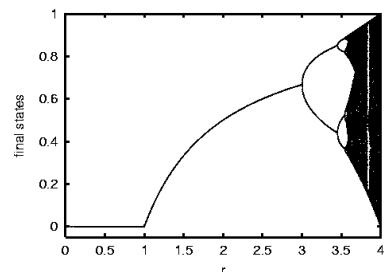
1.  $r = 0.5$ : attracting fixed point at 0.
2.  $r = 2.5$ : attracting fixed point at 0.6.
3.  $r = 3.25$ : attracting cycle of period 2.
4.  $r = 4.0$ : chaos.

Graphically, we can illustrate this as follows:



- What we're doing is, for each  $r$ , iterate and plot the final  $x$  values as dots on the number line.
- Do this for more and more  $r$  values and "glue" the lines together.
- Turn sideways and ...

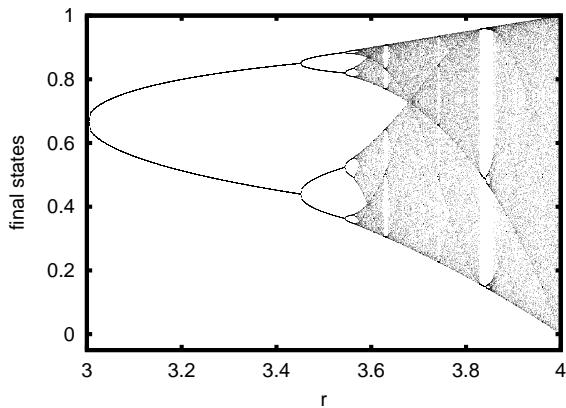
### Bifurcation Diagram



- The bifurcation diagram shows all the possible long-term behaviors for the logistic map.
- $0 < r < 1$ , the orbits are attracted to zero.
- $1 < r < 3$ , the orbits are attracted to a non-zero fixed point.
- $3 < r < 3.45$ , orbits are attracted to a cycle of period 2.
- Chaotic regions appear as dark vertical lines.

### Bifurcation diagram, continued

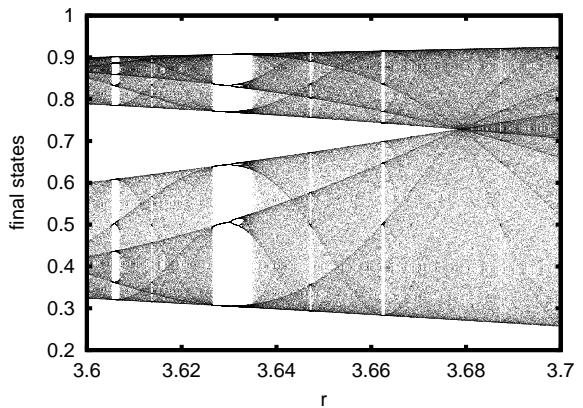
Let's zoom in on a region of the bifurcation diagram:



- The sudden qualitative changes in the long-term behavior are known as **bifurcations**.
- Example: There is a **period-doubling bifurcation** at  $r \approx 3.45$ ,  $r \approx 3.544$ , etc.
- Note the window of period 3 near  $r = 3.83$ .

### Bifurcation diagram, continued

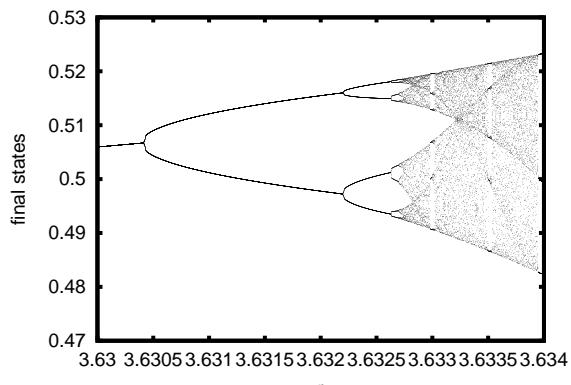
Let's zoom in again:



- Note the sudden changes from chaotic to periodic behavior.

### Bifurcation diagram, continued

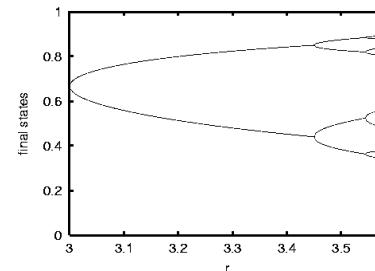
Let's zoom in once more:



- Note the small scales on the vertical axis, and the tiny scale on the y axis.
- Note the self-similar structure. As we zoom in we keep seeing sideways pitchforks.
- As we vary  $r$ , the logistic equation shuffles suddenly between chaotic and periodic behaviors, but the bifurcation diagram reveals that these transitions appear in a structured way.

### Period-Doubling Route to Chaos

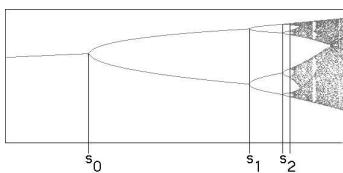
- As  $r$  is increased from 3, a sequence of period doubling bifurcations occur.



- At  $r = r_\infty \approx 3.569945672$  the periods “accumulate” and the map becomes chaotic.
- For  $r > r_\infty$  it has SDIC. For  $r < r_\infty$  it does not.
- This is a type of **phase transition**: a sudden qualitative change in a system’s behavior as a parameter is varied continuously.

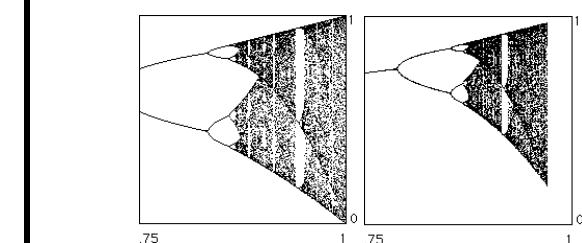
### Period-Doubling Route to Chaos: Geometric Scaling

- Let's examine the ratio of the lengths of the pitchfork tines in the bifurcation diagram.



- The first ratio is:  $\delta_1 = \frac{s_1 - s_0}{s_2 - s_1}$ .
- The  $n^{\text{th}}$  ratio is:  $\delta_n = \frac{s_n - s_{n-1}}{s_{n+1} - s_n}$ .
- This ratio approaches a limit:  $\lim_{n \rightarrow \infty} \delta_n = 4.669201609 \dots$  This is known as **Feigenbaum's constant**  $\delta$ .
- This means that the bifurcations occur in a regular way.
- Amazingly, the value of  $\delta$  is **universal**: it is the same for any period-doubling route to chaos!
- Figure Source: <http://classes.yale.edu/fractals/Chaos/Feigenbaum/Feigenbaum.html>

### Universality



- The figure on the left is the bifurcation diagram for  $f(x) = r \sin(\pi x)$ .
- The figure on the right is the bifurcation diagram for  $f(x) = \frac{27}{4}rx^2(1-x)$ .
- The bifurcation diagrams are very similar.
- Both bifurcation diagrams have  $\delta \approx 4.6692$ .
- Mathematically, things are constrained so that there is, in some sense, only one possible way for a system to undergo a period-doubling to chaos.
- Figure Source: <http://classes.yale.edu/fractals/Chaos/LogUniv/LogUniv.html>

### Experimental Verification of Universality

- Universality isn't just a mathematical curiosity. Physical systems undergo period-doubling order-chaos transitions. Almost miraculously, these systems also appear to have a universal  $\delta$ .
- Experiments have been done of fluids, circuits, acoustics:
  - Water:  $4.3 \pm .8$
  - Mercury:  $4.4 \pm .1$
  - Diode:  $4.5 \pm .6$
  - Transistor:  $4.5 \pm .3$
  - Helium:  $4.8 \pm .6$

Data from Cvitanović, *Universality in Chaos*, World Scientific, 1989.

- A very simple equation, the logistic equation, has produced a quantitative prediction about complicated systems (e.g., fluid turbulence) that has been verified experimentally.
- Nature is somehow constrained.

(The next four slides are about phase transitions and power laws, and are a bit of detour.)

### A Little Bit More About Universality

- The order-disorder phase transition in the logistic map is not the only sort of phase transition that is universal.
- Second order (aka continuous) phase transitions are also universal.
- There are several different universality classes, each of which has different values for quantities analogous to  $\delta$ .
- The symmetry of the order parameter and the dimensionality of the space of the system determine the universality class.
- The order parameter is a quantity which is zero on one side of the transition and non-zero on the other.
- At the transition point, or **critical point**, some quantities (e.g., specific heat) usually diverge. The divergence is described by a power law. The exponents for these power laws are called **critical exponents**.
- At the critical point, the correlations between components of the system usually decay with a power law as the distance increases. Away from the critical point, the decay is exponential—much faster.

### A Little Bit about Power Laws

- At critical points, functions like the specific heat diverge with a power law.
- This divergence arises because the correlations between the system's parts is long range—the corrections decay with a power law, not an exponential.
- Power-law decay of correlations is an indication that the system is organized or complex.
- However, this does not mean that the only way that power law distributions can be formed is via long-range order or correlations or complexity.
- In fact, there are very simple mechanisms for producing complexity.

### Simple Ways to Make A Power Law Distribution

#### Exponentially Observing Exponential Distribution

- Suppose a quantity is growing exponentially:  

$$X(t) = e^{\mu t}.$$
- Suppose we measure the quantity at a random time  $T$ , obtaining the value  $\bar{X} = e^{\mu T}$ .
- Let  $T$  also be exponentially distributed:  

$$Pr(T > t) = e^{-\nu t}.$$
- Then the probability density for  $\bar{X}$  is given by  

$$f_{\bar{X}}(x) = kx^{-\mu/\nu-1}.$$
- Like magic, a power law has appeared.
- In general, there are lots of ways to make power laws by combining exponential distributions in different ways.
- See Reed and Hughes, Why power laws are so common in nature. Physical Review E 66:067103. 2002. <http://www.math.uvic.ca/faculty/reed/>.

## Simple Ways to Make A Power Law Distribution

### Multiplicative Noise:

- Define a random variable  $X$  as the product of a number of other random variables. In many cases  $X$  will be distributed with a power law.
- See, e.g., Sornette, Multiplicative processes and power laws. [arXiv.org/cond-mat/9708213](http://arXiv.org/cond-mat/9708213), 1998.

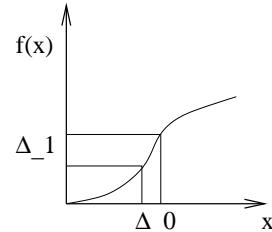
Power law conclusions:

- There are many simple, non-complex ways to make power laws.
- They are not necessarily an indicator of complexity or correlation.
- They are not necessarily an indicator of criticality—of a system on the edge of a phase transition.
- For an excellent review, see Newman, Power laws, Pareto distributions and Zipf's law, [arXiv.org/cond-mat/0412004](http://arXiv.org/cond-mat/0412004)

## Measuring Sensitive Dependence: Lyapunov Exponent

SDIC arises because the function pushes nearby points apart. The Lyapunov exponent measures this pushing.

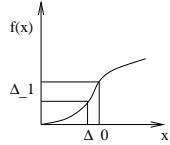
- Consider an initial small interval  $\Delta_0$  of initial conditions centered at  $x_0$ .



- After one iteration, this interval becomes  $\Delta_1 \approx |f'(x_0)|\Delta_0$ .
- $|f'(x_0)|$  is the local stretch (or shrink) factor.
- After  $n$  iterations,  $\Delta_n = \prod_n |f'(x_n)|\Delta_0$ .
- The idea is that for the  $n^{\text{th}}$  iterate interval is getting stretched (shrunk) by the stretch factor  $f'(x)$  evaluated at the  $x_n$ , the location of the  $n^{\text{th}}$  iterate of  $x_0$ .

## Lyapunov Exponent, continued

- We expect the growth of the interval  $\Delta_0$  to be exponential, since we're multiplying the interval at each time step.



- That is, we expect that  $\frac{\Delta_n}{\Delta_0} = e^{\lambda n}$ , where  $\lambda$  is the exponential growth rate.
- The exponential growth is just the product of all the local stretch factors along an itinerary:

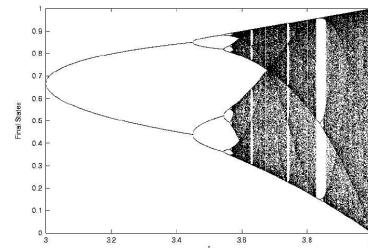
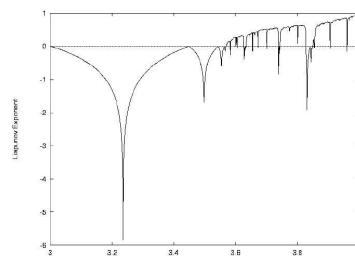
$$e^{\lambda n} = \prod_n |f'(x_n)| .$$

- Solving for  $\lambda$ :

$$\lambda \equiv \lim_{N \rightarrow \infty} \left[ \frac{1}{N} \sum_n \ln |f'(x_n)| \right] . \quad (1)$$

- $\lambda$  is the **Lyapunov exponent**. It measures the degree of SDIC.
- If  $\lambda > 0$ , the function has SDIC.

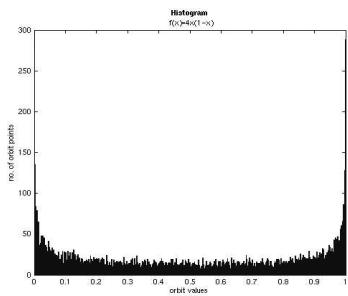
## Lyapunov Exponent for the Logistic Equation



- The top graph shows the Lyapunov exponent as a function of  $r$ .
- Note that  $\lambda > 0$  in the chaotic regions of the bifurcation diagram.

### Initial Conditions?

- It seems like the definition of  $\lambda$  depends on the initial condition. If so,  $\lambda$  is a property of  $x_0$ , and not a global property of  $f$ .
- It turns out that for many dynamical systems you will get the same  $\lambda$  for almost all  $x_0$ . Why is this?
- Imagine building a histogram for orbits. For  $r = 4$ , this will look like:



- <http://www-m8.mathematik.tu-muenchen.de/personen/hayes/chaos/Hist.html>
- Note that the the orbit spends more time near  $x = 0$  and  $x = 1$  than in the middle.

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<http://hornacek.coa.edu/dave>

### Natural Invariant Densities and Ergodicity

- The distribution in this histogram  $\rho(x)$  will be obtained by iterating almost any initial condition  $x_0$ .
- This distribution is known as the **Natural Invariant Density**.
- If we can figure it out, we can determine the Lyapunov exponent by integrating:

$$\lambda = \int \ln(|f'(x)|)\rho(x) dx .$$

- In general, if a dynamical property like the lyapunov exponent can be determined by integrating over  $x$  instead of performing a dynamical average, the system is **ergodic**.
- Proving that a system is ergodic is usually very hard.
- Trivia: for  $r = 4$ ,  $\rho(x) = \frac{\pi}{\sqrt{x(1-x)}}$ .
- For other  $r$  values, an expression for  $\rho(x)$  is not known. Generally,  $\rho(x)$  is non-smooth.

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### Symbolic Dynamics

- It is often easier to study dynamical systems via symbolic dynamics.
- The idea is to encode the continuous variable  $x$  with a discrete variable in some clever way that doesn't entail a loss of information.
- For the logistic equation

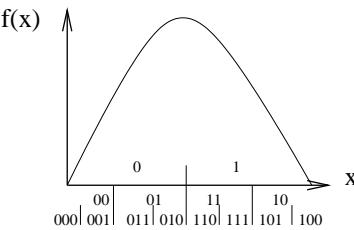
$$s_i = \begin{cases} 0 & x \leq \frac{1}{2} \\ 1 & x > \frac{1}{2} \end{cases} .$$

- Why is this ok? It seems that we're throwing out a lot of information!
  - The function is deterministic, the initial condition contains all information about the itinerary.
  - For the coding above, longer and longer sequences of 1's and 0's code for smaller and smaller regions of initial conditions.
  - Codings that have this property are known as **generating partitions**.

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### Symbolic Dynamics, continued



- If we find a generating partition, we can use the symbols to explore the function's properties.
- The symbol sequences are “the same” as the orbits of  $x$ : they have the same periodic points, the same stability, etc.
- For  $r = 4$ , the symbolic dynamics of the logistic equation produce a sequence of 0's and 1's that is indistinguishable from a fair coin toss.

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### Chaos Conclusions

- Deterministic systems can produce random, unpredictable behavior. E.g., logistic equation with  $r = 4$ .
- Simple systems can produce complicated behavior. E.g., long periodic behavior in logistic equation.
- Some features of dynamical systems are universal—the same for many different systems.

Some of the roots of complex systems are in chaos:

- Universality gives us some reason to believe that we can understand complicated systems with simple models.
- Appreciation that complex behavior can have simple origins.
- Awareness that there's more to dynamical systems than randomness. These systems also make patterns, organize, do cool stuff.
- Is there a way we can describe or quantify these patterns?
- What is a pattern?