Quest 3: Multi-Species Lotka–Volterra

Differential Equations

College of the Atlantic. February 5, 2026

Work on this in a team of four. Be ready to give a 15-20 minute presentation on this on Wednesday 11 February, 2026.

First, let's modify the Lotka-Volterra Equations so in the absence of the other species, each grows logistically. Recall that logistic growth is:

$$\frac{dP}{dt} = rP(1 - \frac{P}{K}) \ , \tag{1}$$

where P is the population, r a growth rate, and K is the carrying capacity.

However, imagine dividing both sides of the above equation by K, then we have

$$\frac{d\frac{P}{K}}{dt} = r\frac{P}{K}(1 - \frac{P}{K}) \ . \tag{2}$$

Let's define a new variable x, which is the population expressed as fraction of the carrying capacity. That is, $x = P/K$. Then the logistic differential equation takes a simpler form:

$$\frac{dx}{dt} = rx(1 - x) \ . \tag{3}$$

The main point is that we can make the K go away by re-scaling the population.

In light of this, we can write generalized Lotka-Volterra equations as:

$$\frac{dx_i}{dt} = r_i x_i \left(1 - \sum_{j=0}^{N-1} \alpha_{ij} x_j\right) \tag{4}$$

We'll consider the case $N = 3$ where we have four populations x_0, x_1, x_2, x_3. So the equations are:

$$\frac{dx_0}{dt} = r_0 x_0 - \alpha_{00} x_0^2 - \alpha_{01} x_0 x_1 - \alpha_{02} x_0 x_2 - \alpha_{03} x_0 x_3 \tag{5}$$

and

$$\frac{dx_1}{dt} = r_1 x_1 - \alpha_{10} x_0^2 - \alpha_{11} x_1 x_1 - \alpha_{12} x_1 x_2 - \alpha_{13} x_1 x_3 \tag{6}$$

and so on. Remember that because we have re-scaled variables by K, each x is always between 0 and 1.

This model has a lot of parameters. There are 4 r's and 16 α's. Gosh. Try these values:

$$r = \begin{bmatrix} 1 \\ 0.72 \\ 1.53 \\ 1.27 \end{bmatrix} \tag{7}$$

and

$$\alpha = \begin{bmatrix} 1 & 1.09 & 1.52 & 0 \\ 0 & 1 & 0.44 & 1.36 \\ 2.33 & 0 & 1 & 0.47 \\ 1.21 & 0.51 & 0.35 & 1 \end{bmatrix} \tag{8}$$

How will you represent α in python? Hmmm....

What does the model do? What behavior do you see? Are there any equilibria? If so, what is the stability? What happens if you plot two of the populations against each other, like we did when we plotted Mice vs. Owls? What if do a 3D plot of three of the variables?