Class Three–Four: Euler's Method and Solving Differential Equations
Computational Physics

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In this part of the course we turn our attention to differential equations (DEs).[1] DEs in physics are very common: Newton's 2 law, Maxwell's Equations, and Schrödinger's Equation, are all DEs. DEs are also very commonly used in mathematical modeling. Most DEs cannot be solved analytically and so we must use a computational approach.

In these notes I describe Euler's method: a simple but important computational method for solving DEs. Euler's method is not particularly fast or efficient. But I think learning about Euler's method is worthwhile because it gives one another way of thinking about what DEs are and what they mean. Euler's method also is a good starting point for thinking about more efficient solution methods.

What are DEs? First, let's think about an equation such as:

$$x^2 - 3x + 2 = 0 . \tag{1}$$

This is not a DE. Solving an equation like this means finding the values of x that make the equation true. In this case, there are two solutions: $x = 2$ and $x = 1$. A solution to an equation like Eq. (1) is a number.

In contrast, here is a differential equation:

$$\frac{d^2y}{dt^2} = -k^2 y . \tag{2}$$

The solution to a differential equation is an entire function. A DE indirectly specifies a function by stating how that function is related to its derivative. In this case, Eq. (2) says: y is a function which, if you take its derivative twice, you end up with the function times $-k^2$. Solving the DE means finding a function $y(t)$ that meets this description. We need to find the function that makes Eq. (2) true. The solution to a DE is a function.

For Eq. (2) we are lucky. It is not too hard to see that there are two solutions:

$$y_1(t) = A\sin(kx) , \quad \text{and} \quad y_2(t) = B\cos(kx) . \tag{3}$$

Here A and B are constants that would be determined by initial conditions or boundary conditions.

Using a Derivative to Approximate Future Values. Let's forget for a moment about differential equations and instead consider a simpler matter. Suppose you know that it is 26 degrees C outside. How hot will it be ten minutes later? Without any additional information, it would be difficult to make a prediction. But suppose you also know that the temperature is currently decreasing at 0.25 degrees per minute? Then you would predict that the temperature ten minutes from now would be 23.5. Why? Because

$$\begin{aligned}
\text{Temp. 10 min. from now} &= \text{Current temp} + \text{Change in temp} && (4)\\
&\approx \text{Current temp} + (\text{Time interval} \times \text{rate of change}) && (5)\\
&\approx 26\text{C} + (10\text{min} \times -0.25\text{C/min}) . && (6)
\end{aligned}$$

[1] We will focus exclusively on *ordinary* differential equations.

This is only an approximation. We know that the temperature is changing *right now* at -0.25 C/min. We don't know that that this rate of change is constant. But this is all the information we have, so the best we can do is assume that it is constant and hope for the best. Intuitively, this approximation will work better for smaller time intervals. I.e., this might be a good way to approximate the temperature in one minute, but would probably be a very bad way to approximate the temperature in one day.

We can generalize the above equations by writing:

$$y(10) \approx y(0) + (10y'(0)) \, , \tag{7}$$

where I am using $y(t)$ for temperature. What if we were interested in the temperature Δt minutes from now? Then we would write:

$$y(\Delta t) \approx y(0) + (\Delta t y'(0)) \, , \tag{8}$$

What we are doing is using a linear function to approximate the unknown function $y(t)$. The right hand side of the above equation is just the equation for the line that is tangent to $y(t)$ at $t = 0$.

Finally, if we were starting at some general t value, not necessarily $t = 0$, then this equation would read:

$$y(t + \Delta t) \approx y(t) + (\Delta t y'(t)) \, , \tag{9}$$

The above several equations are not specific to differential equations. We are just using basic calculus to approximate a curve by a line.

Euler's method. Equation (8) is the key idea behind Euler's method. We will approximate an unknown function—in this case the solution to a DE—by a line. We will do this repeatedly, and in so doing will build up an approximation to the function. Let's start by considering the following DE:

$$y'(t) = -ty \, , \quad y(0) = 2 \, . \tag{10}$$

Our task is to find $y(t)$. Since we are working numerically, we will not get an equation for $y(t)$. Instead, we will represent $y(t)$ by a table of numbers: a list of numerical values for $y(t)$ at different values of t. We are given the initial condition $y(0) = 2$, so we might as well start here. We would like to use Eq. (8) to estimate the value of $y(t)$ at some subsequent time Δt. To do so, we need two additional pieces of information: Δt and $y'(0)$. The value of Δt is up to us. We get to choose it. How do we choose? We want it to be small enough that the derivative is roughly constant over the time interval Δt. Smaller is more accurate, but will take longer for the computer to run. I'll say more about this below. For now, let's choose $\Delta t = 0.5$.

The other thing we need so we can use Eq. (8) is $y'(0)$, the value of the derivative at $t = 0$. To figure this out we just consult the differential equation. A differential equation specifies a function's derivative. So we have easy access to this information. Equation 10 tells us that:

$$y'(0) = -(0)(2) = 0 \, . \tag{11}$$

In the above equation I plugged in $t = 0$ and $y = 2$ into Eq. (10). Thus, the slope at $t = 0$ is zero. So we can now approximate the value of y at $t = 0.25$ via Eq. (8):

$$y(0.25) \approx 2 + (0.25)(0) = 2 \, . \tag{12}$$

So for our approximate solution we have $y(0.25) = 2$.

Time t	$y(t)$ from Euler method	Exact $y(t)$	Percent Error
0	2.000	2.000	0.000
0.25	2.000	1.938	0.032
0.5	1.875	1.765	0.062
0.75	1.641	1.510	0.087
1	1.333	1.213	0.099
1.25	1.000	0.916	0.092
1.5	0.687	0.649	0.059
1.75	0.430	0.433	-0.007

Table 1: Solutions to Eq. (10) via Euler's method with $\Delta t = 0.25$ and the exact solution.

We now repeat this procedure. We know that $y(0.25) = 2$. We use the differential equation to determine $y'(0.25)$:

$$y'(0.25) \;=\; -(0.25)(2) \;=\; -0.5 \;. \tag{13}$$

We use this fact in Eq. (9) to obtain:

$$y(0.5) \;\approx\; y(0.25) + (\Delta t)(y'(0.25)) \;=\; 2 + (0.25)(-0.5) \;=\; 1.875 \;. \tag{14}$$

Let's do one more step. To go from $y(0.5)$ to $y(0.75)$, we determine the derivative at $t = 0.5$ from the differential equation:

$$y'(0.5) \;=\; -(0.5)(1.875) \;=\; -0.9375 \;. \tag{15}$$

And then $y(0.75)$ is given by:

$$y(0.75) \;\approx\; y(0.5) + (\Delta t)(y'(0.5)) \;=\; 1.875 + (0.25)(-0.9375) \;=\; 1.6406 \;. \tag{16}$$

We can continue in this way, figuring out approximate values for $y(t)$ step by step. Doing so, I get the numbers listed in the Table. I would suggest taking a moment to calculate $y(1.0)$ and $y(1.25)$ to be sure that you understand how the method works. (You may get slightly different answers if you round differently.)

In Table 1 I have also shown numerical values for the exact solution.[2] We can see that Euler solution is not too bad. The error is never larger than 10 percent. But the solution isn't great, either. What can we do to get a more accurate solution?

Errors arise from our assumption that the rate of change of the function is constant over the time interval Δt. In truth, the rate of change $y'(t)$ is continually changing. We can thus make our approximation better if make Δt smaller. Then the derivative will change less during Δt, and so Eq. (9) is a better approximation.

Let's try this out for our differential equation. Let's choose a Δt of 0.1. The results of doing so are shown in Table 2.

Convergence and Related Issues. The Euler method will give a solution that converges to the true solution in the limit that $h \to 0$. However, the computer cannot take this limit. So we need to think about how quickly this method converges. How small does h have to be before we start getting good results? It turns out that the convergence is a linear function of h. This means that if we want a solution that is ten times as accurate, then we need to make h ten times as small. This is somewhat bad news, since it means that the computer will need to work ten times as hard, and so it will take ten times as long.

[2] This differential equation is separable, and so is not too difficult to solve. This is the exception, though. Most differential equations cannot be solved via analytic methods.

Time t	$y(t)$ from Euler method	Exact $y(t)$	Percent Error
0	2.000	2.000	0.000
0.1	2.000	1.990	0.005
0.2	1.980	1.960	0.010
0.3	1.940	1.912	0.015
0.4	1.882	1.846	0.019
0.5	1.807	1.765	0.024
0.6	1.717	1.671	0.028
0.7	1.614	1.565	0.031
0.8	1.501	1.452	0.033
0.9	1.381	1.334	0.035
1	1.256	1.213	0.036

Table 2: Solutions to Eq. (10) via Euler's method with $\Delta t = 0.1$ and the exact solution.

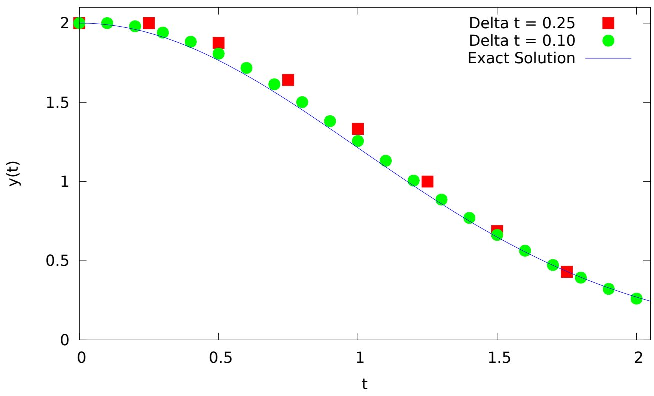

Figure 1: The exact solution to Eq. (10) and approximate solutions obtained via Euler's method with $\Delta t = 0.25$ and 0.10.

In practice, one usually doesn't know the exact solution. After all, if we knew the exact solution then we wouldn't need to use a numerical method like Euler. What can we do to make sure that the method is giving us reasonable solutions? And how should we choose h? Here is some general advice:

1. Start with a reasonably large h and then let it get smaller and smaller. In each case, plot the approximate solution $y(t)$. At some point $y(t)$ should stop changing. This is an indication that your solution has converged.
2. As usual, make use of any limiting cases for which you can figure out an exact solution. This is a good check to make sure the matlab code is working properly.
3. Check to make sure that your solution is behaving as expected from a mathematical and a physical point of view. Use your knowledge of the physics of the situation.
4. Euler's methods (and other solution methods) have a difficult time when the derivative (i.e., the slope) changes quickly. So look out for a rapidly changing slope. Differential equations that have suddenly changing derivatives can be difficult to solve numerically.

We will talk later today or next class about several ways that Euler's method can be improved.