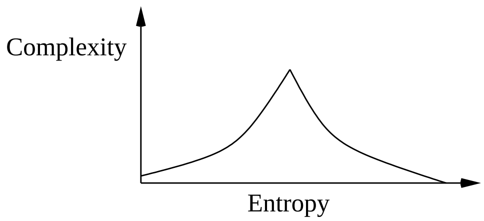
Complexity vs. Entropy

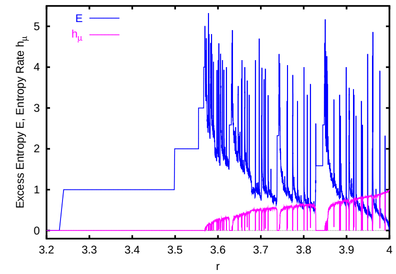
- The main goal of this lecture is to explore the relationships between complexity and entropy.
- It has been conjectured that there is the following general relationship between complexity and entropy:

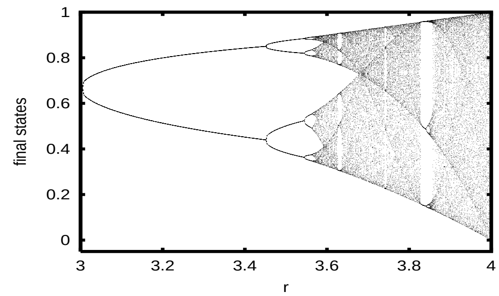


- Intuitively, one might expect that complexity, usually being a combination of order and disorder, is maximized in the middle.
- It is also sometimes hoped that there would be a sharp (perhaps even universal) phase transition in complexity as a function of entropy.
- Let's see what sorts of complexity-entropy relationships there are.

Complexity vs. Entropy: Logistic Equation

Plot the excess entropy $\mathbf{E}$ and the entropy rate h_μ for the logistic equation as a function of the parameter r.





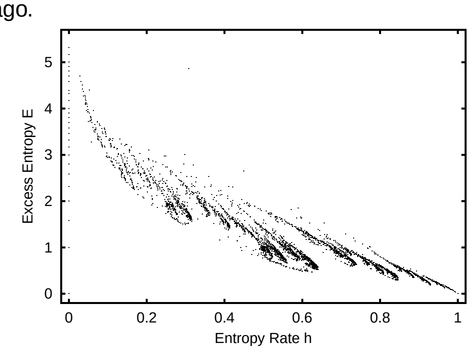
Hard to see how complexity and entropy are related.

Complexity-Entropy Diagrams

- Plot complexity vs. entropy. This will directly reveal how complexity is related to entropy.
- This is similar to the idea behind phase portraits in differential equations: plot two variables against each other instead of as a function of time. This shows how the two variables are related.
- It provides a parameter-free way to look at the intrinsic information processing of a system.
- Complexity-entropy plots allow comparisons across a broad class of systems.
- Most of the figures that follow are from Feldman, et al., "Organization of Intrinsic Computation", in preparation. Available August 2005?

Complexity-Entropy Diagram for Logistic Equation

- Excess entropy $\mathbf{E}$ vs. entropy rate h_μ from two slides ago.



- Structure is apparent in this plot that isn't visible in the previous one.
- Not all complexity-entropy values can occur.
- There seems to be a clear forbidden region.
- Maximum complexity occurs at zero entropy.
- Note the self-similar structure. This isn't surprising, since the bifurcation diagram is self-similar.

Ising Models

Consider a one- or two-dimensional Ising system with nearest and next nearest neighbor interactions:

- This system is a one- or two-dimensional lattice of variables $s_i \in \{\pm 1\}$.
- The energy of a configuration is given by:

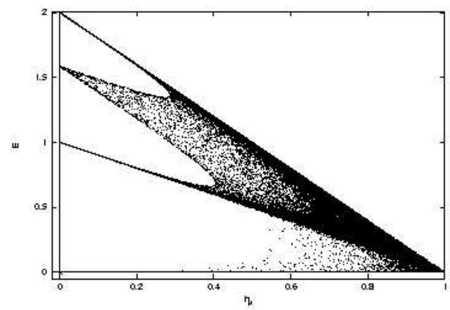
$$\mathcal{H} \equiv -J_1 \sum_i s_i s_{i+1} - J_2 \sum_i s_i s_{i+2} - B \sum s_i \ .$$

- The probability of observing a configuration $\mathcal{C}$ is given by the Boltzmann distribution:

$$\Pr(\mathcal{C}) \propto e^{-\frac{1}{T}\mathcal{H}(\mathcal{C})} \ .$$

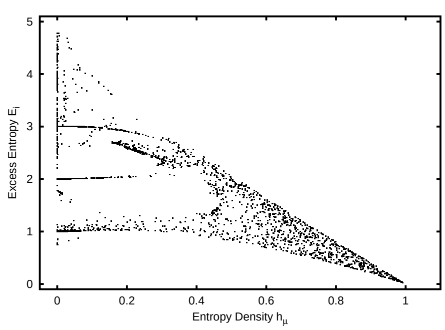
- Ising models are very generic models of spatially extended, discrete degrees of freedom that have some interaction between the variable that makes them want to either do the same or the opposite thing.

Complexity-Entropy Diagram for 1D Ising Models



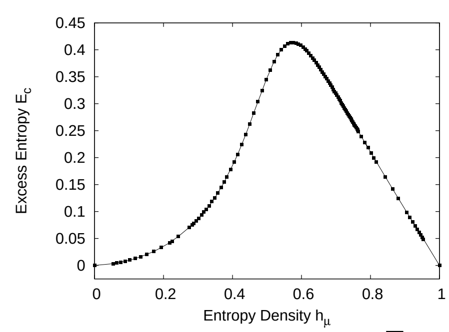
- Excess entropy $\mathbf{E}$ vs. entropy rate $H - \mu$ for the one-dimensional Ising model with anti-ferromagnetic couplings.
- Model parameters are chosen uniformly from the following ranges: $J_1 \in [-8, 0]$, $J_2 \in [-8, 0]$, $T \in [0.05, 6.05]$, and $B \in [0, 3]$.
- Note how different this is from the logistic equation.
- These are exact transfer-matrix results.

Complexity-Entropy Diagram for 2D Ising Models



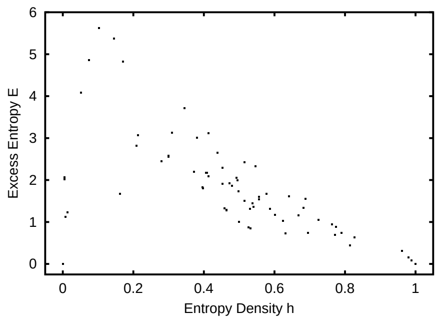
- Mutual information form of the excess entropy $\mathbf{E}_\mathrm{i}$ vs. entropy density h_μ for the two-dimensional Ising model with AFM couplings
- Model parameters are chosen uniformly from the following ranges: $J_1 \in [-3, 0]$, $J_2 \in [-3, 0]$, $T \in [0.05, 4.05]$, and $B = 0$.
- Surprisingly similar to the one-dimensional Ising model.
- Results via Monte Carlo simulation of 100x100 lattice.

Complexity-Entropy Diagram for 2D NN Ferromagnetic Ising Model



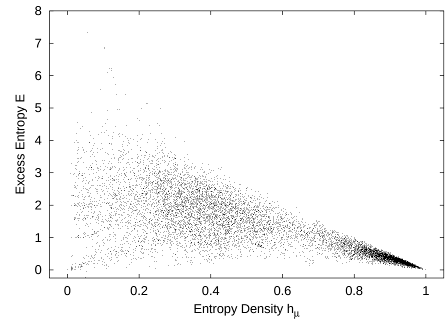
- Convergence form of the excess entropy $\mathbf{E}_\mathrm{i}$ vs. entropy density h_μ for the two-dimensional Ising model with NN couplings.
- Temperature is swept from 0 to 6.
- Model undergoes continuous phase transition at $T \approx 2.269$, or $h_\mu \approx 0.57$.
- There is a peak in the excess entropy, but it is somewhat broad.
- Results via Monte Carlo simulation of 100x100 lattice.

Complexity-Entropy Diagram for Elementary Cellular Automata



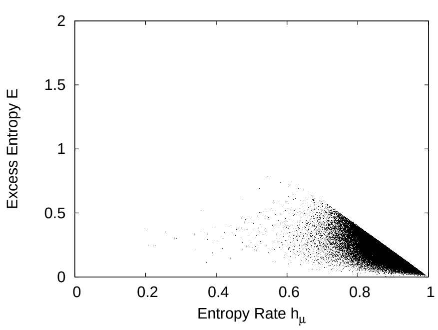
- Excess entropy $\mathbf{E}$ vs. entropy rate h_μ for all elementary cellular automata.
- $\mathbf{E}$ and h_μ from the spatial strings produced by the CAs.

Complexity-Entropy Diagram for Radius-2, 1D CAs



- Excess entropy $\mathbf{E}$ vs. entropy rate h_μ for $10,000$ radius-2, binary cellular automata.
- $\mathbf{E}$ and h_μ from the spatial strings produced by the CAs.
- The CAs were chosen uniformly from the space of all such CAs.
- There are around $10^{30,000}$ such CAs, so it is impossible to sample the entire space.

Complexity-Entropy Diagram for Markov Models



- Excess entropy $\mathbf{E}$ vs. entropy rate h_μ for $100,000$ random Markov models.
- The Markov models here have eight states, corresponding to dependence on the previous three symbols for a binary system.
- Transition probabilities between states were chosen uniformly on $[0,1]$ and then normalized.
- Note that these systems will never have any forbidden sequences.

Edge of Chaos?

Is there an edge of chaos to which systems naturally evolve? My very strong hunch is no, not in general. See the following pair of papers.

- Packard, “Adaptation to the Edge of Chaos” in *Dynamic Patterns in Complex Systems*, Kelso et.al, eds., World Scientific, 1988
- Mitchell, Hraber, and Crutchfield “Revisiting the ‘Edge of Chaos’ ” *Complex Systems*, 7:89-130, 1993. (Response to Packard, 1988).

Transitions in CA Rule Space?

- Is there a sharp complexity transition in CA rule space? No, unless you parametrize the space of CAs in a very particular way. The “transition,” then, is a result of the parametrization and not the space itself.

Transitions in CA Rule Space References

- Langton. “Computation at the Edge of Chaos,” *Physica* D (1990).
- Li, Packard and Langton, “Transition Phenomena in Cellular Automata Rule Space” *Physica D* 45 (1990) 77.
- Wooters and Langton, “Is there a Sharp Phase Transition for Deterministic Cellular Automata?”, *Physica D* 45 (1990) 95.
- Crutchfield, “Unreconstructible at Any Radius”, *Phys. Lett.* A 171: 52-60, 1992.
- Feldman, et al, “Organization of Intrinsic Computation.” In preparation.

Complexity-Entropy Diagrams: Conclusion

- There does not appear to be a universal complexity-entropy curve.
- It is not always the case that there is a sharp complexity-entropy transition.
- It is sometimes but not always the case that complexity is maximized when a phase transition occurs.
- Complexity-entropy diagrams provide a way of comparing the information processing abilities of different systems in a parameter-free way.
- When considering model classes that are very large, how one samples the space is important.
- Nature probably does not choose typical parameter values from model classes.
- In model systems, one observes a broad range of complexity-entropy behaviors.