Stats 110 Strategic Practice and Homework 1

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1 Strategic Practice

1.1 Naive Definition of Probability

1. (a) The probability of rolling 21 with 4 fair dice is > The probability of rolling 22 with for 4 fair dice.

   4 fair dice have a sum in the range $[4, 24]$ with $6^4 = 1296$ combinations. A 21 can be constructed from the partitioned into $(6,6,6,3),(6,6,5,4),(6,5,5,5)$; while 22 can partitioned into $(6,6,6,4),(6,6,5,5)$ For example, for $(6,6,6,3)$ there are 4 possibilities: 4 dice, choose one to be 3, the rest are 6. For example, for $(6,6,5,4)$ there are $\frac{4!}{2!} = 12$ possibilities because there are 4 dice, you must choose one to be 5, one to be 4, and the rest to be 6. Divide by 2! to account for the possible orderings of 6 due to repetition. For example, for $(6,6,5,5)$ there are $\frac{4!}{2!2!} = 6$ possibilities because there are two possible values and order does not matter. Therefore, there are $4 + 4 + 12 = 20$ ways to get 22. Therefore, there are $4 + 6 = 10$ ways to get 21.

   (b) The probability of a random 2 letter word being a palindrome = The probability of a random 3 letter word being a palindrome.

   For a two letter random word, there are 26^2 outcomes. A two letter word is a palindrome when both letters are the same, so there are 26 palindrome and the probability is $\frac{26}{26^2}$. For a three letter random word, there are 26^3 outcomes. A three letter word is a palindrome when the first and third letter are the same, so there are 26 letters, then for each combination, the second letter can be any of the 26 letters. Then by the multiplication rule, there are $26 * 26$ palindrome and the probability is $\frac{26^2}{26^3}$

$$P(2\ \ letter\ \ palindrome) = P(3\ \ letter\ \ palindrome) = \frac{26}{26^2} = \frac{26^2}{26^3}$$

A three letter palindrome is an expansion of a two letter palindrome.

2. Random 5 card poker hand from a standard deck

   (a) A flush (all 5 cards of same suit, excluding royal flush)

   There are 4 suits. For each suit there are 13 cards. There $\binom{13}{5}$ ways to choose a 5 card hand for each suit. 1 hand is excluded, the royal flush.
   The probability is:
$$\frac{(\#suits * (allowable\ \ hands)}{all\ \ possible\ \ hands}$$
$$\frac{4(\binom{13}{5} - 1)}{\binom{52}{5}}$$

   (b) Two pair

   Choose 2 ranks of 13 possible. Choose 2 of the 4 possible cards for each rank. Choose 1 of the remaining cards that is not of the 2 selected ranks. Divide by all possible hands.
$$\frac{\binom{13}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 44}{\binom{52}{5}}$$

3. Paths on a grid from $(0,0)$ to $(110,111)$ where the only moves are one unit up or one unit down.

   (a) Encode the moves as such: U for up and R for right. There are $110-0$ U's and $111-0$ R's. Therefore the set of moves can be written as: $URURURRRR...UR$. There are then $110+111=221$ symbols to be positioned. To determine the number of paths choose a symbol to position:
$$\binom{221}{110} = \binom{221}{111}$$

   (b) Paths from $(0,0)$ to $(210,211)$ that go through $(110,111)$

   From above there are $\binom{221}{110}$ paths from $(0,0)$ to $(110,111)$. Now merely need to find the number of paths from $(110,111)$ to $(210,211)$. This is then $210-110=$

100 more R's and $211 - 111 = 100$ more U's. Therefore $\binom{200}{100}$ paths. By the multiplication rule then there are

$$\binom{221}{110} \cdot \binom{200}{100}$$

total paths.

4. A *no-repeat-word* is a sequence of 26 letters unique and with no repetition. Show that the probability that a *no-repeat-word* using all 26 letters selected at random with all *no-repeat-word* equally likely is very close to $1/e$.

   There are 26! total *no-repeat-words* of length 26. To create a *no-repeat-word* of length k, first select k letters from the 26. $\binom{26}{k}$. Then those letters can be arranged into $k!$ words. Therefore there are $\binom{26}{k}k!$ *no-repeat-words* with k letters. To count all possible *no-repeat-words*, simply sum all the possible words of each possible length: $[1, 26]$. Then, the probability of a *no-repeat-word* having all 26 letters is:

$$= \frac{26!}{\sum_{k=1}^{26} \binom{26}{k} k!} = \frac{26!}{\sum_{k=1}^{26} \frac{26!}{k!(26-k)} k!}$$

$$= \frac{26!}{26! \cdot \sum_{k=1}^{26} \frac{k!}{k!(26-k)!}} = \frac{1}{\sum_{k=1}^{26} \frac{1}{(26-k)!}}$$

$$= \frac{1}{\frac{1}{25!} + \frac{1}{24!} + \cdots + \frac{1}{1!} + 1} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{25!}$$

   which is equivalent to the Taylor series $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{25}}{25!}$ evaluated at $x = 1$.

1.2 Story Proofs

1. Story Proof for: $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. This represents choosing 0 to n from a set of n objects, you are creating all possible sets of combinations. One way to visualize this is as n slots for each object. Then you can specify that an object is contained in a set if it is given a value of 1 and excluded if given a value of 0. Then you are merely creating binary numbers from $0, 0, 0, 0, ..., 0$ (n 0's) to $1, 1, 1, 1, ..., 1$ (n 1's). There are then 2^n combinations.

2. Story Proof for:

$$\frac{(2n)!}{2^n \cdot n!} = (2n-1)(2n-3)\cdots 3 \cdot 1.$$

$$= \frac{(2n)!}{2^n \cdot n!}$$

$$= \frac{(2n)(2n-1)(2n-3)(2n-4)(2n-5)\cdots(3)(2)(1)}{2^n \cdot (n)(n-1)(n-2)(n-3)\cdots(3)(2)(1)}$$

$$= \frac{(2n)(2n-2)(2n-4)\cdots(2)\cdots(2n-1)(2n-3)\cdots(3)(1)}{2^n \cdot (n)(n-1)(n-2)(n-3)\cdots(3)(2)(1)}$$

$$= \frac{(2(n))(2(n-1))(2(n-2))\cdots(2\cdot 1)\cdot(2n-1)(2n-3)(2n-5)\cdots(3)(1)}{2^n \cdot (n)(n-1)(n-2)(n-3)\cdots(3)(2)(1)}$$

$$= \frac{\cancel{2^n \cdot (n)((n-1))((n-2))\cdots(1)} \cdot (2n-1)(2n-3)(2n-5)\cdots(3)(1)}{\cancel{2^n \cdot (n)(n-1)(n-2)(n-3)\cdots(3)(2)(1)}}$$

$$= (2n-1)(2n-3)(2n-5)\cdots(3)(1)$$

3. Story Proof for:

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

This represents the scenario where you have $n+1$ people, where n are members of the club, and 1 is the president. How many ways then are there to select a group of size k? You can decide to select k people from the n, without the president. This would represent groups of size k from n excluding the president. You can decide to select first the president, then the remaining $k-1$ people to form a group of k people from n. The number of groups of size k from $n+1$ people where one is labeled, is the number of groups of size k that exclude the labeled person + the number of groups of size $k-1$ that include the labeled person.

$$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!}$$

$$= \frac{n![(n-k+1)!(k-1)! + (n-k)!k!]}{(n-k)!k!(n-k+1)!(k-1)!}$$

$$= \frac{n![(n-k+1)(n-k)!(k-1)! + (n-k)!k(k-1)!]}{(n-k)!k!(n-k+1)!(k-1)!}$$

$$= \frac{n!(n-k)!(k-1)![(n-k+1)+k]}{(n-k)!k!(n-k+1)!(k-1)!}$$

$$= \frac{n!\cancel{(n-k)!(k-1)!}[(n-k+1)+k]}{\cancel{(n-k)!(k-1)!}k!(n-k+1)!}$$

$$= \frac{n!(n+1)}{k!(n-k+1)!}$$

$$
\begin{aligned}
&= \frac{(n+1)!}{k!(n+1-k)!} \\
&= \binom{n+1}{k}
\end{aligned}
$$

2 Homework

1. 6 Children, 3 boys and 3 girls. There are 6! permutations. For the permutations of consideration, the first 3 must be girls, and the last 3 must be boys. There are 3! orderings for the girls and 3! orderings for the boys. The probability that the 3 eldest are girls is $\frac{(3!)(3!)}{6!}$

2. (a) Splitting 12 people into 3 teams: 2 of size 5 and 1 of size 2. Divide by 2! because there are 2 teams of 5 and order does not matter.

$$= \binom{12}{2} \cdot \binom{10}{5} / 2$$

$$= \frac{12!}{2! \cdot 5! \cdot 5! \cdot 2}$$

   (b) Splitting 12 people into 3 teams of size 4. Divide by 3! because there are 3 teams and order does not matter.

$$= \binom{12}{4} \cdot \binom{8}{4} \cdot \binom{4}{4} / 3!$$

$$= \frac{12!}{4! \cdot 4! \cdot 4! \cdot 3!}$$

3. Probability of conflict. Each of the 10 times slots has the same probability of selection. The probability of a conflict is $1 - P(conflict^c)$. What is the probability of no conflict?

$$= \frac{10 \cdot 9 \cdot 8}{10 \cdot 10 \cdot 10}$$

   Then the probability of conflict is 1 minus that.

$$P(conflict) = 1 - P(conflict^c)$$

$$= 1 - \frac{10 \cdot 9 \cdot 8}{10 \cdot 10 \cdot 10}$$

4. Probability of 1 of 6 districts having more than of the 6 robberies. The probability of no districts having more than 1 robbery is:

$$P(at\ most\ one) = \frac{6!}{6^6}$$

   The probability of districts having more than 1 robbery is $1 - P(at\ most\ one)$.

$$P(some\ with\ more\ than\ one) = 1 - \frac{6!}{6^6}$$