Stats 110 Notes

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1.	1 Notes	

- - A Sample Space is the set of all possible outcomes of an experiment
 - An **event** is a subset of the sample space.
 - Naive Definition of Probability:

$$P(A) = \frac{\# \ of \ favorable \ outcomes}{\# \ possible \ outcomes}$$

• The Multiplication Rule - if you have experiment with n possible outcomes, and for each outcome of the first experiment there are n outcomes for the second experiments, ..., for each r^{th} experiment there are n_r outcomes, then there are $n_1 \cdot n_2 \cdot \cdot \cdot n_r$ overall possible outcomes. Assumes all outcomes are equally likely within a finite sample space

• Binomial Coefficient:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

0 if k > n

How do you get there? Number of subsets of size k of group n. Choosing k from n in order: $n \cdot (n-1)(n-2) \cdot \ldots \cdot (n-k+1)$ Now if order does not matter: $\frac{n \cdot (n-1)(n-2) \cdot \ldots \cdot (n-k+1)}{k!}$ Now through canceling we reach the formula for the binomial coefficient.

1.2 Example: Full House, 5 Card Hand

There are 13 suits. We must choose one suit then 3 of the 4 possible cards. Then we must choose another suit of the remaining 12, then select 2 of the 4 possible cards. Then divide by the sample space which is all possible 5 card hands from a deck of 52 cards.

$$\frac{13\binom{4}{3}\cdot 12\binom{4}{2}}{\binom{52}{5}}$$

1.3 Sampling Table

Sampling Table			
	Order Matters	Order Doesn't Matter	
Replace	n^k	$\binom{n+k-1}{k}$	
Don't Replace	$n(n-1)(n-2)\cdots(n-k+1)$	$\binom{n}{k}$	

2 Lecture 2: Story Proofs, Axioms of Probability

2.1 Notes

• Classic and Fundamental Relationship:

$$\binom{n}{k} = \binom{n}{n-k}$$

• Exploring picking k times from set of n objects with replacement where order doesn't matter. Let's try out a few cases:

Exploration

- Case 1: $k = 0 \to \binom{n-1}{0} = 1$

– Case 2: $k = 1 \rightarrow \binom{n}{1} = n$

- Case 3 (Simplest non-trivial example): $n = 2 \rightarrow \binom{k+1}{k} = \binom{k+1}{1} = k+1$

- This is equivalent to the number of ways to put k indistinguishable particles into n distinguishable boxes, which can be visualized as such:

The | represent partitions for the 'boxes' and the \cdot is a particle. There are then $k \cdot$'s and n-1 |'s. This can be simplified again to be thought of just arranging symbols, and you are choosing to place the symbols. There are k+n-1=n+k-1 symbols, so choose where to place the separators or the particles. Choose where the \cdot 's are, and the |'s are then positioned, and vice versa.

$$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$$

• Story Proof: proof by interpretation.

$$n\binom{n-1}{k-1} = k\binom{n}{k}$$

Pick k people out of n with designated as the president, so you can first pick the president out of n, then select who can be in the club. Or you can pick the club k out of n then choose the president from the k people.

- Vandermonde Identity:

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j} \binom{n}{k-j}$$

You have two groups with labeled people containing m and n people. You need to select k people total from both groups. The summation refers to the combinations of selecting j from m and k-j from n such that you have j+k-j=k total people from the m+n total people in two groups. Each instance is multiplied together due to the multiplication rule.

• The Non-Naive Definition of Probability

- A probability sample consists of **S** and **P** where **S** is a sample space and **P**, a function which takes an event $A \subseteq S$ as input and returns $P(A) \in [0,1]$ as output such that:

* (1)
$$P(\emptyset) = 0$$
, $P(S) = 1$
* (2)
$$P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$$

if $A_1, A_2, ..., A_n$ are disjoint.

3 Lecture 3: Birthday Problem, Properties Probability

3.1 Notes

• Birthday Problem: k people, find the probability that 2 have the same birthday. Exclude February 29, assume other 365 days are equally likely. If k > 365, the probability is 1 by the pigeonhole principle. If $k \leq 365$, the probability of no match:

$$P(M^c) = \frac{365 \cdot 364 \cdot 363 \cdots (365 - k + 1)}{365^k}$$

The probability of a match is: $1 - P(M^c)$

$$P(M) = 1 - P(M^c) = 1 - \frac{365 \cdot 364 \cdot 363 \cdots (365 - k + 1)}{365^k}$$

Imagine pairing up the people. There are $\binom{k}{2}$ pairs.

• Axioms:

1.
$$P(\emptyset) = 0, P(S) = 1$$

2. $P(\bigcup_{n=1}^{\infty} = \sum_{n=1}^{\infty} P(A_n))$ if A_i are disjoint.

• Properties:

1.
$$P(A^c) = 1 - P(A)$$
 Proof: $1 = P(S) = P(A \cup A^c) = P(A) + P(A^c)$ because $A \cap A^c = \emptyset$

2. If
$$A \subseteq B$$
, then $P(A) \le P(B)$. Proof: $B = A \cup (B \cap A^c)$, disjoint, then $P(B) = P(A) + P(B \cap A^c) \ge P(A)$

3.
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
. Proof: $P(A \cup B) = P(A \cup (B \cap A^c)) = P(A) + P(B \cap A^c)$

$$P(A) = P(A) + P(B) - P(A \cap B), P(A \cap B) + P(B \cap A^{c}) = P(B)$$

 $A \cap B$ and $A^c \cap B$ are disjoint and the union is B

• Inclusion-Exclusion: Expanding upon most recent item. Example with 3 sets: A, B, C.

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Generalized:

$$P(A_1 \cup A_2 \cup \cdots \cup A_n)$$

$$= \sum_{j=1}^{n} P(A_j) - \sum_{i < j} P(A_i \cap A_j) + \sum_{i < j < k} P(A_i \cap A_j \cap A_k) + \dots + (-1)^{n+1} P(A_1 \cap \dots \cap A_n)$$

• de Montmort's Matching Problem (link). What is the probability that the j^{th} card of n shuffled cards has the label j?

$$= 1 - 1/e$$