Stats 110 Notes

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1 Lecture 1: Probability and Counting

1.1 Notes

- A **Sample Space** is the set of all possible outcomes of an experiment
- An **event** is a subset of the sample space.
- **Naive Definition of Probability**:

$$P(A) = \frac{\#\ of\ favorable\ outcomes}{\#\ possible\ outcomes}$$

- **The Multiplication Rule** - if you have experiment with n possible outcomes, and for each outcome of the first experiment there are n outcomes for the second experiments, ..., for each r^{th} experiment there are n_r outcomes, then there are $n_1 \cdot n_2 \cdots n_r$

overall possible outcomes. Assumes all outcomes are equally likely within a finite sample space

- **Binomial Coefficient**:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

0 if $k > n$

How do you get there? Number of subsets of size k of group n. Choosing k from n in order: $n \cdot (n-1)(n-2) \cdot ... \cdot (n-k+1)$ Now if order does not matter: $\frac{n \cdot (n-1)(n-2) \cdot ... \cdot (n-k+1)}{k!}$ Now through canceling we reach the formula for the binomial coefficient.

1.2 Example: Full House, 5 Card Hand

There are 13 suits. We must choose one suit then 3 of the 4 possible cards. Then we must choose another suit of the remaining 12, then select 2 of the 4 possible cards. Then divide by the sample space which is all possible 5 card hands from a deck of 52 cards.

$$\frac{13\binom{4}{3} \cdot 12\binom{4}{2}}{\binom{52}{5}}$$

1.3 Sampling Table

Sampling Table		
	Order Matters	**Order Doesn't Matter**
Replace	n^k	$\binom{n+k-1}{k}$
Don't Replace	$n(n-1)(n-2)\cdots(n-k+1)$	$\binom{n}{k}$

2 Lecture 2: Story Proofs, Axioms of Probability

2.1 Notes

- Classic and Fundamental Relationship:

$$\binom{n}{k} = \binom{n}{n-k}$$

- Exploring picking k times from set of n objects with replacement where order doesn't matter. Let's try out a few cases:

  Exploration

  - Case 1: $k = 0 \rightarrow \binom{n-1}{0} = 1$
  - Case 2: $k = 1 \rightarrow \binom{n}{1} = n$
  - Case 3 (Simplest non-trivial example): $n = 2 \rightarrow \binom{k+1}{k} = \binom{k+1}{1} = k + 1$
  - This is equivalent to the number of ways to put k indistinguishable particles into n distinguishable boxes, which can be visualized as such:

    $$\cdots || \cdot\cdot | \cdot$$

    The $|$ represent partitions for the 'boxes' and the $\cdot$ is a particle. There are then k $\cdot$'s and $n-1$ $|$'s. This can be simplified again to be thought of just arranging symbols, and you are choosing to place the symbols. There are $k+n-1 = n+k-1$ symbols, so choose where to place the separators or the particles. Choose where the $\cdot$'s are, and the $|$'s are then positioned, and vice versa.

    $$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$$

- **Story Proof**: proof by interpretation.

  -
    $$n\binom{n-1}{k-1} = k\binom{n}{k}$$

    Pick k people out of n with designated as the president, so you can first pick the president out of n, then select who can be in the club. Or you can pick the club k out of n then choose the president from the k people.

- **Vandermonde Identity**:

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}$$

  You have two groups with labeled people containing m and n people. You need to select k people total from both groups. The summation refers to the combinations of selecting j from m and $k-j$ from n such that you have $j+k-j = k$ total people from the $m+n$ total people in two groups. Each instance is multiplied together due to the multiplication rule.

- **The Non-Naive Definition of Probability**
  - A probability sample consists of **S** and **P** where **S** is a sample space and **P**, a function which takes an event $A \subseteq S$ as input and returns $P(A) \in [0, 1]$ as output such that:
    * (1) $P(\emptyset) = 0$, $P(S) = 1$
    * (2)

$$P(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$$

    if $A_1, A_2, ..., A_n$ are disjoint.

3 Lecture 3: Birthday Problem, Properties Probability

3.1 Notes

- **Birthday Problem**: k people, find the probability that 2 have the same birthday. Exclude February 29, assume other 365 days are equally likely. If $k > 365$, the probability is 1 by the pigeonhole principle. If $k \leq 365$, the probability of no match:

$$P(M^c) = \frac{365 \cdot 364 \cdot 363 \cdots (365-k+1)}{365^k}$$

  The probability of a match is: $1 - P(M^c)$

$$P(M) = 1 - P(M^c) = 1 - \frac{365 \cdot 364 \cdot 363 \cdots (365-k+1)}{365^k}$$

  Imagine pairing up the people. There are $\binom{k}{2}$ pairs.

- **Axioms**:
  1. $P(\emptyset) = 0$, $P(S) = 1$

2. $P(\cup_{n=1}^{\infty} = \sum_{n=1} {}^{\infty} P(A_n)$ if A_i are disjoint.

- **Properties**:
  1. $P(A^c) = 1 - P(A)$ Proof: $1 = P(S) = P(A \cup A^c) = P(A) + P(A^c)$ because $A \cap A^c = \emptyset$
  2. If $A \subseteq B$, then $P(A) \leq P(B)$. Proof: $B = A \cup (B \cap A^c)$, disjoint, then $P(B) = P(A) + P(B \cap A^c) \geq P(A)$
  3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Proof: $P(A \cup B) = P(A \cup (B \cap A^c)) = P(A) + P(B \cap A^c)$

  $$? = P(A) + P(B) - P(A \cap B), P(A \cap B) + P(B \cap A^c) = P(B)$$

  $A \cap B$ and $A^c \cap B$ are disjoint and the union is B

- **Inclusion-Exclusion**: Expanding upon most recent item. Example with 3 sets: A, B, C.

  $$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

  Generalized:

  $$P(A_1 \cup A_2 \cup \cdots \cup A_n)$$

  $$= \sum_{j=1}^{n} P(A_j) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) + \cdots + (-1)^{n+1} P(A_1 \cap \cdots \cap A_n)$$

- **de Montmort's Matching Problem** (link). What is the probability that the j^{th} card of n shuffled cards has the label j?

  $$= 1/e$$

4 Lecture 4: Conditional Probability

4.1 Notes

- Independence
- $P(A \cap B) = P(A) \cdot P(B), P(B) > 0$
- Events A, B, C are independent if they are pairwise independent and completely independent:
  - $P(A \cap B) = P(A) \cdot P(B)$ and $P(A \cap C) = P(A) \cdot P(C)$ and $P(B \cap C) = P(B) \cdot P(C)$ and $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$
- Binomial Probability: Newton-Pepys (1693)
  - What is more likely? At least one six when rolling 6 dice, at least 2 sixes when rolling 12 dice, or at least 3 sixes when rolling 18 dice?
  - For at least 1 in 6, the probability is 1 - the probability of no six.

$$P(A) = 1 - \left(\frac{5}{6}\right)^6$$

  - For at least 2 in 12, the probability is 1 - the probability of no six - the probability of one six

$$P(B) = 1 - \left(\frac{5}{6}\right)^6 - \binom{12}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{11}$$

  - For at least 3 in 18, the probability is 1 - the probability of no six - the probability of one six - the probability of two sixes

$$\begin{aligned} P(C) &= 1 - \left(\frac{5}{6}\right)^6 - \binom{12}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^{15} - \binom{12}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{16} \\ &= 1 - \sum_{k=0}^{2}\binom{18}{k}\left(\frac{1}{6}\right)^k\left(\frac{5}{6}\right)^{18-k} \end{aligned}$$

- Conditional Probability
  -

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, P(B) > 0$$

  - Pebble World: Remove B^c and renormalize to 1
  - Frequentist: Circle reps where B occurred, find the fraction of those where A occurred.

– Theorems:

1. $$P(A \cap B) = P(B)P(A|B) = P(A)P(B|A)$$

2. $$P(A_1 \cap \cdots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap \cdots \cap A_{n-1})$$

3. $$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$