2018 Annual American Control Conference (ACC)
June 27–29, 2018. Wisconsin Center, Milwaukee, USA

B-Splines in Joint Parameter and State Estimation in Linear Time-Varying Systems*

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***Abstract*— A kernel functional representation of linear time-varying systems is employed in conjunction with B-spline functional approximation techniques to construct non-asymptotic state and parameter estimators for LTV systems. Total observability of the estimated system must be assumed. Practical identifiability conditions for parametric estimation are also stated. In the absence of output measurement noise the observer provides almost exact reconstruction of the system state and delivers high fidelity functional estimates of the time varying system parameters. It also shares the usual superior features of algebraic observers such as independence of the initial conditions of the system and good noise attenuation properties. Other advantages of the kernel and B-spline based identification of linear time-varying systems are elucidated.**

I. INTRODUCTION

Non-asymptotic state and parameter estimation is attracting increased interest as the development of more efficient control technologies poses new challenges. Rapidly switching or hybrid control strategies are best matched with dead-beat observers to deliver optimal closed loop performance. Recursive estimation algorithms, with the powerful family of Kalman filters feature good noise rejection properties, but require careful implementation and tuning, as any errors in the assumptions about the system model, noise characteristic, or *initial conditions of the system* can have highly destabilizing effect resulting in their slow convergence.

Non-asymptotic algebraic observers would be an obvious choice if not the consensus of opinion that accurate higher order differentiation of noisy signals is unrealistic. Still, the algebraic differentiation approach based on operational calculus, as first introduced in [1], [2] has remarkably good noise rejection properties as discussed in [3]. Further improvements in the direction of derivative estimation are offered by [4], [5] and lead to the development of high fidelity non-asymptotic state and parameter estimators for LTI systems that do not exhibit singularities and thus deliver fast estimates. A kernel linear system representation is proposed which exploits the obvious differential invariance in homogeneous linear systems that is a direct consequence of the validity of the Cayley Hamilton Theorem. Generally, a differential invariant of a dynamical system is a function $\mathcal{J}(t, y(t), y^{(1)}(t), ..., y^{n}(t))$ of time, output, and a number of its derivatives which remains constant under the action of the flow of the system. Any differential invariant can be regarded as a "deterministic signature" of the system that holds regardless of any uncertainties, a behavioral law that can be harnessed to help eliminate errors in measurement or numerical integration by simply enforcing its validity at all times.

A similar double-sided kernel representation of linear time-invariant systems employs the notion of a controlled invariance and captures the behaviour of systems with exogeuous inputs, as presented in [6]. The LTV kernel system representation is singularity free, delivers a recursive formula for computation of output derivatives and immediately yields a non-asymptotic state observer on any finite observation window, much like the one presented in [7], but is developed entirely in the time domain and exhibits improved performance. The estimation window is dragged forward as new output measurements arrive, which can be carried out safely in the absence of singularities. The observer does not need initialization as the notion of "initial conditions" is removed from the "behavioral" kernel system representation. The measurement noise is "filtered naturally" as all estimates are obtained as output functions of integral operators.

The results presented here build on the ideas of [4], [5], [8], [6] and extend the application of kernel system representations to joint state and functional parameter estimation in linear time-varying systems. The ensemble of results pertaining to parameter estimation in linear time-varying systems is relatively modest, see e.g. [9], [10], [11], [12] and references therein. The prevailing techniques include least squares optimization and involve assumptions about specific model structure e.g. convex polytopic structure (Takagi-Sugeno model) [11].

As the kernel system representation of [6] is linear in the system parameters, the approach presented here can be seamlessly combined with standard B-spline functional approximation. Under mild identifiability assumptions the sampled input–output data is employed in the kernel system representation to deliver linear algebraic equations for the values of the B-spline coefficients thereby reconstructing the unknown functional parameters in finite time with almost uniform accuracy. State reconstruction follows by way of calculating the time derivatives of the system output. The joint B-spline estimation approach is non-asymptotic.

*This work was supported by The National Science & Engineering Research Council of Canada

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978-1-5386-5428-6/$31.00 ©2018 AACC

The paper is organized as follows. The kernel representation of a SISO LTV system is presented first while explaining how it enables full state estimation based on output derivatives. The parameter estimator equations involving B-splines are presented next followed by the joint parameter and state estimation for LTV systems. Numerical example confirms remarkable properties of the proposed observer. The estimation method can use an estimation window of arbitrary length which can be dragged forward as necessary to deliver continued time varying estimates.

II. A KERNEL REPRESENTATION OF AN LTV SYSTEM WITH EXOGENOUS INPUT

A. Model Assumptions

The LTV system considered is assumed to be stated in the state space form:

$$\dot{x} = A(t)x + B(t)u \ ; \quad y = C(t)x \tag{1}$$

where the state, input and output are: $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^d$; with $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$, and $C(t) \in \mathbb{R}^{d\times n}$ are known, $n-1$ times differentiable matrix functions of time.

The following definitions and criteria for complete and total observability of a general system (1) can be found in [13] and [14]:

Definition 1

- System (1) is completely observable on a time interval $[t_0, t_f]$ if any initial state can be determined from the knowledge of the system output and input, $y(t)$ and $u(t)$, on the interval $t \in [t_0, t_f]$.
- System (1) is totally observable on a time interval $[t_0, t_f]$ if it is completely observable on any subinterval of $[t_0, t_f]$.

Theorem 1: [14] System (1) restricted to the time interval $[t_0, t_f]$ is:

- completely observable if $rank\ \mathcal{O}(t) = n$ for $t \in [t_0, t_f]$;
- totally observable if and only if $rank\ \mathcal{O}(t) = n$ on any subinterval of $[t_0, t_f]$;

where the observability matrix is defined by

$$\begin{aligned} &\mathcal{O} = \{S_0(t), S_1(t), \ldots, S_{n-1}(t)\} \\ &S_0(t) = C(t)^T; \\ &S_{k+1}(t) = A(t)^T S_k(t) + \dot{S}_k(t); k = 0, \ldots, n-2 \end{aligned} \tag{2}$$

B. A Controlled Differential Invariant - the Input-Output Equation

Henceforth, only single-input, single-output and strictly proper systems will be considered; i.e. $m = 1$, $m < n$, with $d = 1$ so $u, y \in \mathbb{R}$. With assumption of total observability, the state variable of system (1) can be expressed in terms of the system input, output, and their derivatives; see e.g. [7] for a simple proof of this fact:

Theorem 2: [7] Under the total observability assumption the input-output relation in system (1) is of the form:

$$\sum_{i=0}^{n} a_i(t)y^{(i)} + \sum_{i=0}^{m} b_i(t)u^{(i)} = 0 \quad \text{ for } t \in [t_0, t_f] \tag{3}$$

with $a_n = 1$, $m < n$. The state x can be expressed as a linear function of the input and output and their derivatives:

$$x(t) = \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \Gamma_{n-1} \end{pmatrix}^{-1} \left[\begin{pmatrix} y \\ y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(n-1)} \end{pmatrix} - M \begin{pmatrix} u \\ u^{(1)} \\ u^{(2)} \\ \vdots \\ u^{(n-2)} \end{pmatrix} \right] \tag{4}$$

where

$$\begin{aligned} &\Gamma_0(t) = C(t) \\ &\Gamma_k(t) = \left(\left(A(t)^T + \frac{d}{dt} \right)^k C(t)^T \right)^T \ ; \quad 0 < k < n \end{aligned} \tag{5}$$

$$M = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ \Delta_{11} & 0 & 0 & \ldots & 0 \\ \Delta_{21} & \Delta_{22} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta_{(n-1)1} & \Delta_{(n-1)2} & \Delta_{(n-1)3} & \ldots & \Delta_{(n-1)(n-1)} \end{pmatrix} \tag{6}$$

$$\Delta_{k0}(t) = \Gamma_k(t)B(t) \tag{7}$$

$$\Delta_{kj}(t) = \begin{cases} C(t)B(t); & \text{if } j = k \\ \Delta_{(k-1)(j-1)}(t) + \frac{d}{dt}\Delta_{(k-1)j}(t); & \text{if } 1 \le j < k \end{cases} \tag{8}$$

The invertibility of the matrix in (4) is guaranteed by total observability assumption on the system; see the definition of the observability matrix in Theorem 1.

Equation (3) constitutes a "controlled differential invariant" of system (1) which delivers a non-singular integral kernel representation of the system as well as integral transforms for computation of the output derivatives.

Theorem 3: There exist Hilbert-Schmidt kernels K_y, K_u, K_y^i, K_u^i, $i = 1, \cdots n-1$, and functions f_y^i, f_u^i, $i = 0, \cdots n-2$, defined respectively on $[a, b] \times [a, b]$ and $[a, b]$ such that the output y of system (1) satisfies the following integral equation on $[a, b]$

$$y(t) = \int_a^b K_y(t, \tau)y(\tau)\ d\tau + \int_a^b K_u(t, \tau)u(\tau)\ d\tau \tag{9}$$

while the derivatives of the output $y^{(1)}, \cdots y^{(n-1)}$ for $i = 1, \cdots n-1$ satisfy the recursive relationships :

$$y^{(i)}(t) = \sum_{k=0}^{i-1} f_y^k(t) y^{(k)}(t) + \sum_{k=0}^{i-1} f_u^k(t) u^{(k)}(t)$$
$$+ \int_a^b K_y^i(t,\tau) y(\tau)\ d\tau + \int_a^b K_u^i(t,\tau) u(\tau)\ d\tau \qquad (10)$$

□

See [8] for the complete proof of the above result.

Consider a second order input-output equation on $[a, b]$.

$$y^{(2)}(t) + a_1(t) y^{(1)}(t) + a_0(t) y(t)$$
$$+ b_1(t) u^{(1)}(t) + b_0(t) u(t) = 0 \qquad (11)$$

The "forward" equation for the second order input-output equation (11) on $[a, b]$ is given by

$$(t-a)^2 y(t)$$
$$= \int_a^t K_{Fy}(t,\xi) y(\xi) d\xi + \int_a^t K_{Fu}(t,\xi) u(\xi) d\xi \qquad (12)$$

where the kernel functions K_{Fy} and K_{Fu} have the expressions

$$K_{Fy}(t,\xi) = 4(\xi - a) - (\xi - a)^2 a_1(\xi) + (t - \xi)\{-2$$
$$+ 2(\xi - a) a_1(\xi) + (\xi - a)^2 a_1^{(1)}(\xi) - (\xi - a)^2 a_0(\xi)\} \qquad (13)$$

$$K_{Fu}(t,\xi) = -(\xi - a)^2 b_1(\xi) + (t - \xi)\{2(\xi - a) b_1(\xi)$$
$$+ (\xi - a)^2 b_1^{(1)}(\xi) - (\xi - a)^2 b_0(\xi)\} \qquad (14)$$

Similarly, the "backward" equation is given by

$$(b-t)^2 y(t)$$
$$= \int_t^b K_{By}(t,\zeta) y(\zeta) d\zeta + \int_t^b K_{Bu}(t,\zeta) u(\zeta) d\zeta \qquad (15)$$

where the kernel functions K_{By} and K_{Bu} are

$$K_{By}(t,\xi) = 4(b - \zeta) + (b - \zeta)^2 a_1(\zeta) + (t - \zeta)\{2$$
$$+ 2(b - \zeta) a_1(\zeta) - (b - \zeta)^2 a_1^{(1)}(\zeta) + (b - \zeta)^2 a_0(\zeta)\} \qquad (16)$$

$$K_{Bu}(t,\xi) = (b - \zeta)^2 b_1(\zeta) + (t - \zeta)\{2(b - \zeta) b_1(\zeta)$$
$$- (b - \zeta)^2 b_1^{(1)}(\zeta) + (b - \zeta)^2 b_0(\zeta)\} \qquad (17)$$

Adding equations (12) and (15) side by side while dividing both sides by $C_{[a,b]}(t) = [(t-a)^2 + (b-t)^2]$ yields the desired result (9) with the kernels given by

$$K_y(t,\tau) = \frac{1}{C_{[a,b]}(t)} \begin{cases} K_{Fy}(t,\tau) & \text{for } \tau \le t \\ K_{By}(t,\tau) & \text{for } \tau > t \end{cases} \qquad (18)$$

$$K_u(t,\tau) = \frac{1}{C_{[a,b]}(t)} \begin{cases} K_{Fu}(t,\tau) & \text{for } \tau \le t \\ K_{Bu}(t,\tau) & \text{for } \tau > t \end{cases} \qquad (19)$$

The kernels for the derivative expression is obtained similarly; see [8] for details.

It follows from equations (13) - (19) that the kernels K_y and K_u of Theorem 3 are linear functions of the system parameters $a_0(t), a_1(t), ..., a_{n-1}(t)$, $b_0(t), b_1(t), ..., b_{n-1}(t)$, and their time derivatives up to order $n-1$.

At this point the unknown time-varying parameters will be represented by their approximations - linear combinations of B-splines in appropriate spline spaces (where the B-splines constitute a functional basis). The construction of B-spline functions and a formula for their differentiation is provided next.

C. B-splines

A function $\mathrm{B}_{i,p,t}(x)$, $B_{i,p,t}(\cdot) : \mathbb{R} \to \mathbb{R}$ is called the i-th B-spline of order p with a knot sequence $t := \{t_1, \cdots, t_{p+1}\}, t_j \le t_{j+1}$; see [15] for a comprehensive introduction to B-spline spaces, their properties and applications.

The B-splines have the property that any spline function of order p on a given set of knots t can be expressed as a linear combination

$$S_{p,t}(x) = \sum_i \alpha_i B_{i,p,t} \qquad (20)$$

hence lending themselves well to function approximation or interpolation. To approximate a desired function over some inteval $[a, b]$ the splines should cover a superset of $[a, b]$.

Expressions for the polynomial pieces $B(i, p, t)$ can be derived by means of the Cox-de Boor recursion formula:

$$\mathrm{B}_{i,1,t}(x) = \begin{cases} 1 & \text{if } t_i \le x \le t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

Higher order B-splines are constructed using the recursion

$$\mathrm{B}_{i,p,t}(x) = \omega_{i,p}(x) \mathrm{B}_{i,p-1,t}(x) + (1 - \omega_{i+1,p}(x)) \mathrm{B}_{i+1,p-1,t}(x)$$

where $\omega_{i,p}(x)$ is given by

$$\omega_{i,p}(x) = \frac{x - t_i}{t_{i+p-1} - t_i}$$

The following differentiation formula is valid for B-splines

$$\frac{d}{dx}\left(\sum_{i=1}^n \alpha_i \mathrm{B}_{i,p,t}(x)\right) = \sum_{i=1}^n \alpha_i \mathrm{B}'_{i,p,t}(x) \qquad (21)$$

where $\mathrm{B}'_{i,p,t}(x)$ is given by

$$\mathrm{B}'_{i,p,t}(x) = \frac{p-1}{t_{i+p-1} - t_i} \mathrm{B}_{i,p-1,t}(x)$$
$$- \frac{p-1}{t_{i+p} - t_{i+1}} \mathrm{B}_{i+1,p-1,t}(x) \qquad (22)$$

While writing function approximations by B-splines it is convenient to omit indicating the order and knot sequence in the index of the B-splines, thus denoting the k-th B-spline by B_k while letting the order and knot sequence be clear from the context.

III. PARAMETER ESTIMATION FOR LTV SYSTEMS

For the purpose of estimation, the system parameters $a_0, \cdots, a_{n-1}$ and $b_0, \cdots, b_{n-1}$, are first expressed by their respective approximations in the B-spline bases:

$$a_j(t) = \sum_{i \in I_j^a} \alpha_i^j B_i(t), \; b_j(t) = \sum_{i \in I_j^b} \gamma_i^j B_i(t) \quad (23)$$

$$j = 0, \cdots, n-1$$

Their derivatives are then also linear combinations of B-splines by virtue of the differentiation formula (21) - (22). This further implies that the kernels in the system representation of Theorem 3, which are themselves linear functions of these parameters and their derivatives, then appear in the form of linear combinations involving the ensemble of the coefficients in the B-spline approximations (23). More precisely, there exist functions $g_{y,k}, k \in \mathcal{S}_y$ and $g_{u,j}, j \in \mathcal{S}_u$, such that the kernels of the system representation can be expressed in the following form

$$\begin{aligned} K_y(t,\tau) &= \sum_{k \in \mathcal{S}_y} \beta_k g_{y,k}(t,\tau) B_k(\tau) \\ K_u(t,\tau) &= \sum_{j \in \mathcal{S}_u} \beta_j g_{u,j}(t,\tau) B_j(\tau) \end{aligned} \quad (24)$$

where the index sets $\mathcal{S}_y$, $\mathcal{S}_u$ contain the ensemble of indices of B-splines that appear in the B-spline approximations of the system parameters $a_0, \cdots, a_{n-1}$ and $b_0, \cdots, b_{n-1}$, respectively. It further follows that the evaluation functional (9) of Theorem 3 can be written as

$$\begin{aligned} y(t) = &\sum_{k \in \mathcal{S}_y} \beta_k \int_a^b g_{y,k}(t,\tau) B_k(\tau) y(\tau) d\tau \\ &+ \sum_{j \in \mathcal{S}_u} \beta_j \int_a^b g_{u,j}(t,\tau) B_j(\tau) u(\tau) d\tau \end{aligned} \quad (25)$$

Defining

$$\begin{aligned} h_k(y,t) &:= \int_a^b g_{y,k}(t,\tau) B_k(\tau) y(\tau) d\tau; \quad k \in \mathcal{S}_y \\ h_j(u,t) &:= \int_a^b g_{u,j}(t,\tau) B_j(\tau) u(\tau) d\tau; \quad j \in \mathcal{S}_u \end{aligned} \quad (26)$$

and stacking the vectors

$$\begin{aligned} &\beta := [\beta_k; k \in \mathcal{S}_y \mid \beta_j; j \in \mathcal{S}_u]^T; \\ &h(y,u,t) := [h_k(y,t); k \in \mathcal{S}_y \mid h_j(u,t); j \in \mathcal{S}_u] \end{aligned} \quad (27)$$

gives $\beta, h(y,u,t) \in \mathbb{R}^s; s := \text{card}\{\mathcal{S}_u \cup \mathcal{S}_y\}$ and

$$y(t) = h(y,u,t)\beta \quad (28)$$

Let the output y in (28) be the measured output y^M. Given distinct time instants $t_1, \cdots, t_m \in (a,b]$, $m \geq 2n$, the last equation is now re-written point-wise as a system of linear algebraic equations

$$q(y^M) = P(y^M,u)\beta \quad (29)$$

$$q(y^M) \stackrel{def}{=} \begin{bmatrix} y^M(t_1) \\ \vdots \\ y^M(t_m) \end{bmatrix}; P(y^M,u) \stackrel{def}{=} \begin{bmatrix} h(y^M,u,t_1) \\ \ddots \\ h(y^M,u,t_m) \end{bmatrix} \quad (30)$$

where, clearly, $q \in \mathbb{R}^m$ and $P \in \mathbb{R}^m \times \mathbb{R}^s$ while $\beta \in \mathbb{R}^s$.

As no assumptions are made about the noise which may determine the invertibility of the matrix P, the following practical definition of linear identifiability is introduced

A. Practical Linear Identifiability

Definition 1. The linear time-varying system (1) is practically linearly identifiable on $[a,b]$ with respect to a particular realization of the output measurement, $y^M(t), t \in [a,b]$ corresponding to a known input u , if there exist distinct time instants $t_1, \cdots, t_m \in (a,b]$ such that the matrix $P(y^M,u)$ has rank $2n$.

By analogy with the nomenclature used in [16] output trajectories which render the satisfaction of the condition rank$P(y^M,u) = 2n$ are called *persistent*. The estimation of the β coefficient vector that deliver the best fitting B-splines approximations of the sytem parameters is best performed solving (29) in terms of the pseudo-inverse $P^\dagger$ of P:

$$\beta = P^\dagger(y^M,u) \; q(y^M) \quad (31)$$

The coefficients β so obtained are then used in (23) to deliver the estimates of the time-varying system parameters.

IV. JOINT STATE AND PARAMETER ESTIMATION

The results in Theorem 3 readily deliver a state estimator for system (1) on an arbitrary window $[a,b]$. Given a measured output $y_M : [a,b] \to \mathbb{R}$ in response to a known system input $u_K : [a,b] \to \mathbb{R}$ the estimator equations are:

$$\begin{aligned} y_E(t) &= \int_a^b K_y(t,\tau) y_M(\tau) \; d\tau + \int_a^b K_u(t,\tau) u_K(\tau) \; d\tau \\ y_E^{(i)}(t) &= \sum_{k=0}^{i-1} f_y^k(t) y_E^{(k)}(t) + \sum_{k=0}^{i-1} f_u^k(t) u_K^{(k)}(t) \\ &+ \int_a^b K_y^i(t,\tau) y_M(\tau) \; d\tau + \int_a^b K_u^i(t,\tau) u_K(\tau) \; d\tau \end{aligned} \quad (32)$$

$$x_E(t) = \begin{pmatrix} \Gamma_0 \\ \vdots \\ \Gamma_{n-1} \end{pmatrix}^{-1} \left[\begin{pmatrix} y_E(t) \\ \vdots \\ y_E^{(n-1)}(t) \end{pmatrix} - M \begin{pmatrix} u_K(t) \\ \vdots \\ u_K^{(n-2)}(t) \end{pmatrix} \right]$$

where the input derivatives : $u_K^{(k)} : [a,b] \to \mathbb{R}$ are also considered known, and where $y_E, y_E^{(i)}; i = 1, \ldots n-1$, and x_E are the estimated outputs, output derivatives, and the estimated states, over the estimation window $[a,b]$, respectively.

Joint state and parameter estimation is carried out by using the raw measurement data y^M to first obtain the estimates for the time-varying system parameters employing the approach described in section III. The parameter estimates are then used to carry out state estimation as in section IV.

V. EXAMPLE

The performance of the proposed estimator is assessed on a simple model:

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -(t^2+1)x_1(t) - x_2(t) + \frac{1}{t+1000}u(t)$$

with $y = x_1$ as the measured output, x_1, x_2 as the states, and with u as the control input.

The controlled system invariance is:

$$y^{(2)}(t) + y^{(1)}(t) + (t^2+1)y(t) - \frac{1}{t+1000}u(t) = 0 \quad (33)$$

where the known input function is assumed to be given by $u(t) = 120\sin(t)$.

A. Noiseless Case

Let n be the number of B-splines used over the entire time horizon and let p be the order of the B-spline. The results for the noiseless case, were obtained using $n = 86$ and $p = 3$ (quadratic B-spline) and uniform knot sequence. Due to the inherent properties of B-splines, the approximation precision degrades at the ends of any given estimation interval. In the case when estimation is required over extended periods of time, good approximation precision can be maintained by employing a dragging estimation window. Here, as a proof of concept, the total length of the estimation window is $[0, 20s]$, but the results are shown on an inner window of $[5, 15s]$ where the B-spline estimation is considered to be reliable.

The estimated constant value of a_1 is 0.9617 while the actual value of a_1 is 1. The estimated time-varying coefficient of the system is shown in Fig. 1, where it is compared against the true system coefficient. Fig. 2 and Fig. 3 show the corresponding estimates of the system states that coincide with the system output and its derivative. The true system state trajectories are overlayed on the corresponding plots of the estimated trajectories. It is then seen that the estimates and their true counterparts are almost identical.

B. The Case of Noisy Measurement

The estimation was also carried out for the case when the system output is perturbed by 30 SNR (Signal-to-Noise Ratio) gaussian noise. A shorter total window of $[0, 9s]$ is used to display the estimation errors better. Again quadratic B-splines were used ($p = 3$) with $n = 16$ and a uniform knot sequence. The inner estimation window is chosen as $[1.5, 7.5s]$.

The estimated constant value of a_1 is 0.8321 while the actual value of a_1 is 1. The estimated time-varying coefficient for the noisy case is shown in Fig. 4, where it is compared against the true system coefficient. Fig. 5 and Fig. 6 show the corresponding estimates of the system states and its derivative which do not coincide with the system output asserting the fact that the estimation is sensitive to noise.

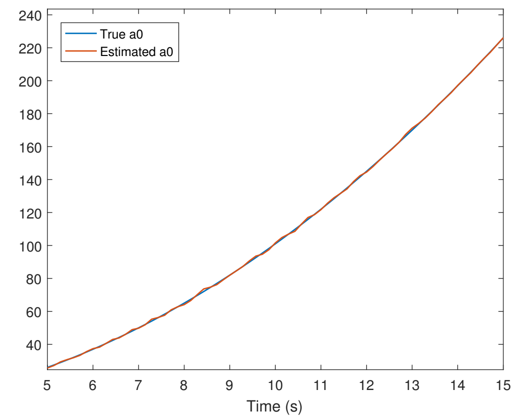

Fig. 1. Estimated parameter a_0 for Noiseless Case

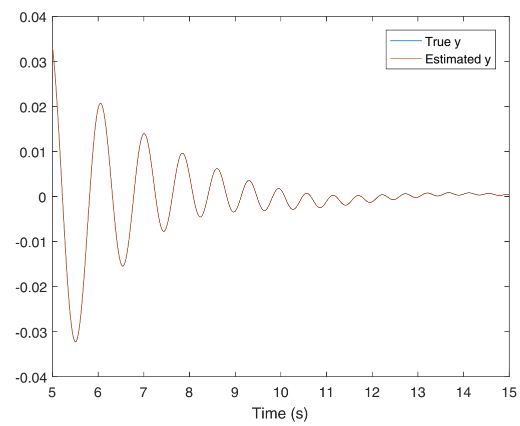

Fig. 2. Estimated output y for Noiseless Case

VI. CONCLUSIONS

The approach presented here is more sensitive to noise than the state and parameter estimation method of [5] as shown in the Fig. 4, Fig. 5, and Fig. 6, because it deals with a time-varying case which poses the challenge of functional estimation. Additionally, only the homogeneous (input free system) was considered in [5]. Many diverse improvements can be envisaged. For example, adequately adapted sentinel tests as suggested in [5] can be introduced for validation of the estimation correctness and accuracy. Also, it is expected that the computational burden associated

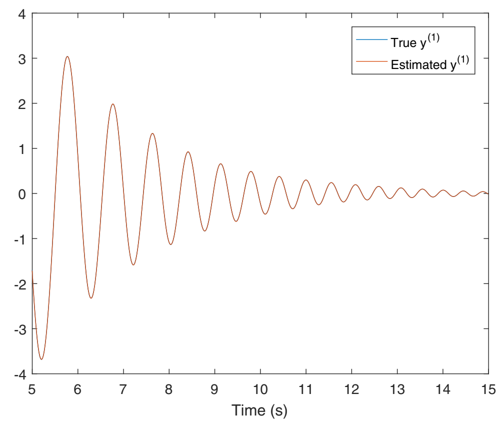

Fig. 3. Estimated derivative of y for Noiseless Case

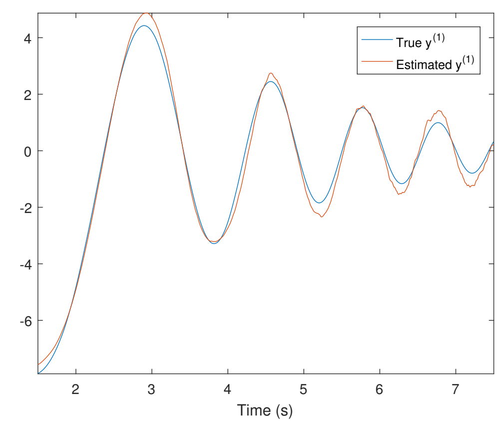

Fig. 6. Estimated derivative of the ouput y for the noisy case

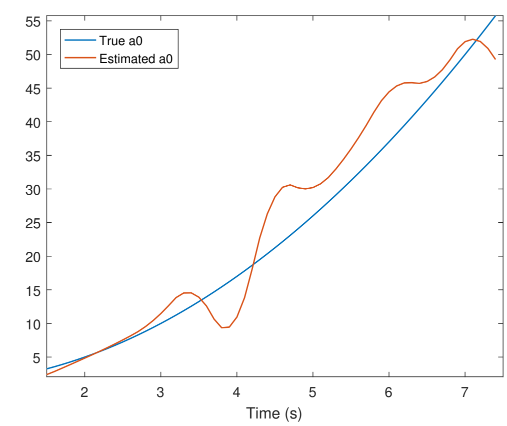

Fig. 4. Estimated parameter a_0 for the noisy case.

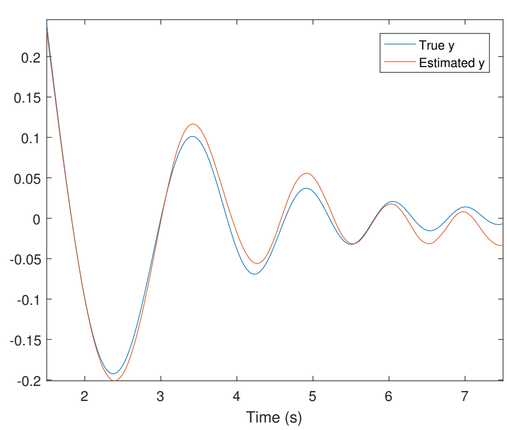

Fig. 5. Estimated system output y for the noisy case.

with the functional parameter estimation can be somewhat decreased by employing P-splines in place on B-splines as P-spline methods usually require fewer knots to achieve a comparable approximation accuracy while avoiding over-fitting. The method in this paper can be made more robust to noise by carrying out simultaneous denoisification and parameter estimation in LTV systems using adequately penalized P-splines with the optimal selection of knots.

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