2017 IEEE Conference on Control Technology and Applications (CCTA)
August 27-30, 2017. Kohala Coast, Hawai'i, USA

Kernel-Based Adaptive Multiple Model Target Tracking*

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Abstract— **The novel adaptive multiple-model target tracking algorithm presented here employs a non-asymptotic state and parameter estimator whose design hinges on a non-standard integral system representation. The same estimator can be used for target maneuver detection and isolation and hence constitutes the principal ingredient of the tracking algorithm. The algorithm does not maintain a model bank, but creates and identifies new models in an attempt to best track the measurement data. Such an approach is rendered uniquely possible by the fact that the state and parameter estimator is essentially dead-beat. Practical model identifiability, persistent excitation condition for the measured signal are discussed. Although this first version of the algorithm is deterministic and employs threshold-based maneuver detection, it exhibits good robustness with respect to Gaussian measurement noise.**

I. INTRODUCTION

Target tracking is a type of hybrid estimation problem involving both continuous and discrete components. Due to its many multi-disciplinary applications, it has been a subject of intense research, [1], [2]. If a target is performing maneuvers its dynamical model changes, at times abruptly. The discrete component is then concerned with the detection and isolation of such behavioural changes while the task of the continuous component is to estimate the state of the target. The term "isolation" pertains to estimation of the moment of the abrupt change and to identification of the new target model after the change. It is now acknowledged that the main challenge in this process arises in the target model uncertainty as the use of an incorrect model can result in unacceptable state estimation errors. The use of multiple model (MM) target tracking approaches is hence a well motivated, prevailing trend in modern applications; see the extensive survey on these methods [3]. Three generations of MM tracking algorithms have been identified in [4]: (1) autonomous, [5], [6]; (2) cooperating or interacting MM [7], [8], [9], [10]; and variable structure, [11], [12], [13], [14]. The three generations differ in the way in which the key components of MM estimation are performed, these being listed as: (a) model-set determination; (b) cooperation strategy (dealing with hypothesis about the model sequences); (c) conditional filtering; (d) output processing (generation of overall estimates). The autonomous generation is characterized by the fact that each of its elemental filters operates individually and independently of all the other elemental filters and hence exhibits superior output processing. In the cooperating generation the elemental filters work together as a team via effective internal (probabilistic) cooperation - competition which includes re-initialization of state estimating filters and other measures to achieve improved performance. The model groups or teams in the first two generations of MM algorithms have a fixed membership over time and thus have a fixed structure while the model set is allowed to vary in the third generation. For the reason of increased adaptability the variable structure family of algorithms is gaining momentum rapidly and is becoming the state of the art of MM estimation. It is worth pointing out that most, if not all, of the algorithms mentioned above employ some form of Kalman filter as their continuous state-estimating component by which to achieve measurement noise attenuation. With all its superior qualities the Kalman filter is sensitive to tuning and initialization so the use of its valued, recursive, asymptotically converging formula may be put to question in fast switching environments. This may be of concern as the variable structure MM algorithms delete and create members in their model banks that require frequent re-initialization.

While all of the above references point to probabilistic MM algorithms, the MM approach to target tracking is a general methodology which is not limited to statistical setting. Many non-statistical methods for maneuvering target tracking have been proposed and include algorithms based on evidential reasoning, neural networks, fuzzy logic, genetic algorithms, or deterministic settings; see eg. [15], [16], [17].

This note presents a deterministic version of a MM algorithm for target tracking with the following features.

(i) The algorithm creates its models as is necessary to achive adequate matching with the measurement data (target position); the bank of models is not maintained or remembered as no particular (e.g. Markov chain) model sequencing is assumed; multiple models are only considered in the stage of maneuver isolation to determine the order of the model that best fits the data after the change;

(ii) A non-asymptotic state and parameter estimation algorithm is employed that is based on a kernel (integral) representation of the system introduced in [18], later used for parameter estimation in [19]. The kernel representation springs from a modification of the idea of Fliess and Sira-Ramirez pertaining to differentiation of noisy signals as employed in [20], [21], [22], [23], and [24]. The novel algorithm uses a sliding estimation window whose size is adapted according to the task performed by the algorithm.

*This work was supported by The National Science & Engineering Research Council of Canada

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No restriction exists pertaining to the window size except the availability of measurement data.

(iii) The maneuver isolation is performed using the same estimation algorithm as in (ii) to determine the time instant at which an abrupt change in the residual error (between the measured and the estimated target position) occurs; alternative methods for detection are possible.

The advantages of the novel non-asymptotic estimator for the purpose of model identification and state estimation, as demonstrated in [18], [19], are re-iterated here for completeness:

(a) the estimation algorithm does not require initialization as the influence of the initial conditions on the state as well as parameter estimates has been removed by design of the kernel system representation;

(b) the estimation horizon can be chosen arbitrarily (subject to the availability of measured data and the satisfaction of the persistance of excitation assumption (PE) on the measured signal);

(c) in the absence of measurement noise the estimation of the state and system parameters is exact on the entire estimation window provided that the PE identifiability condition is satisfied; the algorithm features a level of robustness with respect to Gaussian measurement noise - this property is attributed to the smoothing/averaging property of iterated integration as discussed in [25].

(d) the algorithm simultaneously delivers the estimates of the parameters, the target state, and the time derivatives of the state corresponding to the order of the system; the latter can be used to enhance the accuracy of the isolation of the maneuver.

In summary, the purpose of this introductory paper is to show that the novel non-asymptotic estimator is perfectly suited to play a prominent role in adaptive MM target tracking algorithms such as the one presented here. The present deterministic version of the algorithm, with its simplistic threshold-based detection, can be extended to a probabilistic setting to achieve yet better robustness to measurement noise if the noise characteristics is known or can be identified. In the enhanced version, the isolation of a maneuver can be performed employing the usual canons of hypothesis testing to assess the significance of changes in the residual error during state/derivatives estimation.

II. Assumed Target Dynamics and its Kernel Representation

As the control input employed by a maneuvering target is usually unknown, it will be assumed that the target dynamics is limited to that described by a linear time invariant homogeneous equation of the kind:

Torsion Target Model

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1 y^{(1)}(t) + a_0 y^{(0)}(t) = 0 \tag{1}$$

where y represents the measured output (or state of the target), n is the order of the target model, and $a_{n-1}, ..., a_0$ are the unknown model parameters, and $y^{(0)} := y$ for consistency of notation.

It was shown in [18] that the ordinary differential equation (1), which is assumed to be valid on an interval $t \in [a, b]$, can be equivalently represented as an integral equation as described below.

A. Kernel Representation of a Linear System

Algebraic manipulation and repeated forward-backward integration of the system (1) leads to an alternative integral system representation in a reproducing kernel Hilbert space (RKHS).

Theorem 1: [18] There exist Hilbert-Schmidt kernels K, K^i, $i = 1, \cdots n-1$, such that the output function y of (1) is reproduced on any given interval $[a, b]$ in accordance with the action of the evaluation functional

$$y(t) = \int_a^b K(t, \tau) y(\tau) \ \mathrm{d}\tau \ ; \ \forall t \in [a, b] \tag{2}$$

Additionally, the derivatives of the output $y^{(1)}, \cdots y^{(n-1)}$ can be computed *recursively* by way of output integration, so that for $i = 1, \cdots n-1$ and for all $t \in [a, b]$:

$$y^{(i)}(t) = \sum_{k=0}^{i-1} b_k(t) y^{(k)}(t) + \int_a^b K^i(t, \tau) y(\tau) \ \mathrm{d}\tau \tag{3}$$

where $y^{(0)} \equiv y$ and $b_k(\cdot)$ are rational functions of t. Hilbert-Schmidt kernels are square integrable functions on $L^2[a, b] \times L^2[a, b]$. The kernels K, K^i, $i = 1, \cdots n-1$ and functions $b_k, k = 0, n-2$ are linear functions of the system parameters $a_{n-1}, a_{n-2}, ..., a_1, a_0$.

An output function $y : [a, b] \rightarrow \mathbb{R}$ satisfies the homogeneous equation (1) on the interval $[a, b]$ if an only if it is reproduced by the evaluation functional in (2).

The exact formulae for the integral kernel in Theorem 1 for $n = 3$ expressed in terms of the unknown parameters a_2, a_1, a_0, are given in the Appendix. The procedure for their computation is explained in [18].

III. Parameter Identifiability and Persistent Excitation

Following [26], the system (1) of order n is linearly identifiable because the linear system of equations

$$P(y)\begin{bmatrix} a_{n-1} \\ \vdots \\ a_0 \end{bmatrix} = \begin{bmatrix} -y^{(n)} \\ \vdots \\ -y^{(2n-1)} \end{bmatrix} \tag{4}$$

$$\text{with } P(y) := \begin{bmatrix} y^{(n-1)} & \dots & y^{(1)} & y^{(0)} \\ y^{(n)} & \dots & y^{(2)} & y^{(1)} \\ \vdots & \vdots & \vdots & \vdots \\ y^{(2n-2)} & \dots & y^{(n)} & y^{(n-1)} \end{bmatrix} \tag{5}$$

has a unique algebraic solution $[a_{n-1}, \dots, a_0]$ as the algebraic expression for the determinant det$P$ is not zero. Worth noticing is the way in which the identification equation (4) is formed: its first row is the re-arranged system equation (1) while the remaining rows are obtained by consecutive differentiation of (1).

Linear identifiability implies identifiability only if *persistent trajectories* are available.

Definition 1. Persistent Trajectory
A system trajectory $y(t), t \in [a, b]$, is called persistent if the matrix $P(y(t))$ is nonsingular for some $t \in [a, b]$, i.e. when the determinant det$P(y(t))$ does not vanish identically on $[a, b]$.

It is easy to see that det$P(y(t))$ is the Wronskian of the set of functions $\mathcal{S} := \{y^{n-1}, \dots, y^{(1)}, y^{(0)}\}$, so the existence of a $t \in [a, b]$ for which det$P(y(t)) \neq 0$ implies linear independence of functions in this set. Immediate is the conlusion that a trajectory of (1) is persistent as long as the system is moving (as then its output y is a non-trivial linear combination of the (linearly independent) fundamental solutions of (1) which in turn implies linear independence of the functions in $\mathcal{S}$). Trivial trajectories $y(t) \equiv 0$ are not persistent as then det$P(0) \equiv 0$.

It is also important to notice that a system model of the order lower than n is not identifiable from the n-dimensional identification equation (4) as then the lower order system equation is of the type $y^{(n-1)} + b_{n-2}y^{(n-2)} + \dots + b_0 y^{(0)} \equiv 0$ which implies linear dependence of the functions in $\mathcal{S}$ so det$P(y) \equiv 0$. In other words, full identification of a model (including its order) requires calculation of parameter estimates from a set of estimation equations of different dimensions.

IV. Implementable Simultaneous State and Parameter estimation

The identifiability equation (4) involves the derivates of the measured signal y so it cannot be applied directly. It is the results of Theorem 1 that renders it practically implementable as follows.

Since each row of (4) is a re-arranged homogeneous linear differential equation it is possible to represent it equivalently in an integral form with a kernel that is linear in the parameters $[a_{n-1}, \dots, a_0]$. For example, the first row is replaced by

$$y(t) = \int_a^b K(t,\tau) y(\tau)\ \mathrm{d}\tau = \sum_{i=0}^{n-1} a_i g_i(t, y)$$
$$\text{where } g_i(t, y) \overset{def}{=} \int_a^b k_i(t,\tau) y(\tau)\ \mathrm{d}\tau;\ t \in [a, b] \tag{6}$$

where the $k_i; i = 0, \dots, n-1$ are "component kernels" of K.

As parameter estimation will employ the measured values of y it is viewed practical to integrate (6) at least once more to secure a degree of robustness to measurement errors by the smoothing action of iterated integration. Equation (6) then becomes

$$\int_0^t y(\sigma) d\sigma = \sum_{i=0}^{n-1} a_i \int_0^t g_i(\sigma, y)\ d\sigma\ ;\quad t \in [a, b] \tag{7}$$

Similar equations as the above can then be derived by applying Theorem 1 to all remaining rows of the identification equation (4) that would lead to its direct correspondent in integral form. The linear set of equations so constructed could then be solved if a measured trajectory is persistent on $[a, b]$ and the time instants $t \in [a, b]$ at which det$P(y(t)) \neq 0$ are somehow available. An obvious drawback of such an approach is the computational effort in deriving all the integral kernels for the n rows of (4).

In the particular case at hand, a simpler method comes to mind. Assuming that a trajectory y is persistent, its corresponding set $\mathcal{S}$ is linearly independent. In the absence of any measurement noise, implied is the existence of distinct time instants $t_1, \cdots, t_n \in (a, b]$, such that the matrix

$$P' \overset{def}{=} \begin{bmatrix} y^{(n-1)}(t_1) & \cdots & y^{(0)}(t_1) \\ \dots & \ddots & \dots \\ y^{(n-1)}(t_n) & \cdots & y^{(0)}(t_n) \end{bmatrix} \tag{8}$$

is non-singular. It follows that the identification equation can be replaced by swapping P for P' and substituting its right hand side with the vector $[y^{(n)}(t_1), \dots, y^{(n)}(t_n)]^T$. Without the loss of generality, and since the time instant are not known a priori, it is expeditious to use a large number of possibly equidistant time instants $t_k, k = 1, \dots m; m >> n$, thus enlarging P' to a non-square matrix $\tilde{P} \in \mathbb{R}^{n \times m}$. The operations just described have the following correspondent using integral system representation on the identification interval $[a, b]$.

$$\tilde{P}(y_M)a = Q(y_M) \tag{9}$$

$$Q \stackrel{def}{=} \begin{bmatrix} q(t_1) \\ \vdots \\ q(t_m) \end{bmatrix}; a \stackrel{def}{=} \begin{bmatrix} a_{n-1} \\ \vdots \\ a_0 \end{bmatrix}; \tilde{P} \stackrel{def}{=} \begin{bmatrix} p_1(t_1) \cdots p_n(t_1) \\ \ddots \\ p_1(t_m) \cdots p_n(t_m) \end{bmatrix}$$

$$q(t_i) = \int_0^{t_i} y(\sigma)d\sigma;\ p_k(t_i) = \int_0^{t_i} g_k(\sigma, y)\ d\sigma \tag{10}$$

Equation (9) employs only the measured trajectory $y_M(t), t \in [a, b]$ and is hence subject to noise perturbations. Clearly, this can adversely affect the rank of the matrix $\tilde{P}$ that leads to the following practical definition of identifiability.

A. Practical Identifiability

Definition 2. Practical Identifiability
The homogeneous system (1) is practically identifiable on $[a, b]$ with respect to a particular realization of the output measurement, $y_M(t), t \in [a, b]$, if and only if there exist distinct time instants $t_1, \cdots, t_m \in (a, b]$ which render the matrix $\tilde{P}(y_M)$ to be full rank; i.e. Rank$\tilde{P}(y_M) = n$. Such measured output trajectories will be called *practically persistent*.

B. Practical Parameter and State Estimation

The parameter identification problem (9) can now be solved using standard linear regression by way of solving the normal equation: $\tilde{P}^T(y_M)\tilde{P}(y_M) = \tilde{P}^T(y_M)Q(y_M)$ delivered in terms of the pseudo-inverse $\tilde{P}^\dagger$ of $\tilde{P}$ while assuming that Rank$\tilde{P} = n$.

$$\hat{a} = \tilde{P}^\dagger(y_M)Q(y_M) \tag{11}$$

The estimated parameter can then be used to calculate a smoothed state estimate $\hat{y}$:

$$\hat{y}(t) = \int_a^b K(t, \tau)y_M(\tau)\ \mathrm{d}\tau \tag{12}$$

in which $K(t, \tau)$ is a linear function of the estimate $\hat{a}$.

V. The Multiple-Model Tracking Algorithm

The non-asymptotic estimator can be employed to carry out all the tasks of target tracking i.e. detection of maneuver, isolation of maneuver by model identification and determination of maneuver onset instant, and state estimation after maneuver. It employs a sliding estimation window $[t_i, t_i+T]$, $i = 1, \ldots,$ which is shifted forward by an interval of length Δ. Both T and Δ can be adjusted dynamically, according to the task performed. The constants T and Δ are measured in units of sampling points of the target position y_M.

A. Maneuver Detection

Assume that a target model has been identified prior to the sliding window $[t_i, t_i + T]$, as reflected by the identified model order n and the estimated model parameters $\hat{a}^i$. The window is then forwarded incrementally to new positions $t_k, k = i + 1, i + 2, \ldots,$ by steps of length $\Delta << T$. The parameter estimation is repeated in each new window using the same model order n, thus producing a sequence of estimated model parameter values $\hat{a}^k, k = i + 1, \ldots$. A maneuver is considered detected when the smallest value of k is found such that the difference in values of the estimated parameters exceeds a prescribed threshold $\beta > 0$:

$$||\hat{a}^{k+1} - \hat{a}^k||^2 > \beta \tag{13}$$

Denoting such smallest value by k_d, the maneuver has been located within the interval $[t_a, t_b] := [t_{k_d} + T, t_{k_d+1} + T]$ of length Δ. The interval containing the onset of the maneuver can be refined by using a smaller window (of length equal to a fraction of T) and back-tracking from $t_{k_d+1} + T$ using the same threshold criterion (13).

B. Isolation of Maneuver

(1) Maneuver Onset Time
Maneuver onset time is estimated after a maneuver is detected within a window $[t_a, t_b]$ whose length is much shorter than T. A window $[t_i, t_i + T]$ is then located which contains $[t_a, t_b]$ with instant t_b located closest to $t_i + T$. The onset time $t_o \in [t_a, t_b]$ is calculated as the first time when an abrupt change is detected in the residual function value

$$R_o(t) := \int_{iT}^{t} (\hat{y}(\sigma) - y_M(\sigma))^2\ d\sigma; t \in (t_i, t_b] \tag{14}$$

where $\hat{y}$ is calculated as in (12) and is the state estimate corresponding to model parameters $\hat{a}$ which were identified to be valid for times prior to t_a. The detection of abrupt change can be carried out using a threshold value.

(2) Model Identification
Model identification is carried out after locating a maneuver onset time t_o. An initial assumption about the model order, n, is made first. The parameter identification is performed simultaneously for all model dimensions $k = 1, \cdots, n$ by using equations (9), (11), and producing different sets of estimates $\hat{a}_k$. The best model is selected by the formula involving the n residuals R_k from the respective state estimators over a critical window length $[t_o, t_c]$:

$$R_k = \int_{t_0}^{t_c} (\hat{y}^k(\sigma) - y_M(\sigma))^2\ d\sigma \tag{15}$$

where $\hat{y}^k$ is the state estimate corresponding to model parameters $\hat{a}_k$ and calculated as in (12). The best model is selected as the one that yields the smallest residual, i.e. it is the system of order k^* and parameters $\hat{a}^*$ such that

$$k^* := \arg\ \min\{R_k; k = 1, \ldots, n\} \tag{16}$$

C. State Estimation

Over any window $[t_i, t_i + T]$ with no maneuvers detected and the model parameters identified, the state estimation is performed as in (12).

VI. EXAMPLE

Three different linear time-invariant systems are used to assess the performance of the proposed adaptive MM target tracking algorithm by way of simulation. The outputs are switched every 2 seconds (i.e. maneuvers happen at $2s$ and $4s$) and generate the "ground truth" composite y trajectory as in Figure 1.

The model equations of the first system are:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= x_3(t)\\ \dot{x}_3(t) &= 3x_1 - 100x_2\end{aligned}$$

The model equations of the second system are:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= x_3(t)\\ \dot{x}_3(t) &= x_1 - 10x_2\end{aligned}$$

The model equations of the third system are:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= x_3(t)\\ \dot{x}_3(t) &= 1.5x_1 - 50x_2\end{aligned}$$

All three systems have $y = x_1$ as the measured output trajectory.

The results show exact estimation of the trajectory (after maneuver detection and model identification) in the absence of measurement noise on the entire estimation horizon; see Figure 2. The sampling resolution used in the simulation is $0.001s$. In the absence of any measurement noise, the maneuver onset times are detected as $2.001s$ and $4.001s$.

Even if the output is perturbed with Gaussian noise with SNR (Signal-to-Noise Ratio) = 40 (see Figure 3), the estimation error is remarkably small and is uniformly bounded; see Figure 4. The maneuver onset times are however detected as $2.297s$ and $4.285s$.

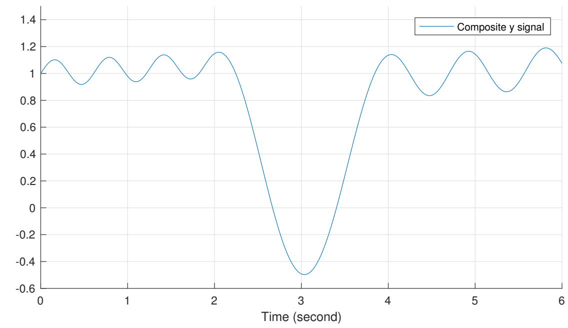

Fig. 1. Composite trajectory y

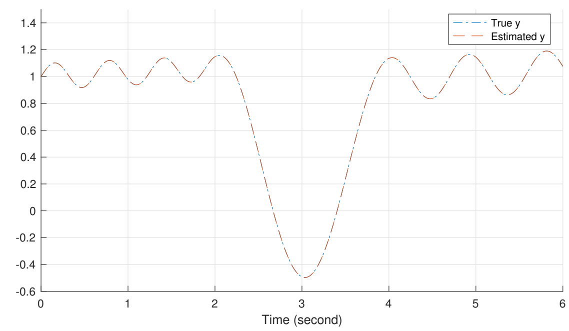

Fig. 2. Estimated trajectory y for noiseless case

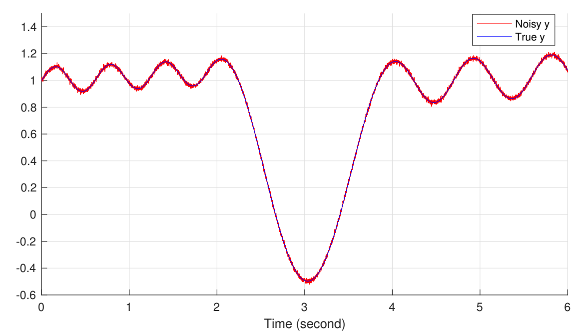

Fig. 3. Noisy y versus the true y

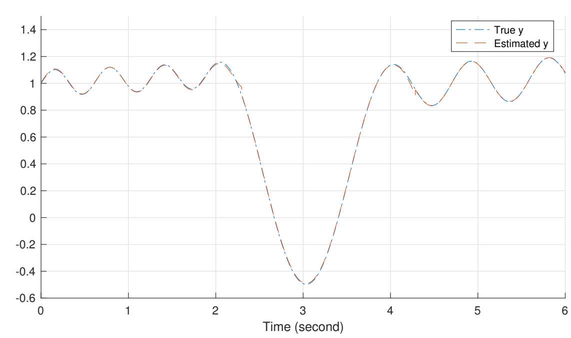

Fig. 4. Estimated trajectory y for 40 SNR noise

VII. Conclusions

The multiple-model target tracking approach presented here employs a novel non-asymptotic state and parameter estimation which permits to abandon the classical idea of maintaining a model bank. New models are created instead as dictated by the value of the error residuals in the estimated state. More importantly, the estimator does not require initialization and does not involve the notion of convergence. Enhanced probabilistic versions of the approach presented here may give rise to yet a new generation of target tracking algorithms characterized by increased tracking accuracy.

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Appendix

The exact formulae for the integral kernel in Theorem 1 for $n = 3$ expressed in terms of the unknown parameters a_2, a_1, a_0 :

$$y(t) = \int_a^b K(t,\tau) y(\tau)\,\mathrm{d}\tau$$

$$K(t,\tau) \triangleq \begin{cases} \frac{1}{[(t-a)^3+(b-t)^3])} K_F(t,\tau) & for \quad \tau \le t \\ \frac{1}{[(t-a)^3+(b-t)^3])} K_B(t,\tau) & for \quad \tau > t \end{cases}$$

$$\begin{aligned} K_F(t,\tau) \triangleq\ & 9(\tau-a)^2 - a_2(\tau-a)^3 \\ & + (t-\tau)\Big[-18(\tau-a) + 6a_2(\tau-a)^2 \\ & \qquad - a_1(\tau-a)^3\Big] \\ & + \frac{1}{2}(t-\tau)^2\Big[6 - 6a_2(\tau-a) + 3a_1(\tau-a)^2 \\ & \qquad - a_0(\tau-a)^3\Big] \end{aligned}$$

$$\begin{aligned} K_B(t,\tau) \triangleq\ & 9(b-\tau)^2 + a_2(b-\tau)^3 \\ & + (t-\tau)\Big[18(b-\tau) + 6a_2(b-\tau)^2 \\ & \qquad + a_1(b-\tau)^3\Big] \\ & + \frac{1}{2}(t-\tau)^2\Big[6 + 6a_2(b-\tau) + 3a_1(b-\tau)^2 \\ & \qquad + a_0(b-\tau)^3\Big] \end{aligned}$$