Institute for Logic,
Language and Computation

University of Amsterdam

Justified Belief Based on Uncertain and Contradictory Evidence

LILaC Seminar
Apr. 2, 2024

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joint work with

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and

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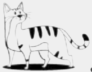
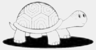
"Computing (degree of) belief

based on evidence."

Suspects = { , , ,  }

 := {Dog, Cat, Rabbit}

 := {Dog}

 := {Rabbit, Turtle}

Images: Freepik.com, Flaticon.com

Suspects $= \{$  $\}$

 $:= \{\text{Dog}, \text{Cat}, \text{Rabbit}\}$

 $:= \{\text{Dog}\}$

 $:= \{\text{Rabbit}, \text{Turtle}\}$

Do I believe in *P*?

Solution: Topological Models of Evidence

evidence, argument, justification

Baltag, A.; Bezhanishvili, N.; Özgün, A.; and Smets, S. 2022. Justifed belief, knowledge, and the topology of evidence. *Synthese* 200(6):1–51.

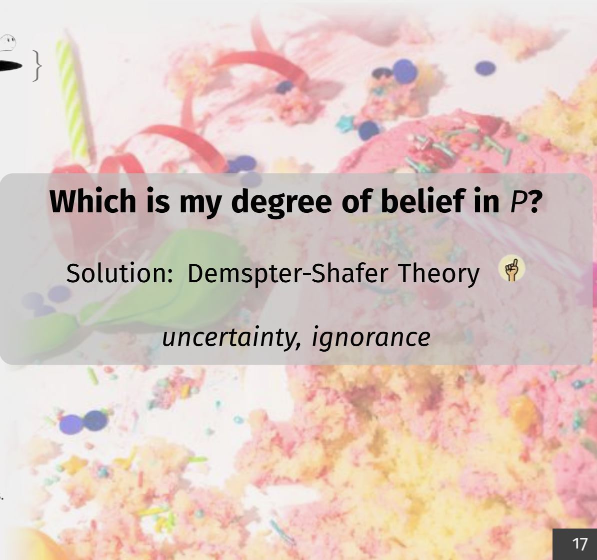
Images: Freepik.com, Flaticon.com

Suspects = {  }

 := ({Dog, Cat, Rabbit}, 0.95)

 := ({Dog}, 0.70)

 := ({Rabbit, Turtle}, 0.45)



Which is my degree of belief in *P*?

Solution: Demspter-Shafer Theory

uncertainty, ignorance

Shafer, G. 1976. *A Mathematical Theory of Evidence*. Princeton, NJ: Princeton University Press.

Images: Freepik.com, Flaticon.com

INPUT

$$S = \{\text{Dog}, \text{Cat}, \text{Rabbit}, \text{Turtle}\}$$

$$E_1 = (\{\text{Dog}, \text{Cat}, \text{Rabbit}\}, 0.95)$$

$$E_2 = (\{\text{Dog}\}, 0.70)$$

$$E_3 = (\{\text{Rabbit}, \text{Turtle}\}, 0.45)$$

OUTPUT

Degree_of_Belief(P)

where P is a subset of S

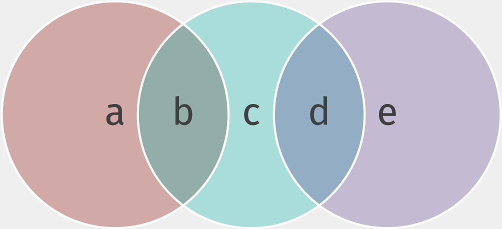
How to combine *imperfect* evidence and compute degree of belief based on it?
(Capturing both Topological Models of Evidence and Dempster-Shafer Theory)

Our Solution: Multi Layer Belief Model

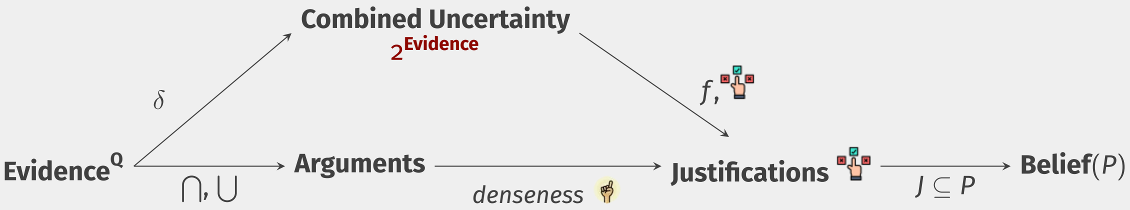
Evidence $\xrightarrow[\cap,\,\cup]{}$ **Arguments** $\xrightarrow[\textit{denseness}]{}$ **Justifications** $\xrightarrow[J \subseteq P]{}$ **Belief**(P)

Evidence	Arguments	Justifications
{Dog, Cat, Rabbit}		
{Dog}	{Rabbit}	{Dog, Rabbit}
{Rabbit, Turtle}		

$S = \{a, b, c, d, e\}$



$B(\{b, d, e\}) = 1$ $\quad$ $B(\{d, e\}) = 0$



$(\{\text{Dog}, \text{Cat}, \text{Rabbit}\}, 0.95)$

$(\{\text{Dog}\}, 0.70)$ $(\{\text{Rabbit}\}, \mathbf{?})$ $(\{\text{Dog}, \text{Rabbit}\}, \mathbf{?})$

$(\{\text{Rabbit}, \text{Turtle}\}, 0.45)$

Multi Layer Belief Model I

Input: Universe S, Evidence + Certainty Degree $\mathcal{E} = \{(E_i, p_i)\}_{i=1,\dots,k}$, 2 Parameters

Output: Degree of belief for every $P \in 2^S$

Justification Frame (Qualitative Layer)

$\mathcal{J}_1$:= Every element of the topology of evidence (except $\emptyset$).

e.g., $\mathcal{J}_1 =$
$\{\{\text{Dog}, \text{Cat}, \text{Rabbit}\}, \{\text{Dog}\}, \{\text{Rabbit}, \text{Turtle}\}, \{\text{Rabbit}\}, \{\text{Rabbit}, \text{Dog}\}, \{\text{Dog}, \text{Rabbit}, \text{Turtle}\},$
$\{\text{Dog}, \text{Cat}, \text{Rabbit}, \text{Turtle}\}\}$

$\mathcal{J}_2$:= Dense elements of the topology of evidence.

e.g.,
$\mathcal{J}_2 = \{\{\text{Dog}, \text{Cat}, \text{Rabbit}\}, \{\text{Dog}, \text{Rabbit}\}, \{\text{Dog}, \text{Rabbit}, \text{Turtle}\}, \{\text{Dog}, \text{Cat}, \text{Rabbit}, \text{Turtle}\}\}$

Input: Universe S, Evidence + Certainty Degree $\mathcal{E} = \{(E_i, p_i)\}_{i=1,\ldots,k}$, 2 Parameters
Output: Degree of belief for every $P \in 2^S$

δ**-function** (Quantitative Layer)

$\delta : 2^{\mathcal{E}} \to [0, 1]$

$$\delta(\boldsymbol{E}) = \prod_{\substack{(E_i, p_i) \text{ s.t.} \\ E_i \in \boldsymbol{E}}} p_i \prod_{\substack{(E_i, p_i) \text{ s.t.} \\ E_i \notin \boldsymbol{E}}} (1 - p_i)$$

e.g., $\delta(\{\{\text{Dog}, \text{Cat}, \text{Rabbit}\}, \{\text{Rabbit}, \text{Turtle}\}\}) = 0.95 \cdot 0.45 \cdot (1 - 0.70) = 0.13$

✓ The Certainty values are combined and distributed over $2^{\mathcal{E}}$.

✗ Our justifications are in 2^S, not in $2^{\mathcal{E}}$.

Input: Universe S, Evidence + Certainty Degree $\mathcal{E} = \{(E_i, p_i)\}_{i=1,\ldots,k}$, 2 Parameters
Output: Degree of belief for every $P \in 2^S$

Allocation Function (Bridging Layer)

$f : 2^{\mathcal{E}} \to$ Topology of Evidence

Intersection, Union

e.g., if $f =$ intersection
then $f(\{\{\text{Dog}, \text{Cat}, \text{Rabbit}\}, \{\text{Rabbit}, \text{Turtle}\}\}) = \{\text{Rabbit}\}$
and the δ-value of $\{\{\text{Dog}, \text{Cat}, \text{Rabbit}\}, \{\text{Rabbit}, \text{Turtle}\}\}$ goes for $\{\text{Rabbit}\}$.

- ✓ The certainty values are combined and distributed over 2^S.
- ✓ After some operations, the certainty values are combined and distributed over $\mathcal{J}$.

Our Solution: Multi Layer Belief Model

Input: Universe S, Evidence + Certainty Degree $\mathcal{E} = \{(E_i, p_i)\}_{i=1,\dots,k}$, 2 Parameters

Output: Degree of belief for every $P \in 2^S$

e.g., $\textsc{belief}(\{\text{Dog}, \text{Rabbit}\}) = 0.81$; $\textsc{belief}(\{\text{Dog}\}) = 0.40$ under certain parameters.

e.g., $\textsc{belief}(\{\text{Dog}, \text{Rabbit}\}) = 0$; $\textsc{belief}(\{\text{Dog}\}) = 0$ under certain parameters.

Layers

Justification Frame

δ-function

Allocation Function

Belief Function

$\textsc{Belief}(P) \in [0, 1]$

adding δ-values of justifications contained in P

✓Partial evidence ✓Uncertain evidence ✓Inconsistent evidence

✓It replicates Topological Models of Evidence and Demspter's rule of combination

✓It does not increase the computational complexity of the problem (#P-COMPLETE)

Pinto Prieto, D.; de Haan, R.; and Özgün, A. 2023. A belief model for conficting and uncertain evidence. In *KR 2023*, 552–561.

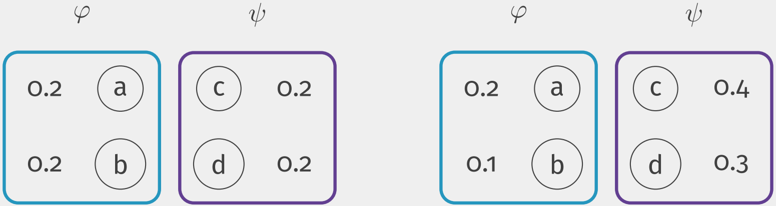
- Computational Complexity analysis of Demspter’s rule of combination, Topological Models of Evidence and Multi-Layer Belief Model.
- Knowledge Compilation approach for Demspter’s rule of combination.
- Logic for reasoning about comparative strengths of evidence and belief under uncertainty.

Starting point:

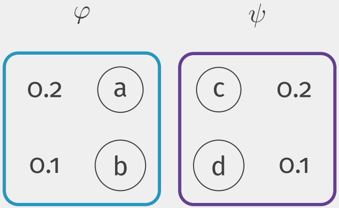
Harmanec, David; Hájek, Petr. 2011. **A qualitative belief logic**. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*.

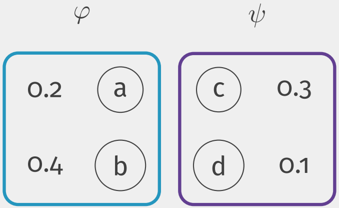
$\mathcal{M} = \langle S, \mathsf{bel}, \mathcal{V} \rangle$	$\mathcal{M} = \langle S, \mathcal{E}^{Q}, \mathsf{Bel}, \mathcal{V} \rangle$
$\mathsf{Bel}(\varphi) = \mathsf{bel}(\llbracket \varphi \rrbracket)$	$\mathcal{M}, s \models \varphi \triangleleft_{\mathsf{Bel}} \psi$ if and only if $\mathsf{Bel}(\varphi) \leq \mathsf{Bel}(\psi)$
$\mathcal{M}, s \models \varphi \triangleleft \psi$ if and only if $\mathsf{Bel}(\varphi) \leq \mathsf{Bel}(\psi)$	$\mathcal{M}, s \models \varphi \triangleleft_{\bullet} \psi$ if and only if **?**

Defining a element-wise comparison operator

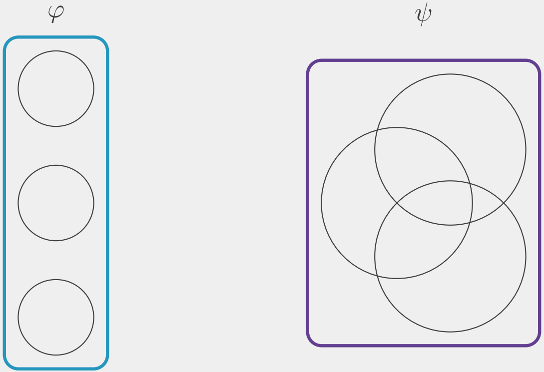


$\varphi \lhd_{\bullet} \psi$ if and only if for every $X, Y \in \mathcal{E}$ such that $X \subseteq [\![\varphi]\!]$ and $Y \subseteq [\![\psi]\!]$, $q_X \leq q_Y$.





$\varphi \lhd_{\bullet} \psi$ if and only if
for every $X \in \mathcal{E}$ such that $X \subseteq [\![\varphi]\!]$ there exists $Y \in \mathcal{E}$ such that $Y \subseteq [\![\psi]\!]$ and $q_X \leq q_Y$,
and for every $Y \in \mathcal{E}$ such that $Y \subseteq [\![\psi]\!]$ there exists $X \in \mathcal{E}$ such that $X \subseteq [\![\varphi]\!]$ and $q_X \leq q_Y$.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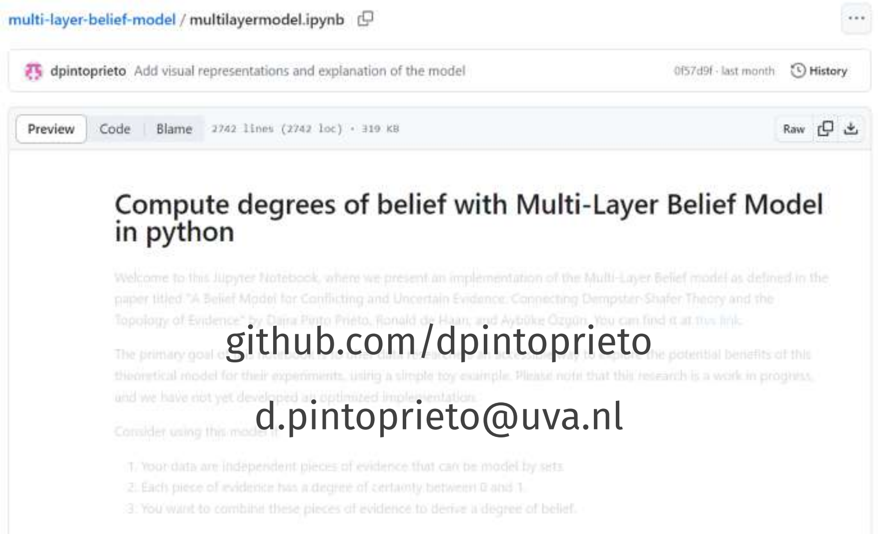
References

- Shafer, G. 1976. *A Mathematical Theory of Evidence*. Princeton, NJ: Princeton University Press.
- Baltag, A.; Bezhanishvili, N.; Özgün, A.; and Smets, S. 2022. Justifed belief, knowledge, and the topology of evidence. *Synthese* 200(6):1–51.
- Pinto Prieto, D.; de Haan, R.; and Özgün, A. 2023. A belief model for conficting and uncertain evidence – connecting dempster-shafer theory and the topology of evidence. In *Proceedings of the 20th International Conference on Principles of Knowledge Representation and Reasoning (KR 2023)*, 552–561.
- Harmanec, David; Hájek, Petr. 2011. A qualitative belief logic. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*.



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Solution Quantitative: Dempster-Shafer Theory

Input: Universe S, Evidence + Certainty Degree $\mathcal{E} = \{(E_i, p_i)\}_{i=1,\dots,k}$
Output: Degree of belief for every $P \in 2^S$

e.g., $\textsc{bel}(\{\text{Dog}, \text{Rabbit}\}) = 0.75$; $\textsc{bel}(\{\text{Dog}\}) = 0.56$

Dempster's Rule of Combination

$E_1 \oplus E_2$

Belief Functions

$\textsc{bel}(P) \in [0, 1]$

✓ Partial evidence ✓ Uncertain evidence $\sim$ Inconsistent evidence

Computational Complexity: #P-COMPLETE

Shafer, G. 1976. *A Mathematical Theory of Evidence*. Princeton, NJ: Princeton University Press.

Definition (Basic probability assignment)

$m : 2^S \to [0,1]$ such that $m(\emptyset) = 0$ and $\sum_{A \subseteq S} m(A) = 1$.

Definition (Dempster's rule of combination)

Let m_1 and m_2 be b.p.a. Then $m_1 \oplus m_2$, is:
$m(\emptyset) = 0$ and
$m(C) = \frac{\sum_{A_i \cap B_j = C} m_1(A_i) m_2(B_j)}{K}$, where K is $1 - \sum_{A_i \cap B_j = \emptyset} m_1(A_i) m_2(B_j)$, for all nonempty sets $C \subseteq S$.

Definition (Belief function)

$\textsc{Bel} : 2^S \to [0,1]$ such that
$\textsc{Bel}(\emptyset) = 0$, $\textsc{Bel}(S) = 1$ and $\textsc{Bel}\big(\cup_1^n A_i\big) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \textsc{Bel}\big(\cap_{i \in I} A_i\big)$.

Solution qualitative: Topological Models of Evidence

Input: Universe S, Evidence $\mathcal{E} = \{E_i\}_{i=1,\dots,k}$ (no uncertainty!)
Output: Boolean belief operator for every $P \in 2^S$

e.g., $\mathrm{B}(\{\mathrm{Dog}, \mathrm{Rabbit}\}) = 1$; $\mathrm{B}(\{\mathrm{Dog}\}) = 0$

Topology of Evidence

Evidence set
closed under finite intersections
and arbitrary unions

Belief Operator

$\mathrm{B}(P) = 1$ iff
there is a *dense* set in the topology
contained in P

✓ Partial evidence ✗ Uncertain evidence ✓ Inconsistent evidence

Computational Complexity: EXP (applying formal definition straightforward)

Baltag, A.; Bezhanishvili, N.; Özgün, A.; and Smets, S. 2022. Justifed belief, knowledge, and the topology of evidence. *Synthese* 200(6):1–51.

Definition (Topological Space)

A *topological space* is a pair (S, τ) , where S is a nonempty set and τ is a family of subsets of S such that $S, \emptyset \in \tau$, and τ is closed under finite intersections and arbitrary unions.

Definition (Topology of Evidence)

Given a set of pieces of evidence $\mathcal{E}$, the topology $\tau_{\mathcal{E}}$ generated by $\mathcal{E}$ is called the *topology of evidence*.

Definition (Mutually inconsistent/consistent)

We say that two pieces of evidence $E, E' \in \tau\mathcal{E}$ are *mutually inconsistent* if $E \cap E' = \emptyset$, and mutually consistent otherwise.

Definition (Belief Operator)

A proposition P *is believed* if and only if there is a $T \in \tau_{\mathcal{E}} \setminus \{\emptyset\}$ and $T \subseteq P$, such that T is consistent with any piece of evidence in $\tau\mathcal{E} \setminus \{\emptyset\}$, i.e., T is dense in S w.r.t. $\tau_{\mathcal{E}}$.

Definition (Set of evidence allocation functions)

Let $(S, \mathcal{E})$ be a qualitative evidence frame. A *set of evidence allocation functions* $\mathfrak{F}$ on $(S, \mathcal{E})$ is a set of of functions from $2^{\mathcal{E}}$ to $\tau_{\mathcal{E}}$ (the topology generated by $\mathcal{E}$) such that for all $f, g \in \mathfrak{F}$:

1. $f(\emptyset) = S$,
2. for all $\boldsymbol{E} \subseteq \mathcal{E}$, $f(\boldsymbol{E}) \in \tau_{\boldsymbol{E}}$ (the topology generated by $\boldsymbol{E}$) and it is dense in $\cup\boldsymbol{E}$ w.r.t. $\tau_{\boldsymbol{E}}$; or $f(\boldsymbol{E}) = \emptyset$.[a]
3. for all $\boldsymbol{E} \subseteq \mathcal{E}$ and every f, g in $\mathfrak{F}$, $f(\boldsymbol{E}) \subseteq g(\boldsymbol{E})$ or $g(\boldsymbol{E}) \subseteq f(\boldsymbol{E})$.

[a] Note that since $\boldsymbol{E} \subseteq \mathcal{E}$, we have $\tau_{\boldsymbol{E}} \subseteq \tau_{\mathcal{E}}$ (follows immediately by the definitions of generated topologies).

Definition (Minimal dense set allocation function)

$d : 2^{\mathcal{E}} \to \tau_{\mathcal{E}}$ such that

$$d(\boldsymbol{E}) = \begin{cases} \min((\mathrm{dense}(\boldsymbol{E}), \subseteq)) & \text{if } \boldsymbol{E} \neq \emptyset, \\ S & \text{otherwise.} \end{cases}$$

Multi-Layer Belief Model: Formulas

Input:

$S \qquad \mathcal{E} \times [0,1]$

Justification frame:

$\mathcal{J}$

Allocation Function:

$f : 2^{\mathcal{E}} \rightarrow \tau$

Mass Function:

$\delta : 2^{\mathcal{E}} \rightarrow [0,1]$

$$\delta_{\tau}(T) = \sum_{\boldsymbol{E}:f(\boldsymbol{E})=T} \delta(\boldsymbol{E}) \qquad \delta_{\mathcal{J}}(A) = \frac{\delta_{\tau}(A)}{\sum_{T\in\mathcal{J}} \delta_{\tau}(T)} \qquad Bel^{\mathcal{J}}(P) = \sum_{J\subseteq P} \delta_{\mathcal{J}}(J)$$

Multi-Layer Belief Model: Full Example

{Dog, Rabbit}	$\mathcal{J}_1$	$\mathcal{J}_2$
Intersection	0.75	0
Union	0.02	0
Minimal dense set	0.81	0.62

{Dog}	$\mathcal{J}_1$	$\mathcal{J}_2$
Intersection	0.56	0
Union	0.02	0
Minimal dense set	0.40	0

Zadeh's paradox

$S = \{A, B, C\}$

$m_1(A) = 0.99, m_1(B) = 0.01,$

$m_2(C) = 0.99, m_2(B) = 0.01$

$$\boldsymbol{m_1 \oplus m_2(B) = 1}$$

$$\boldsymbol{m_1 \oplus m_2(\{A, C\}) = 0}$$