Institute for Logic, Language and Computation

University of Amsterdam

Belief Model for Conflicting and Uncertain Evidence

FOAM Seminar
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joint work with

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"Computing (degree of) belief based on evidence."


BLIND

Suspects $= \{\text{Fox}, \text{Cat}, \text{Pinocchio}, \text{Parrot}\}$

 $= \{\text{Fox}, \text{Cat}, \text{Parrot}\}$

 $= \{\text{Cat}\}$

 $= \{\text{Fox}, \text{Pinocchio}\}$

Suspects $= \{\text{Fox}, \text{Cat}, \text{Pinocchio}, \text{Parrot}\}$

 $= (\{\text{Fox}, \text{Cat}, \text{Parrot}\}, 0.95)$

 $= (\{\text{Cat}\}, 0.70)$

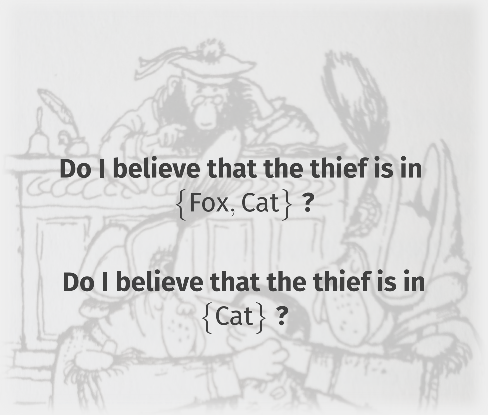
 $= (\{\text{Fox}, \text{Pinocchio}\}, 0.45)$

Suspects $= \{\text{Fox, Cat, Pinocchio, Parrot}\}$

 $= (\{\text{Fox, Cat, Parrot}\}, 0.95)$

 $= (\{\text{Cat}\}, 0.70)$

 $= (\{\text{Fox, Pinocchio}\}, 0.45)$



Do I believe that the thief is in $\{\text{Fox, Cat}\}$ **?**

Do I believe that the thief is in $\{\text{Cat}\}$ **?**

INPUT

$$S = \{\text{Fox}, \text{Cat}, \text{Pinocchio}, \text{Parrot}\}$$

$$E_1 = (\{\text{Fox}, \text{Cat}, \text{Parrot}\}, 0.95)$$

$$E_2 = (\{\text{Cat}\}, 0.70)$$

$$E_3 = (\{\text{Fox}, \text{Pinocchio}\}, 0.45)$$

OUTPUT

Degree_of_Belief({Fox, Cat})

Degree_of_Belief({Cat})

How to combine *imperfect* evidence and compute degree of belief based on it?

Input: Universe S, Evidence + Certainty Degree $\mathcal{E} = \{(E_i, p_i)\}_{i=1,\ldots,k}$
Output: Degree of belief for every $P \in 2^S$

e.g., $\textsc{bel}(\{\text{Fox}, \text{Cat}\}) = 0.75$; $\textsc{bel}(\{\text{Cat}\}) = 0.56$

Dempster's Rule of Combination

$E_1 \oplus E_2$ ☝

Belief Functions

$\textsc{bel}(P) \in [0, 1]$ ☝

✓ Partial evidence ✓ Uncertain evidence ∼ Inconsistent evidence

Computational Complexity: #P-COMPLETE

Shafer, G. 1976. *A Mathematical Theory of Evidence*. Princeton, NJ: Princeton University Press.

Input: Universe S, Evidence $\mathcal{E} = \{E_i\}_{i=1,\ldots,k}$ (no uncertainty!)
Output: Boolean belief operator for every $P \in 2^S$

e.g., $\mathrm{B}(\{\mathrm{Fox}, \mathrm{Cat}\}) = 1$; $\mathrm{B}(\{\mathrm{Cat}\}) = 0$

Topology of Evidence

Evidence set
closed under finite intersections
and arbitrary unions

Belief Operator

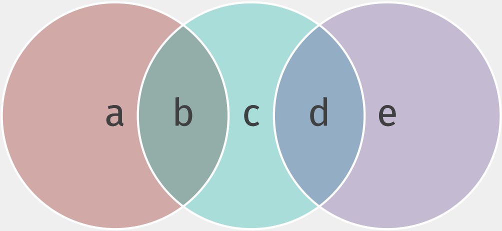
$\mathrm{B}(P) = 1$ iff
there is a *dense* set in the topology
contained in P

✓ Partial evidence ✗ Uncertain evidence ✓ Inconsistent evidence

Computational Complexity: EXP (applying formal definition straightforward)

Baltag, A.; Bezhanishvili, N.; Özgün, A.; and Smets, S. 2022. Justifed belief, knowledge, and the topology of evidence. *Synthese* 200(6):1–51.

$S = \{a, b, c, d, e\}$



$B(\{b, d, e\}) = 1$ $B(\{d, e\}) = 0$

Input: Universe S, Evidence + Certainty Degree $\mathcal{E} = \{(E_i, p_i)\}_{i=1,\dots,k}$, 2 Parameters
Output: Degree of belief for every $P \in 2^S$

Justification Frame (Qualitative Layer)

$\mathcal{J}_1 :=$ Every element of the topology of evidence (except $\emptyset$).

e.g., $\mathcal{J}_1 = \{\{\text{Fox}, \text{Cat}, \text{Parrot}\}, \{\text{Cat}\}, \{\text{Fox}, \text{Pinocchio}\}, \{\text{Fox}\}, \{\text{Fox}, \text{Cat}\}, \{\text{Fox}, \text{Cat}, \text{Pinocchio}\}, \{\text{Fox}, \text{Cat}, \text{Pinocchio}, \text{Parrot}\}\}$

$\mathcal{J}_2 :=$ Dense elements of the topology of evidence.

e.g., $\mathcal{J}_2 = \{\{\text{Fox}, \text{Cat}, \text{Parrot}\}, \{\text{Fox}, \text{Cat}\}, \{\text{Fox}, \text{Cat}, \text{Pinocchio}\}, \{\text{Fox}, \text{Cat}, \text{Pinocchio}, \text{Parrot}\}\}$

Input: Universe S, Evidence + Certainty Degree $\mathcal{E} = \{(E_i, p_i)\}_{i=1,\dots,k}$, 2 Parameters
Output: Degree of belief for every $P \in 2^S$

δ**-function** (Quantitative Layer)

$\delta : 2^{\mathcal{E}} \rightarrow [0, 1]$

$$\delta(\boldsymbol{E}) = \prod_{\substack{(E_i,p_i) \text{ s.t.} \\ E_i \in \boldsymbol{E}}} p_i \prod_{\substack{(E_i,p_i) \text{ s.t.} \\ E_i \notin \boldsymbol{E}}} (1 - p_i)$$

e.g., $\delta(\{\{\text{Fox}, \text{Cat}, \text{Parrot}\}, \{\text{Fox}, \text{Pinocchio}\}\}) = 0.95 \cdot 0.45 \cdot (1 - 0.70) = 0.13$

✓ The Certainty values are combined and distributed over $2^{\mathcal{E}}$.

✗ Our justifications are in 2^S, not in $2^{\mathcal{E}}$.

Input: Universe S, Evidence + Certainty Degree $\mathcal{E} = \{(E_i, p_i)\}_{i=1,\dots,k}$, 2 Parameters
Output: Degree of belief for every $P \in 2^S$

Allocation Function (Bridging Layer)

$f : 2^{\mathcal{E}} \to$ Topology of Evidence

Intersection, Union

e.g., if $f =$ intersection then $f(\{\{\text{Fox}, \text{Cat}, \text{Parrot}\}, \{\text{Fox}, \text{Pinocchio}\}\}) = \{\text{Fox}\}$ and the δ-value of $\{\{\text{Fox}, \text{Cat}, \text{Parrot}\}, \{\text{Fox}, \text{Pinocchio}\}\}$ goes for $\{\text{Fox}\}$.

- ✓ The certainty values are combined and distributed over 2^S.
- ✓ After some operations, the certainty values are combined and distributed over $\mathcal{J}$.

Input: Universe S, Evidence + Certainty Degree $\mathcal{E} = \{(E_i, p_i)\}_{i=1,\dots,k}$, 2 Parameters
Output: Degree of belief for every $P \in 2^S$

e.g., $\textsc{belief}(\{\text{Fox}, \text{Cat}\}) = 0.81$; $\textsc{belief}(\{\text{Cat}\}) = 0.40$ under certain parameters.
e.g., $\textsc{belief}(\{\text{Fox}, \text{Cat}\}) = 0$; $\textsc{belief}(\{\text{Cat}\}) = 0$ under certain parameters.

Layers

Justification Frame
δ-function
Allocation Function

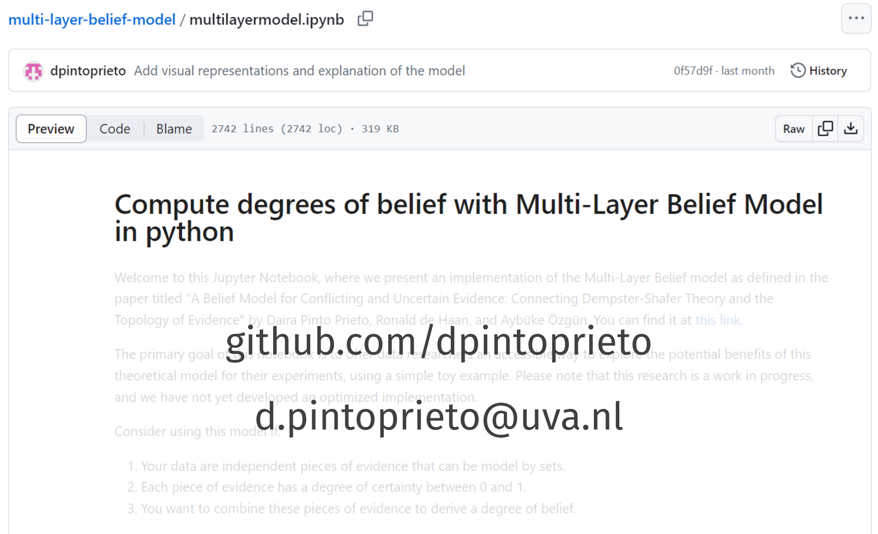
Belief Function

$\textsc{Belief}(P) \in [0, 1]$
adding δ-values of justifications contained in P

✓ Partial evidence ✓ Uncertain evidence ✓ Inconsistent evidence

Computational Complexity: #P-COMPLETE

Pinto Prieto, D.; de Haan, R.; and Özgün, A. 2023. A belief model for conficting and uncertain evidence. In *KR 2023*, 552–561.



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(Possible) Discussion Topics

- **Why is this model interesting?** Flexibility.

Self-driving car

Driver assistance system

- **Assessment of the model:** Generalization of DST and TME, preserving computational complexity.

- **Next steps:** Assessing the model for real uses (*volunteers?*), formal comparison with other belief models, logic of evidence based on this model, computational complexity of adding or removing evidence...

References

- Shafer, G. 1976. *A Mathematical Theory of Evidence*. Princeton, NJ: Princeton University Press.
- Baltag, A.; Bezhanishvili, N.; Özgün, A.; and Smets, S. 2022. Justifed belief, knowledge, and the topology of evidence. *Synthese* 200(6):1–51.
- Pinto Prieto, D.; de Haan, R.; and Özgün, A. 2023. A belief model for conficting and uncertain evidence – connecting dempster-shafer theory and the topology of evidence. In *Proceedings of the 20th International Conference on Principles of Knowledge Representation and Reasoning (KR 2023)*, 552–561.

Definition (Basic probability assignment)

$m : 2^S \to [0, 1]$ such that $m(\emptyset) = 0$ and $\sum_{A \subseteq S} m(A) = 1$.

Definition (Dempster's rule of combination)

Let m_1 and m_2 be b.p.a. Then $m_1 \oplus m_2$, is:
$m(\emptyset) = 0$ and
$m(C) = \frac{\sum_{A_i \cap B_j = C} m_1(A_i) m_2(B_j)}{K}$, where K is $1 - \sum_{A_i \cap B_j = \emptyset} m_1(A_i) m_2(B_j)$, for all nonempty sets $C \subseteq S$.

Definition (Belief function)

$\textsc{Bel} : 2^S \to [0, 1]$ such that
$\textsc{Bel}(\emptyset) = 0$, $\textsc{Bel}(S) = 1$ and $\textsc{Bel}\big(\cup_1^n A_i\big) \geq \sum_{\emptyset \neq I \subseteq \{1, \ldots, n\}} (-1)^{|I|+1} \textsc{Bel}\big(\cap_{i \in I} A_i\big)$.

Definition (Topological Space)

A *topological space* is a pair (S, τ) , where S is a nonempty set and τ is a family of subsets of S such that $S, \emptyset \in \tau$, and τ is closed under finite intersections and arbitrary unions.

Definition (Topology of Evidence)

Given a set of pieces of evidence $\mathcal{E}$, the topology $\tau_{\mathcal{E}}$ generated by $\mathcal{E}$ is called the *topology of evidence*.

Definition (Mutually inconsistent/consistent)

We say that two pieces of evidence $E, E' \in \tau\mathcal{E}$ are *mutually inconsistent* if $E \cap E' = \emptyset$, and mutually consistent otherwise.

Definition (Belief Operator)

A proposition P *is believed* if and only if there is a $T \in \tau_{\mathcal{E}} \setminus \{\emptyset\}$ and $T \subseteq P$, such that T is consistent with any piece of evidence in $\tau\mathcal{E} \setminus \{\emptyset\}$, i.e., T is dense in S w.r.t. $\tau_{\mathcal{E}}$.

Definition (Set of evidence allocation functions)

Let $(S, \mathcal{E})$ be a qualitative evidence frame. A *set of evidence allocation functions* $\mathfrak{F}$ on $(S, \mathcal{E})$ is a set of of functions from $2^{\mathcal{E}}$ to $\tau_{\mathcal{E}}$ (the topology generated by $\mathcal{E}$) such that for all $f, g \in \mathfrak{F}$:

1. $f(\emptyset) = S$,
2. for all $\boldsymbol{E} \subseteq \mathcal{E}$, $f(\boldsymbol{E}) \in \tau_{\boldsymbol{E}}$ (the topology generated by $\boldsymbol{E}$) and it is dense in $\cup \boldsymbol{E}$ w.r.t. $\tau_{\boldsymbol{E}}$; or $f(\boldsymbol{E}) = \emptyset$.[a]
3. for all $\boldsymbol{E} \subseteq \mathcal{E}$ and every f, g in $\mathfrak{F}$, $f(\boldsymbol{E}) \subseteq g(\boldsymbol{E})$ or $g(\boldsymbol{E}) \subseteq f(\boldsymbol{E})$.

[a] Note that since $\boldsymbol{E} \subseteq \mathcal{E}$, we have $\tau_{\boldsymbol{E}} \subseteq \tau_{\mathcal{E}}$ (follows immediately by the definitions of generated topologies).

Definition (Minimal dense set allocation function)

$d : 2^{\mathcal{E}} \to \tau_{\mathcal{E}}$ such that

$$d(\boldsymbol{E}) = \begin{cases} \min((\mathrm{dense}(\boldsymbol{E}), \subseteq)) & \text{if } \boldsymbol{E} \neq \emptyset, \\ S & \text{otherwise.} \end{cases}$$

Multi-Layer Belief Model: Formulas

Input:

$S \qquad \mathcal{E} \times [0, 1]$

Justification frame:

$\mathcal{J}$

Allocation Function:

$f : 2^{\mathcal{E}} \to \tau$

Mass Function:

$\delta : 2^{\mathcal{E}} \to [0, 1]$

$$\delta_\tau(T) = \sum_{\boldsymbol{E}:f(\boldsymbol{E})=T} \delta(\boldsymbol{E})$$

$$\delta_{\mathcal{J}}(A) = \frac{\delta_\tau(A)}{\sum_{T\in\mathcal{J}} \delta_\tau(T)}$$

$$Bel^{\mathcal{J}}(P) = \sum_{J\subseteq P} \delta_{\mathcal{J}}(J)$$

{Fox, Cat}	$\mathcal{J}_1$	$\mathcal{J}_2$
Intersection	0.75	0
Union	0.02	0
Minimal dense set	0.81	0.62

{Cat}	$\mathcal{J}_1$	$\mathcal{J}_2$
Intersection	0.56	0
Union	0.02	0
Minimal dense set	0.40	0

$S = \{A, B, C\}$

$m_1(A) = 0.99, m_1(B) = 0.01,$

$m_2(C) = 0.99, m_2(B) = 0.01$

$$\boldsymbol{m_1 \oplus m_2(B) = 1}$$

$$\boldsymbol{m_1 \oplus m_2(\{A, C\}) = 0}$$