1 Pontryagin's Maximum Principle

The function b defines how controls work:

$$b : \mathbb{R}^+ \times \mathbb{R}^d \times A \to \mathbb{R}^d$$

$$\dot{x}_t = b(t, x_t, u_t), \quad t \geq s, \quad x_s = x$$

We have a stopping set $D \subset \mathbb{R}^d$ and time horizon T.
Cost function:

$$c : [0, T) \times \mathbb{R}^d \times A \to \mathbb{R}, \quad C : \{T\} \times D \to R$$

Total cost:

$$V^u(s, x) = \int_s^\tau c(t, x_t, u_t)dt + C(\tau, x_\tau)$$

Infimal cost function:

$$V(s, x) = \inf_u V^u(s, x)$$

Define Hamiltonian:

$$H(t, x, u, \lambda) = \lambda^T b(t, x, u) - c(t, x, u)$$

Assume stopping set is some hyperplane $D = \{y\} + \Sigma$.

Theorem 1 *Pontryagin's Principle. If $(x_t, u_t)_{t \leq \tau}$ is optimal then there exist adjoint paths $(\lambda_t)_{t \leq \tau}$ in $\mathbb{R}^d$ and $(\mu_t)_{t \leq \tau}$ in $\mathbb{R}$ with the following properties: for all $t \leq \tau$:*

1. *$H(t, x_t, u, \lambda_t) + \mu_t$ has maximum value 0, achieved at $u = u_t$,*
2. *$\dot{\lambda}_t^T = -\lambda_t^T \nabla b(t, x_t, u_t) + \nabla c(t, x_t, u_t)$,*
3. *$\dot{\mu}_t = -\lambda_t^T \dot{b}(t, x_t, u_t) + \dot{c}(t, x_t, u_t)$,*
4. *$\dot{x}_t = b(t, x_t, u_t)$.*

Moreover we have

5. *$(\lambda_\tau^T + \nabla C(\tau, x_\tau))\sigma = 0$ for all $\sigma \in \Sigma$,*

and in the time-unconstrained case,

6. $\mu_\tau + \dot{C}(\tau, x_\tau) = 0,$

In time unconstrained case, $\mu_t = 0$ *for all* t*.*

Alternatively

1. $0 = \partial H/\partial u$, note: this may not apply, fundamentally we're minimising wrt u,
2. $\dot{\lambda} = -\partial H/\partial x$,
3. $\dot{\mu} = -\partial H/\partial t$,
4. $\dot{x} = \partial H/\partial \lambda$.