1 The Adiabatic Theorem

Schrödinger equation:

$$i\hbar\partial_t|\psi_n(t)\rangle = H(t)|\psi_n(t)\rangle \tag{1}$$

Consider $e^{-i\int_0^t E_n(t')dt'}|\psi_n(t)\rangle$. Not a solution as

$$i\partial_t e^{-i\int_0^t E_n(t')dt'}|\psi_n(t)\rangle \neq H(t)e^{-i\int_0^t E_n(t')dt'}|\psi_n(t)\rangle$$

LHS:

$$\begin{aligned} & (i\cdot(-i)E_n(t)e^{-i\int_0^t E_n(t')dt'} + ie^{-i\int_0^t E_n(t')dt'}\partial_t)|\psi_n(t)\rangle \\ = \; & e^{-i\int_0^t E_n(t')dt'}(E_n(t)+i\partial_t)|\psi_n(t)\rangle \end{aligned}$$

RHS:

$$E_n(t)e^{-i\int_0^t E_n(t')dt'}|\psi_n(t)\rangle$$

Mismatch of: $e^{-i\int_0^t E_n(t')dt'}i\partial_t|\psi_n(t)\rangle$.
Expand wavefunction $|\Psi(t)\rangle$ in terms of $e^{-i\int_0^t E_n(t')dt'}|\psi_n(t)\rangle$ anyway:

$$|\Psi(t)\rangle = \sum_n c_n(t)e^{-i\int_0^t E_n(t')dt'}|\psi_n(t)\rangle$$

Substitute in 1:
LHS:

$$\sum_n c_n(t)e^{-i\int_0^t E_n(t')dt'}E_n(t)|\psi_n(t)\rangle$$

RHS:

$$\begin{aligned} & i\hbar\sum_n((\partial_t c_n(t))e^{-i\int_0^t E_n(t')dt'}|\psi_n(t)\rangle + c_n(t)(-iE_n(t))e^{-i\int_0^t E_n(t')dt'}|\psi_n(t)\rangle + \\ & +c_n(t)e^{-i\int_0^t E_n(t')dt'}\partial_t|\psi_n(t)\rangle \\ = \; & (i\hbar\partial_t c_n(t) + c_n(t)E_n(t))e^{-i\int_0^t E_n(t')dt'}|\psi_n(t)\rangle + c_n(t)e^{-i\int_0^t E_n(t')dt'}\partial_t|\psi_n(t)\rangle \end{aligned}$$

$$\sum_n c_n(t)e^{-i\int_0^t E_n(t')dt'}i\hbar|\psi_n(t)\rangle + i\hbar(\partial_t c_n(t))e^{-i\int_0^t E_n(t')dt'}|\psi_n(t)\rangle = 0$$

1.1 Differentiating the energy

$$H(t)|\psi_n(t)\rangle = E_n(t)|\psi_n(t)\rangle$$

Differentiating this:

$$(\partial_t H(t))|\psi_n(t)\rangle + H_n(t)\partial_t|\psi_n(t)\rangle = (\partial_t E_n(t))|\psi_n(t)\rangle + E_n(t)\partial_t|\psi_n(t)\rangle$$

LHS:

$$\begin{aligned} & \langle\psi_n(t)|(\partial_t H(t))|\psi_n(t)\rangle + \langle\psi_n(t)|H_n(t)\partial_t|\psi_n(t)\rangle \\ = & \langle\psi_n(t)|(\partial_t H(t))|\psi_n(t)\rangle + E_m(t)\langle\psi_n(t)|\partial_t|\psi_n(t)\rangle \end{aligned}$$

RHS:

$$\partial_t E_n(t) + E_n(t)\langle\psi_n(t)|\partial_t|\psi_n(t)\rangle$$

Therefore:

$$\langle\psi_n(t)|(\partial_t H(t))|\psi_n(t)\rangle + (E_m(t) - E_n(t))\langle\psi_n(t)|\partial_t|\psi_n(t)\rangle = \delta_{mn}\partial_t E_n(t)$$

If $m \neq n$:

$$\langle\psi_n(t)|\partial_t|\psi_n(t)\rangle = \frac{\langle\psi_n(t)|(\partial_t - H(t))|\psi_n(t)\rangle}{E_n(t) - E_m(t)}$$