1 Four types of fourier transform

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dx e^{ikx} = \delta(k) \tag{1}$$

$$\frac{1}{2\pi}\int_{0}^{2\pi} dx e^{ikx} = \delta_{k0},\ k \text{ integer} \tag{2}$$

$$\frac{1}{N}\sum_{m=0}^{N-1} e^{2\pi ikm/N} = \sum_{i=-\infty}^{\infty} \delta_{k,iN},\ k \text{ integer} \tag{3}$$

$$\frac{1}{L}\sum_{k=-\infty}^{\infty} e^{2\pi ikx/L} = \sum_{i=-\infty}^{\infty} \delta(x-kL) \tag{4}$$

$$\tag{5}$$

2 Fourier transform of operators

$$\begin{aligned}
& \int d\alpha d\beta \exp(i\alpha\hat{p} + i\beta\hat{x}) \operatorname{tr}(f(\hat{x},\hat{p}) \exp(-i\alpha\hat{p} - i\beta\hat{x}))|x\rangle && (6) \\
= & \int d\alpha d\beta \operatorname{tr}(f(\hat{x},\hat{p}) \exp(-i\alpha\hat{p} - i\beta\hat{x})) \exp(i\alpha\hat{p} + i\beta\hat{x})|x\rangle && (7) \\
= & \int d\alpha d\beta \operatorname{tr}(f(\hat{x},\hat{p}) \exp(-i\alpha\hat{p}) exp(-i\beta\hat{x})) \exp(i\beta\hat{x}) \exp(i\alpha\hat{p})|x\rangle && (8) \\
= & \int d\alpha d\beta \operatorname{tr}(f(\hat{x},\hat{p}) \exp(-i\alpha\hat{p}) exp(-i\beta\hat{x})) \exp(i\beta\hat{x})|x+\alpha\rangle && (9) \\
= & \int d\alpha d\beta \operatorname{tr}(f(\hat{x},\hat{p}) \exp(-i\alpha\hat{p}) exp(-i\beta\hat{x})) \exp(i\beta(x+\alpha))|x+\alpha\rangle && (10) \\
= & \int d\alpha d\beta dx' \langle x'|f(\hat{x},\hat{p}) \exp(-i\alpha\hat{p}) exp(-i\beta\hat{x})|x'\rangle \exp(i\beta(x+\alpha))|x+\alpha\rangle && (11) \\
= & \int d\alpha d\beta dx' \langle x'|f(\hat{x},\hat{p}) \exp(-i\alpha\hat{p}) exp(-i\beta x')|x'\rangle \exp(i\beta(x+\alpha))|x+\alpha\rangle && (12) \\
= & \int d\alpha d\beta dx' \langle x'|f(\hat{x},\hat{p}) \exp(-i\beta x')|x'-\alpha\rangle \exp(i\beta(x+\alpha))|x+\alpha\rangle && (13) \\
= & \int d\alpha d\beta dx' \langle x'|f(\hat{x},\hat{p})|x'-\alpha\rangle \exp(i\beta(x-x'+\alpha))|x+\alpha\rangle && (14) \\
= & \int d\alpha dx' \langle x'|f(\hat{x},\hat{p})|x'-\alpha\rangle \delta(x-x'+\alpha)|x+\alpha\rangle && (15) \\
= & \int dx' \langle x'|f(\hat{x},\hat{p})|x'-x'+x\rangle \delta(x-x'+\alpha)|x+x'-x\rangle && (16) \\
= & \int dx' \langle x'|f(\hat{x},\hat{p})|x\rangle|x'\rangle && (17) \\
= & \int dx' |x'\rangle\langle x'|f(\hat{x},\hat{p})|x\rangle && (18) \\
= & \; f(\hat{x},\hat{p})|x\rangle && (19) \\
& && (20)
\end{aligned}$$

3 Fermi-Dirac Statistics

$$\begin{aligned}
g_i &= \text{number of possible states at level } i && (21)\\
n_i &= \text{number of particles at level } i && (22)\\
n_i \text{ particles into } g_i \text{ states} &= \frac{g_i!}{n_i!(g_i - n_i)!} && (23)\\
Z &= \prod_i \frac{g_i!}{n_i!(g_i - n_i)!} && (24)\\
\log Z &= \text{const} - \sum_i (\log n! - \log(g_i - n_i)!) && (25)\\
&\approx \text{const} - \sum_i (n_i \log n_i - n_i + (g_i - n_i)\log(g_i - n_i) - g_i + n_i) && (26)\\
& && (27)
\end{aligned}$$

$$\begin{aligned}
W &= \text{const} - \sum_i (n_i \log n_i - n_i + (g_i - n_i)\log(g_i - n_i) - g_i + n_i) + \alpha(N - \sum_i n\\
\frac{\partial W}{\partial n_i} &= -(\log n_i + 1 - 1 - \log(g_i - n_i) - 1 + 1) - \alpha - \beta\epsilon_i\\
\log(g_i - n_i) - \log n_i &= \alpha + \beta\epsilon_i\\
\log(g_i/n_i - 1) &= \alpha + \beta\epsilon_i\\
g_i/n_i - 1 &= \exp(\alpha + \beta\epsilon_i)\\
n_i &= \frac{g_i}{1 + \exp(\alpha + \beta\epsilon_i)}
\end{aligned}$$

4 The Adiabatic Theorem

Schrödinger equation:

$$i\hbar\partial_t|\psi_n(t)\rangle = H(t)|\psi_n(t)\rangle \quad (35)$$

Consider $e^{-i\int_0^t E_n(t')dt'}|\psi_n(t)\rangle$. Not a solution as

$$i\partial_t e^{-i\int_0^t E_n(t')dt'}|\psi_n(t)\rangle \neq H(t)e^{-i\int_0^t E_n(t')dt'}|\psi_n(t)\rangle$$

LHS:

$$\begin{aligned} & (i \cdot (-i) E_n(t) e^{-i\int_0^t E_n(t')dt'} + i e^{-i\int_0^t E_n(t')dt'} \partial_t) |\psi_n(t)\rangle \\ = \; & e^{-i\int_0^t E_n(t')dt'} (E_n(t) + i\partial_t) |\psi_n(t)\rangle \end{aligned}$$

RHS:

$$E_n(t) e^{-i\int_0^t E_n(t')dt'} |\psi_n(t)\rangle$$

Mismatch of: $e^{-i\int_0^t E_n(t')dt'} i\partial_t |\psi_n(t)\rangle$.

Expand wavefunction $|\Psi(t)\rangle$ in terms of $e^{-i\int_0^t E_n(t')dt'}|\psi_n(t)\rangle$ anyway:

$$|\Psi(t)\rangle = \sum_n c_n(t) e^{-i\int_0^t E_n(t')dt'} |\psi_n(t)\rangle$$

Substitute in 35:

LHS:

$$\sum_n c_n(t) e^{-i\int_0^t E_n(t')dt'} E_n(t) |\psi_n(t)\rangle$$

RHS:

$$\begin{aligned} & i\hbar \sum_n ((\partial_t c_n(t)) e^{-i\int_0^t E_n(t')dt'} |\psi_n(t)\rangle + c_n(t)(-iE_n(t)) e^{-i\int_0^t E_n(t')dt'} |\psi_n(t)\rangle + \\ & + c_n(t) e^{-i\int_0^t E_n(t')dt'} \partial_t |\psi_n(t)\rangle \\ = \; & (i\hbar\partial_t c_n(t) + c_n(t) E_n(t)) e^{-i\int_0^t E_n(t')dt'} |\psi_n(t)\rangle + c_n(t) e^{-i\int_0^t E_n(t')dt'} \partial_t |\psi_n(t)\rangle \end{aligned}$$

$$\sum_n c_n(t) e^{-i\int_0^t E_n(t')dt'} i\hbar |\psi_n(t)\rangle + i\hbar (\partial_t c_n(t)) e^{-i\int_0^t E_n(t')dt'} |\psi_n(t)\rangle = 0$$

4.1 Differentiating the energy

$$H(t)|\psi_n(t)\rangle = E_n(t)|\psi_n(t)\rangle$$

Differentiating this:

$$(\partial_t H(t))|\psi_n(t)\rangle + H_n(t)\partial_t|\psi_n(t)\rangle = (\partial_t E_n(t))|\psi_n(t)\rangle + E_n(t)\partial_t|\psi_n(t)\rangle$$

LHS:

$$
\begin{aligned}
& \langle\psi_n(t)|(\partial_t H(t))|\psi_n(t)\rangle + \langle\psi_n(t)|H_n(t)\partial_t|\psi_n(t)\rangle \\
= \quad & \langle\psi_n(t)|(\partial_t H(t))|\psi_n(t)\rangle + E_m(t)\langle\psi_n(t)|\partial_t|\psi_n(t)\rangle
\end{aligned}
$$

RHS:

$$\partial_t E_n(t) + E_n(t)\langle\psi_n(t)|\partial_t|\psi_n(t)\rangle$$

Therefore:

$$\langle\psi_n(t)|(\partial_t H(t))|\psi_n(t)\rangle + (E_m(t) - E_n(t))\langle\psi_n(t)|\partial_t|\psi_n(t)\rangle = \delta_{mn}\partial_t E_n(t)$$

If $m \neq n$:

$$\langle\psi_n(t)|\partial_t|\psi_n(t)\rangle = \frac{\langle\psi_n(t)|(\partial_t - H(t))|\psi_n(t)\rangle}{E_n(t) - E_m(t)}$$

5 Kubo Formula

$$
\begin{aligned}
i\frac{\partial}{\partial t}|\psi\rangle &= H|\psi\rangle \\
i\frac{\partial}{\partial t}|\psi'\rangle &= H'|\psi'\rangle \\
|\psi'\rangle &= |\psi\rangle + |\delta\psi\rangle \\
H' &= H + \delta H(t) \\
i\frac{\partial}{\partial t}(|\psi\rangle + |\delta\psi\rangle) &= (H + \delta H(t))|(\psi\rangle + |\delta\psi\rangle) \\
i\frac{\partial}{\partial t}(|\psi\rangle + |\delta\psi\rangle) &= H|(\psi\rangle + |\delta\psi\rangle) + \delta H(t)|(\psi\rangle + |\delta\psi\rangle) \\
i\frac{\partial}{\partial t}|\delta\psi\rangle &= H|\delta\psi\rangle + \delta H(t)|\psi\rangle + O(\delta^2) \\
i(\frac{\partial}{\partial t} + iH)|\delta\psi\rangle &= \delta H(t)|\psi\rangle \\
\exp(iHt)i(\frac{\partial}{\partial t} + iH)|\delta\psi\rangle &= \exp(iHt)\delta H(t)|\psi\rangle \\
i\frac{\partial}{\partial t}\exp(iHt)|\delta\psi\rangle &= \exp(iHt)\delta H(t)|\psi\rangle \\
i\exp(iHt)|\delta\psi\rangle &= \int dt'\exp(iHt')\delta H(t')|\psi\rangle \\
|\delta\psi\rangle &= -i\int dt'\exp(iH(t'-t))\delta H(t')|\psi\rangle \\
\langle\psi'|B|\psi'\rangle &= (\langle\psi| + \langle\delta\psi|)B(|\psi\rangle + |\delta\psi\rangle) \\
\delta\langle\psi'|B|\psi'\rangle &= \langle\psi|B|\delta\psi\rangle + \langle\delta\psi|B|\psi\rangle \\
\delta\langle\psi'|B|\psi'\rangle &= -i\int dt'\langle\psi(t)|B\exp(iH(t'-t))\delta H(t')|\psi(t')\rangle + \langle\delta\psi|B|\psi\rangle \\
\delta\langle\psi'|B|\psi'\rangle &= -i\int dt'\langle\psi(t)|B\exp(iH(t'-t))\delta H(t')\exp(-iH(t'-t))\exp(iH(t'-t \\
\delta\langle\psi'|B|\psi'\rangle &= -i\langle\psi(t)|\int dt' B\exp(iH(t'-t))\delta H(t')\exp(-iH(t'-t))|\psi(t)\rangle + \langle\delta\psi|I \\
\delta\langle\psi'|B|\psi'\rangle &= -i\langle\psi(t)|\int dt'[B,\exp(iH(t'-t))\delta H(t')\exp(-iH(t'-t))]|\psi(t)\rangle \\
\delta\langle\psi'|B|\psi'\rangle &= -i\langle\int dt'[B,\exp(iH(t'-t))\delta H(t')\exp(-iH(t'-t))]\rangle
\end{aligned}
$$

$$
\begin{aligned}
i\frac{\partial}{\partial t}|\psi\rangle &= H|\psi\rangle \\
i\exp(iHt)|\delta\psi\rangle &= \int dt' \exp(iHt')\delta H(t')|\psi\rangle \\
|\delta\psi\rangle &= -i\int dt' \exp(iH(t'-t))\delta H(t')|\psi\rangle \\
\langle\psi'|B|\psi'\rangle &= (\langle\psi| + \langle\delta\psi|)B(|\psi\rangle + |\delta\psi\rangle) \\
\delta\langle\psi'|B|\psi'\rangle &= \langle\psi|B|\delta\psi\rangle + \langle\delta\psi|B|\psi\rangle \\
\delta\langle\psi'|B|\psi'\rangle &= -i\int dt'\langle\psi(t)|B\exp(iH(t'-t))\delta H(t')|\psi(t')\rangle + \langle\delta\psi|B|\psi\rangle \\
\delta\langle\psi'|B|\psi'\rangle &= -i\int dt'\langle\psi(t)|\exp(iH(t'-t))\exp(-iH(t'-t))B\exp(iH(t'-t))\delta H(t')|\psi(t')\rangle \\
\delta\langle\psi'|B|\psi'\rangle &= -i\int dt'\langle\psi(t')|\exp(-iH(t'-t))B\exp(iH(t'-t))\delta H(t')|\psi(t')\rangle + \langle\delta\psi|B|\psi\rangle \\
\hat{B}(t,t') &= \exp(-iH(t'-t))B\exp(iH(t'-t)) \\
\delta\langle\psi'|B|\psi'\rangle &= -i\int dt'\langle\psi(t')|\hat{B}(t,t')\delta H(t')|\psi(t')\rangle + \langle\delta\psi|B|\psi\rangle \\
\delta\langle\psi'|B|\psi'\rangle &= -i\langle\int dt'[\hat{B}(t,t'),\delta H(t')]\rangle \\
C(t,t') &= -i\langle[\hat{B}(t,t'),\delta H(t')]\rangle \\
\delta\langle\psi'(t)|B|\psi'(t)\rangle &= \int dt' C(t,t')
\end{aligned}
$$

6 Position operator in bloch representation

$$\Psi(x) = \sum_n \int_{BZ} dk \Psi_n(k) u_{n,k}(x) e^{ik\cdot x} \tag{69}$$

$$\text{want } x\Psi(x) = \sum_n \int_{BZ} dk X \Psi_n(k) u_{n,k}(x) e^{ik\cdot x} \tag{70}$$

$$\begin{aligned}
x\Psi(x) &= x \sum_n \int_{BZ} dk \Psi_n(k) u_{n,k}(x) e^{ik\cdot x} && (71)\\
&= \sum_n \int_{BZ} dk \Psi_n(k) u_{n,k}(x) x e^{ik\cdot x} && (72)\\
&= \sum_n \int_{BZ} dk \Psi_n(k) u_{n,k}(x)(-i\partial_k) e^{ik\cdot x} && (73)\\
&= \sum_n \int_{BZ} dk i\partial_k(\Psi_n(k) u_{n,k}(x)) e^{ik\cdot x} && (74)\\
&= \sum_n \int_{BZ} dk i\partial_k \Psi_n(k) u_{n,k}(x) + \Psi_n(k) i\partial_k u_{n,k}(x)) e^{ik\cdot x} && (75)\\
&= \sum_n \int_{BZ} dk i\partial_k \Psi_n(k) u_{n,k}(x) + \Psi_n(k) \int dx' \delta(x-x') i\partial_k u_{n,k}(x')) e^{ik\cdot x} && (76)\\
&= \sum_n \int_{BZ} dk i\partial_k \Psi_n(k) u_{n,k}(x) + \Psi_n(k) \int dx' \sum_m u^*_{m,k}(x') u_{m,k}(x) i\partial_k u_{n,k}(x')) e^{ik\cdot x} && (77)\\
&= \sum_n \int_{BZ} dk i\partial_k \Psi_n(k) u_{n,k}(x) + \Psi_n(k) \sum_m u_{m,k}(x) \int dx' u^*_{m,k}(x') i\partial_k u_{n,k}(x')) e^{ik\cdot x} && (78)\\
&= \sum_n \int_{BZ} dk i\partial_k \Psi_n(k) u_{n,k}(x) + \Psi_n(k) \sum_m u_{m,k}(x)(-A_{mn}(k))) e^{ik\cdot x} && (79)\\
& && (80)
\end{aligned}$$

So in Bloch representation, $X = \partial_k - (A_{mn})$

7 Orthogonality in kq-representation

$$
\begin{aligned}
|q-q'| &< a \text{ by definition} \\
|k-k'| &< b \text{ by definition} \\
\psi_{kq}(x) &= \sum_n \delta(x-q-na)e^{inka} \\
\int dx\psi^*_{kq}(x)\psi_{k'q'}(x) &= \int dx \sum_{n,n'} \delta(x-q-na)e^{-inka}\delta(x-q'-n'a)e^{in'k'a} \\
&= \int dx \sum_{n,n'} e^{in'k'a-inka}\delta(x-q-na)\delta(x-q'-n'a) \\
&= \sum_{n,n'} e^{ia(n'k'-nk)}\delta(q-q'-(n+n')a) \\
&= \sum_{n,n'} e^{ia(n'k'-n'k+n'k-nk)}\delta(q-q'-(n-n')a) \\
&= \sum_{n,n'} e^{ian'(k'-k)+ia(n'-n)k)}\delta(q-q'-(n-n')a) \\
&= \sum_{n,n'} e^{ian'(k'-k)+i(q'-q)k)}\delta(q-q'-(n-n')a) \\
&= \sum_{n,n'} e^{ian'(k'-k))}\delta(q-q'-(n-n')a), \text{ because } |q-q'|<a \text{ and } q-q'=(n-
\end{aligned}
$$

8 Position operator in Zak basis

$$\begin{aligned}
\langle k'q'|p|kq\rangle &= \int dpdp' \langle kq|p\rangle\langle p|p|p'\rangle\langle p'|l'q'\rangle && (92)\\
&= \int dpp \langle kq|p\rangle\langle p|k'q'\rangle && (93)\\
&= \int dpp \psi^*_{kq}(p)\psi_{k'q'}(p) && (94)\\
&= \int dpp \frac{1}{\sqrt{a}} e^{ikq} \sum_n \delta(p-k-nb) e^{iqnb} \frac{1}{\sqrt{a}} e^{-ik'q'} \sum_{n'} \delta(p-k'-n'b) e^{-iq'n'b} && (95)\\
&= \int dpp \frac{1}{a} \sum_{n,n'} \delta(p-k-nb)\delta(p-k'-n'b) e^{ib(qn-q'n')} e^{i(kq-k'q')} && (96)\\
&= \frac{1}{a} \sum_{n,n'} (k+nb)\delta(k+nb-k'-n'b) e^{ib(qn-q'n')} e^{i(kq-k'q')} && (97)\\
&= \frac{1}{a} \sum_{n,n'} (-i)\frac{\partial}{\partial q}\delta(k+nb-k'-n'b) e^{ib(qn-q'n')} e^{i(kq-k'q')} && (98)\\
&= -i\frac{\partial}{\partial q}\frac{1}{a} \sum_{n,n'} \delta(k+nb-k'-n'b) e^{ib(qn-q'n')} e^{i(kq-k'q')} && (99)\\
&= -i\frac{\partial}{\partial q}\delta(q-q')\delta(k-k') \text{ because we've already proved orthogonality} && (100)\\
& && (101)
\end{aligned}$$

9 Zak combinatorial formula

$$
\begin{aligned}
& [\prod_{p=1}^{m} K_{i_p}][\prod_{q=1}^{n} K_{i_{m+q}}] && (102)\\
= & [\prod_{p=1}^{m+n} K_p] + \frac{1}{(m+n)!}\sum_{p=1}^{m}\sum_{q=1}^{n}[K_{i_p}, K_{i_{m+q}}] \sum_{\sigma\in S_{m+n}, \sigma^{-1}(p)>\sigma^{-1}(m+q)} K_{i_{\sigma(1)}}\ldots\hat{K}_{i_{m+q}}\hat{K}_{i_p}\ldots K_{i_{\sigma(m+n)}} && (103)\\
= & [\prod_{p=1}^{m+n} K_p] + \frac{(m+n)!}{(m+n-2)!}\frac{(m+n-2)!}{(m+n)!}\sum_{p=1}^{m}\sum_{q=1}^{n}\frac{1}{2}[K_{i_1}\ldots\hat{K}_{i_{n+q}}\ldots\hat{K}_p\ldots K_{i_{m+n}}] + \cdots && (104)\\
= & [\prod_{p=1}^{m+n} K_p] + \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3}\frac{1}{2}[K_\alpha, K_\beta][(\frac{\partial}{\partial K_\alpha}\prod_{s=1}^{m} K)(\frac{\partial}{\partial K_\beta}\prod_{t=1}^{n} K_t)] && (105)
\end{aligned}
$$

10 Kronig-Penney

$$
\begin{aligned}
T(k,z) = & \begin{pmatrix} e^{ikz} & 0 \\ 0 & e^{-ikz} \end{pmatrix} \text{ translate by } z \\
S(k) = & \begin{pmatrix} 1 & 1 \\ ik & -ik \end{pmatrix} \text{convert from } \{e^{ikx}, e^{-ikx}\} \\
S(k)T(k,z)S(k)^{-1} = & \begin{pmatrix} \cos(kz) & \sin(kz)/k \\ -k\sin(kz) & \cos(kz) \end{pmatrix} \\
0 = & |S(\beta)T(\beta,b)S(\beta)^{-1}S(\alpha)T(\alpha,a)S(\alpha)^{-1} - e^{ik(a+b)}\mathbb{1}| \\
\cos((a+b)k) = & -\frac{\alpha^2+\beta^2}{2\alpha\beta}\sin(a\alpha)\sin(b\beta) + \cos(a\alpha)\cos(b\beta)
\end{aligned}
$$

11 Su-Schrieffer-Heeger Model

$$
\begin{aligned}
H(\Delta) = & t(1+\Delta)\sum_i c^\dagger_{i,A}c_{i,B} + t(1-\Delta)\sum_i c^\dagger_{i+1,A}c_{i,B} + \text{h.c.} + u\Delta^2 && (106)\\
& && (107)
\end{aligned}
$$