# AUTOENCODERS FOR DYNAMIC SYSTEM IDENTIFICATION

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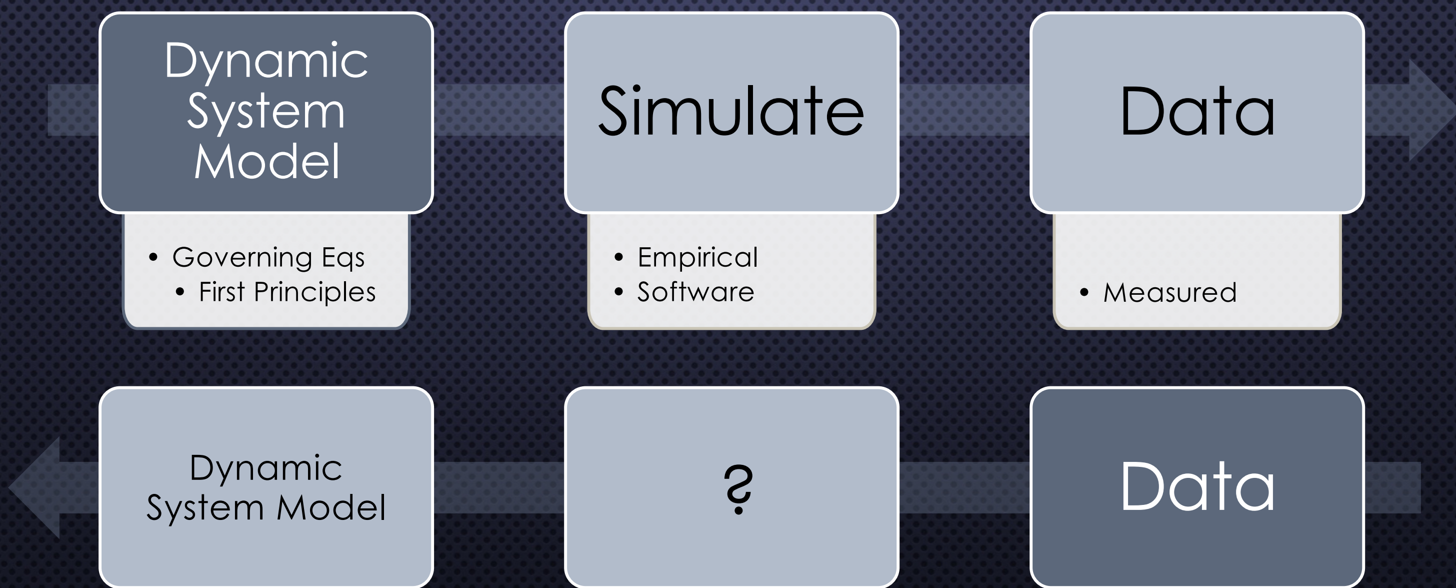
DS 615 – FALL 2022

# DYNAMIC SYSTEM MODELING



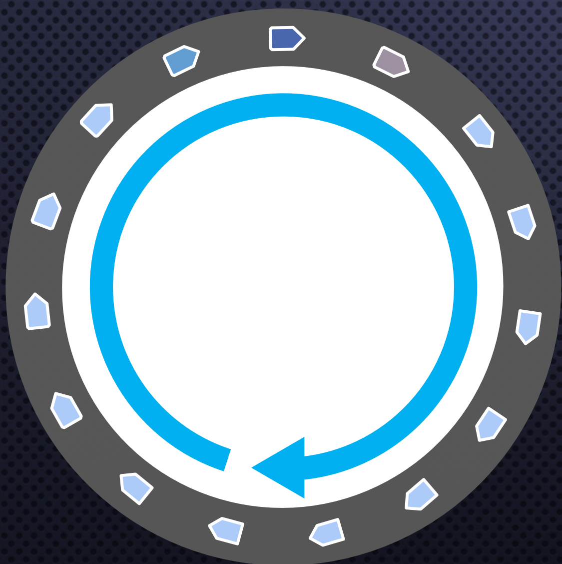
# DYNAMIC TRAFFIC MODELING

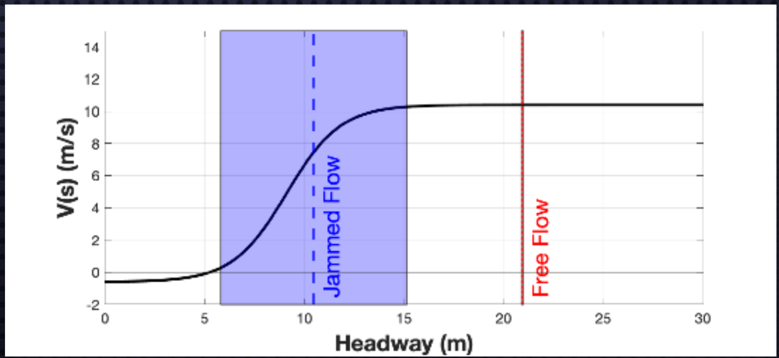
## Optimal Velocity Model:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = a[V(h) - x_2]$$

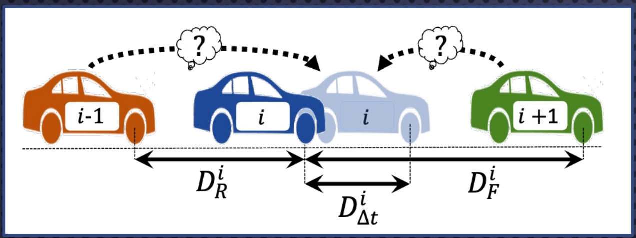
$$h = x_1^{k+1} - x_1^k$$

$$V(h) = \alpha \tanh[\beta(h - s_0)] + v_o$$





# DATA GENERATION

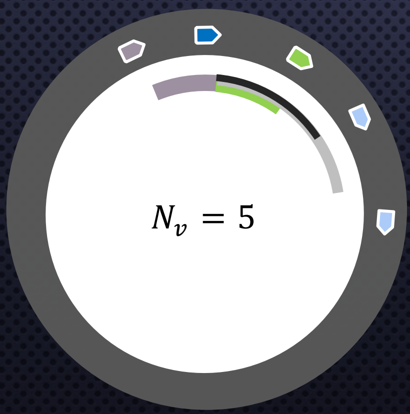



$\dot{x}_1 = x_2$

$\dot{x}_2 = a(V(h) - x_2)$

$\boldsymbol{u} = N_v$ - 1 trajectories

$\boldsymbol{u} = ?$




- 5 vehicles: 2 state variables, 4 inputs
- 5 seconds ($\Delta$t = 0.2)
  - 1500 time intervals
- x1000 instances:

Data tensor: [5000,2,4,1500]

# SPARSE IDENTIFICATION OF N DYNAMICS

$\dot{x}_1 = x_2$

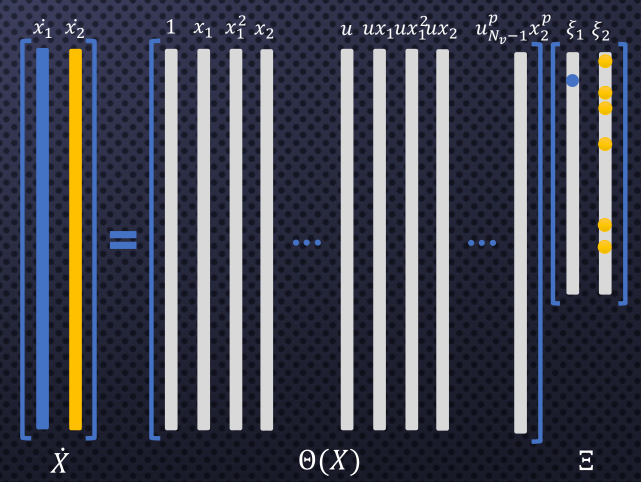
$\dot{x}_2 = a(V(h) - x_2)$

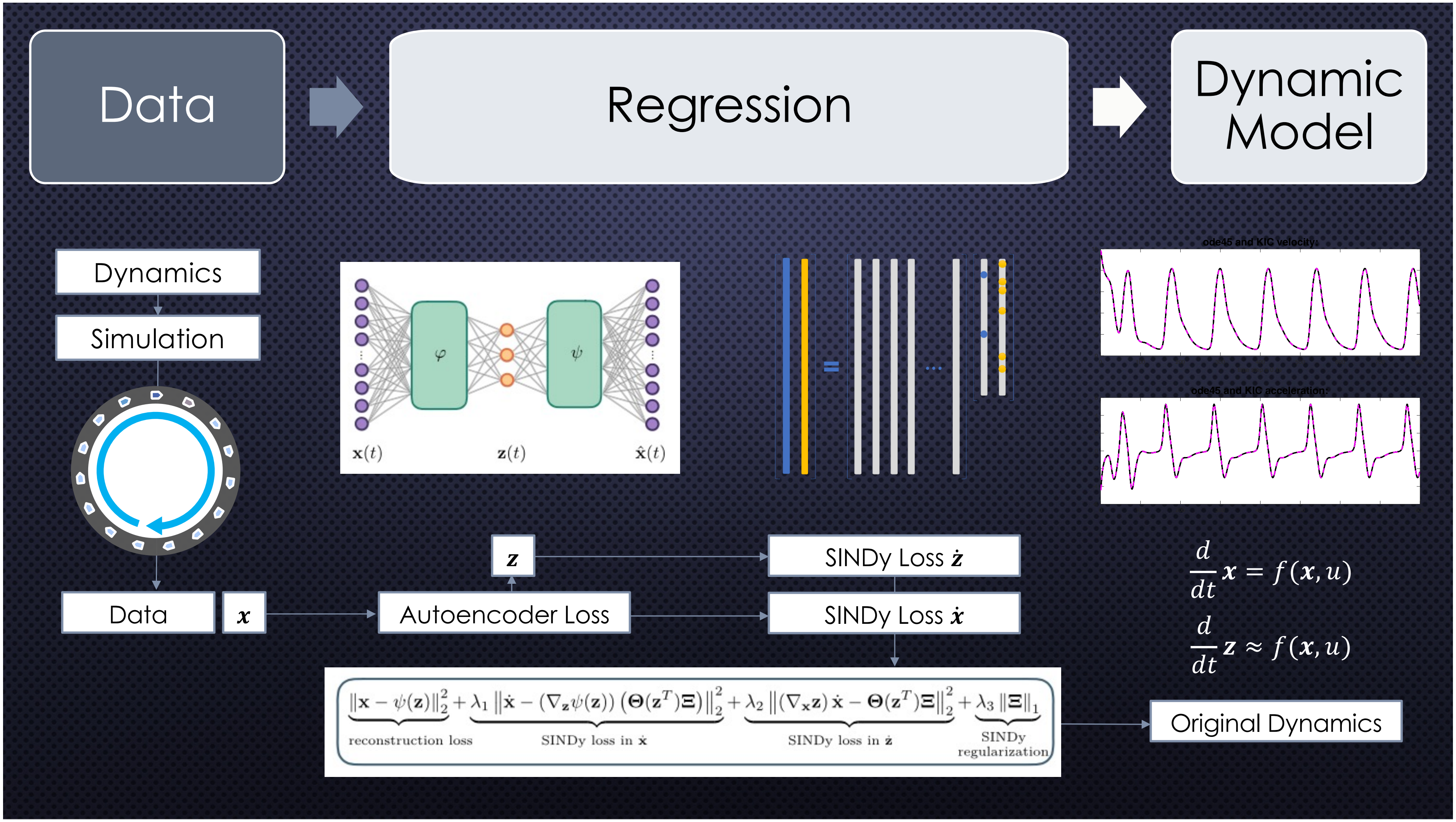
$\boldsymbol{u} = N_v$ - 1 trajectories

$$\dot{\mathrm{X}} = \boldsymbol{\Theta}(\mathbf{X}, \mathbf{U})\boldsymbol{\Xi}$$

$$(\mathbf{Y} = \boldsymbol{X}\beta)$$

$$\boldsymbol{\xi}_k = \operatorname{argmin}_{\boldsymbol{\xi}_k} \|\dot{\mathbf{X}}_k - \boldsymbol{\xi}_k \boldsymbol{\Theta}^T(\mathbf{X})\|_2 + \alpha \|\boldsymbol{\xi}_k\|_1,$$




Data
Regression
Dynamic Model
Dynamics
Simulation
Data
$\boldsymbol{x}$
$\boldsymbol{z}$
Autoencoder Loss
SINDy Loss $\dot{\boldsymbol{z}}$
SINDy Loss $\dot{\boldsymbol{x}}$
$\mathbf{x}(t)$
$\mathbf{z}(t)$
$\hat{\mathbf{x}}(t)$
$\varphi$
$\psi$
$\underbrace{\|\mathbf{x}-\psi(\mathbf{z})\|_2^2}_{\text{reconstruction loss}} + \underbrace{\lambda_1 \left\|\dot{\mathbf{x}} - (\nabla_{\mathbf{z}}\psi(\mathbf{z}))\left(\boldsymbol{\Theta}(\mathbf{z}^T)\boldsymbol{\Xi}\right)\right\|_2^2}_{\text{SINDy loss in } \dot{\mathbf{x}}} + \underbrace{\lambda_2 \left\|(\nabla_{\mathbf{x}}\mathbf{z})\,\dot{\mathbf{x}} - \boldsymbol{\Theta}(\mathbf{z}^T)\boldsymbol{\Xi}\right\|_2^2}_{\text{SINDy loss in } \dot{\mathbf{z}}} + \underbrace{\lambda_3 \|\boldsymbol{\Xi}\|_1}_{\text{SINDy regularization}}$
$\frac{d}{dt}\boldsymbol{x} = f(\boldsymbol{x}, u)$
$\frac{d}{dt}\boldsymbol{z} \approx f(\boldsymbol{x}, u)$
Original Dynamics

# LINEARIZING THE OVM

## Optimal Velocity Model:

$$\dot{x_1} = x_2$$
$$\dot{x_2} = a[V(h) - x_2]$$

$$h = x_1^{k+1} - x_1^k$$

$$V(h) = \alpha \tanh[\beta(h - s_0)] + v_o$$

$$V(h) = \alpha \frac{3\beta(h - s_0)}{(\beta h - \beta s_0)^2 + 3} + v_o$$

**Exact**

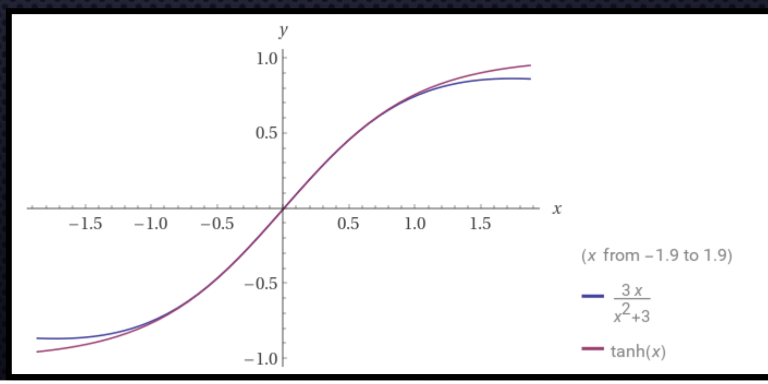
$$\tanh(z) \equiv \frac{\sinh z}{\cosh z} = \frac{e^z - e^{-z}}{e^z + e^{-z}} = \frac{e^{2z} - 1}{e^{2z} + 1},$$

**Taylor Series**

$$= \sum_{n=0}^{\infty} \frac{2^{2n}(2^{2n} - 1) B_{2n}}{(2n)!} z^{2n-1}$$
$$= z - \frac{1}{3} z^3 + \frac{2}{15} z^5 - \frac{17}{315} z^7 + \frac{62}{2835} z^9 - \ldots$$

**Pade' approx (2,2)**

$$= \frac{3z}{z^2 + 3}$$



# LINEARIZING THE OVM

## ILL-POSED FUNCTION:

$$\dot{x_1} = x_2$$
$$\dot{x_2} = a[V(h) - x_2]$$

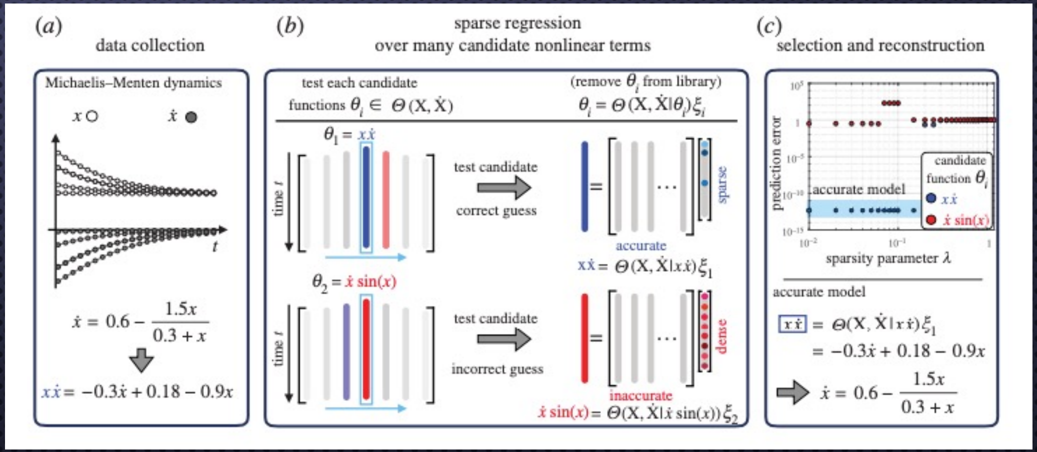
$$h = x_1^{k+1} - x_1^k$$

$$V(h) = \alpha \tanh[\beta(h - s_0)] + v_o$$

$$\Downarrow$$

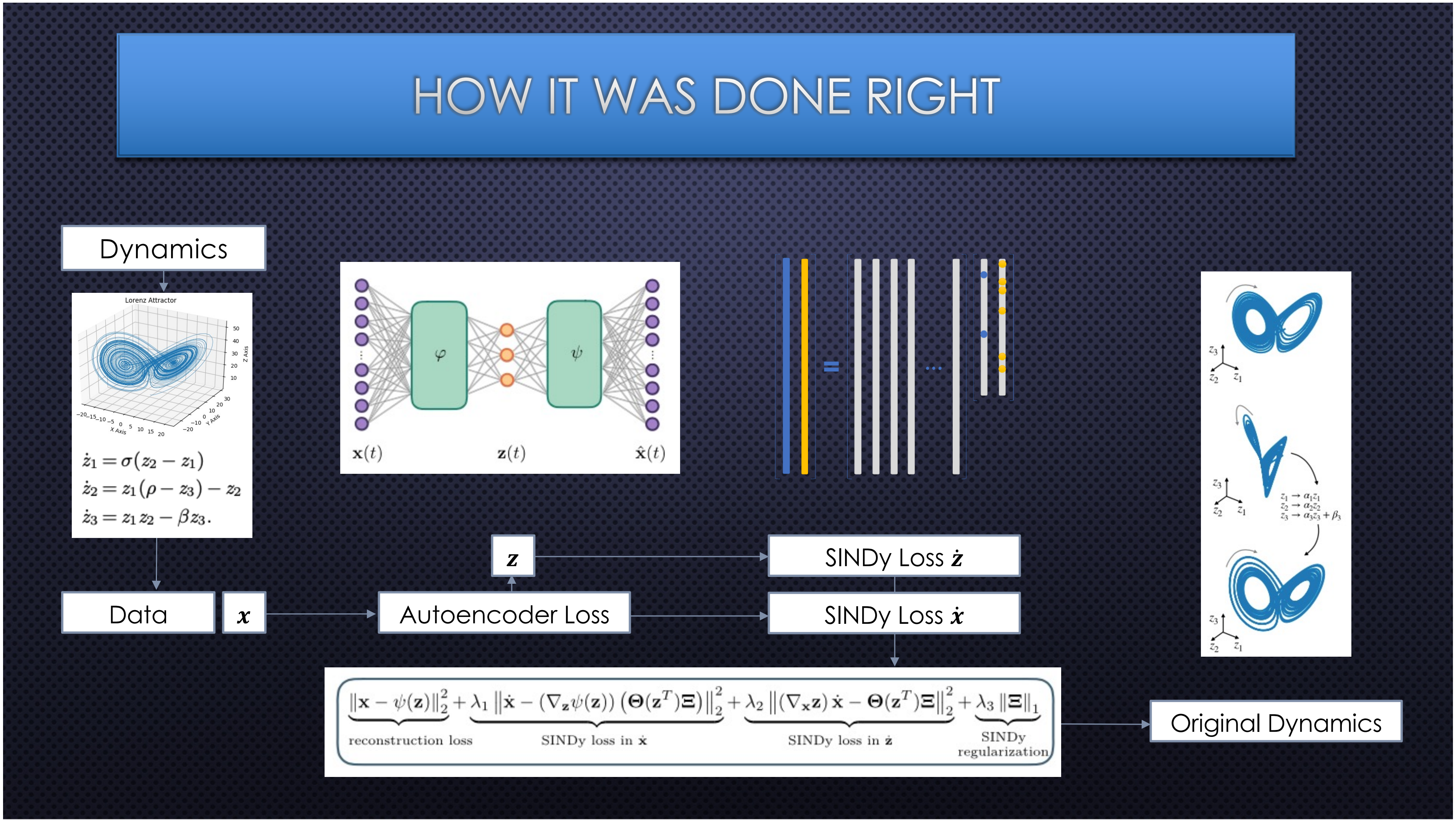
$$V(h) = \alpha \frac{3\beta(h - s_0)}{(\beta h - \beta s_0)^2 + 3} + v_o$$

## THE PROBLEM:

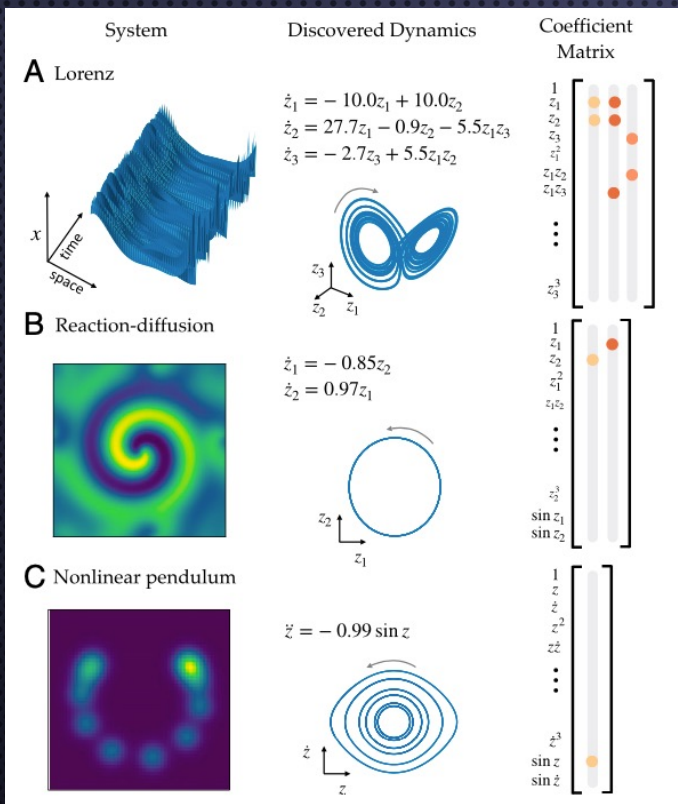


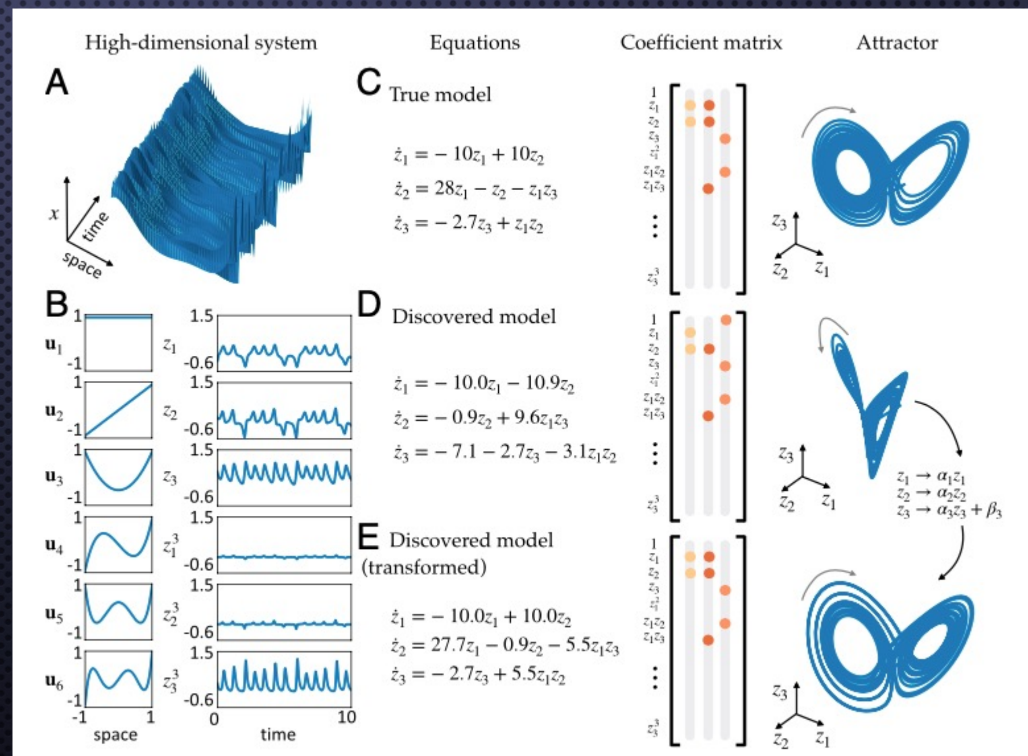

$$\theta_j(\mathbf{X}, \dot{\mathbf{X}}) = \boldsymbol{\Theta}(\mathbf{X}, \dot{\mathbf{X}} | \theta_j(\mathbf{X}, \dot{\mathbf{X}})\boldsymbol{\xi}_j,$$

$$\|\theta_j(\mathbf{X}, \dot{\mathbf{X}}) - \boldsymbol{\Theta}(\mathbf{X}, \dot{\mathbf{X}} | \theta_j(\mathbf{X}, \dot{\mathbf{X}})\boldsymbol{\xi}_j\|_2 + \beta \left\|\boldsymbol{\xi}_j\right\|_0,$$

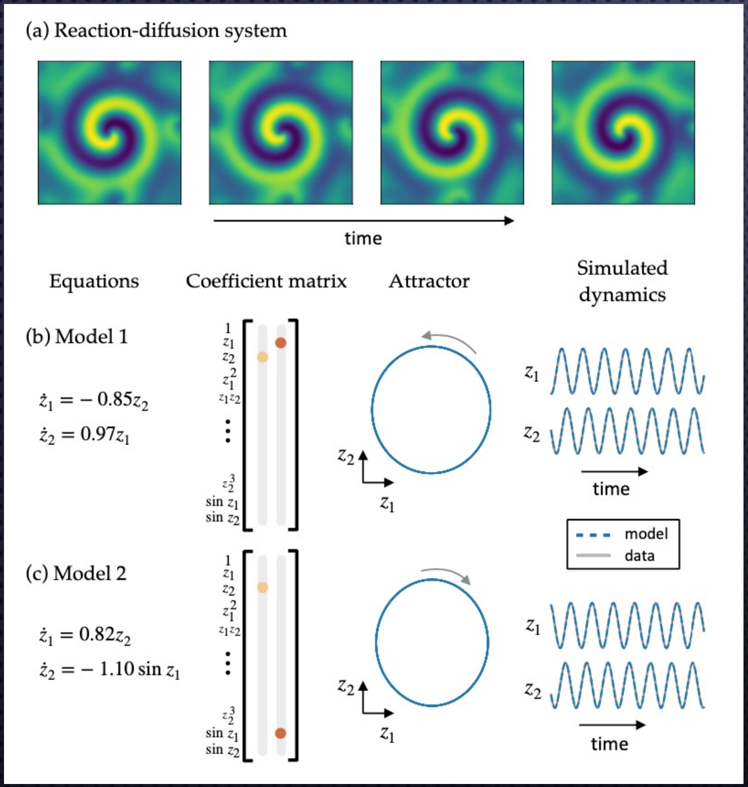

HOW IT WAS DONE RIGHT
Dynamics
Lorenz Attractor
$\dot{z}_1 = \sigma(z_2 - z_1)$
$\dot{z}_2 = z_1(\rho - z_3) - z_2$
$\dot{z}_3 = z_1 z_2 - \beta z_3.$
$\varphi$
$\psi$
$\mathbf{x}(t)$
$\mathbf{z}(t)$
$\hat{\mathbf{x}}(t)$
$\boldsymbol{z}$
SINDy Loss $\dot{\boldsymbol{z}}$
Data
$\boldsymbol{x}$
Autoencoder Loss
SINDy Loss $\dot{\boldsymbol{x}}$
$\underbrace{\|\mathbf{x}-\psi(\mathbf{z})\|_2^2}_{\text{reconstruction loss}} + \lambda_1 \underbrace{\|\dot{\mathbf{x}} - (\nabla_{\mathbf{z}}\psi(\mathbf{z}))(\boldsymbol{\Theta}(\mathbf{z}^T)\boldsymbol{\Xi})\|_2^2}_{\text{SINDy loss in } \dot{\mathbf{x}}} + \lambda_2 \underbrace{\|(\nabla_{\mathbf{x}}\mathbf{z})\dot{\mathbf{x}} - \boldsymbol{\Theta}(\mathbf{z}^T)\boldsymbol{\Xi}\|_2^2}_{\text{SINDy loss in } \dot{\mathbf{z}}} + \lambda_3 \underbrace{\|\boldsymbol{\Xi}\|_1}_{\text{SINDy regularization}}$
Original Dynamics

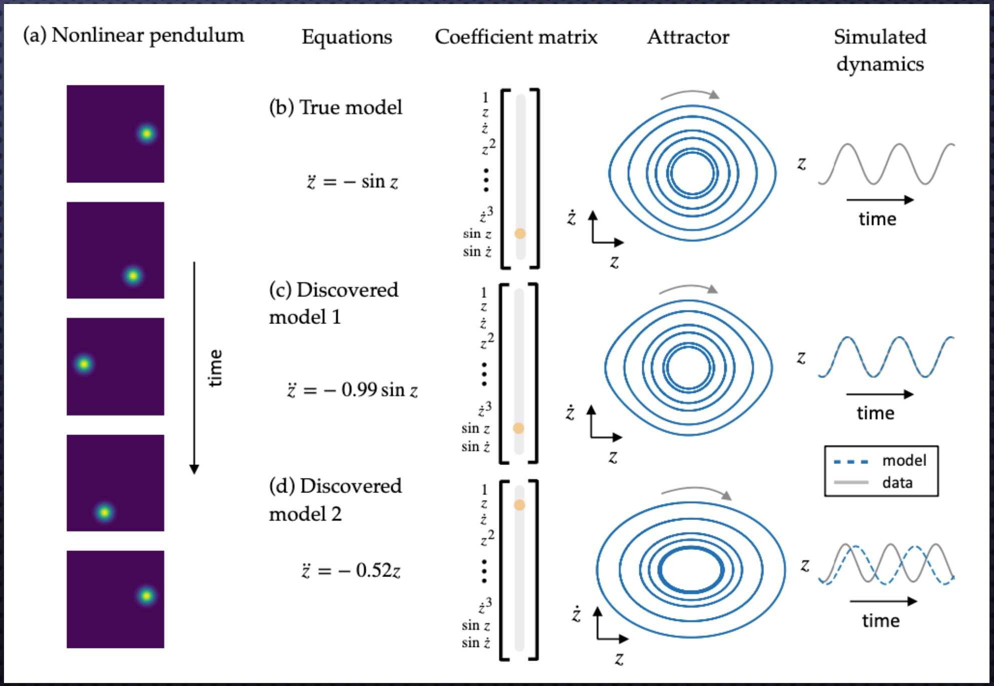
# HOW IT WAS DONE RIGHT



# HOW IT WAS DONE RIGHT



# POSSIBLE SOLUTIONS?

1. Select a traffic model that is not rational/implicit
2. Configure SINDy Library to also use negative exponents
3. Use attention/embedding layer to select correct inputs
4. Use a different structured NN to obtain dynamics

   (graph Network with symbolic regression, or similar)

# CONCLUSIONS

1. Autoencoders are powerful for reducing the dimensionality of a system.
2. With proper regression, Dynamic models can be discovered.
3. Every problem is unique, and its solution may require a different NN architecture to solve.

# REFERENCES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