Saturation of the Magnetorotational Instability

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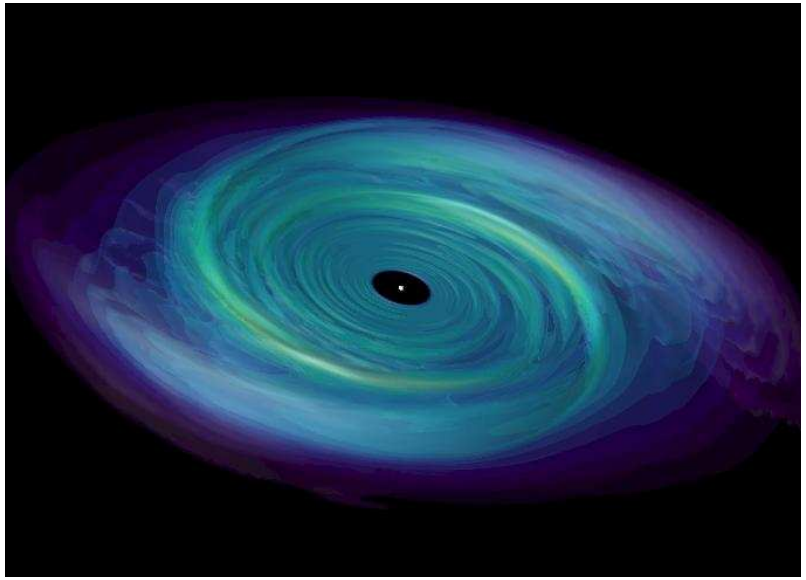
Talk Outline

- Motivation
  - Accretion disks
  - Laboratory experiments
- Model Problem
  - Local shearing sheet approximation
- Scaling
- Asymptotic solution
- Sample solutions
- Interpretation of the solution
- Discussion

Motivation



- Turbulent accretion disks require the presence of an efficient mechanism for ang. mtm. transport
- MRI: Linear instability occurring in Rayleigh-stable regime in the presence of a weak poloidal B-field
  - Thin sheets of matter moving radially inwards and outwards
  - Couette-Taylor geometry: Velikhov 1959, Chandrasekhar 1960
  - Keplerian disks: Balbus-Hawley 1991, 1998
- Efficiency of ang. mtm. transport depends on saturation of MRI

- Numerical investigations: shearing sheet geometry
  - Balbus-Hawley 1991: evol'n to solid body rotation, X-points suggest reconnection process important to saturation
  - Sano *et al.* 1998: whether saturation occurs depends on the Elsasser number $\Lambda(> or < 1)$
  - Goodman & Xu 1994, Fleming *et al.* 2000; saturation $\forall\Lambda$ if non-axisymmetric instability included
- Numerical investigations: global cylindrical geometry
  - Kersalé *et al.* 2004, 2006
  - Cattaneo *et al.* 2005

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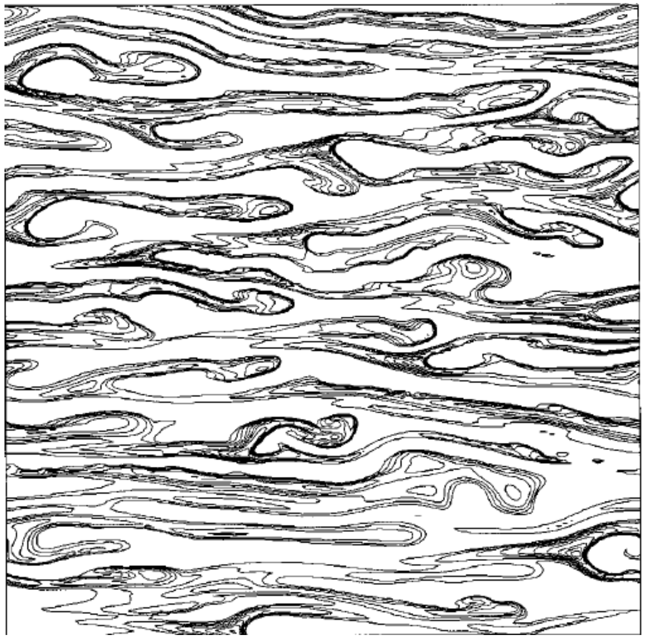

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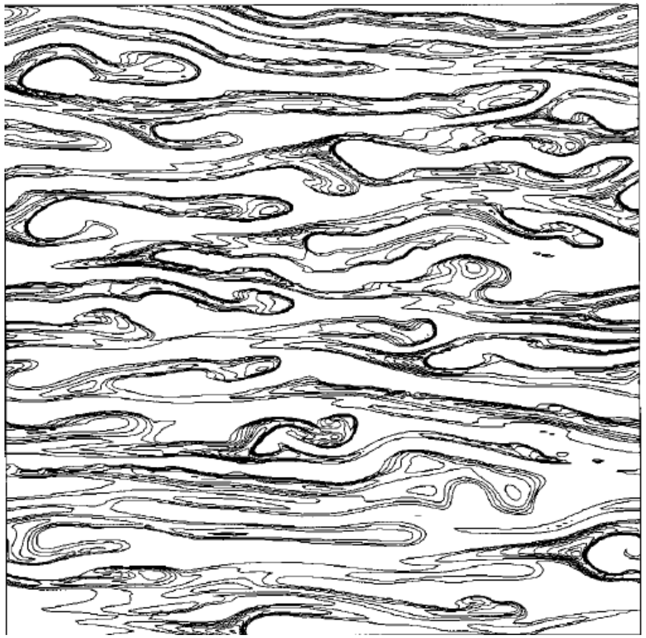

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- Laboratory experiments: global cylindrical geometry
  - Sisan *et al.* 2004
  - Stefani *et al.* 2006

Formulation of a Model Problem

- Shearing sheet approx'n at r^* with local ang. velocity $\Omega^*(r^*)\widehat{\mathbf{z}}$:
- Straight channel: $-L^*/2 \leq x^* \leq L^*/2$, $-\infty < y^* < \infty$, $-\infty < z^* < \infty$
- Linear shear: $\mathbf{U_0}^* = (0, \sigma^* x^*, 0)$
- Constant B-Field: $\mathbf{B_0}^* = (0, B^*_{tor}, B^*_{pol})$
- Perturbations: $\mathbf{u} \equiv (u, v, w) = (-\psi_z, v, \psi_x)$, $\mathbf{b} \equiv (a, b, c) = (-\phi_z, b, \phi_x)$

  <u>Axisymmetric Equations</u>

$$\nabla^2\psi_t + 2\Omega v_z + J(\psi, \nabla^2\psi) = v_A^2\nabla^2\phi_z + v_A^2 J(\phi, \nabla^2\phi) + \nu\nabla^4\psi, \tag{1}$$

$$v_t - (2\Omega + \sigma)\psi_z + J(\psi, v) = v_A^2 b_z + v_A^2 J(\phi, b) + \nu\nabla^2 v, \tag{2}$$

$$\phi_t + J(\psi, \phi) = \psi_z + \eta\nabla^2\phi, \tag{3}$$

$$b_t + J(\psi, b) = v_z - \sigma\phi_z + J(\phi, v) + \eta\nabla^2 b, \tag{4}$$

where $J(f, g) \equiv f_x g_z - f_z g_x$.

- $v_A \equiv B^*_{pol}/\sqrt{\mu_0\rho^*}U^*$, Ω, ν, η are the *dimensionless* Alfvén speed, rotation rate, kinematic viscosity and ohmic diffusivity

Remarks on Model Problem

Local shearing sheet approx'n $\Rightarrow$ special properties of model eqs:

- Toroidal field B^*_{tor} drops out
  - suppression of hoop stresses
  - toroidal field remains in the radial pressure balance

$$2\Omega^* V_0^* + \frac{V_0^{*2}}{r^*} = \frac{1}{\rho^*}\frac{dP_0^*}{dr^*} + \frac{d}{dr^*}\left(\frac{B^{*2}_{tor}}{2\mu_0\rho^*}\right) + \frac{B^{*2}_{tor}}{\mu_0\rho^* r^*} \qquad (5)$$

- no distinction between inward and outward directions
  - symmetry $x \to -x$, $(\psi, v, \phi, b) \to -(\psi, v, \phi, b)$
  - direction of accretion and angular mtm. flux must be imposed externally
- MRI is an exponentially growing instability
  - this is not the case in polar coordinates with nonzero B^*_{tor}: Knobloch 1992

Linear Theory

- Linearization about the trivial state $\psi = v = \phi = b = 0$:
- Perturbation $\exp[\lambda t + ikx + inz]$, $p = k^2 + n^2 \Rightarrow$ dispersion rel'n

$$p[(\lambda+\nu p)(\lambda+\eta p)+v_A^2 n^2]^2 + 2\Omega n^2[(\lambda+\eta p)^2(2\Omega+\sigma)+\sigma v_A^2 n^2] = 0. \quad (6)$$

- Conventional view of MRI: positive growth rate λ achieved for sufficiently large vertical wavenumbers n whenever $\sigma < 0$, $v_A \neq 0$, provided only that ν, and η are sufficiently small
  - For $\nu = \eta = 0$

$$\lambda^2 = -\frac{v_A^2 n^2 \sigma}{2\Omega+\sigma} + O(v_A^4 n^4). \quad (7)$$

  - For $\lambda = 0$ threshold for instability exists. For small ν, η critical Elsasser number

$$\Lambda_c \equiv v_A^2/\Omega\eta = \eta\left(\frac{2\Omega+\sigma}{\Omega\sigma}\right)\frac{p^2}{n^2} + O(\nu,\eta)^3. \quad (8)$$

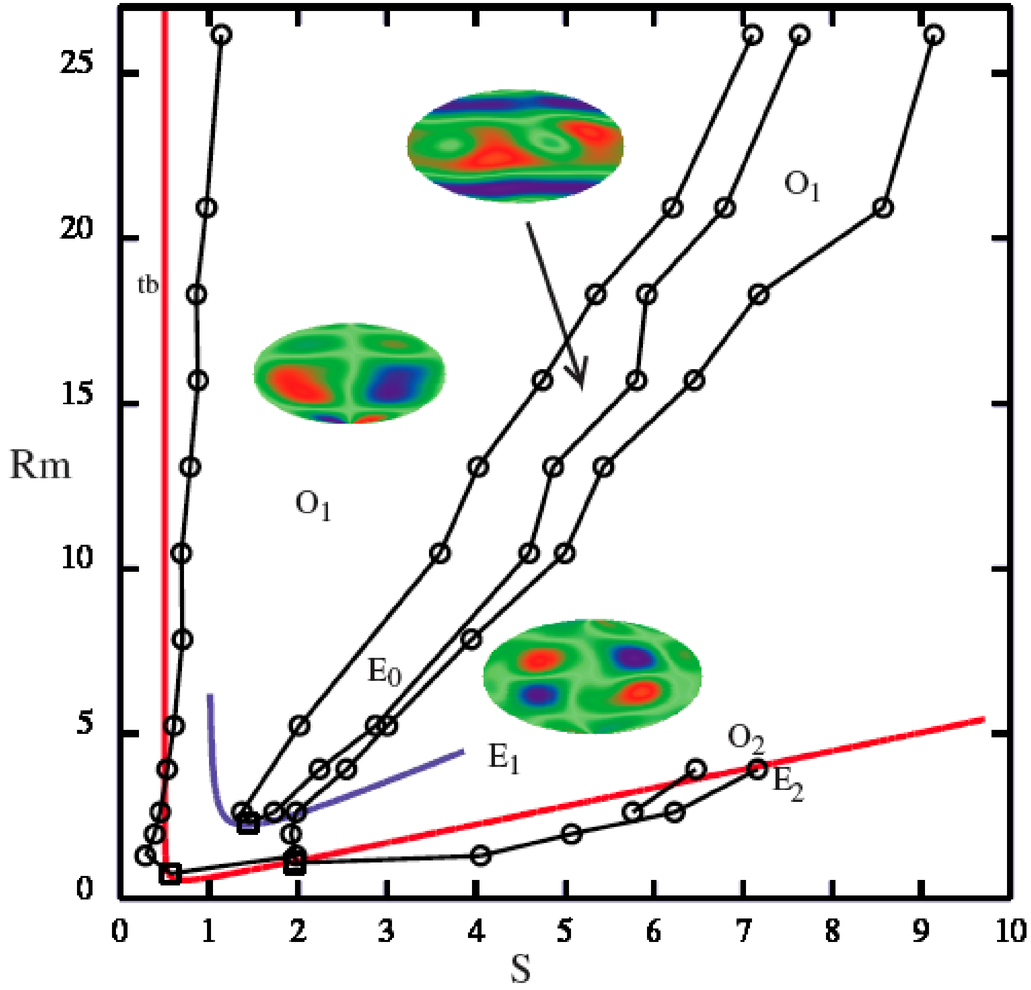
  - Reconnection effects described by finite η are more important for stabilizing the system against the MRI than viscosity

Scaling Assumptions

- Traditional approach to nonlinear saturation: weakly nonlinear theory with $(\Lambda - \Lambda_c)/\Lambda_c \ll 1$.
- Our approach: strongly nonlinear theory
  - shear is the dominant source of energy for the MRI
  - MRI itself requires the presence of a (weaker) vertical magnetic field
  - dissipative effects are weaker still but cannot be ignored since they are ultimately responsible for the saturation of the instability
- Hence scaling:
  - rapid rotation, strong shear: $(\Omega, \sigma) = \epsilon^{-1}(\hat{\Omega}, \hat{\sigma})$
  - magnetic field: $v_A = 1$ *i.e.*, $U^* = v_A^* \equiv B_{pol}^* / \sqrt{\mu_0 \rho^*}$
  - weak dissipative processes: $(\nu, \eta) = \epsilon(\hat{\nu}, \hat{\eta})$
  - thin fingers, strong growth: $\partial_x \to \partial_x, \quad \partial_z \to \epsilon^{-1}\partial_z, \quad \partial_t \to \epsilon^{-1}\partial_t$
- In the following we take $\epsilon \ll 1$, or equivalently $Rm \gg S \gg \max(1, Pm)$, while $\Lambda = O(1)$. Here $Rm = |\sigma^*|L^{*2}/\eta^*$, $Pm = \nu^*/\eta^*$, $S \equiv v_A^* L^*/\eta^*$ are the magnetic Reynolds, magnetic Prandtl and Lundquist numbers.

Relation to Recent Experiment

- Lathrop group (Sisan *et al.* PRL 2004)



- Our scaling goes through the plane as $Rm \sim S^2$

Scaled Equations

- In parallel with the above assumptions we need to make further assumptions about the relative magnitude of the various fields:
- we find $(\psi, \phi) \to \epsilon(\psi, \phi)$, $(v, b) \to \epsilon^{-1}(v, b)$ leads to a self-consistent set of reduced pdes
- scaled pdes:

$$\epsilon \frac{D}{Dt}\left(\partial_x^2 + \epsilon^{-2}\partial_z^2\right)\psi + 2\epsilon^{-3}\hat{\Omega}v_z = v_A^2\left(\partial_x^2 + \epsilon^{-2}\partial_z^2\right)\phi_z + \epsilon v_A^2 J\left(\phi, \left(\partial_x^2 + \epsilon^{-2}\partial_z^2\right)\phi\right) + \epsilon^2\hat{\nu}\left(\partial_x^2 + \epsilon^{-2}\partial_z^2\right)^2\psi \tag{9}$$

$$\epsilon^{-1}\frac{D}{Dt}v - \epsilon^{-1}(2\hat{\Omega} + \hat{\sigma})\psi_z = \epsilon^{-2}v_A^2 b_z + \epsilon^{-1}v_A^2 J(\phi, b) + \hat{\nu}(\partial_x^2 + \epsilon^{-2}\partial_z^2)v \tag{10}$$

$$\epsilon\frac{D}{Dt}\phi = \psi_z + \epsilon^2\hat{\eta}(\partial_x^2 + \epsilon^{-2}\partial_z^2)\phi \tag{11}$$

$$\epsilon^{-1}\frac{D}{Dt}b = \epsilon^{-2}v_z - \epsilon^{-1}\hat{\sigma}\phi_z + \epsilon^{-1}J(\phi, v) + \hat{\eta}(\partial_x^2 + \epsilon^{-2}\partial_z^2)b, \tag{12}$$

where $D/Dt = \partial_t + J[\psi, \bullet]$.

Derivation of Reduced PDEs

- To solve the scaled equations we suppose $\psi(x,z,t) = \psi_0(x,z,t) + \epsilon\psi_1(x,z,t) + \ldots$, etc.
- Deduction: Leading order azimuthal fields v_0, b_0 represent large-scale adjustment to background shear and toroidal field due to MRI
  - From Eqs for azimuthal fields v, b at $O(\epsilon^{-2})$ and poloidal fields ψ at $O(\epsilon^{-3})$

$$v_A^2 b_{0z} + \hat{\nu} v_{0zz} = 0, \qquad v_{0z} + \hat{\eta} b_{0zz} = 0, \qquad 2\widehat{\Omega} v_{0z} = 0 \tag{13}$$

  - Hence

$$v_0 = V(x,t), \qquad b_0 = B(x,t) \tag{14}$$

  - Averaging in t at $O(\epsilon^{-1}) \Rightarrow$ slow time evolution. Hence

$$v_0 = V(x), \qquad b_0 = B(x) \tag{15}$$

Reduced PDE's

- From Eqs for azimuthal fields v, b at $O(\epsilon^{-1})$ and poloidal fields ψ, ϕ at $O(\epsilon^{-2})$, $O(1)$

$$\psi_{0zzt} + 2\widehat{\Omega} v_{1z} = v_A^2 \phi_{0zzz} + \widehat{\nu} \psi_{0zzzz} \tag{16}$$

$$v_{1t} - (2\widehat{\Omega} + \widehat{\sigma} + V'(x))\psi_{0z} = v_A^2 b_{1z} - v_A^2 B'(x)\phi_{0z} + \widehat{\nu} v_{1zz} \tag{17}$$

$$\phi_{0t} = \psi_{0z} + \widehat{\eta}\phi_{0zz} \tag{18}$$

$$b_{1t} - \psi_{0z} B'(x) = v_{1z} - (\widehat{\sigma} + V'(x))\phi_{0z} + \widehat{\eta} b_{1zz} \tag{19}$$

- Closure requires determination of $V'(x), B'(x)$.
  - Averaging Eqs for azimuthal fields v, b at $O(1)$ in z, t and integrating gives

$$\widehat{\nu} V'(x) = \overline{\psi_0 v_{1z}} - v_A^2 \overline{\phi_0 b_{1z}} + C_1 \tag{20}$$

$$\widehat{\eta} B'(x) = \overline{\psi_0 b_{1z}} - \overline{\phi_0 v_{1z}} + C_2 \tag{21}$$

- C_1 is determined by BC's; $0 < C_2 < C_{max}$ range of total to zero support of disk by radial pressure gradient.

Strongly Nonlinear Single-Mode Solutions

These equations have stationary solutions of the form

$$\psi_0 = \frac{1}{2}(\Psi(x)\, e^{inz} + \text{c.c.}), \quad v_1 = \frac{1}{2}(\mathcal{V}(x)\, e^{inz} + \text{c.c.}), \tag{22}$$
$$\phi_0 = \frac{1}{2}(\mathcal{F}(x)\, e^{inz} + \text{c.c.}), \quad b_1 = \frac{1}{2}(\mathcal{B}(x)\, e^{inz} + \text{c.c.}),$$

where

$$\mathcal{F} = \frac{i\Psi}{\hat{\eta} n}, \tag{23}$$

$$\mathcal{V} = \frac{(v_A^2 + \hat{\eta}^2 n^2)V' + \hat{\eta}^2 n^2 (2\hat{\Omega} + \hat{\sigma}) + v_A^2 \hat{\sigma}}{n\hat{\eta}(v_A^2 + \hat{\nu}\hat{\eta} n^2)}\, i\Psi, \tag{24}$$

$$\mathcal{B} = \frac{i(v_A^2 + \hat{\nu}\hat{\eta} n^2)B' + n(\hat{\nu}(\hat{\sigma} + V') - \hat{\eta}(2\hat{\Omega} + \hat{\sigma} + V'))}{n\hat{\eta}(v_A^2 + \hat{\nu}\hat{\eta} n^2)}\Psi, \tag{25}$$

and we obtain the nonlinear dispersion relation

$$2\hat{\Omega}[(v_A^2 + \hat{\eta}^2 n^2)V' + (2\hat{\Omega} + \hat{\sigma})\hat{\eta}^2 n^2 + \hat{\sigma} v_A^2] + n^2 (v_A^2 + \hat{\nu}\hat{\eta} n^2)^2 = 0. \tag{26}$$

- Except for the presence of the additional shear rate V' this is nothing but the dispersion relation for the MRI in our scaling regime

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where

$$\mathcal{F} = \frac{i\Psi}{\hat{\eta} n}, \quad \mathcal{V} = Func\left[V'; \widehat{\Omega}, \widehat{\sigma}, v_A, \widehat{\nu}, \widehat{\eta}\right] i\Psi, \quad \mathcal{B} = Func\left[V', B'; \widehat{\Omega}, \widehat{\sigma}, v_A, \widehat{\nu}, \widehat{\eta}\right] i\Psi,$$

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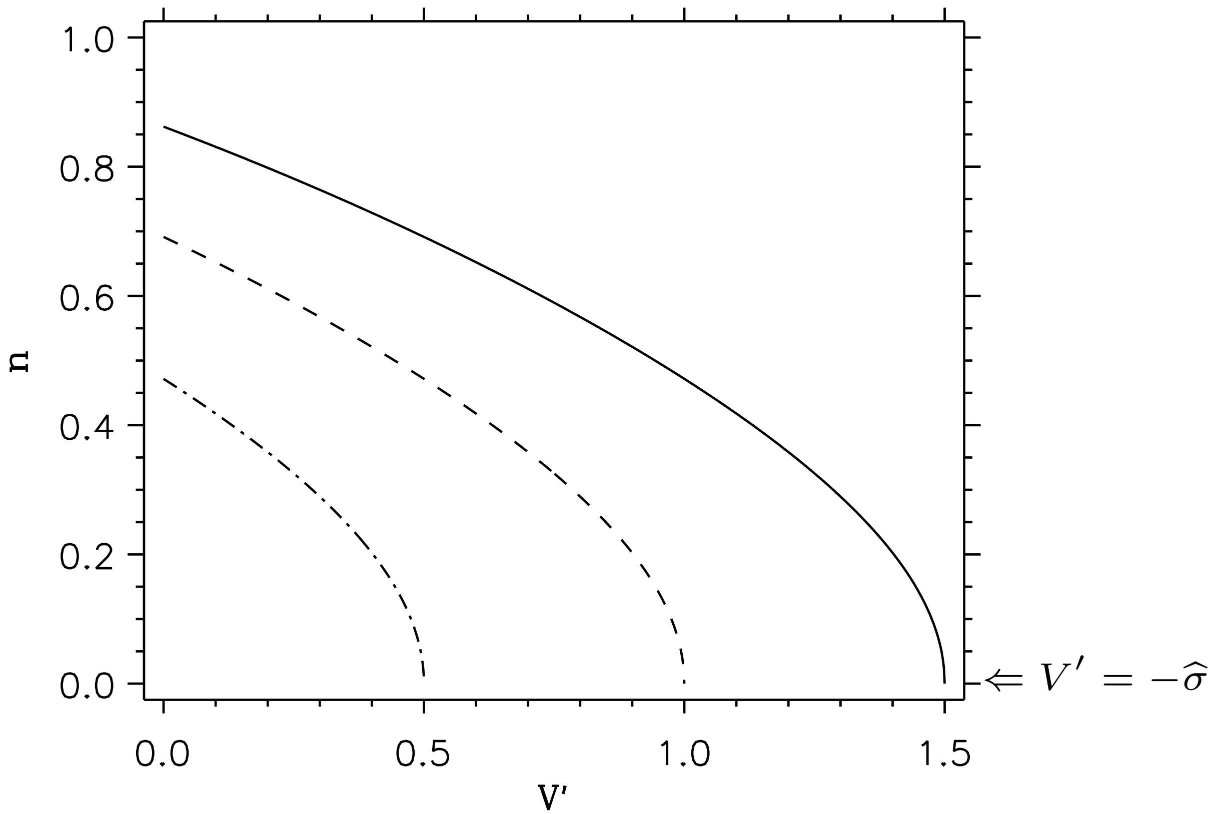
$$2\hat{\Omega}[(v_A^2 + \hat{\eta}^2 n^2)V' + (2\hat{\Omega} + \hat{\sigma})\hat{\eta}^2 n^2 + \hat{\sigma} v_A^2] + n^2 (v_A^2 + \hat{\nu}\hat{\eta} n^2)^2 = 0. \tag{28}$$

- Except for the presence of the additional shear rate V' this is nothing but the dispersion relation for the MRI in our scaling regime
- For each wavenumber n dispersion relation determines V' (see Figure)
- Closure requires the determination of V', B' as a function of Ψ

Nonlinear Dispersion Relation

- nonlinear dispersion relation: $V' =$ const. as function of n

$$2\hat{\Omega}[(v_A^2 + \hat{\eta}^2 n^2)V' + (2\hat{\Omega} + \hat{\sigma})\hat{\eta}^2 n^2 + \hat{\sigma} v_A^2] + n^2(v_A^2 + \hat{\nu}\hat{\eta} n^2)^2 = 0. \quad (29)$$



$\widehat{\Omega} = 1, v_A = 1, \widehat{\nu} = \widehat{\eta} = 1$, and $\widehat{\sigma} = -1.5, -1, -0.5$ (solid, dashed, dashed-dot)

- n decreases from its linear theory value indicating increase in the vertical wavelength as MRI saturates. Selected n vanishes when solid body rotation reached

Single-Mode Solutions: Closure

- Closure requires the determination of V', B' as a function of Ψ. Given

$$\psi_0 = \frac{1}{2}(\Psi(x)\, e^{inz} + \text{c.c.}), \quad v_1 = \frac{1}{2}(\mathcal{V}(x)\, e^{inz} + \text{c.c.}), \tag{30}$$
$$\phi_0 = \frac{1}{2}(\mathcal{F}(x)\, e^{inz} + \text{c.c.}), \quad b_1 = \frac{1}{2}(\mathcal{B}(x)\, e^{inz} + \text{c.c.}),$$

we find

$$V'(x) = \frac{C_1 - \frac{1}{2}\beta|\Psi|^2}{\widehat{\nu} + \frac{1}{2}\alpha|\Psi|^2}, \quad \alpha, \beta = Func\left[\widehat{\Omega}, \widehat{\sigma}, v_A, \widehat{\nu}, \widehat{\eta}\right], \tag{31}$$

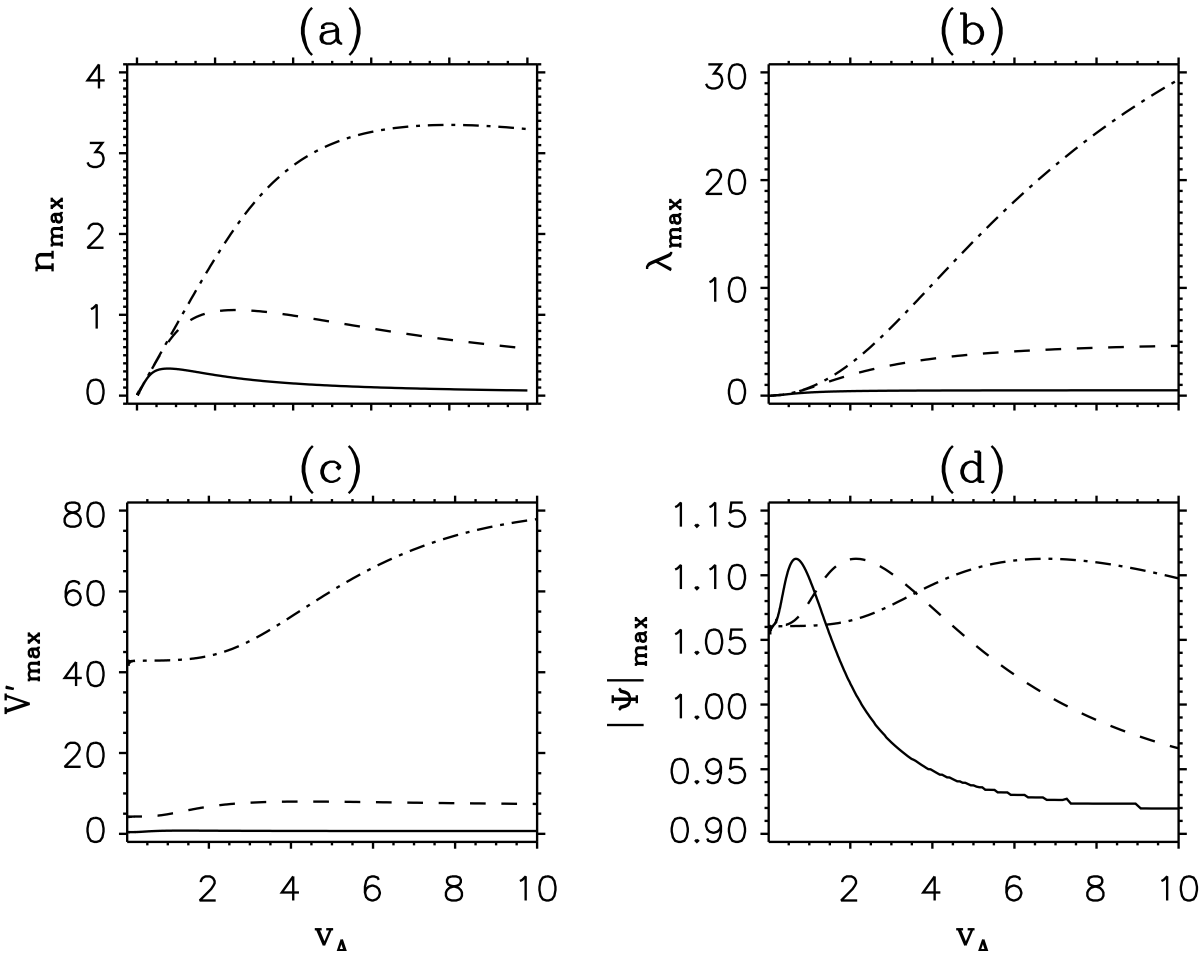
$$B'(x) = \frac{\widehat{\eta} C_2}{\widehat{\eta}^2 + \frac{1}{2}|\Psi|^2}. \tag{32}$$

- MRI requires $C_1 = 0$ for nonzero V' and Ψ
- Nonlinear dispersion relation then gives the saturated value of $|\Psi|$:

$$|\Psi|^2 = -\frac{2\widehat{\nu}\widehat{\eta}^2\left[n^2(v_A^2 + \widehat{\nu}\widehat{\eta}n^2)^2 + 2\widehat{\Omega}\widehat{\sigma}v_A^2 + 2\widehat{\Omega}(2\widehat{\Omega} + \widehat{\sigma})\widehat{\eta}^2 n^2\right]}{\left[4\widehat{\Omega}^2 v_A^2 \widehat{\eta} + n^2\left(v_A^2 + \widehat{\nu}\widehat{\eta}n^2\right)\left(\widehat{\nu}v_A^2 + \widehat{\eta}^3 n^2\right)\right]} \tag{33}$$

  - This is a bifurcation equation with saturation determined by bifurcation parameter $\widehat{\sigma}$ or v_A (equivalently, the Elsasser number Λ)

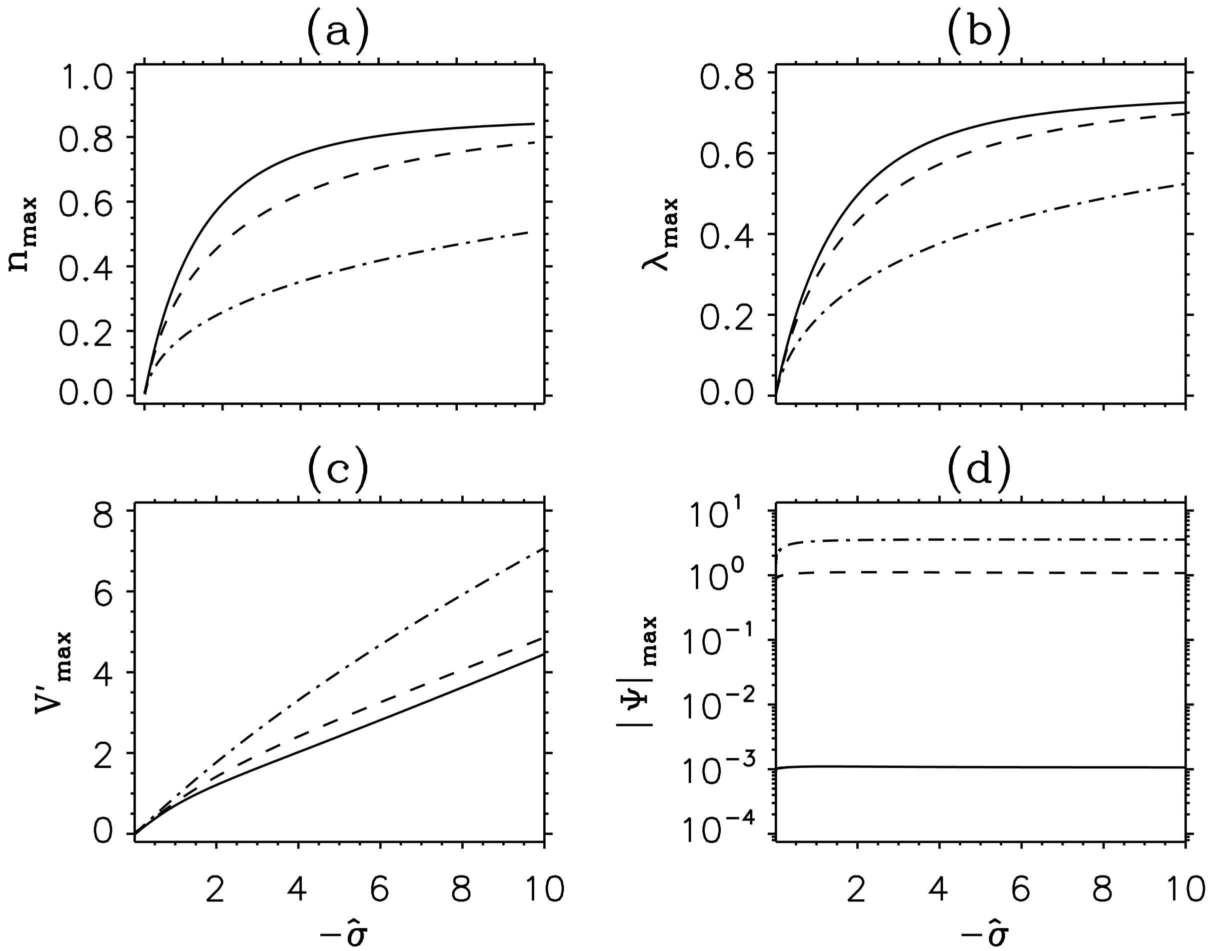
Single Mode Results I



$\widehat{\Omega} = -2\widehat{\sigma}/3$, $\widehat{\nu} = \widehat{\eta} = 1$, and $\widehat{\sigma} = -100, -10, -1$ (dashed-dot, dashed, solid)

- Maximum growth rate λ and V' increases with v_A, whereas associated wavenumber n and saturation level $|\psi|$ peaks
- Increasing n initially gets around stabilizing Lorentz force but once MRI flow is capable of slipping through the field further increase in n is of no benefit

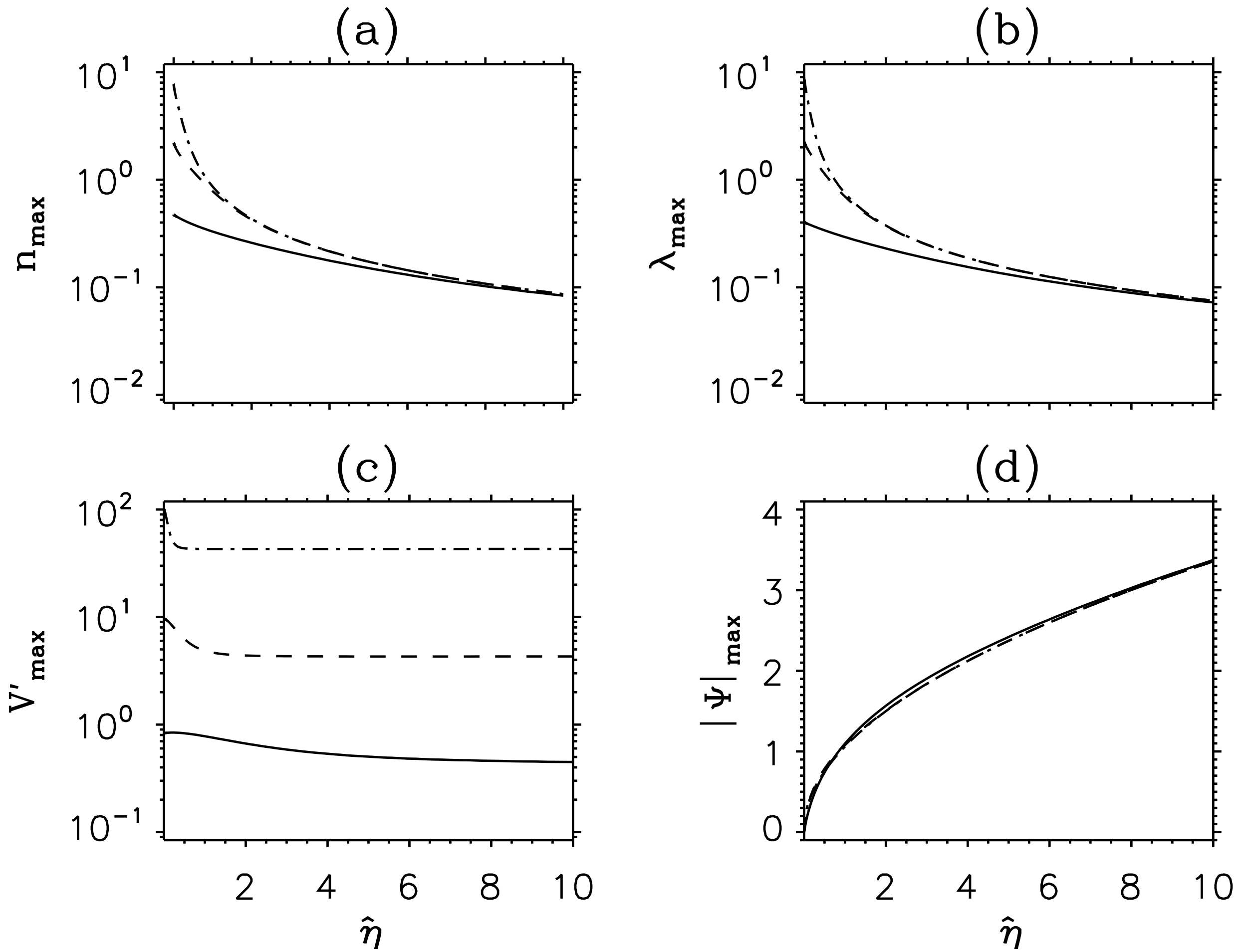
Single Mode Results II



$\widehat{\Omega} = -2\widehat{\sigma}/3$, $v_A = \widehat{\eta} = 1$, and $\widehat{\nu} = 10^{-6}, 1, 10$ (solid, dashed, dashed-dot)

- V' increases rapidly with shear rate $|\widehat{\sigma}|$ while $n, \lambda, |\Psi|$ saturate. This is a consequence of the reduced role of the Coriolis force
- Saturation values increase with ν indicating subtle role of viscosity in nonlinear regime, c.f. linear regime. Larger viscosity transports more ang. mtm., competing with magnetic stresses

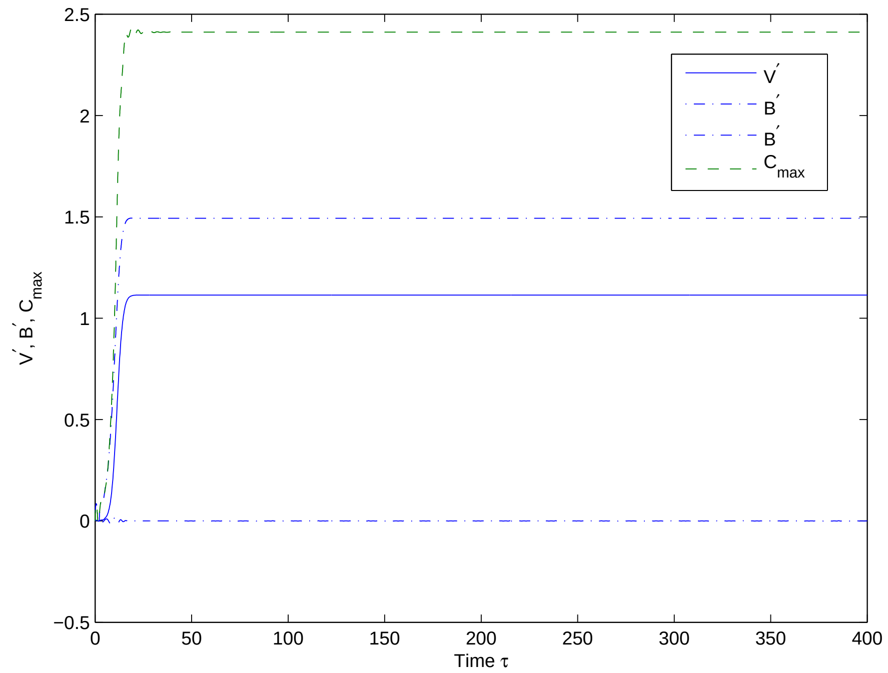
Single Mode Results III

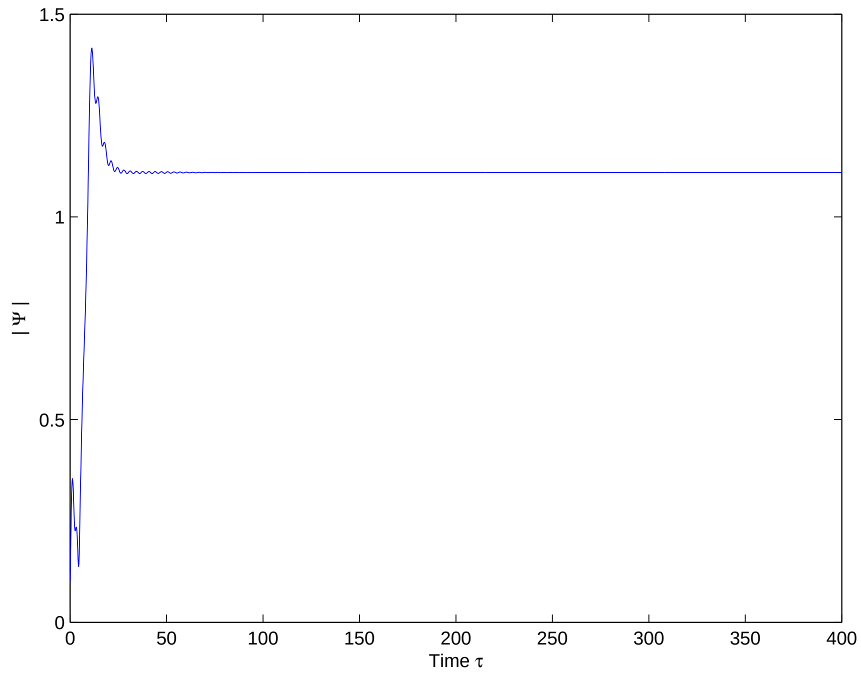


$\widehat{\Omega} = -2\widehat{\sigma}/3$, $v_A = \widehat{\nu} = 1$, and $\widehat{\sigma} = -100, -10, -1$ (dashed-dot, dashed, solid)

- For small $\widehat{\eta}$, $\widehat{\nu}$ MRI grows on the dynamical timescale. As $\widehat{\eta}$ increases growth and wavenumber decrease but saturation level of $|\Psi|$ increases
- Behavior consistent with the idea that reconnection reduces the effect of Lorentz force and thus enhances the amplitude of MRI. This does not translate into increased V' (*i.e.* modification of background shear)

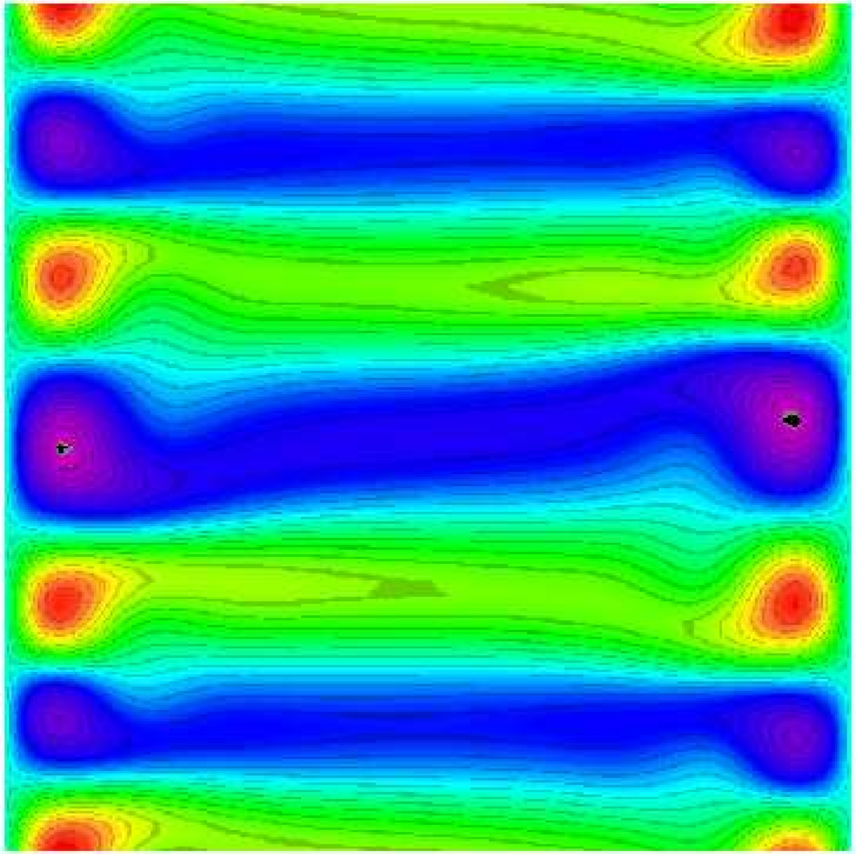
Approach to Saturated State





- Time-dependent evolution of an x-invariant single-mode perturbation indicates approach to predicted stationary solution
- Above results display extreme cases: disks supported entirely by mechanical $(B' = 0)$ or magnetic $(B' \neq 0)$ pressure
- $\nu_t = 2\pi\epsilon|\Psi| \sim O(\epsilon)$: turbulent viscosity associated with developed MRI

Simulation of Full PDEs



$\widehat{\Omega} = -2\widehat{\sigma}/3$, $\widehat{\sigma} = -4$, $v_A = \widehat{\nu} = \widehat{\eta} = 1$, and $\epsilon = 0.1$

- For impenetrable bc's interior flow approaches x-invariant state.
- Inclusion of boundary layers requires extension of theory.

Summary

- Simple scaling suffices to characterize one-parameter family of self-consistent equilibrated states
  - Strong modification of the background shear that feeds the MRI
  - Equilibration ultimately determined by ohmic + viscous dissipation
- Comparison with shearing sheet simulations (BH 1991, HGB 1995, Sano *et al.* 1998)
  - With resistive effects included but viscosity excluded no saturation occurs for $\Lambda > 1$. Our theory indicates viscosity plays an important role in this regime
  - Saturated MRI speed is $O(1)$, but the effective viscosity is $O(\epsilon)$
  - Simulations show tendency to solid body rotation and increased wavelength of MRI. This is also consistent with theory

Ongoing Research

- Comparison of reduced and full axisymmetric equations
- Extension of asymptotic theory to cylindrical geometries to overcome degeneracy in direction of ang. mtm. transport

- Details in:
  - Phys. Fluids 17, 094106 (2005);
  - Stellar Fluid Dynamics and Numerical Simulations: From the Sun to Neutron Stars, M. Rieutord and B. Dubrulle (eds), EAS Publication Series 21 (2006);
  - J. Math. Phys. 48, 065405 (2007).