Magnetorotational instability: a review

Joyce Shi Sim, David P. Larson, and Wonjae Lee
University of California, San Diego

(Received 26 May 2013)

The abstract goes here like this.

1. Introduction

Background info

What is MRI? Accretion disks? Maybe provide references to explain relevance of MRI tobirth of stars, geophysics, etc.

Also outline what the report will cover (probably as the last paragraph).

1.1. *Example citations*

General papers: Julien & Knobloch (2010), Chandrasekhar (1960), Acheson & Hide (1973), Balbus & Hawley (1991)

Experiments: Gailitis *et al.* (2002), Sisan *et al.* (2004), Stefani *et al.* (2006), Stefani *et al.* (2007), Ji (2010), Seilmayer *et al.* (2012)

Numerical simulations: Kageyama *et al.* (2004), Liu (2008), Gissinger *et al.* (2011), Travnikov *et al.* (2011), Kirillov *et al.* (2012), Zhao & Zikanov (2012)

2. Physical Explanation

Physical explanation of magnetohydrodynamic (MHD) equations, concept of "frozen-in-field", analogy of magnetic and string tension, main idea of MRI

3. Theoretical Work

Acheson & Hide (1973) and Knobloch (1992) showed the linear stability analysis of rotating magneto-fluid bounded in coaxial cylinder. For the case of gaseous astrophysical disks in unbounded geometry was shown in Balbus & Hawley (1991). The condition for stability was shown to be radially increasing angular velocity profile. We present a linear stability analysis of MRI for bounded case following Acheson (1972), Acheson (1973), Knobloch (1992) and Julien & Knobloch (2010).

3.1. *Governing equation*

The wave dispersion equation of a cylindrical magneto-fluid can be obtained from the magnetohydrodynamic (MHD) equations. Assuming the fluid is inviscid and perfectly conducting, the ideal MHD equations are

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right)=-\nabla p+\mathbf{J}\times\mathbf{B} \tag{3.1}$$

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})=0 \tag{3.2}$$

$$\mathbf{E}+\mathbf{u}\times\mathbf{B}=0 \tag{3.3}$$

$$\nabla\times\mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t} \tag{3.4}$$

$$\nabla\times\mathbf{B}=\mu\mathbf{J}. \tag{3.5}$$

To investigate MRI, one can consider homogeneous incompressible fluid rotating with angular velocity $\Omega(r)=\frac{V(r)}{r}$ in externally imposed magnetic fields $\mathbf{B_0}=[0,B_\phi(r),B_z(r)]$. Combining electromagnetic equations with momentum relation, we get the appropriate MHD equations,

$$\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}=-\frac{1}{\rho}\nabla\left(P+\frac{\mathbf{B}^2}{2\mu}\right)+\frac{1}{\mu\rho}(\mathbf{B}\cdot\nabla)\mathbf{B} \tag{3.6}$$

$$\frac{\partial \mathbf{B}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{B}=(\mathbf{B}\cdot\nabla)\mathbf{u} \tag{3.7}$$

$$\nabla\cdot\mathbf{u}=0 \tag{3.8}$$

$$\nabla\cdot\mathbf{B}=0 \tag{3.9}$$

3.2. *Linear perturbation equation and eigenvalue problem*

We can get linearized equations by perturbing the basic state by small amount of $\mathbf{u_1}$ and $\mathbf{b_1}$ for velocity and magnetic fields. The axisymmetric perturbation is assumed to have the form

$$\psi=\Re\left[\hat{\psi}(r)e^{i(kz-\omega t)}\right]. \tag{3.10}$$

Accoring to Acheson (1972), the normal mode equations are

$$-i\omega\hat{b}_r=ikB_z\hat{u}_r \tag{3.11}$$

$$-i\omega\hat{b}_\phi=-\frac{d}{dr}(\hat{u}_rB_\phi)+ik(\hat{u}_\phi B_z-\hat{u}_zB_\phi) \tag{3.12}$$

$$-i\omega\hat{b}_z=-\frac{1}{r}\frac{d}{dr}(r\hat{u}_rB_z) \tag{3.13}$$

$$ik\hat{u}_z=-\frac{1}{r}\frac{d}{dr}(r\hat{u}_r) \tag{3.14}$$

$$i(\omega^2-k^2V_{Az}^2)\hat{u}_\phi=\frac{\hat{u}_r}{r}(2\Omega r\omega+2kV_{A\phi}V_{Az}) \tag{3.15}$$

where $V_{A\phi}=\frac{B_\phi(r)}{\sqrt{\mu\rho}}$, $V_{Az}=\frac{B_z(r)}{\sqrt{\mu\rho}}$ are Alfvén speeds for associated external magnetic field components.

Solving the normal mode equation set for radial velocity perturbation $\hat{u}_r=u$, Acheson (1973) obtained following eigenvalue problem,

$$\frac{d}{dr}\left[(\omega^2-k^2V_{Az}^2)\left(\frac{du}{dr}+\frac{u}{r}\right)\right]-k^2\left[\omega^2-k^2V_{Az}^2+r\frac{d}{dr}\left(\frac{V_{A\phi}^2}{r^2}-\frac{V^2}{r^2}\right)\right]u$$
$$=-\frac{4k^2}{r^2}\frac{(kV_{A\phi}V_{Az}+\omega V)^2}{(\omega^2-k^2V_{Az}^2)}u. \tag{3.16}$$

3.3. *Stability criterion*

3.3.1. *Standard magnetorotational instability*

Consider a standard MRI of radially bounded coaxial fluid cylinder with externally imposed axial magnetic field but without axial current flowing. Therefore, we have $V_{Az} = \frac{B_z}{\sqrt{\rho\mu}} = const \neq 0$ and $V_{A\phi} = \frac{B_\phi}{\sqrt{\rho\mu}} = 0$. Considering boundary condition $u(r_1) = u(r_2) = 0$, we multiply the eigenvalue equation by complex conjugate of u and integrate over radial coordinate,

$$(\omega^2 - k^2V_{Az}^2)^2 = \frac{k^2}{D}\int_{r_1}^{r_2}\left[\frac{\omega^2}{r^2}\frac{d}{dr}r^2V^2 - r^2k^2V_{Az}^2\frac{d}{dr}\left(\frac{V^2}{r^2}\right)\right]|u|^2dr \tag{3.17}$$

where

$$D \equiv \int_{r_1}^{r_2}\left(r\left|\frac{du}{dr}\right|^2 + \frac{|u|^2}{r} + k^2r|u|^2\right)dr > 0 \tag{3.18}$$

Accoring to Chandrasekhar (1960), ω^2 must be real. We get stable modes with $\omega^2 > 0$ and unstable modes with $\omega^2 < 0$. If the angular velocity increases radially outward, $\frac{d}{dr}\left(\frac{V^2}{r^2}\right) > 0$, the system is stable because ω^2 is bounded from below by positive number,

$$\omega^2 > \frac{r^2k^2V_{Az}^2\frac{d}{dr}\left(\frac{V^2}{r^2}\right)}{4\frac{V^2}{r} + r^2\frac{d}{dr}\left(\frac{V^2}{r^2}\right)} > 0. \tag{3.19}$$

If we have radially decreasing anguar velocity profile, $\frac{d}{dr}\left(\frac{V^2}{r^2}\right) < 0$, somewhere $r_1 < r < r_2$, then ω^2 may have negative solution which makes the system unstable.

3.3.2. *Helical magnetorotational instability*

When the external nonzero magnetic fields in axial and azimuthal directions are considered, it was found that the eigenvalue equation can be written as

$$\frac{d}{dr}r\frac{du}{dr} - \frac{u}{r} - k^2ru = \frac{k^2}{(\omega^2 - k^2V_{Az}^2)^2}\left[r^2\frac{d}{dr}\left(\frac{V_{A\phi}^2 - V^2}{r^2}\right)(\omega^2 - k^2V_{Az}^2) - \frac{4}{r}(kV_{A\phi}V_{Az} - \omega V)^2\right]u \tag{3.20}$$

According to Knobloch (1992) and Julien & Knobloch (2010), the exponentially growing mode $\omega = -i\lambda$, $\lambda > 0$ is possilbe when the eigenvalue relation has following form

$$(\lambda^2 + k^2V_{Az}^2)^2 = \frac{k^2}{D}\int_{r_1}^{r_2}\left[r^2\frac{d}{dr}\left(\frac{V_{A\phi}^2 - V^2}{r^2}\right)(\lambda^2 + k^2V_{Az}^2) + \frac{4}{r}(kV_{A\phi}V_{Az} - i\lambda V)^2\right]|u|^2dr. \tag{3.21}$$

Considering the imaginary part of the equation, we have

$$\int_{r_1}^{r_2}\frac{1}{r}V_\phi V|u|^2dr = 0 \tag{3.22}$$

Knobloch (1992) shows exponentially growing instability is only possible when $V_{A\phi}$ or V changes sign somewhere in $r_1 < r < r_2$.

4. Laboratory Experiments

Discuss on experimental work that has been done related to this topic.

Try to categorize experiments so we can group the discussion. Also, the main focus is to explain what has been done to attempt to recreate MRI in the lab.

NOTE: most literature on MRI experiments seem to be about either 1) axial-only magnetic fields (standard MRI) or 2) axial and azimuthal magnetic fields (helical MRI).

5. Numerical Work

Have there been any numerical experiments done that involve the topic.

There's probably numerical simulations of both standard and helical MRI that have already been tested in the lab, but there may also be more large-scale simulations (e.g. planet-scale).

6. Conclusion

Concluding remarks

REFERENCES

Acheson, D. J. 1972 On the hydromagnetic stability of a rotating fluid annulus. *Journal of Fluid Mechanics* **52**, 529–541.

Acheson, D. J. 1973 Hydromagnetic wavelike instabilities in a rapidly rotating stratified fluid. *Journal of Fluid Mechanics* **61**, 609–624.

Acheson, D J & Hide, R 1973 Hydromagnetics of rotating fluids. *Reports on Progress in Physics* **36** (2), 159.

Balbus, S. A. & Hawley, J. F. 1991 A powerful local shear instability in weakly magnetized disks. i - linear analysis. ii - nonlinear evolution. *Astrophysical Journal* **376**, 214–233.

Chandrasekhar, S. 1960 The stability of non-dissipative couette flow in hydromagnetics. *Proceedings of the National Academy of Science* **46**, 253–257.

Gailitis, Agris, Lielausis, Olgerts, Platacis, Ernests, Gerbeth, Gunter & Stefani, Frank 2002 Colloquium: Laboratory experiments on hydromagnetic dynamos. *Rev. Mod. Phys.* **74**, 973–990.

Gissinger, Christophe, Ji, Hantao & Goodman, Jeremy 2011 Instabilities in magnetized spherical couette flow. *Phys. Rev. E* **84**, 026308.

Ji, Hantao 2010 Current status and future prospects for laboratory study of angular momentum transport relevant to astrophysical disks. *Proceedings of the International Astronomical Union* **6**, 18–25.

Julien, K. & Knobloch, E. 2010 Magnetorotational instability: recent developments. *Philosophical Transactions of the Royal Society A* **368**, 1607–1633.

Kageyama, Akira, Ji, Hantao, Goodman, Jeremy, Chen, Fei & Shoshan, Ethan 2004 Numerical and experimental investigation of circulation in short cylinders. *Journal of the Physical Society of Japan* **73** (9), 2424–2437.

Kirillov, Oleg N., Stefani, Frank & Fukumoto, Yasuhide 2012 A unifying picture of helical and azimuthal magnetorotational instability, and the universal significance of the liu limit. *The Astrophysical Journal* **756** (1), 83.

Knobloch, E. 1992 On the stability of magnetized accretion discs. *Monthly Notices of the Royal Astronomical Society* **255**, 25P–28P.

Liu, Wei 2008 Numerical study of the magnetorotational instability in princeton mri experiment. *The Astrophysical Journal* **684** (1), 515.

Seilmayer, Martin, Stefani, Frank, Gundrum, Thomas, Weier, Tom, Gerbeth, Gunter, Gellert, Marcus & Rüdiger, Günther 2012 Experimental evidence for a transient tayler instability in a cylindrical liquid-metal column. *Phys. Rev. Lett.* **108**, 244501.

Sisan, Daniel R., Mujica, Nicolás, Tillotson, W. Andrew, Huang, Yi-Min, Dorland, William, Hassam, Adil B., Antonsen, Thomas M. & Lathrop, Daniel P. 2004 Experimental observation and characterization of the magnetorotational instability. *Phys. Rev. Lett.* **93**, 114502.

Stefani, Frank, Gundrum, Thomas, Gerbeth, Gunter, Rüdiger, Günther, Schultz, Manfred, Szklarski, Jacek & Hollerbach, Rainer 2006 Experimental evidence for magnetorotational instability in a taylor-couette flow under the influence of a helical magnetic field. *Phys. Rev. Lett.* **97**, 184502.

Stefani, Frank, Gundrum, Thomas, Gerbeth, Gunter, Rdiger, Gnther, Szklarski, Jacek & Hollerbach, Rainer 2007 Experiments on the magnetorotational instability in helical magnetic fields. *New Journal of Physics* **9** (8), 295.

Travnikov, V., Eckert, K. & Odenbach, S. 2011 Influence of an axial magnetic field on the stability of spherical couette flows with different gap widths. *Acta Mechanica* **219** (3-4), 255–268.

Zhao, Yurong & Zikanov, Oleg 2012 Instabilities and turbulence in magnetohydrodynamic flow in a toroidal duct prior to transition in hartmann layers. *Journal of Fluid Mechanics* **692**, 288–316.