Branching Processes: Simulating Birth and Assassinations …. update this

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## Abstract

Monte Carlo simulation is an extremely useful method to analyze complex mathematical relationships. When used to investigate a stochastic process, Monte Carlo simulation allows a researcher to analyze thousands of independent random outcomes given underlying distributional assumptions. In this paper, we model computer job queueing as a Poisson process. We use a Poisson distribution to estimate the number of jobs and an exponential distribution to estimate the lifetime of each respective job. We investigate the number of jobs created given these assumptions by simulating outcomes using different values of the Poisson lambda and exponential rate parameters. We find that the ratio of the two distribution parameters is more important than two discrete values in determining the number of jobs created. Thus, simulation parameters can be simplified to use only the ratio of the two parameters instead of iterating over hundreds of combinations.

## Introduction

In the context of software engineering, it is good practice to anticipate the likelihood of a release date, the cost of a project, or the impact of implementing new code base. These types of exercises usually fall into the category of risk analysis. However, it is often extremely difficult to successfully identify or anticipate outcomes if those outcomes are considered mostly random. One such software engineering example proudcing random outcomes is computer job queueing.

Even with parallel processing, a queue of interdependent tasks can form, often slowing down processing time (NTL 1). The number of these tasks can be completely random, specifically when considering the amount and type of requests the computer must consider. Typically, a parent job generates children which then generate additional children recursively. Each one of these jobs have a distinct run time that begins after the completion of their respective parent job. Obviously, this queueing process can result in significant slow down times, therefore it is often useful to simulate the outcomes of this branching process to optimize code and job processing logic.

In order to model the queueing process, we must utilize probability distributions to estimate the number of jobs created and their associated lifetimes. Given the randomness of the queueing process, we explore many different outcomes utilizing these probability distributions via Monte Carlo simulation. These simulations allow us to explore the distributions of job counts our queue model produces.

Of particular importance are the associated distributional parameters that must be identified as part of our simulation analysis. Our objective in this paper is to determine if these parameters and their values are related to the longevity of the queueing process. Specifically, we investigate whether discrete parameter values or the ratio of the two parameters is more important to the longevity of the queueing process.

In the following section, we review academic literature on stochastic processes and Monte Carlo simulation. In the methods section, we provide a description of the probability distribution assumptions used to model the CPU queueing process. We also identify the parameter selections and logic used to test the sensitivity of our model given the distributional parameters. Results of Monte Carlo simulation for the queueing process and associated parameters are then presented and compared. We conclude this paper by summarizing results and considering future work.

## Background

Gabbiani and Cox (2) define a stochastic process as a collection of random variables indexed by a variable t, usually representing time. These processes are grouped into two categories: discrete-time and continuous-time stochastic processes. They are studied intensely as part of mathematical models to describe systems that occur in a random manner. The applications of these models can be found in computer science, cryptography, telecommunication, finance and many other academic disciplines and industries.

One of the easiest stochastic processes to understand is the Bernoulli process (3) as it only has two possible outcomes: success or failure, given probabilities of {p, 1-p}. For example, tossing a fair coin ten times can be modeled as a Bernoulli process with a sequence of ten coin-flip results and probabilities of 0.5 for head and 0.5 for tail (since it is a fair coin). Each coin-flip is a binary-valued random variable.

Another example of a stochastic process is the Poisson process which we use in this paper to estimate the number and run time of jobs in a queueing process. It is used in instances when we need to count the number of occurrences of an event that happens at random. In practice, (4) the Poisson process is used to model the number of car accidents at a site, the location of users in a wireless network, the request for individual documents on a web server and the outbreak of wars.

The Poisson process with rate is defined as (4): Let > 0 be fixed. The counting process {N(t), t $\elementof$ [0,???)} is called a Poisson process with rate if all the following conditions hold: 1. N(0) = 0; 2. N(t) has independent increments; 3. the number of arrivals in any interval of length ??>0 has Poisson(????) distribution.

Modeling a random process given only a few samples of that respective process can lead to inconclusive or erroneous assumptions. In order to empirically analyze these random processes, Monte Carlo simulations are used frequently. This method is used to generate random variables for modeling risk or uncertainty of a system (5). Monte Carlo simulations have some key advantages over more deterministic analyses (6) including simulated outcomes and probabiities for each outcome. It is easy to generate graphical representations based on the results, and one can easily spot which output has the largest effect on the bottom line results. Monte Carlo simulations are used extensively to simulate outcomes, especially for electoral processes and in sports betting.

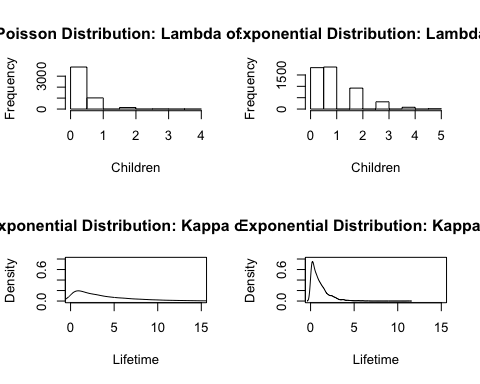
## Methods

We must identify a method to model the previously described queue forming process in order to analyze many random outcomes given a set of distribution parameters. Our model assumptions are as follows:

1. The queueing process begins with a single parent
2. This parent (and all subsequent parents) have a randomly generated run time or “lifetime”
3. Each parent can randomly generate n amount of children jobs
4. All child jobs are “protected” and do not run until their parent job completes or “dies”
5. Each child can spawn n amount of child jobs
6. The process runs until no additional children are generated or infinitely

Starting with the parent, each job has a fixed lifetime that is determined randomly. Within this fixed interval of time, a parent can generate n amount of child jobs. Thus, we can use a Poisson process to model the counts of jobs over each fixed time interval. Specifically, we use the Poission distribution to randomly generate the number of children for each parent. Finally, each job runs for a certain amount of time and defines the time boundaries between different events. We can randomly generate these run times using a random variable from the exponential distribution to determine the lifetime of each job.

The Poisson and exponential distributions allow us to parameterize our queueing process. Each distribution contains a rate parameter, which represent the number of events per unit time. Specifically, we use lambda and kappa for our rate designations for the Poisson and exponential distributions, respectively. These rate parameters can elongate or shorten the queueing process dramatically. Obviously, if a parent generates many child jobs, the probability of a process continuing is greater. Further, if a job has a longer run time, the probability of a process continuing is also greater. It is the interaction of these two parameters we are interested in analyzing. As an example, we randomly generate 5000 samples from a Poisson and exponential distribution with rate parameters of 0.25 and 1 below.

Figure 1: Rate Parameter Comparisons from 5000 Randomly Generated Variables 

As can be seen in Figure 1, each distribution’s rate parameter can significantly change the outcome for the number of children and the lifetime for each job in our model. We seek to understand if the ratio of these parameters are of importance, or if one parameter dominates when considering the longevity of our queueing process.

In order to investigate the relationship between lambda and kappa in our queueing model, we carry out a Monte Carlo simulation study by fixing kappa, which represents the parameter for the lifetime of each job, to be one. We run 400 simulations to determine the number of child jobs created using each of six values of lambda, which represents the rate parameter for the number of offspring for each parent. We then run another set of simulations where kappa is not equal to one, however, the ratio of lambda to kappa is equal to one of the lambda values from the earlier simulation where kappa was equal to one. This allows us to analyze whether the relationship between these two parameters are important or if one or more parameters dominates the longevity of our queueing process.

## Results

We then identified several values for both lambda and kappa across both cases where kappa is kept constant at 1 and where kappa = c and does not equal 1. These values are indicated in the table below.

We selected several values for lambda and kappa with matching ratios to determing first in the test case to see the overall patterns that would emerge from the changes, and then the simulation case, iterating over 40 times to provide a representative sample. Our first attempt was to change the lambda while keeping kappa constant at 1 as shown below.

The first sample simulation over 40 iterations for the requisite sample size was then completed.

From this, we then changed the values for kappa and lambda to reflect the ratios used in the first example while ensuring that kappa did not equal 1.

And then the full run of 40 iterations of the simulation for our sample size.

## Conclusions and Future Work

The use of simulation, and especially Monte Carlo simulation, has become an increasing part of the public lexicon in recent years. Many models that are visible to the public include at some level a Monte Carlo simulation. These include election results modeling, as well as the outcomes of major sports leagues. Fivethirtyeight.com’s models for both the election, as well as for results of sports leagues, include Monte Carlo simulations. [1] The method has also been mentioned in the context of quantative finance, genetics and many other fields.

The pitfalls of Monte Carlo simulation may be somewhat obvious. It relies on an assumption that the future follows some probabalistic distribution, and that the future distribution is consistent with the past values of the variable. Like most statistically-based forecast methods, there’s some expectation that major shifts in the underlying causes of change in the behavior of the variable will remain the same. In the example of a model for a Financial Market, while the major shifts and catastrophic market events are possible in the simulation, the actual probability of the event might differ from the *n*-standard deviations from the mean that would be expected.

In this case, we simulated a process of birth and death of jobs, with dependencies between the start and finish, to understand the total time required. In the context of the case, these events were referenced as jobs or processes that are broken up for parallel computing, and the process by which the later jobs must wait for the completion of earlier jobs. This same method could be employed for the analysis of a manufacturing line. In manufacturing, the Monte Carlo process can be used to simulate a process with interdependencies, and variability.

The use of the simulation can be used to identified bottlenecks in the process.[3]

## References

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## APPENDIX - BOOK CODE