MATHEMATICS MADE DIFFICULT

A Handbook for the Perplexed

Carl E. Linderholm

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WOLFE PUBLISHING LTD
10 Earlham Street London WC2

ISBN 72340415 1

Printed in Great Britain by
Butler & Tanner Ltd, Frome and London

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To

CLEMENT V. DURELL, M.A.

WITHOUT WHOM THIS BOOK
WOULD NOT HAVE BEEN NECESSARY

(*With apologies to St Thomas Aquinos*)

INTRODUCTION

One of the great Zen masters had an eager disciple who never lost an opportunity to catch whatever pearls of wisdom might drop from the master's lips, and who followed him about constantly. One day, deferentially opening an iron gate for the old man, the disciple asked, 'How may I attain enlightenment?' The ancient sage, though withered and feeble, could be quick, and he deftly caused the heavy gate to shut on the pupil's leg, breaking it.

When the reader has understood this little story, then he will understand the purpose of this book. It would seem to the unenlightened as though the master, far from teaching his disciple, had left him more perplexed than ever by his cruel trick. To the enlightened, the anecdote expresses a deep truth. It is impossible to spell out for the reader what this truth is; he can only be referred to the anecdote.

Simplicity is relative. To the great majority of mankind—mathematical ignoramuses—it is a simple fact, for instance, that $17 \times 17 = 289$, and a complicated one that in a principal ideal ring a finite subset of a set E suffices to generate the ideal generated by E. For the reader and for others among a select few, the reverse is the case. One needs to be reminded of this fact especially as it applies to mathematics. Thus, the title of this book might equally well have been *Mathematics Made Simple*; whereas most books with that title might equally well have been called *Mathematics Made Complicated.* The simplicity or difficulty depends on who is reading the book.

There is no doubt that an absolute ignoramus (not a mere qualified ignoramus, like the author) may become slightly confused on reading this book. Is this bad? On the contrary, it is highly desirable. Mathematicians always strive to confuse their audiences; where there is no confusion there is no prestige. Mathematics is prestidigitation. Confusion itself may be taken as the guiding principle in what is done here—if there is a principle. Just as the fractured leg confused the Zen disciple, it is hoped that this book may help to confuse some uninitiated reader and so put him on the road to enlightenment, limping along to mathematical *satori.* If confusion is the first principle here, beside it and ancillary to it is a second: pain. For too long, educators have followed blindly the pleasure principle. This over-simplified approach is rejected here. Pleasure, we take it, is for the initiated; for the ignoramus, if not precisely pain, then at least a kind of generalized *Schmerz.*

1. The structure of the book

The following account of the structure of the book makes no pretence to be complete; nor is it in every respect completely accurate. It is a naïve account, along taxonomic lines, which it is hoped will be of use to the reader. The book is divided into chapters, which are divided into sections, which are divided into paragraphs, which are divided into sentences, which are divided into words, which are divided into letters. For reference purposes, each letter in the book has a number (actually a finite sequence of positive integers) attached to it. The finite sequence has exactly six (the first perfect number) terms, so that the book, considered as a finite sequence of letters, is mapped to N_0^6. It has been found very convenient both for author and reader to do this abstractly, rather than concretely.

Walking home along a deserted street late at night, the reader may imagine himself to feel in the small of his back a cold, hard object; and to hear the words spoken behind him, 'Easy now. This is a stick-up. Hand over your money.' What does the reader do? He attempts to generate the utterance. He says to himself, now if I were standing behind someone holding a cold, hard object against his back, what would make me say that? What would I mean by it? The reader is advised that he can only arrive at the deep structure of this book, and through the deep structure arrive at the semantics, if he attempts to generate the book for himself. The author wishes him luck.

2. How to read this book

Reading books, like mathematics, is an art to which many pretend, to which some aspire, and to which few attain. Nothing outside the realm of mathematics is more beautiful to behold than a beautifully read book. The art of reading books beautifully, like mathematics, cannot be imparted in a few easy lessons. It is difficult to give hints without incurring the danger of becoming fatally lucid. The following advice, it is hoped, avoids this deadly lucidity as effectively as it is avoided in the book itself. Here as in so many things, it is really the man who is totally at sea who has got both feet firmly on the ground.

It is the author's general impression that, at first reading at least, it is unwise to read the book on too many levels at once. For most readers it will in any case be impossible to penetrate beyond the lowest level or two; these should be quite satisfactory for a beginning.

Consider a category (small, if you wish) in which there is at most one arrow between two objects, and in which every isomorphism is an identity. Products and coproducts exist; moreover, product distributes over coproduct in the sense that the dotted arrow exists:

$$\begin{tikzcd}
 & (A \Pi B) \amalg (A \Pi C) & \\
A \Pi B \arrow[ur] \arrow[r] \arrow[d] & A & A \Pi C \arrow[ul] \arrow[l] \arrow[d] \\
B \arrow[dr] & A \Pi (B \amalg C) \arrow[u] \arrow[d] \arrow[uu, dashed, bend right] & C \arrow[dl] \\
 & B \amalg C &
\end{tikzcd}$$

Further, assume initial and final objects 0, 1 and a contravariant functor from the category to itself $\cdot'$ that is alternating on objects and preserves no arrow except $0 \longrightarrow 1$. Then it is possible to think, because the category is a boolean algebra. Certain obvious refinements will bring the reader face-to-face with logic. A little logic of this naïve sort may occasionally be found helpful in reading the book.

It is finally suggested that the reader begin reading at some definite point; say, the first chapter.

1
ARITHMETIC

1. So you think you can count?

QUESTION 1. Whether anybody really counts?

Objection 1. In order really to count, the counter must know the names of the numbers. But no-one really knows the true names of the numbers, since Englishmen call them one thing, Chinamen another; therefore, no-one really counts.

Reply to Objection 1. It is not necessary to know the real names of the numbers in order to count. You may call the numbers anything you like; moreover, you need not use what are usually called numbers at all in counting—or to put it otherwise, almost anything can serve as a number if you so desire. (Simply transfer the structure of the counting system to another set by means of a bijection.) Mathematicians never claim to know about the elements of a system, only about relations between them and structures imposed on them.

Objection 2. Furthermore, although in a certain language a man might be said to count as far as a certain number, after that he must stop. For instance, in English [1] no-one can count higher than nine hundred and ninety-nine thousand nine hundred and ninety-nine decillion nine hundred and ninety-nine thousand nine hundred and ninety-nine nonillion . . . nine hundred and ninety-nine thousand nine hundred and ninety-nine. In Melanesian Pidgin English many speakers cannot count higher than *one-fellow man.* But in counting, if the counter stops at a certain point he is said not really to count.

Reply to Objection 2. It is true that most languages have only a few names for numbers. Whether there are any exceptions to this rule is a question I leave to linguists. [2] The remark about Melanesian Pidgin may be misleading, since [3] *one-fellow man* means twenty, just as *two-fellow hand one-fellow foot* means fifteen. As far as it concerns the English, and most civilised, 'counting systems' the objection is a valid one. The trouble is that the rule for naming the successor of a number after you have named the number is not everywhere defined; there is no successor function. Thus the English system is mathematically backward, compared with certain 'primitive' systems. This is the reason why children ask 'What is the greatest number?' but it does not prove that no-one can count; it proves only that aged, civilised non-mathematicians do not count. This we knew already.

Objection 3. If anybody could count, then certainly mathematicians would be able to count. But mathematicians pretend to count by means of a system supposed to satisfy the so-called Peano axioms. In fact, there cannot be any Peano axioms, since they were really invented by Dedekind. Hence even mathematicians cannot count. Furthermore, the piano has only 88 keys; hence, anyone counting with these axioms is soon played out.

Reply to Objection 3. Dedekind's version [4] is indeed equivalent to Peano's [5] although it was stated differently. Dedekind's paper is still of great historical interest, and contains many suggestive ideas. But Peano (G. for Giuseppe, not Grand) was an influential innovator in ideas, and in the mathematical notation of this century; he deserves the renown for his beautifully stated axioms. He invented a system of stenographic shorthand, but I am unaware of any musical works. In his system (without shorthand) 89 is written

0+++++++++++++++++++++++++++++
+++++++++++++++++++++++++++++
+++++++++++++++++++++++++++++
++.

Objection 4. Furthermore, Peano's axioms imply a supposed well-ordering, whereby there is a least number with a given property, provided any number possesses the property. But the least number that cannot be named in fewer than twenty-one English syllables has just been named in fewer than twenty-one English syllables; hence no number exists that cannot be named in fewer than twenty-one English syllables. But there are at most 10,000 such syllables, hence at most $21^{10,000}$ numbers. But then mathematicians cannot count to $21^{10,000} + 1$.

Reply to Objection 4. This logical paradox is called the Richard paradox. [6] It is a question of foundations of mathematics, rather than mathematics itself; or, at least, I hope so. The reply is left to the reader as an exercise. (This phrase always means that the writer cannot do the problem himself.)

Objection 5. Only mathematicians really count; but they are disembodied minds, hence a mathematician is not any body.

Reply to Objection 5. So how did I write this book?

Objection 6. Really to count is to count *realiter*; that is, by means of things (*rebus*). But only children enjoy rebusses, and children do not count in a serious argument. Moreover, counting by means of things usually means counting fingers, which are not bodies but parts of a body. Those who count on their fingers admit they cannot count well.

Reply to Objection 6. On the contrary, counting on the fingers is excellent exercise in finite cardinal arithmetic. This subject is treated in a later section; at present we are concerned with counting *per se*.

Objection 7. Mathematicians assert that the set of numbers is a non-empty infinite set. But that which is infinite has no parts, since a part is divided from the whole by a boundary (*fine*). But only the empty set has no proper subsets; i.e., no parts. Hence there are no numbers.

Reply to Objection 7. A set is infinite in mathematics *not* if it is infinite unconditionally, but (according to one definition) if the set of cardinal numbers of finite subsets is unbounded as a set of natural numbers. This is obvious for the set of natural numbers itself, since the cardinal number of

$$\{0, 1, \ldots, n - 1\}$$

is n. According to another definition, a set is infinite if the set of natural numbers can be injected into it. The identity function is such an injection for the set of natural numbers itself. Another possible definition requires a proper injection from the set to itself: such is

$$n \mapsto n+$$

for N_0.

PEANO'S AXIOMS

(0) *N_0 is a set;*
(1) $0 \in N_0$*;*
(2) $+ : N_0 \to N_0$ *is a function;*
(3) *If* $0 \in E \subset N_0$ *and if* $(E)+ \subset E$ *then* $E = N_0$.

(Note that $+$ is written on the right of its argument; (3) is the induction axiom.)

(4) $+$ *is injective* $(a+ = b+ \Rightarrow a = b)$*;*
(5) $0 \notin (N_0)+$.

Axioms (0), (1), (2) just say that

$$\{0\} \hookrightarrow N_0 \xrightarrow{+} N_0$$

is a counting system. The child's system

$$\begin{tikzcd} {*0} \arrow[r, mapsto] & 1 \arrow[r, mapsto] & 2 \arrow[loop right] \end{tikzcd}$$

previously mentioned is inductive but not injective; the Chinese system of naming years, isomorphic to

$$\begin{tikzcd} {*\,0} \arrow[r, mapsto] & 1 \arrow[r, mapsto] & 2 \arrow[r, mapsto] & \ldots \arrow[r, mapsto] & 58 \arrow[r, mapsto] & 59 \arrow[lllll, mapsto, bend left] \end{tikzcd}$$

is inductive and injective, but fails (~~4~~ 5); non-inductive systems like

(1)
$$\begin{tikzcd} {*\text{Away}} \arrow[r, mapsto] & \text{away} \arrow[r, mapsto] & \text{with} \arrow[r, mapsto] & \text{rum} \arrow[d, mapsto] \\ & & \text{gum} \arrow[u, mapsto] & \text{by} \arrow[l, mapsto] \end{tikzcd}$$

or like

(2)

$$\begin{tikzcd}\text{Never} \arrow[r, mapsto] & {}^*\text{say} \arrow[r, mapsto] & \text{die} \arrow[l, bend left, mapsto]\end{tikzcd}$$

$$\begin{tikzcd}\text{I} \arrow[rr, mapsto] & & \text{give} \arrow[dl, mapsto] \\ & \text{up} \arrow[ul, mapsto] & \end{tikzcd}$$

provide numbers that are never used in counting.

I ANSWER THAT some people really count, namely mathematicians, some savages, and some two-year-olds. Anyone counts who possesses a counting system. Mathematicians have a universal counting system N_0 called the natural numbers. But he who counts both universally and naturally counts really. Also, some two-year-olds count as follows:

one, two, three, three, three, ...

But this implies the existence of a counting system

$$\begin{tikzcd}{}^*\text{one} \arrow[r, mapsto] & \text{two} \arrow[r, mapsto] & \text{three} \arrow[loop right, mapsto]\end{tikzcd},$$

since if $s : X \to X$ is defined by

$$X = \{\text{one, two, three}\}$$

$$\begin{aligned} s : \text{one} &\mapsto \text{two} \\ \text{two} &\mapsto \text{three} \\ \text{three} &\mapsto \text{three} \end{aligned}$$

then s is a function and hence

$$\{\text{one}\} \hookrightarrow X \xrightarrow{s} X$$

is a counting system (this is discussed further below). Moreover, many so-called 'primitive' languages have such a system.

QUESTION 2. Whether it is natural to use the natural numbers?

Objection 1. They would seem to be very unnatural, since they involve sets and functions; these are abstractions and innovations, probably irreligious, impractical, and disrespectful to the flag.†

† In Britain, the Queen.

Reply to Objection 1. Sets and functions are more natural than arrangements of n like objects in a line without repetitions, independent variables, and such paraphernalia, [7] all of which belong in the Natural History Museum because they used to be natural but now they are history. These relics are equally abstractions. Mathematics is not meant to be practical. Sets and functions are intensely pious and loyal.

Objection 2. What is natural can be easily understood by children; but only a genius could make sense of all this stuff.

Reply to Objection 2. Children aged twelve to eighteen can easily learn the Schröder–Bernstein theorem and the proof that finite fields have prime-power order. [8]

Objection 3. The Peano axioms are already seventy years old; but it is unnatural in science to hang on to what is *passé* or old hat.

Reply to Objection 3. At present the Peano axioms are still found to be useful.

Objection 4. The Peano axioms describe the 'natural numbers' internally, rather than as a universal object in the category of pointed functions. This would seem unnatural, since categories are the mathematics of the future.

Reply to Objection 4. This is partly true; the future will tell.

Objection 5. The additive group of integers is basic, and includes the natural numbers as well as the negative numbers. Why not start with that group, and kill two birds with one stone?

Reply to Objection 5. You do it your way, I'll do it my way.

Objection 6. Everybody does arithmetic in decimal, or arabic, numerals. This seems a perfectly good system, and does not require all this abstraction. You are just making mathematics difficult!

Reply to Objection 6. First of all, there are no arabic numerals since they were invented by the Hindus. [9] Secondly, they involve the arbitrary choice of ten as a base. Mathematics requires considering numbers in infinite sets as well as one at a time, and for this purpose numerals are insufficient. A numeral is a sequence of digits; i.e., a function

$$N_0 \longrightarrow \{0, 1, 2, \ldots, 9\}.$$

What does this mean if N_0 is not already available?

Objection 7. Before you study *Numbers* you should read *Genesis*, *Exodus*, and *Leviticus*. [10].

Reply to Objection 7. I did: 'Look now toward heaven, and tell the stars, if thou be able to number them: . . .' [11] Since the number of stars is not known and the naked eye not mentioned, it is natural to establish N_0 before beginning the project.

Objection 8. Peano's axioms are clearly nonsense: Let f be a bijection from the supposed N_0 to a set of filing cabinets. Lock $f(0)$ and put the key in $f(1)$, lock $f(1)$ and put the key in $f(2)$, and continue by induction. Now all the cabinets are locked, with their keys locked inside their neighbours. That is obviously impossible. Hence N_0 does not exist.

Reply to Objection 8. Filing cabinets are not mathematics.

Objection 9. Induction is nonsense, since it purports to prove that all girls are blondes, as follows: It is sufficient to show that if one of any n girls is a blonde, then they all are blondes. This is clear if $n = 1$. With $n \geqslant 1$, suppose the proposition is true for n. Then it is also true for $n + 1$. Suppose one of $n + 1$ girls is a blonde. She is either among the first n of the girls, or among the last n—say the first n. These, by the hypothesis about n, are all blondes. Then at least one of the last n is a blonde, and they, too, are all blondes. Now all the $n + 1$ girls are blondes.

Reply to Objection 9. This is ineptly stated, since the same thing can be proved more easily with hydrogen peroxide. Also, it does not work if $n + 1 = 2$.

Objection 10. The natural numbers bore me to tears. But it would seem unnatural to use anything uninteresting in mathematics, since this subject is intensely interesting through and through.

Reply to Objection 10. All numbers are interesting, since the first uninteresting number would be interesting. Or better; no number is interesting but the system of all natural numbers is very interesting.

I ANSWER THAT it is very natural for mathematicians to use the natural numbers, since without them they could not do mathematics. The natural number system is a universal pointed function and so must be natural, since universal objects are a natural idea.

If $\{x\} \hookrightarrow X \xrightarrow{s} X$ (abbreviated X) and $\{y\} \hookrightarrow Y \xrightarrow{t} Y$ are pointed functions a morphism $X \xrightarrow{f} Y$ is a function $X \longrightarrow Y$ such that the squares of

$$\begin{tikzcd} \{x\} \arrow[r, hook] \arrow[d] & X \arrow[r, "s"] \arrow[d, "f"] & X \arrow[d, "f"] \\ \{y\} \arrow[r, hook] & Y \arrow[r, "t"] & Y \end{tikzcd}$$

commute (which means that $x \mapsto y$ and $fs = tf$). If X has the property that from X to any system Y there is exactly one morphism, then X is universally repelling and mathematicians universally find X attractive.

AXIOM [12]. *There exists a universally repelling pointed function*

$$\{0\} \hookrightarrow N_0 \xrightarrow{+} N_0.$$

QUESTION 3. Whether $2 + 2 = 4$?

Objection 1. When I say to a mathematician, 'You've come a long way since you learned that $2 + 2 = 4$,' he always makes a wry face. Maybe $2 + 2$ is not 4?

Reply to Objection 1. The comments people make when they find out you are a mathematician are always painful to hear. The average man's notion of what a microbiologist or an anthropologist does is relatively accurate compared with his absurd fancies about mathematical work. Popular notions regarding the spirit of mathematical inquiry are definitely stuck in the Babylonian era. This is especially the case with respect to things like '$2 + 2 = 4$'; for this reason, the latter is a principal mathematical shibboleth that others will do well to avoid mentioning in our company.

Objection 2. '$2 + 2 = 4$' is an obvious fact of everyday life, but everyday life is not mathematics. Hence in mathematics it would seem that $2 + 2$ is not 4.

Reply to Objection 2. If people choose to use addition in everyday life, we do not mind. This makes mathematics neither more nor less valid in its own sphere.

Objection 3. Solomon, a very wise king, says 'There be three things which are too wonderful for me, yea, four which I know not: . . . .' [13] Thus $2 + 2$ would appear to be 3, and not four.

Reply to Objection 3. This author is not writing about mathematics; moreover, he explicitly disclaims mathematical knowledge: 'four, which I know not'.

Objection 4. What is 2? Define your terms!

Reply to Objection 4. The number 2 will be defined by the above equation $1 + 1 = 2$, as soon as I get around to defining 1 and $+$.

Objection 5. This would seem to imply that two games of chess equal a game of bridge. But chess is mathematical in nature, whereas bridges are an engineering problem and merely an application of mathematics.

Reply to Objection 5. We do not yet assert that two *players* plus two other players make four players; this is adding *things*, which goes beyond the scope of this section.

Objection 6. Furthermore, when people put two and two together, admittedly they usually get four, but sometimes they get other numbers. Hence we cannot be absolutely sure that $2 + 2 = 4$. But mathematics deals in absolute certainty, so '$2 + 2 = 4$' is not mathematically valid.

Reply to Objection 6. Our assertion bears absolutely no relation to the data of practical experience. Moreover, most people cannot count, let alone add.

Objection 7. That $2 + 2 = 4$ is a consequence of the so-called associative law, obeyed by all monoids including that of the natural numbers. But natural numbers have no organs of communication. Hence they cannot form associations, groups, or societies. Hence it would seem that $2 + 2 \neq 4$.

Reply to Objection 7. It is the mathematicians who group the numbers, not the numbers themselves. But monoids have the property that it makes no difference whether we group their elements one way or another, which property we call the associative law.

Objection 8. It would be unnatural to add numbers, since we have already seen that mathematicians make no claim to know what those are in themselves. If you do not know what 2 is, how can you know how to add it to anything?

Reply to Objection 8. It is true that it is unnatural to add numbers. The natural thing to do is to add, or compose, functions, and the objector will be relieved to learn that this is what we shall do. Since the functions are induced by the numbers in a natural way we may speak, *par abus de langage*, of adding numbers.

Objection 9. $2 + 2$ can be anything you like, since we may use $+$ for any law of composition. For instance, if you count

$$0 \mapsto 1 \mapsto 2 \mapsto 3 \circlearrowleft$$

then probably you would say $2 + 2 = 3$.

Reply to Objection 9. It is true that the symbol '$+$', or any symbol, may be used for any law of composition; though '$+$' is usually reserved for commutative ones. Those who count with 0, 1, 2, 3 would indeed be naturally led to the commutative monoid

$$\begin{array}{c|cccc}
3 & 3 & 3 & 3 & 3 \\
2 & 2 & 3 & 3 & 3 \\
1 & 1 & 2 & 3 & 3 \\
0 & 0 & 1 & 2 & 3 \\
\hline
+ & 0 & 1 & 2 & 3
\end{array}$$

whereby $2 + 2 = 3$; but this counting system, and this monoid, can be obtained from N_0 by identifying all numbers, except 0, 1, and 2 with each other; hence 3 is just another name for 4 (as Solomon, of course, realised).

I ANSWER THAT $2 + 2 = 4$. Mathematicians have established the associative law for the binary operation $+$, whereby

$$(a + b) + c = a + (b + c)$$

for any natural numbers a, b, and c. But by definition

$$\begin{aligned}
2 &= 1 + 1, \\
3 &= 2 + 1, \\
\text{and} \qquad 4 &= 3 + 1.
\end{aligned}$$

$$\begin{aligned}
\text{Hence} \qquad 2 + 2 &= 2 + (1 + 1) \text{ by definition} \\
&= (2 + 1) + 1 \text{ by the associative law} \\
&= 3 + 1 \text{ by definition} \\
&= 4 \text{ by definition.}
\end{aligned}$$

The associative law will be made clear later.

2. Add astra per aspirin

This section is about addition. The fact that the reader has been told this does not necessarily mean that he knows what the section is about, at all. He still has to know what addition is, and that he may not yet know. It is the author's fond hope that he may not even know it after he has read the whole section. Though addition, in general, is a special case of multiplication, here it is thought enough to consider only the addition of numbers, a very special case.

In the dim ages of the half-forgotten past, in Babylon and ancient Egypt, most people seem to have proceeded to learn to do what they thought was adding after having mastered counting; and they then 'added' *with the same numbers they 'counted' with.* The childhood of the man is said to mirror the childhood of the race, and among children this traditional course is indeed still followed by many. Should this be so, or ought we rather to add before we can count? We no longer dwell in caves, and the fact that counting has already been *mentioned* in this book before adding need hardly prejudice the reader to think that it *comes* first.

'Generations have trod, have trod, have trod;' but pity the poor ordinary sod, the present-day beneficiary of all their traditional treading! You have only to pronounce in his hearing the word 'addition' to conjure up before his fevered imagination a frantic nightmare. He sees the numerals in a black whose blackness has no bottom to its depth:

	\$179.63
	£26·96
	£1066–1–5
	1–0–1
	205–12–71
TOTAL:	13250–2–9½

He knows for sure the sum is wrong. But how? Where? The digits loom larger and nearer; he falls into the blackness; at last, mercifully, he faints. Alas! Yet, in his benighted way, he too has entered ploddingly on the great quest. He is trying, unconsciously perhaps but struggling for all that, to make mathematics difficult. His best effort is a nasty long sum; very well: let him whose mathematical erudition is perfect cast the first aspersion.

With a little more mathematics, it is easy to produce *more* difficulty with *less* effort; this principle of efficiency is basic, and is discussed further in the Appendix.† There is no need to dirty our fingers with sums—they are beneath our dignity. The horrid difficulties of infinite series, of such things as

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \ldots$$

or

$$0{\cdot}333 \ldots$$

—put these aside. There is a great lot of difficulty to be got out of much, much simpler things—for the simpler the things a man gets difficulty out of, the better his mathmanship. Do you suppose there is much difficulty in such a thing as

$$\begin{array}{r} 12 \\ +25 \\ \hline =37 \end{array} \;?$$

Indeed there is! To be honest, the previous discussion of $2 + 2 = 4$ was mere fools' play. Let us be fair, and start with

$$\begin{array}{r} 1 \\ +1 \\ \hline =2. \end{array}$$

A bit too hasty, perhaps. The trouble starts with that '1'. Remember that the experts are less than sanguine about giving any real, obvious meaning to such a thing as 1. What is the real meaning of 1? Who can say? It is impossible to agree on any meaning; and if the deepest thinkers could agree on some idea of 1, that would be of no real help at all. But let it pass. What can be meant by '+'?

Since it appears to be difficult beyond all difficulty for the greatest philosophers to say what anything *is* 'in itself', mathematicians customarily come to some sort of agreement about what they will

† See p. 25, footnote.

get together and 'believe'† about things. Since nobody has the slightest idea—least of all the poor mathematicians themselves—what the things *are* about which the mathematicians entertain their so-called 'beliefs', who can blame them for their harmless fancy? Like the world of a science-fiction story, a system of beliefs need not be highly credible—it may be as wild as you like, so long as it is not self-contradictory—and it should lead to some interesting difficulties, some of which should, in the end, be resolved.

$$a + 0 = a = 0 + a. \qquad \text{(M1)}$$

This is a shorthand way of saying that if you add 0 to any number whatever, you get the same number. Since nobody knows what a number is, it might begin to appear as if this rule had a very limited applicability. Supposing Mao Tse-Tung to be a number, for example, one could write the sum

$$\text{Mao Tse-Tung} + 0 = \text{Mao Tse-Tung}.$$

On the other hand, if he is not a number, it does not say if you can, or not. May we put the beloved chairman into our sums, or not? Is it a friendly or an unfriendly act? Will he back us up if we do it? Will he turn the cold shoulder and apologise to our head of state for our bad behaviour? Is he really a number, or is it only propaganda? Naturally, the reader shall not find out from me. Partly, what is involved here is the 'belief':

Everything, even Chairman Mao, either is or is not a number. (S1)

There are a few other little pieces of etiquette, such as never writing $+$ between any two things except two numbers, and that 0 is a number. (The reader had better get used to the idea of not knowing whether Mao is a number. But if he is not one, the reader must not write him into any sums.‡)

Part of what has just been said is contained in writing:§

$$+ : N_0 \times N_0 \to N_0;$$

this also tells you that adding two numbers produces a number; not,

† The expression ' "believe" ', as distinct from 'believe', is used in this book for something like 'pretend to pretend to believe'. See *Hypocritic Oath*, Appendix.§

‡ The author is unaware if he has ever written Chairman Mao into a sum. On the other hand, if he is not a number then I haven't.

§ See p. 24.

for instance, a suit of cards—unless the suit of cards is a number. For example, if it is the case (as is much rumoured of late) that Mao Tse-Tung and Spiro Agnew are numbers, and if it is also the case that

$$\text{Mao Tse-Tung} + \text{Spiro Agnew} = \text{Hearts}$$

(which Heaven forfend), then Hearts must be a number. Nevertheless, being himself no searcher of Hearts, the author does not wish to insist on this conclusion; it is stated only tentatively and subject to several conditions.

QUESTION 4. Whether 1 is a number?

Objection 1. Numbers are either politicians or suits of cards. But 1 is neither. Therefore 1 is not a number.

Reply to Objection 1. Maybe 1 is also a politician, a young lady, or a suit of cards. I did not say it was not. The set of politicians, young ladies, and suits of cards cannot be properly injected into itself; the set of numbers can; therefore some numbers are none of these things. Armies and political parties are given to claiming that numbers are on their side; they cannot mean *all* numbers. The allegiance of the divine and mystical 1 has often been claimed on both sides of a dispute, but proofs are usually lacking.

Objection 2. Back when you were teaching us how to count, it was made very clear that 1 was a number. There was no doubt about it whatsoever. This would make it seem natural that now, when we are doing addition, it should still be a number. So most likely you will say that 1 is *not* a number, just to make us look silly.

Reply to Objection 2. This is an example of 'second-guessing'. It looks as if I am about to do one thing, and therefore it is likely that I may do the opposite. The difficulty is that I may go ahead and do the first thing, thus second-guessing the second-guesser. So I have, and this is one of the attractive things about mathematics: sometimes the answers turn out to be just what you thought they would be. Mathematics is a tricky subject.

Objection 3. If we say, 'A number of old men were affected with apoplexy, and took a number of valerian drops,' we always mean several old men and several valerian drops, and never one. Since a number never means one, one is not a number.

Reply to Objection 3. The fact that we do not say 'a number' when we mean '1' is a *singularity* of usage.

I ANSWER THAT 1 is indeed a number. There is no special reason why; it is just a number because I say so.

Since 1 is a number, there is no harm in forming the sum '1 + 1'; and if we choose to call the resulting number 2, no-one can argue. Of course, it is not fair to use the name '2' for the result of this sum if that name already has another, possibly quite different meaning. Before writing

$$1 + 1 = 2$$

it is necessary to divest oneself of all preconceptions about 1 and about 2. In other words, it is fair to write it only in case it is meaningless. Goodbye to the idea that 2 is company! Adieu to death and taxes, for they are no longer certain to be 2 things in this life that are certain! We must be ignorant of such fancies. What is 2, but the other end of an arrow?

$$(1, 1) \mapsto$$

Let us be generous with ourselves. Unstinting, profligate, prodigal, let us write

$$\begin{aligned} 2 + 1 &= 3 \\ 3 + 1 &= 4 \\ 4 + 1 &= 5 \\ 5 + 1 &= 6 \end{aligned}$$

Unflinching in our audacity, with lunatic abandon let us dare to inscribe:

$$\begin{aligned} 6 + 1 &= 7 \\ 7 + 1 &= 8 \\ 8 + 1 &= 9. \end{aligned}$$

With the famous axiom of Associativity

$$a + (b + c) = (a + b) + c \tag{M2}$$

one may then prove

$$\begin{aligned} 5 + 4 &= 5 + (3 + 1) = (5 + 3) + 1 = (5 + (2 + 1)) + 1 = \\ &= ((5 + 2) + 1) + 1 = ((5 + (1 + 1)) + 1) + 1 = \\ &= (((5 + 1) + 1) + 1) + 1 = ((6 + 1) + 1) + 1 = \\ &= (7 + 1) + 1 = 8 + 1 = 9. \end{aligned}$$

What glorious certainty! What a happy ending! Does the reader feel a bit relaxed, cosy, warm, satisfied? A common reaction. It may even be worth the taxpayers' money, paying all those university mathematicians; at least they are busy making sure of the facts of arithmetic. Without them, who knows, perhaps $5 + 4$ might slip and become 6 instead of 9. It's good to know that things are all right, after all. Or are they?

Far from it. Mathematical ideas, as they trickle down to the popular mind, become diluted. A sneaking suspicion is abroad that mathematicians, some few evil ones at least, are casting doubt on eternal verities, like $5 + 4 = 9$ or $2 + 2 = 4$. Fie! There is not a grain of truth in it. The calumny is too low, too base. This vulgar reaction is an excellent case in point for the novice mathsman to sharpen his wits on. Popular imagination conjures up the myth of the wicked professor who teaches that $2 + 2$ is not 4 precisely because it cannot conceive the truth. The truth is much stranger, more monstrous, more impressive. It is not scepticism about $5 + 4 = 9$ that exists, but scepticism about 5, about 4, and about 9.

With a few brackets it is easy enough to see that $5 + 4$ is 9. What is not easy to see is that $5 + 4$ is not 6. (Great delicacy and tact are needed in presenting this idea in conversation, if the aim is, as it should be, to bewilder and frighten the opponent. His level of sophistication is very important. He may know all about it; then he will utter some crushing reply, like 'So what else is new?' He may even add, 'Just finding out about cyclic groups?'—or mention some other concept you yourself have never heard of; if he does so, you have lost the advantage and may not get out of it without a few scratches on your own escutcheon. On the other hand he may be so ignorant as to be impervious to doubt; you will be laughed at. The idea is much more useful if the intention is merely to annoy; and is often used for this purpose by children to teachers. One strong point is the amount of time that can sometimes be wasted merely making clear just what is meant; another is that it undermines authority, since very few teachers can answer this question sensibly.)

When we said that

$$1 + 1 = 2$$

this meant only that $1 + 1$ was to be called 2. It may perhaps be, for all we know, that $1 + 1$ is really 0, so that we have agreed to call 0 by the name '2'. In other words, it may be that $2 = 0$. Who can tell? It was agreed that all prejudices and preconceptions about 2 were to be given up. So if we say $0 = 2$ we are hardly saying that none are company or that nothing is certain in life. The addition table below satisfies the axioms so far; there are not many numbers, which greatly simplifies all the effort expended in teaching arithmetic.

$0 + 1 = 1$	$1 + 1 = 0$
$0 + 0 = 0$	$1 + 0 = 1$

It still remains to be settled whether

$$1 + 1 = 0.$$

All we know so far is that it cannot be settled at all if all that is known about the system of numbers is that it is a monoid. Obviously, the thing to do is to assume a universal property:

The† monoid N_0 is universally repelling.

(It is self-evident that this has reference to the subcategory of the category of all categories; but to obviate any doubt, the morphisms are the functors.)

Note: The reader may well object to the use of monoids and not groups. He should be reminded that this book is not a first text in

† pointed, with 1.

algebra, and that the difficulties of negative numbers are sufficient unto another section. It must be admitted that monoids are logically prior.

If

$$S = \{0, \alpha\} \quad (0 \neq \alpha)$$

is the monoid with table

$0 + \alpha = \alpha$	$\alpha + \alpha = \alpha$
$0 + 0 = 0$	$\alpha + 0 = \alpha$

then the unique monoid homomorphism

$$\varphi : N_0 \to S$$

such that

$$1 \mapsto \alpha$$

satisfies

$$\varphi(1 + 1) = \varphi(1) + \varphi(1) = \alpha + \alpha = \alpha;$$

if now

$$1 + 1 = 0$$

then

$$0 = \varphi(0) = \varphi(1 + 1) = \alpha,$$

which is false. The contradiction establishes that

$$1 + 1 \neq 0.$$

Back in section 1, when the subject of discussion was counting, the phrase 'natural numbers' was mentioned several times. Now that addition is all the rage, the very same phrase has cropped up. Moreover, the set of numbers used has both times been written N_0. But is this really legitimate? Is there really such a close relationship between the counting system and the adding system? Granted that many of us are wont to use the same words in ordinary discourse— we say 'one, two, three, fiddle dee dee' as readily as 'two plus two is four'— might not this be a convenient shorthand for a situation that is, perhaps, in reality much more complicated, and fraught with difficulty?

QUESTION 5. Whether one should count with the same numbers he adds with, up to isomorphism?

Objection 1. Of course you must use the same numbers in both cases! Pshaw, when it comes down to practical applications, despite all this falferal about morphisms and whatnot, people count things to see how many they are. Then they put two sets of things alongside of one another, and use addition to see how many they have got altogether. Then they count again to see if the answer came out right. Bring in two separate kinds of numbers, and all that becomes impossibly difficult. Isomorphism has nothing whatever to do with it. Ever heard of Steamomorphism? That's the place I go up to when I want to do some arithmetic. Warm and cosy, not at all like up to Isomorphism.

Reply to Objection 1. It is far beneath us to meddle with what anyone finds it convenient to do when he is counting, or adding. All we want is for everybody to be happy.

Objection 2. Mathematicians have the sort of orderly minds that want the right tool for the job. It is certainly very unmathematical to count with one and the same system that is used for adding. It is like using the very same screwdriver both for putting in screws and for pulling them out. And mark my words, serious trouble is going to come when you run into transfinite numbers.

Reply to Objection 2. If anyone is superstitious, he may count with finite ordinals and add with finite cardinals. It is possible to avoid encountering those (shudder) transfinite numbers by resolutely not counting, or adding, anything at all, but just counting and adding pure and simple.

Objection 3. If you use a number for several different purposes, who can blame it for becoming a little suspicious? It will begin to resent such treatment. Especially in these days of 'bloodymindedness' and continual unrest, we must guard against the development of system-consciousness among numbers. There is the ever-present danger that they will begin to ask themselves '*Was sind und was sollen wir Zahlen?*' Perhaps all numbers will desire to be perfect, or at least prime, or even transfinite. Be kind, but firm, to your numbers; write them on paper that is neither too fine nor too coarse. Do not let them get together in large sets. Above all, give them a sense of identification with one specific job.

Reply to Objection 3. Some people seem to like bringing politics into mathematics. There is really nothing to fear from these radicals. If all the partially ordered sets of the universe should one day become discrete, thus discarding their chains, new structures could be manufactured on demand.

I ANSWER THAT one may count with the very same numbers he adds with.

PROPOSITION. *If you can add, you can count.*

Proof. In counting with the additive monoid N_0, we start at 0; after saying n, we say $n + 1$; the few diehards still around who prefer to begin counting at 1 instead of 0 must make the adjustment for once and go along with the growing trend, because it will be much more convenient in the present instance to start at 0. Thus we already have a counting system.

$$\{0\} \hookrightarrow N_0 \xrightarrow{n \mapsto n+1} N_0.$$

But having a counting system is not enough. What you must have in order to assure success in all your counting endeavours is a real, true, initial counting system. So let

$$\{x\} \hookrightarrow X \xrightarrow{f} X$$

be any counting system. The set of all functions

$$X \to X$$

is easily verified to be a monoid $\mathscr{X}$. Hence there is (since adding is one thing we feel absolutely confident about and can do blindfolded with our hands tied behind our backs) a unique monoid homomorphism

$$N_0 \to \mathscr{X}$$

sending $$1 \mapsto f;$$

which is written

$$n \longmapsto f^n.$$

Now it is possible to define a mapping

$$N_0 \longrightarrow X$$

by writing

$$n \longmapsto f^n(x).$$

We could go on to show that $n \longmapsto f^n(x)$ is the only morphism of counting systems from N_0 to X. This would complete the proof that you can count with the same numbers that you add with. In fact, the very same set can serve for both counting and adding. The qualifying phrase 'up to isomorphism' is not strictly necessary. But some of us consider it a virtue to specify which among isomorphic systems is in use, only when it is absolutely necessary; and to be as vague as possible as often as possible. This is a feeling that merges into the psychological. When two mathematicians discuss the natural numbers, do they both have in mind the same numbers, or are there two distinct but isomorphic systems? It is not the sort of question to ask very often, but perhaps one may be permitted to ask such a question once in, say, a sidereal decade.

Addition, like mathematics, occurs on various cultural levels. A perfect example is provided by a certain meal in a restaurant in Athens. The diners at the table near the back are mathematicians. According to the custom of the place, when they have finished the waiter asks them what they have had, takes down the items on his pad at dictation, affixing the prices, and adds up. At that point, for some reason, one of the mathematicians remembers—'Oh, yes. And besides all that, I also had a beer.' In such an eventuality, even in Athens, a waiter will commonly add the price of one beer to the sum already obtained and present the corrected bill. This waiter, instead, tore up the incorrect bill and added up the whole meal again with the extra beer included. When the diners explained what was the more usual procedure in such cases, and suggested that it also produced the correct sum, the man in question admitted that that might theoretically be as they said. But he still stuck fast to his own method. 'I have a restaurant to run; I am not a philosopher.'

On the lowest level of mathematical culture, adding up is a ritual, part of the daily ritual of life; it affects the pattern of events somehow, and any departure from the ritual pattern produces an aberration in the course of events. Something might go wrong with the restaurant. Most waiters live somewhere above the ritual or superstitious plane of mathematical thought. They understand that the purpose of adding the bill is a fair exchange of money for goods. Because they are capable of considerations of this nature, they acquire a certain arithmetical dexterity. Sometimes you have to watch them carefully. This is the level of applied mathematics.

The mathematicians at the table are at the highest level, that of pure mathematics. To them, the meal is a function

$$f: M \rightarrow N_0,$$

where M is the menu and $f(m)$ is the number of items m consumed; and the adding up is a homomorphism

$$\beta : N_O{}^M \rightarrow N_0$$

(the bill function) from the meal-monoid to the Greek currency monoid (which is isomorphic to N_0). The bill $\beta(f)$ for a meal f is a natural number of drachmas. Since the extra beer is in itself a small meal, and since

(incorrect hypothetical meal) + (extra beer) = (corrected meal)

in $N_O{}^M$, the additivity of β implies that the usual procedure of waiters is correct.

One very small difficulty is the commutative law. Many students of ordinary arithmetic have noticed that

$$7 + 5 = 12;$$

and also that

$$5 + 7 = 12;$$

they may, indeed, have observed that the equation

$$a + b = b + a$$

holds true in a large number of cases in ordinary arithmetic. Now, of course, ordinary arithmetic is a complicated subject—much more complicated than the simplified version considered here. It involves digital representations and quite a large number of ideas all mixed up together. There are lots of numbers. How many people have actually checked that

$$19667 + 543628 = 543628 + 19667?$$

And how many cases of the rule

$$a + b = b + a$$

must be verified in order to rule out even one single exception? Obviously, no number of cases will suffice. The commutative law, as any schoolchild knows today, does not hold for every conceivable monoid. An example is a monoid in which 'addition'† acts by absorbing the right-hand (or dexter) element into the left-hand (or sinister) one:

$$x + y = x$$

—an exception must be made if one of the elements is the neutral element, since in any monoid this element must always be absorbed. To be specific:

$$\begin{array}{c|cccc}
\gamma & \gamma & \alpha & \beta & \gamma \\
\beta & \beta & \alpha & \beta & \gamma \\
\alpha & \alpha & \alpha & \beta & \gamma \\
0 & 0 & \alpha & \beta & \gamma \\
\hline
 & 0 & \alpha & \beta & \gamma
\end{array}$$

(The table is read like a map, in analytic geometry fashion.)

To prove that the natural numbers commute, one need only consider the centre; this is obviously a submonoid, and contains 1 if the centraliser of 1 is N_0. Since the centraliser of 1 is a submonoid, one now need only show that 1 is an element of this centraliser. This follows from the fact that

$$1 + 1 = 1 + 1.$$

Now the proof is complete, depending only on the universal property

† Admittedly, this behaviour is so queer that convention would prefer to call it multiplication, not addition. This is because ordinary multiplication of natural numbers is just as commutative as ordinary addition, but is more complicated.

of N_0: since the centre N contains 1, the left-hand (or sinister) homomorphism in

$$\begin{gathered} 1 \mapsto 1 \\ N_0 \rightarrow N \rightarrow N_0 \end{gathered}$$

can be defined. The composition must be the identity, so

$$N = N_0$$

and

$$a + b = b + a$$

for every pair of natural numbers a and b.

The addition of columns like

$$\begin{array}{r} 1 \\ 174 \\ 9 \\ 34 \\ 9 \\ \hline ? \end{array}$$

leads to the general associative laws, which are too deep to consider as yet.

3. Subtraction

Marco (the waiter) Cameriere meets Pigritio (Piggy) Risorgiamento in the automat. 'Say, Marco. You know that guy Watermaker?'

'Watermaker the differential geometer?'

'Yeah. Know something? He uses infinitesimals. Infinitesimals aren't rigorous.'

'Aw, c'mon, Piggy. Everybody uses infinitesimals these days. They can be made rigorous. They are rigorous.'

'They're not rigorous, Marco. Watermaker needs a lesson in rigour.' Idly chopping his cigar in bits with poultry shears, Pigritio emphasises the point. 'What he needs is rigour.'

The following day, a shocking announcement is made at the conference on singularities at New Rochelle. 'Dr Watermaker is in hospital, apparently with a shoulder wound caused by a sawn-off shotgun or some similar instrument. Fortunately, we have been able to fill in a replacement speaker. Professor P. Risorgiamento will talk on "The treatment of obstructions".'

There are many practical applications of the idea of subtraction in everyday life. The general idea may be used for shortening cigars, for treating obstructions, for removing corns. At present, consideration is restricted to subtraction of one natural number from another. As this is accomplished in ordinary arithmetic, it is a complicated operation involving the borrowing of ones and some rather fancy pencilwork, based on fossilised ritual technique.

A generally accepted idea about subtraction is that it works only in one direction. For instance,

$$\begin{array}{r} 327 \\ -143 \\ \hline ??? \end{array}$$

has an answer, and

$$\begin{array}{r} 143 \\ -327 \\ \hline ??? \end{array}$$

has none. The only case when both '$a - b$' and '$b - a$' are considered to be subtractions that can be carried out with natural numbers, and giving natural numbers as answers as well, is when a and b are both exactly the same number:

$$a = b.$$

The standard line of argument is this: If you could take a from b and still have some left over, that would mean that b was bigger than a. Similarly, if b can be taken from a then b is bigger. But it is impossible for each to be bigger than the other. So the subtraction cannot be carried out in both directions. If that were all there was to it, the question would be simple indeed; but, unfortunately, there is a flaw in the reasoning. To say that one number is bigger than another is no more or less than to say that the second can be subtracted from the first, leaving something over. Hence, to say that each of two numbers cannot be bigger than the other is to repeat the statement that is to be proved. It is not correct in logic to prove something by saying it over again; that only works in politics, and even there it is usually considered desirable to repeat the proposition hundreds of times before considering it as definitely established.

A slightly more mathematical idea is this: Suppose it were possible to subtract a from b, and also possible to subtract b from a. To make the idea realistic, write

$$a - b = p$$

and

$$b - a = q.$$

These equations give

$$\begin{aligned} a &= b + p \\ b &= a + q \end{aligned}$$

and hence

$$a = b + p = (a + q) + p = a + (q + p).$$

If a is cancelled from the above equation, the result is

$$0 = q + p.$$

But then it is known that both q and p are 0, so $a = b$. That is just what everyone has always thought—the subtraction works in both directions only if the two numbers are the same number.

The last-mentioned argument appears so finical and precise that it is in danger of being considered correct. It has one mistake, however.

QUESTION 6. Whether, in an equation setting the sum of two natural numbers equal to another, if the same number appear on either side, one may cancel it?

Objection 1. The word *cancel* comes from the Latin *cancello*, to make like a lattice; to cut crosswise; to deface, rase, or cross out, to cancel. This in turn is derived from *cancelli, -orum*, meaning lattices, balisters or rails to compass in; windows, casements, or peeping-holes; also little crevisses or crab-fishes. [14] 'And all that have not fins and scales in the seas, and in the rivers, of all that move in the waters, and of any living thing that is in the waters, they shall be an abomination unto you . . .' [15] Since *cancelli* are an abomination, one may not cancel.

Reply to Objection 1. The Latin word for 'scales' in the passage quoted is *squamas*. Anything that has *squamas* is perfectly all right, as the immediately preceding verse states explicitly. But *écrevisses* or crawfish are extremely squeamish, as any little girl will tell you. The other meaning of *cancelli*, about ladders and banisters, directly points to the idea of a ladder or *scala*, similarly constructed with rails and used for scaling walls. Hence by another proof, *cancelli* have scales.

Objection 2. If something is to be defaced or rased, it should not have been written in the first place. But in mathematics, anything may be written in a proof that is an axiom, a hypothesis, or that follows from previous steps in the proof by *modus ponens*. If something should not have been written, it did not satisfy these conditions. Hence any proof containing a cancellation also contains an error.

Reply to Objection 2. One does not actually need to cross out anything in order to perform the abstract mathematical act of cancellation. What would be done in a formal proof is to copy the equation over, omitting the terms that are considered to be cancelled. If this is too difficult, the actual cancellation may be done on a piece of scratch paper and immediately burned. Then who will know?

Objection 3. Cancellation is not always correct. For example, $2 \times 0 = 0$ and $314209 \times 0 = 0$. Hence

$$2 \times 0 = 314209 \times 0.$$

If cancellation were always correct, then by cancelling 0 on both sides we should obtain

$$2 = 314209.$$

But this is false. Hence it is incorrect to cancel in mathematics.

Reply to Objection 3. It is true that cancellation is not always correct; but it is correct under the conditions stated in the question. The example given is multiplication.

I ANSWER THAT one may cancel.

Suppose that x is a cancellable number; i.e., suppose that

$$x + p = x + q$$

never holds true unless

$$p = q$$

also holds true, no matter what the numbers p and q are. Suppose that y is also cancellable. Since

$$(x + y) + p = x + (y + p)$$

from

$$(x + y) + p = (x + y) + q$$

one can infer that

$$y + p = y + q$$

and hence that

$$p = q.$$

It has just been proved that the sum of cancellable numbers is itself cancellable. If

$$\{x\} \hookrightarrow X \xrightarrow{f} X$$

is any counting system whatsoever, it can be augmented by writing

$$X_0 = X \cup \{x_0\}$$

with $x_0 \notin X$, and by defining

$$f_0 : X_0 \longrightarrow X_0$$

so that $f_0(y) = f(y)$ for $y \in X$ and by $f_0(x_0) = x$. Then the morphism $N_0 \longrightarrow X_0$ provides by restriction a morphism

$$\varphi : N_0 + 1 \longrightarrow X.$$

With X set equal to N_0, necessarily

$$\varphi(n + 1) = n;$$

which shows that

$$n \longmapsto n + 1$$

is injective and that 1 is a cancellable number. Now one may conclude that all numbers are cancellable.

Thus, the question whether two quite distinct and unequal numbers exist each of which can be subtracted from the other, either way round you like, has definitely and positively been settled in the negative. The answer is no. The next question would seem to be, are there two numbers, neither of which can be subtracted from the other, no matter how the problem is attempted? These would be two numbers, neither of which is any bigger than the other, nor are they the same number. The two numbers could not be compared in size; they would be mutually incomparable.

QUESTION 7. Whether any two numbers are comparable?

Objection 1. The kinds of comparison—the absolute, comparative, and superlative, are called degrees. Hence the question refers to the numbers of degrees used in recording temperatures. But if the temperature exceeds a certain level, mathematics is impossible, since mathematicians require pencils and paper and these would ignite. Hence only small numbers are comparable to other numbers.

Reply to Objection 1. A mathematician can deal with arbitrarily large temperatures under the most comfortable working conditions, simply by inventing new scales for the measurement of temperature. This is done by means of a 'scaling factor' which converts one degree Fahrenheit (or Celsius) into as many degrees as one likes of the new scale.

Objection 2. Some numbers are so large that no-one can form an idea of their magnitude. But mathematicians cannot deal with that of which they can form no idea; hence to compare two such great magnitudes is beyond them.

Reply to Objection 2. Mathematical statements about natural numbers are really, or can be converted into, statements about the system of natural numbers, obviating the necessity for actual contemplation of huge magnitudes and other such arcane trash.

Objection 3. If any two numbers can be compared so that one is greater, then there must be a greatest number n. But then $n + 1$ is greater than n, contradicting the supremacy of n. Since this is nonsense, there must be two mutually incomparable numbers.

Reply to Objection 3. It is inaccurate to infer that if any two things of a certain kind are comparable then one of the things is greatest. The argument of this objection shows correctly that no natural number is greatest, despite which fact it is the case that any two of them are comparable.

I ANSWER THAT any two numbers are comparable.

A number is *subtractable*, for the nonce, if it is comparable to any other natural number whatsoever. If x is subtractable and if y is any other number whatsoever, then either the subtraction '$x - y$' can be done, or else the subtraction '$y - x$' can be done. (In '$x - y$', the number x is the *minuend* and y is the *subtrahend*.†) Suppose that each of x and y is a subtractable number and that z is any other number at all. If z can be subtracted from x then

$$x = z + p$$

whence

$$x + y = (z + p) + y = z + (p + y)$$

so that z can be subtracted from $x + y$. Otherwise, since x can be subtracted from z, the equation

$$z = x + p$$

holds for suitable p. Since y is subtractable, either

$$y = p + q$$

or

$$p = y + r;$$

yielding either

$$x + y = z + q$$

or

$$z = (x + y) + r.$$

Since all numbers of the form

$$1 + p,$$

together with 0, form a submonoid of N_0 containing 1, the number 1 is subtractable, and so are all natural numbers.

† This terminology is of no use whatsoever.

4. The negative–positive diathesis

QUESTION 8. Whether negative numbers are fictions?

Objection 1. Negative numbers are fictions. The expression

$$'-3 + 4'$$

has a meaning, since it is just

$$'4 - 3'$$

written backwards. It is really possible to take 3 from 4. Since it is not at all possible to take 4 from 3, the expression

$$'3 - 4'$$

is a mere fiction. In practice, it is sometimes convenient to use such a fiction; the procedure for working out the sum is as follows: Since no-one could ever take 4 from 3, change the numbers about and take 3 from 4. The answer is 1. But it would be wrong to say that 1 is the answer to '3 − 4', since a fictional problem cannot have a true numerical answer. Hence, the minus sign is affixed to the 1 and −1 is given as the answer. Since both sides of the equation

$$'3 - 4 = -1'$$

are nonsense, the equation is true.

Reply to Objection 1. Good fiction is never nonsense. The mistake here is in thinking positive numbers are more real or actual or concrete than negative numbers. Both are equally fictitious.

Objection 2. Negative numbers are not fictions. They represent such things as debts (to take an example from the field of commerce). Debts are only fictions if the borrower does not pay them; in that case they are fictions because that is dishonest. But failure to pay debts is punishable by law; hence it is wrong; and when something is wrong, mathematicians need not pay attention to it.

Reply to Objection 2. Money is probably fictitious, too.

I ANSWER THAT negative numbers are fictions, but they are not nonsense.

The system of all positive and negative numbers is the group of *integers*. It is indeed possible to construct the group of integers more or less along the lines of Objection 1. The danger that one encounters is that of being led to take natural numbers as prior, or natural, and negative numbers as secondary, or artificial. This is the diathesis that must be avoided at all costs. There are already too many temptations to fall into this grievous error. One of the worst is the usual method of working out sums such as $3 - 4 + 2$ or $-17 + 4 - 3 + 21 - 13$. A better method, and one less prejudiced against the negative numbers, is suggested later, under division.

The widespread prejudice against the negative numbers is totally unjustified by science. At some point, a mathematician had to point to either 1 or -1 as being the distinguished element. He did so quite arbitrarily, but ever since then the positive numbers have felt superior—just because they have been placed on the right-hand (or dexter) side. Their opposite numbers on the left-hand side of the origin are pushed aside when selection is made for the important jobs. They are considered to be abnormal, peculiar, inferior. How unjust!

The integers form a group—as every undergraduate, even every sharp third-former, knows, a monoid in which every element has an opposite or inverse. Thus

$$-x + x = 0 = x + (-x).$$

In other words, a group is a category with one object in which every morphism is an isomorphism. Just about the simplest category is $\mathcal{O}$,

$$\begin{tikzcd} O \arrow[loop above, "\iota"] \end{tikzcd}$$

which has one object O and one morphism $\iota : O \longrightarrow O$. Slightly more fancy is $\mathcal{I}$:

$$\begin{tikzcd}
A \arrow[loop right, "\iota_A"] \arrow[d, bend left, "f"] \\
B \arrow[loop right, "\iota_B"] \arrow[u, bend left, "g"]
\end{tikzcd}$$

with $gf = \iota_A$, $fg = \iota_B$. From $\mathscr{O}$ to $\mathscr{I}$ there exist exactly two functors and no more, namely $a : \iota \mapsto \iota_A$ and $b : \iota \mapsto \iota_B$. Then the equaliser $z : \mathscr{I} \to Z$ of

$$\begin{tikzcd}
\mathscr{O} \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & \mathscr{I}
\end{tikzcd}$$

is such that

$$\begin{tikzcd}
\mathscr{O} \arrow[r, "a"] \arrow[d, "b"'] & \mathscr{I} \arrow[d, "z"] \\
\mathscr{I} \arrow[r, "z"] & Z
\end{tikzcd}$$

commutes, and that if

$$\begin{tikzcd}
\mathscr{O} \arrow[r, "a"] \arrow[d, "b"'] & \mathscr{I} \arrow[d, "c"] \\
\mathscr{I} \arrow[r, "c"'] & \mathscr{C}
\end{tikzcd}$$

commutes then there exists a unique functor $d: Z \to \mathscr{C}$ such that

$$\begin{tikzcd}
& \mathscr{S} \arrow[dl, "z"'] \arrow[ddr, "c"] & \\
Z \arrow[drr, "d"'] & & \\
& & \mathscr{C}
\end{tikzcd}$$

commutes.

Since $z(A) = z(B)$, it is easily seen that Z is a monoid. The invertible morphisms in Z form a submonoid and include $z(f)$ and $z(g)$; hence Z is a group. This group is called the group of integers. The composition of the morphisms in Z is usually written additively: $m + n$ and not mn or $m \circ n$. The usual notation for $z(\iota_A)$ or $z(\iota_B)$—they are the same—is 0. It is necessary to choose, if the usual notation is preferred or a distinguished element is desired, whether to use $z(f)$ or $z(g)$ for 1. Whichever is 1, the other must of course be -1. There are many ways of making this choice. One might, e.g., ask $z(f)$ and

$z(g)$ both to bring offerings. Then one could have respect unto one of their offerings, and not have respect unto the other. [16] The morphism unto whose offering you have not respect would then get a mark set upon it, namely the minus sign. Or, one might flip a coin.

It is possible to get the natural numbers in a similar way. The category $\mathscr{I}$ is replaced by $\mathscr{K}$,

$$\begin{tikzcd} A \arrow[loop right, "\iota_A"] \arrow[d, "f"] \\ B \arrow[loop right, "\iota_B"] \end{tikzcd}$$

There was a time when an *axiom* was a general ground or rule of any art, and when the axioms of mathematics were supposed to be incontrovertible propositions, so obvious as not to require any proof. That day is past. [17] Nowadays one experiments with new art forms, in mathematics as in other arts. A new art requires a new basis, and one may say that at the present time an axiom is a proposition so useful that it must be put beyond question. The *truth* of an axiom, in any absolute sense, is not a matter of interest. Mathematics has been called the art in which men do not know what they are talking about and cannot tell if what they say is true. The ultimate judge of the mechanical correctness of a mathematician's work is a mathematical proof-checking machine; the ultimate judge of its rightness and suitability is the opinion of the other mathematicians. As an axiom on which to base the positive numbers and the integers, which have in the past produced much harmless amusement and are still widely accepted as useful by most mathematicians, some such proposition as the following is sometimes considered as being pleasant, elegant, or at least handy:

AXIOM: *Equalisers exist in the category of categories.*

It is worth emphasising that according to the approach outlined above—or any other enlightened approach—there is nothing negative about the negative numbers. The notion is entirely false that negative numbers are positive numbers turned the wrong way; or rather, this notion, and the notion that *positive* numbers are *negative* numbers turned the wrong way are equally slanderous. In the *group* of integers there is absolutely nothing to tell which numbers are positive and which are negative. Psychological tests under stringent conditions of probabilistic and experimental rigour have shown that subjects, being shown photographs of numbers, did so badly at identifying the sign of the number that the correlation between the answer given and the true sign of the number was nil. More surprisingly, leftists and rightists showed no more aptitude at this exercise than people of moderate directional tendencies. What we have got is not that positive integers can be distinguished in some special, magical way from other integers; we have got a *pointed* group

$$\{1\} \hookrightarrow Z$$

in which the submonoid generated by 1 can of course easily be distinguished.

On the other hand, it is usually taken for granted that the natural numbers are part of the integers. The integer 1, the famous and distinguished element of the group of integers, is written exactly the same as the distinguished *natural number* 1. Looking at the numbers on the pages of this book, the man in the street will seldom ask, is it 48 the natural number, or 48 the integer that is being used to number this page? Yet the mathematical system within which the page numbering is done is surely of the greatest importance. It would appear to be wrong to do anything at all with numbers without paying close attention to the question of what number system one is working in. The numbers themselves, indeed, are of secondary importance. It is the mathematical system that makes all the difference. Moreover, natural numbers, as opposed to integers, have political overtones: grocers, classicists, schoolteachers prefer natural numbers, while technicians and pseudo-intellectuals like integers better. Failure to make the distinction could lead, not only to confusion, but to a delicate situation requiring the utmost tact, and a spirit of tolerant give-and-take if the inevitable differences of opinion based on such a wide mathematical discrepancy are to be avoided. After all, the system of integers and even the natural number 0 are still regarded in many quarters as contrivances of the devil.

Fortunately, the difficulty just mentioned is not entirely insurmountable. Popular culture, in this permissive age, has mellowed toward integers; they are still regarded by many as effete intellectual trash, but it is now felt more and more that effete intellectual trash has its place. Being effete is increasingly regarded as just another way of doing your thing. It is admitted by many that the world would be a duller place without the prime number theorem.

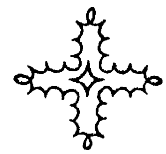
When tempers run high on the question of page numbering in integers as opposed to natural numbers, there is mathematical oil to pour on the troubled waters. The fact is that there is a natural, down-to-earth homomorphism from the naturals to the integers, and that this homomorphism, sending as it does the natural number 1 to the integer 1 and being so natural and injective as it certainly is, is just perfect for being considered as a mere inclusion and nothing else. In other words, one might as well pretend that some of the integers actually *are* natural numbers, and that natural numbers are integers. The integers one pretends this about are called *positive* (or non-negative).

5. Multiplication

One of the heresiarchs of Uqbar declared mirrors and fatherhood to be abominable, because they *multiply* the visible universe. [18] Multiply it by what, one may wish to ask? Here we pass over such questions belonging principally to heretical theology and the investigations of *Tlönistas,* just as we ignore the whole matter of the multiplication table, since this belongs properly to division.

By '*multiplication*', properly speaking, a mathematician may mean practically anything. By '*addition*' he may mean a great variety of things, but not so great a variety as he will mean by 'multiplication'. What, then, is the main difference between addition and multiplication? The most important difference is that addition is always denoted by '$+$'. This is not quite true. Occasionally, for decorative purposes primarily, one will see something like

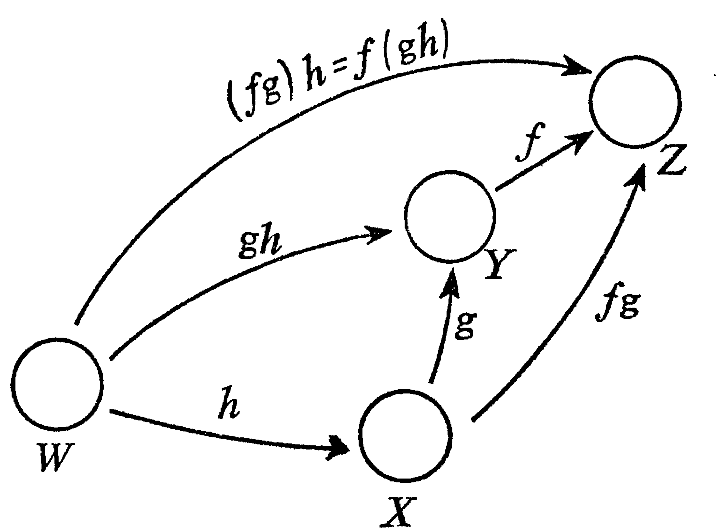
$$\bigoplus \quad \text{or}$$



Multiplication may be denoted by $\times$ or by $*$, or by something more monstrous, or by nothing at all. There are other, less important differences. Usually, addition is commutative. Multiplication may also be commutative. Addition is usually associative. So is multiplication (usually, but not always). The main thing to remember, for the beginner, is that the things multiplied need not be numbers; nor need multiplying things make them more multiple, or multiplex. (It is advisable for the beginner not to try too hard to remember isolated facts. On a first reading, or possibly for a few readings thereafter, it may be desirable to forget temporarily everything that the reader has just learnt.)

Bricks are made of clay which is brought to a plastic consistency by the admixture of a suitable quantity of water, formed into shapes established by custom and precedent, dried in the sun, and finally baked. They differ from the loaves of Beaulieu Derrière in that their dimensions never vary from day to day, and in that they are not in general square or cubical. The brick, because of its shape and because of the fact that the ratio of its sides remains constant, may be used for the computation of volumes. Bricks have, of course, other uses. Who has not seen, outside the tent of a particularly cheap sideshow at a travelling carnival, an advertisement of the presence within of that destructive creature, the *red Irish batt*? The uses of the brick in cosmetics and in the art of preparing camels for the journey across the desert are too universally known to merit discussion in a work of this sort. Slightly more mathematical is the use of bricks in connection with simplified versions of the three-body problem: by computing parabolic trajectories in intersection, one may practise the applied mathematical art of *trashing*. None of the foregoing uses, it may fairly be objected, is a proper use of bricks; in all these instances we see bricks used, yes—but they are not used *qua* bricks. When bricks are used *qua* bricks, for constructional purposes, three properties at least which bricks enjoy are of great importance: bricks are movable, rigid, and take up space. If it were not for these properties, among others, bricks would be useless for building walls.

But are we safe in relying on these ideas? May we feel secure in our houses, if we have allowed them to pass unquestioned? Is there really such a thing as the volume of a brick, remaining unchanged however the brick is displaced by rotation, translation, and the zig-zag-and-swirl of Lawsonomic motion? [19] It hardly seems a blessed hope. One way, of course, of dealing at least partially with this frightening problem is the standard one, familiar to every school algebraist. (Here we assume bricks whose sides are whole numbers. Although this rules out the lovely bricks of nineteenth-century Surrey malt factories, which had faces that were golden rectangles [20], our result still holds good if applied to the more mundane bricks of commerce.) The school algebraist will remind his hearer of the commutativity and associativity of the ring Z. He will point out that integers, when you are multiplying them, are functions, and that the composition of functions is about the most associative thing we've got in this sublunary vale. No doubt he will exhort you to remember from your schooldays that the centre of a ring is a subring, and that it contains 1. By a universal argument, he can then demonstrate beyond chance of contradiction that the volume of a brick of integer sides does not depend on how you turn the brick—even if you turn it inside out. (On the question of turning *cubes* inside out see [21].)



THE SIDES OF A BRICK ARE (ESSENTIALLY) HOMOMORPHISMS

That is, as I may have hinted, the old-fashioned, crotchety schoolboy approach to bricks. A more sensible, and not very different, way of looking at the problem is this one: The volume of a brick of dimensions (a, b, c) is the value at (a, b, c) of a trilinear symmetric function

$$Z^3 \rightarrow Z$$

taking

$$(1, 1, 1) \mapsto 1.$$

(The latter says that a brick whose dimensions are 1 inch in one direction, again 1 inch in a perpendicular direction, and once more 1 inch in a third direction perpendicular to them both, is 1 cubic inch —*provided the measurements* (1, 1, 1) *are written down in the correct order*! Note that the non-existence of such bricks is entirely immaterial; indeed, the whole question only becomes intelligible if all the bricks involved are themselves wholly immaterial.) Obviously, exactly one trilinear function $Z^3 \rightarrow Z$ sending $(1, 1, 1) \mapsto 1$ exists; just as obviously, the uniqueness makes it symmetric.

EXERCISE. Show that four-dimensional bricks behave in the same way.

EXERCISE. A brick is thrown by an extraverted undergraduate genius through the window of the mathematics common-room at the University of Both Putfords, Devon. It strikes the head of the Senior Lecturer in Analytic Manipulations and shatters, falling together on the floor in such a way that the new volume of the brick exceeds that of the mathematics building. [22] Indeed, it is now bigger than the Putfords. Since the mathematics department is now a vacuous body of men, what degree must be awarded posthumously to the student? Submit detailed plans for archaeological excavations and reconstruction of the villages and the new university, bearing in mind the reduced density of the brick.

QUESTION 9. Whether $2 \times 2 = 4$?

Objection 1. Geometrically, multiplication is carried out by using rectangles. For example, if a line segment 3 units in length is constructed perpendicular to another segment of length 2 units, and if certain further lines parallel to these are constructed, one obtains a rectangle divided into 6 unit squares. The fact that from a length of 2 units and a length of 3 units one obtains a *rectangle* of 6 units is the reason for saying that $2 \times 3 = 6$. But from segments 2 units long each, one obtains not a rectangle at all but a square. Thus one may not say '$2 \times 2 = 4$,' but only '$2^2 = 4$'; i.e. 'two, squared, is four'.

Objection 2. Moreover, 2 units multiplied by 2 units is not 4 units but 4 *squared* units. But 4 squared is 16. Hence $2 \times 2 = 16$.

Replies to Objections 1 and 2. Squares are rectangles; in any case, we are doing arithmetic, not geometry. Hence both arguments are irrelevant. Besides, it is not at all obvious *a priori* that 16 is not exactly 4.

I ANSWER THAT $2 \times 2 = 4$. The first step is to replace '2' by '$1 + 1$'. Few mathematicians doubt that this can be done, but few except the subtlest logicians can put up a good defence for it against determined opposition, at least unless they are given adequate time to prepare a defence. The principle involved is the replacement of one thing by something equal to it. No doubt there may be contexts in which this may safely be done; but how hasty can we be in assuming that we have one here? Difficulties can certainly arise in the application of the principle here employed. The standard example is [23]. Editions of works by this author can easily be found in second-hand bookstores that bear on the spine the words 'by the author of *Waverley*'. It is not at all unlikely that someone, somewhere, somewhen has looked at such a book and said, 'I wonder if the author of *Waverley* is Scott?' Now, as a matter of fact, it is very definitely the case that the author of *Waverley* is indeed Scott himself. That being the case, by the application of the famous principle that equals may be substituted for equals, anyone whatever, be he ever so slow-witted, may arrive at the conclusion that the speaker might equally well, and just as truly, have said 'I wonder if Scott is Scott?'—or, 'I wonder if the author of *Waverley* is the author of *Waverley*?' Some mockers have the temerity to suggest that they, at least, do not intend to be caught pondering such idiotic tautologies. So, if equals may not *always* be substituted for equals, when may they be?

The answer is, of course, that we shall have to invent a special reason why one can substitute '$1 + 1$' for '2', at least just this once. Or else, we must just do it and worry about it later, if at all.

Having got past this hurdle, we now have

$$2 \times 2 = 2 \times (1 + 1).$$

The brackets indicate that we add 1 and 1, and then take 2 times the result. The next step is to say

$$2 \times (1 + 1) = (2 \times 1) + (2 \times 1).$$

Here, the argument is that multiplication is a bilinear map,

$$Z \times Z \to Z.$$

It is a triviality, arising from the definition of multiplication, that

$$2 \times 1 = 2.$$

Since $2 + 2 = 4$, we are done.

Note. One may also attack the problem via the natural isomorphism

$$n \mapsto (1 \mapsto n)$$

from the additive group Z to the *additive group* **End**(Z), whereby the obvious ring structure of **End**(Z) may be transferred back to Z. It is left to the reader to consider which of the two approaches may be the more elegant.

6. Division

So far in this present work, it has not been possible to mention certain very interesting, natural numbers. At least, this is true on the hypothesis that there are any interesting natural numbers. Students of the question, whether there are any interesting natural numbers, are divided on what answer to give. Some experts believe that *all* natural numbers are interesting; others, perhaps more cautious of hasty generalisation or less sanguine of temperament, hold the wiser and more temperate opinion that *no natural number at all* is of any interest. (This view is preferred by the author of the present work, which should settle the matter.) Whether or not there are any interesting natural numbers, then, it has not been possible to mention any of them so far in this book; because so far in the book, no means of mentioning natural numbers greater than 9 has been established. (This statement is made rather informally, and is open to all sorts of quibbling. Some people would say that it has not been established what is meant by *mentioning* something. It would be hard to show, perhaps, that no mention has been made herein of the ninth reindeer in Santa Claus' team. A good lawyer would point to all the occurrences of the letters 'red', in that sequence; the allusions to the idea of nose; the fact that Carnap is a logician; that this book dwells heavily on the subject of logic; that Carnap's first name is *Rudolf*. Once it is established that Rudolph, the red-nosed reindeer, has been mentioned, then it becomes clear that almost anything may have been mentioned. In particular, anything nonexistent may have been mentioned—Rudolph is known to be apocryphal, to be a later accretion on the fixed body of *mythos septentrionalis*. [24] If you can mention him, you can mention anything or anybody—even all those nonexistent interesting natural numbers. And very likely they have been mentioned. The good lawyer might get away with the argument just outlined, just because nobody knows what it means to mention a thing, anyway.)

Because of all this quibble, let us merely say that there has been up to this point no way at all to mention numbers bigger than 9, except by circumlocution; it was always possible to talk about the sum of 6 and 7, and even to show that this number was bigger than 9; what was not possible was to write this number as 13. It is still not possible to write it as 13; or rather, it is just as possible to write it as 13 as to write it as 31 or 995. There is no system for writing these numbers. So far as this book is concerned, we do not yet know *how* to write them. Equally well, we do not know, mathematically and within the system of this book, how to name them. We cannot yet count up to a hundred, though we can count to 9.

The ability to count to a hundred is part of *numeration*, or the study of number numbering. Traditionally, one learns the art of numeration before the other arts of arithmetic. How this can be is of course beyond the ability of the present author to account; but perhaps one may suppose a latent ability in the human soul to count, somewhat on the lines of the supposed ability latent in the human soul to learn a human language. [25] After all, the old philosophers

[26] are said to have taught that man's soul is a number numbering itself. In the study of mathematics, this pedagogical order must in any case be reversed. In mathematical terms, if you are to learn to give names to the numbers in something like the usual system, and so to count aloud, you must first learn to divide; and that cannot be done until you have already mastered addition, subtraction, and multiplication. Fortunately, anyone who has carefully and attentively read the preceding pages of this book has already mastered the arts of adding, subtracting, and multiplying. (Human beings appear to have had since the earliest times the ability to multiply; indeed this faculty seems to extend even to the higher animals, though with them it appears to be a seasonal activity. Division, strangely, appears to be very much an acquired taste among the human race, though oddly enough it is commonly observed under the microscope as a natural activity of the lowest forms of life—bacteria, protozoa, and such. The old philosophers do not appear to have considered the protozoan soul, but if they did they might well think of it as a number dividing itself.)

Before it will be possible to go any further, it is necessary to dispose of a ridiculous quibble, which the author's mountainous wealth of practical teaching experience teaches him is bound to arise at this point. Little minds love to ask big questions, or what appear to them as big questions; never stopping to reflect how trivial the answer must be, if only the questioner would take the trouble to think it through. Sometimes it is necessary for the writer of such a serious work as the present one to call a halt in the consideration of matters of real weight and interest and to remember how weak and frail are the reasoning powers of his lowly readers. Someone, somewhere, has asked, 'What about the numbers at the foot of the pages? You really had no right to have them there, since in a work of this nature, they have no meaning as yet.' The answer will be obvious, if the reader will only think. It is true that we have already come to pages of the book with numbers on them that are bigger than 9. In reading a scientific treatise, however, it is well to remember that patience is a virtue. One could have compressed all the material of this book, up to and including the present section, into one page. Very often, in a book, the first page of text has no number on it at all. One could have done this simply by telling the printer to use very tiny type. It would have been hard to read, of course. But this was not done. Why? Because the publisher insisted. Publishers simply insist on numbering their pages. The author protests, expostulates, threatens—all to no avail. The author emphatically assures the troubled reader that absolutely none of the mathematical material of this book depends essentially on the numbering of the pages. If the pages were not numbered everything herein contained would be as firm, as crystal clear as it is now. And even if it were not so, the reader will soon learn all about these numbers; they are to be thoroughly explained, and any nagging doubts about them will be dispelled.

But above and beyond these rather trite and obvious remarks, it is certain from principles of aesthetic economy that the numerals at the foot of all the pages preceding this page cannot be empty of meaning; since as we have just seen they do not mean numbers, we may infer that they mean something else. Hence we have shown that the numerals have meanings other than their numerical referents: thus establishing the bases of the science of *numerology*. This is of course a very interesting remark indeed, and having turned aside for a moment to make it, we may now smugly return to the main subject at hand.

QUESTION 10. Whether the ordinary method of numeration, used by book publishers and other normal people for writing numbers, makes sense?

Objection 1. The question is irrelevant to a mathematical discussion, and does not belong in a book of the present sort. Publishers and grocers, and other simple folk of that ilk, are free to follow the rules of their respective callings. They may decorate their books and vegetables with whatever arcane symbols they find suitable, whether from reasons of custom, cunning, or artistic satisfaction. Such decorations may quite legitimately include strings of denary digits, like 666 (a number used to designate certain pages in long books, and certain beasts). The approval of the mathematical societies is neither demanded nor required.

Reply to Objection 1. In and of itself, of course, the use of digits and the paraphernalia of numeration is not proscribed, and is open to any class of people, no matter how base. Occasionally, however, such uses have become connected by custom in the popular mind with notions of a quasi-mathematical nature; such as, that a book containing a page 666 must be a longish book, as books go; or that if a revue advertises '28 beautiful girls 28' there is available a moderately wide sample of feminine pulchritude. Now these uses of numbers are so exceedingly various that only in mathematics can we hope to find any underlying system explaining all at once the common idea present in each of the ramifications of applied numeration—vegetable shop numeration, strip-show numeration, lottery numeration, etc.

Objection 2. Let us suppose, then, that the question is admissible, and that one may legitimately ask if there is sense in the ordinary way of writing numbers by means of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, all combined in artful arrays to form figures, such as 343, 78, or 5629. Then the overwhelming weight of the total experience of civilisation, from the earliest days right up to the present, is that it does indeed make perfect sense. Otherwise, everything would have broken down long ago.

Reply to Objection 2. This most absurd idea is unlikely to tempt any but the rank mathematical outsider. The whole question clearly turns on the age of civilisation. Now just here we must be ever so very careful, as this is a mathematical discussion, and one or two readers may not be expert palaeontologists. But the author has asked around among his eminent palaeontological acquaintances, and almost all of them agree that civilisation had a beginning; that there used to be a time when there was no civilisation. There was even a time when there was nobody around that counted. Now, in a discussion of this kind, we cannot just come straight out with a non-mathematical statement, like, 'People have only been able to count for a finite length of time, and before that time there were no people able to count (or no people at all, or no universe and no time for them to count in, etc.)' because that is beyond our competence. We are not supposed to know about all that. But the fact that lots of people *do* say all that, and not just people, either, but people who are supposed to know, does at least put the burden on the other fellow to show that it is *not* so. So we may fairly *assume* that people have so far only had a finite length of time to do their counting in. That will be one of the cardinal points of our argument. Another cardinal point, which I think you will find the most eminent psychologists, graphologists, and linguists will grant, is that it takes a certain minimum length of time to say, write, or think a number. Finally, physicists will admit that it is quite customary to use the real numbers as a mathematical model for the time axis. Since the reals obey the archimedean property, one is led to the conclusion that *only finitely many numbers exist that have ever actually been considered in terms of a digital representation* and hence there is a biggest such number. Now of course the above considerations come immediately to the mind of a mathematician, and he does not need to be reminded of them. (It should be noted, of course, that even if civilisation, including the use of numbers, has always existed it does not *necessarily* follow that all the numbers have been written and all the arithmetic problems done. Some of them could still have been left out, by accident as it were.) Nor does he deny that for practical purposes the experience of the ages provides reasonable grounds for caballists and stratarithmeticians to continue their work without the necessity of a theoretical study of mathematics. Mathematically, however, this is no indication that the idea of digital numeration makes sense. 'Probability and sensible proof, may well serve in things naturall, and is commendable: In Mathematicall reasonings, a probable Argument, is nothing regarded: nor yet the testimony of sens, any whit credited: But onely a perfect demonstration, of truths certain, necessary, and invincible: universally and necessarily concluded: is allowed as sufficient for an Argument exactly and purely Mathematicall.' [27]

Objection 3. Granted that experience is not a sure and certain guide in high theoretical matters. It may still be maintained on purely mathematical grounds that the digital system of numeration is mathematically correct, and far preferable to the effete intellectual rubbish often set forth in pretentious articles and lectures, and in the present preposterous book. If a numeral is given as a string of digits, and a large number of people are each asked to add 1 to it so as to get the next number, they will usually all get the same answer. This shows that they have been taught a trick for adding 1 to any string of digits; and if they have been taught a trick then such a trick must exist. This means they have got a counting system.

Reply to Objection 3. What this argument shows is that the ordinary system of arithmetic must involve some kind of counting system. What it does not show is that it is a universal counting system. It could be, for instance, that if you start counting at 0, and keep going long enough, you will find yourself going round in circles, coming back to the same old numbers that you have already counted before. If you take a public opinion poll on this subject, only a small percentage of the people asked will tell you that this is at all likely; most will say they never heard of such a ridiculous idea, if you can get them to understand the idea at all. A large number will suppose that you are proselytising for a new Californian religion, or selling encyclopedias. But public opinion is almost always wrong. Admittedly, no-one has ever actually had the experience of counting until he came back to the same numbers again, and recorded the fact. But this might be because nobody ever went on long enough. To be absolutely sure, one would have to do the counting on paper, which is laborious. Then again, has anyone actually counted up to

197493827593452987248674329878756545785789456345438753286392 7497856?

If not, how do we know that in the system that is in popular use this string of digits ever gets used? If there are any extra strings of digits that do not in fact come into the counting when we start at 0 but are left out no matter how long we go on trying to reach them, then the system fails to be a universal counting system.

Moreover, even to say what a string of digits is one needs N_0 since the strings of digits are the elements of

$$\{0\} \cup \bigcup_{n \in N_0} \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}^n \times \{1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

Objection 4. Perhaps it must be admitted that the ordinary numbers written with digits have their weak points from a theoretical point of view. On the other hand, the natural numbers, being an abstract system, have a disadvantage or two from a practical point of view. As an example, take the numbering of pages in books. Everybody knows how to number them in the good old everyday system that all the publishers use. How does one number pages mathematically?

Reply to Objection 4. It will be seen below that numbering things is just as hard, or just as easy, whether you do it with the natural numbers or with the numerals of commerce—except that if a book is too long there may be technical difficulties in getting the numerals on the pages towards the end of the book. There would not necessarily be room to print the numerals on the pages. This difficulty does not occur with natural numbers, which need not be printed on the pages at all. For this reason the first printed books and many old manuscripts were numbered with natural numbers, causing certain ignorant modern philologists to suppose that the pages were not numbered at all; which is quite false and merely shows the decline in abstract knowledge that has accompanied the spread of mere arithmetical technique.

Nevertheless, the problem of numbering the pages in a book is a very hard one indeed, and is treated in another place. The point here is merely that the difficulty does not depend on using, or not using, digits.

I ANSWER THAT the system of decimal, or denary, numeration† is quite all right. Mathematically it is sound, and the difficulty of explaining how it works, which *a priori* would make it seem unlikely that many people could be taught to use it, may safely be disregarded since (rather amazingly) people do seem to be able to pick it up somehow, and even to do arithmetic in it.

A *string of digits* has already been defined in reply to Objection 3. It is more convenient to write numbers backwards, and of course we shall find it helpful to use the standard notation for an element of the Cartesian product. Thus instead of 1975 we shall write (5, 7, 9, 1). Note that empty strings like( ) have been excluded.‡ It will now be explained how to get $s + 1$, where s is a string of digits; in other words the successor function will be defined. If s is a string, it is useful to augment s by tacking on a 0 at the end (in ordinary terms, the beginning): if $s = (s_0, s_1, \ldots, s_{n-1})$ then s is replaced by $s' = (s_0, s_1, \ldots, s_{n-1}, 0) = (s'_0, s'_1, \ldots, s'_{n-1}, s'_n)$.

We then define $s + 1$ as follows: let k be the first natural number such that $0 \leqslant k \leqslant n$ and such that $s'_k \neq 9$. If $0 \leqslant j < k$ replace

† Also called decadic [28].

‡ See p. 158.

s'_j by $0 = t'_j$; and replace s'_k by $s_k + 1 = t'_k$. If now $t'_n = 0$, remove t'_n to get

$$s + 1 = t = (t_0, t_n, \ldots, t_{n-1}) = (t'_0, t'_1, \ldots, t'_{n1}).$$

Otherwise take

$$s + 1 = t = t'.$$

It is clear that we have defined a function from the strings of digits to themselves. In order to show that this system is mathematically sound, we must establish the universal property. To do so it is necessary to map the denary numerals to the naturals homomorphically. The map is of course

$$s \mapsto \sum_{j=0}^{n-1} s_j X^j,$$

where X is $9 + 1$ (the natural number). We shall be able to establish that this is a homomorphism if we can show that the equation

$$\sum_{j=0}^{n-1} 9X^j + 1 = X^n$$

holds for natural numbers.

But this is equivalent to the polynomial identity

$$x^n - 1 = (x - 1) \sum_{j=0}^{n-1} x$$

evaluated at $x = X$. The remaining details of the proof are trivial. Since the homomorphism just given is obviously surjective, it need only be shown to be injective.

From the above discussion it is now clear that every number can be written in denary notation; indeed, in only one way. Since

$$\begin{aligned}\sum_{j=0}^{n-1} s_j X^j &= s_0 + X \sum_{j=1}^{n-1} s_j X^{j-1} \\ &= s_0 + X \sum_{j=0}^{n-2} s_{j+1} X^j,\end{aligned}$$

we see that it is *possible* to divide by X. If X is any natural number bigger than 1, and if we replace $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ by $\{n \in N_0 : n < X\}$ then all the above arguments apply, and it is possible to divide by X. By this we mean, of course, division with quotient and remainder.

It is not at all obvious why people are so fond of writing numbers to base ten; in other words, why do they not use some other system such as the binary or the duodecimal base? One explanation is that in former times, when such customs became fixed, people were not so terribly broad-minded and tolerant as they have lately become. Those of us lucky enough to have twelve fingers, instead of being praised for our cleverness and admired for our piano playing, were shunned—despised—rejected. We became pariahs and outcasts. No one would give us so much as the time of day. Now it was we, the twelve-fingered supermen, who naturally counted by the duodecimal system. When we wrote the numbers, they began

1, 2, 3, 4, 5, 6, 7, 8, 9, X, E, 10, 11, . . .

(Note that the last few are commonly pronounced 'ten, eleven, twelve, thirteen' by you starfish—as we call you among ourselves; and you write them differently.) Only in the last few decades—or duodecades, it is roughly the same thing—have we been able to lift the veil with any degree of safety. Now we are beginning to come out into the open, and the more broad-minded of you starfish are beginning to admit that we are infinitely superior to you, and deserve to take over. By the end of the century it is my own private opinion, which I seldom dare to reveal publicly, that those of you who do not knuckle under will be ruthlessly exterminated. We have already disposed of grosses and grosses of the most virulent decimalists.

If one does accept ten as the base for the expression of numbers in digits, it is by no means immediate that one must use the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 as the digits and not some other set of numbers. In fact, the numbers

$$-4, -3, -2, -1, 0, 1, 2, 3, 4, 5$$

will also do as digits. In some ways, they do better. Let us agree to write Ɩ, ↊, ↋, ᔭ as being more compact than the usual -1, -2, -3, -4. Then the number usually written 171 is $200 - 30 + 1$, and hence in the new system must not be written 171 at all; it must only be written 2↋1.

On the other hand, -171 is $-200 + 30 - 1$; that is, we must not write -171, but only ᄅƐƖ.

The perceptive reader perceives that one may write both positive and negative integers in this system without the encumbrance of minus signs. The multiplication table is easier to learn too. One may do division by the usual tricks, suitably modified, but only by the use of an extra symbol, ϛ.

Here is a worked example, showing how to divide the number ordinarily written 4056 by 3:

```
     14ϛ2 quot.
   3)41ㄣㄣ
     3
     11
     12
      1ㄣ
      1ϛ
       1ㄣ
       1ㄣ
        0 rem.
```

It is now necessary to note that 14ϛ2 = 1352. Note that in dividing by 3, one must not accept 2 as a remainder; the only remainders allowed are Ɩ, 0, and 1. The general rule for an odd divisor is that the modulus of the remainder must be smaller than half the divisor. Leaving for posterity this marvellous simplification and generalisation of the workings of ordinary arithmetic problems,† the author passes on to the more mundane question below.

QUESTION 11. Does plain long division, as it was before all these elegant variations were introduced, give the right answer?

I ANSWER THAT it does. The first step in ordinary long division, as taught in school, says: 'To divide N by d, choose k so that

$$10^{k-1}d \leqslant N < 10^k d.$$

† Posterity will wish to know how to pronounce the new digits; the suggestion that they be called *ruof, eerht, owt, eno* (and *evif*) seems to me to have merit. It is due to L. Sabroc.

Taking $N = n.10^{k-1} + s$ where $0 \leqslant s < 10^{k-1}$, since

$$d \leqslant n + s/10^{k-1} < 10d,$$

and since $s/10^{k-1}$ is less than 1, the division of n by d gives a quotient q and remainder r such that q is a digit and $r < d$. Take q as the first digit of the answer, and continue the problem by dividing d into $r.10^{k-1} + s$.' That is how one is told to do it; and by following these instructions (which may sometimes be couched in slightly different language) we have quite generally got good results. But is it mathematically correct? Certainly—the mathematically inclined reader has probably found the proof by now, but for the record one remarks that $r.10^{k-1} + s < d.10^{k-1}$, so that induction on k is available.

QUESTION 12. Whether it is true that all numbers that will divide exactly by 2 end in an even figure, 0, 2, 4, 6, 8?

There are numerous possible objections: a number has two ends, the left-hand end and the right-hand end. In a Hebrew book the numbers are still written the right way around even though the writing in the book is written backwards (and is in a foreign language). Not only that, but the book itself begins at the end, not at the beginning. Perhaps one should say that the number *either* begins *or* ends in an even figure, depending on your point of view? Then there is the whole question of numbers written to base eleven, and such. One mode of proof is the following (the objections are left for the reader to dispose of as he sees fit). Every even number $2a$ is a zero-divisor modulo ten. By inspection, the only zero-divisors among residue classes modulo ten are 0, 2, 4, 5, 6, 8. Thus we have shown that every even number either ends in one of the digits we require it to end in, or else ends in 5. We must now show that no even number actually ends in 5. If $2a$ ends in 5 then of course so does $10a$, since 5 is idempotent modulo 10. But $10a$ ends in 0. Hence no even number ends in 5, and we are done.

7. Casting out nines

Enneekbole, or the casting out of nines, is an ancient and interesting method of checking or verifying the accuracy of an arithmetical computation. Any addition, subtraction, or multiplication can be checked by the method of casting out nines. That is, it is certainly feasible to check the problem in this way. It was customary among the ancient Romans to check up on the correctness of their decisions in difficult matters by the method of inspecting the insides of birds, to see what they had been eating. It would be feasible to try to find out if sums had been worked correctly by opening up chickens. Occasionally, in wild and woolly colonial places, if the general storekeeper demanded payment in excess of what one suspected he was reasonably entitled to, one might test his sincerity by purchasing two hunting knives and offering him one of them. This might be called checking an arithmetic problem by the method of opening up general storekeepers. Although both of these methods have had their day and for all we know may or may not become very fashionable again, it would be expecting too much to demand a mathematical proof of their effectiveness; too little is known, mathematically speaking, about the inward parts of birds and merchants. Can the same be said of the method of casting out of nines?

Just what is the casting out of nines? Let us consider the multiplication

$$14 \times 11 = 154.$$

To cast the nines out of 14, we add all the digits of the number, obtaining $1 + 4 = 5$. Similarly for 11 we get $1 + 1 = 2$. Casting nines out of 154, we see that $5 + 4 = 9$, so we cast out the 5 and the 4 (sending them, if you will, into a sort of *limbus enneadum*). This leaves only the 1. Now instead of the problem we started with we have got the simpler multiplication

$$5 \times 2 = 1.$$

Since 5×2 is known to be 10, and since $1 + 0$ is indeed 1, we see that the original working was correct, and that indeed

$$14 \times 11 = 154.$$

(It should be remembered that our present purpose is not to defend the method of casting out nines, but merely to show how it is done.) To make perfectly sure the method is understood, let us contemplate another worked example:

$$\begin{array}{r} 3991 \\ \times\ 19 \\ \hline 75829 \end{array}$$

By casting out the two nines, the number 3991 becomes $3 + 1 = 4$. Similarly, 19 becomes 1. Casting out the nine, the answer 75829 becomes $7 + 5 + 8 + 2 = 22$, and by casting nines out of this it becomes 4. Since the problem has reduced to $4 \times 1 = 4$, it must be that the answer was correct.

Here then is the general rule for checking any arithmetic problem by the method of casting out nines: To cast nines out of a number, be thorough. First cast out all the nines you can see—any nines that occur as digits, and any sets of digits that add up to nine. Now add up the digits of the number. But do not stop there, if you can help it. If the new number so obtained has any nines in it, cast them out. If there remain more than one digit, add up the digits. Continue in this way, by repeated application of casting nines and summing digits, until all that is left of the number is a single digit, which will not be 9 but may be 0. Now do this to every single number that occurs in the problem. Finally, do the problem all over again, but using the new numbers that you have obtained as replacements for the old numbers by the method of casting out all the nines in the old numbers. When you have got the answer to the new problem that you have done with the new numbers, cast the nines out of it, too. Compare this with the number that was obtained from the original answer by the casting out of nines. They should be the same.

No-one has ever seriously asserted that by the casting out of nines you will necessarily detect any error you might care to make in arithmetic. For instance, 17×17 reduces to 8×8 or 64, and this gives $6 + 4$ which is 10, which gives 1. If someone, by some slip, obtained the answer

$$\begin{array}{r} 17 \\ \times\ 17 \\ \hline 100 \end{array}$$

then casting nines out of 100 gives 1 as well, and the method fails to detect the error. It is not a perfect method for detecting any error, no matter how gross. But there is the question of the accuracy of the procedure when it seems to have detected an error.

QUESTION 13. Whether, when a problem has been checked by the method of casting out nines, and the two answers disagree, the problem is necessarily wrong?

I ANSWER THAT the method is completely accurate. The cyclic group of order 9, which we write Z_9, can be taken as the set of natural numbers $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ with addition modulo 9. If n is any natural number, write $S(n)$ for the digit sum of n, taken according to the rules mentioned above—in other words, $S(n)$ is n with the nines cast out. One considers S as a map $N_0 \longrightarrow Z_9$, and it is easy to see that

$$S(n + 1) = S(n) + 1;$$

of course the $+$ and the 1 on the right-hand side mean something quite different from the $+$ and the 1 on the left-hand (or sinister) side. Hence by induction, the map S is a homomorphism. Since $S(1) = 1$, it is the standard homomorphism from N_0 to Z_9, which extends to the standard homomorphism from Z to Z_9. Since the latter is a ring homomorphism, not only do we have

$$S(m + n) = S(m) + S(n)$$

but also

$$S(mn) = S(m)\, S(n).$$

Note that the first equation above could have been proved directly, if we had had at our disposal the methods of ordinary school addition, with carrying and all the rest of it. This subject has been excluded from the present work, as being somewhat too abstruse for treatment in an elementary work of this kind.

How effective is the method of *enneekbole*? If an arbitrary wrong answer is given to an arithmetic problem, the casting out of nines will show up the error about 8/9ths of the time—in the sense that the proportion p_N of wrong answers such that $1 \leqslant A \leqslant N$ that are shown up, taken among all wrong answers, obeys

$$\forall_{\varepsilon>0} \; \exists_{M \in N_0} \ni \forall_{N \in N_0} \; (N > M \Rightarrow |p_N - 8/9| < \varepsilon).$$

From this it would appear to be obvious that to check an arithmetical problem, one ought to look merely at the last digit, and see if that is right. The reason for preferring this very simple method to casting out nines is not only its greater simplicity, but also the fact that it gives a greater chance of detecting an error: 9/10 as opposed to 8/9. There may be special cases where the casting out of nines is to be preferred, however. In the calculations just made, it was tacitly assumed that the worker of the problem was just guessing. Although this is not impossible, it is more likely that he has learned a few tricks about working arithmetic problems. One of the most useful tricks for faking up answers to arithmetic problems is to be sure and get the first and last digits of the answer correct. This succeeds in giving the appearance of a correct computation, without requiring much effort; and anyone poking his nose over your shoulder will probably not go further than checking the last digit of the answer. To counter the fact that the problem-solver has probably thought such thoughts as these, the person checking the problem will be on his guard, and may find the method of casting out nines to be of use. Faking up a wrong answer so that it passes the nines test is so laborious that it is often easier simply to do the sum honestly.

For problems involving rather big numbers, the method of casting out *ninety-nines* is useful; it is harder to do than the nines test, but it exposes all but 1·010101 . . . % of the wrong answers. To check

$$\begin{array}{r} 54679 \\ \times\ 2604 \\ \hline 142384116, \end{array}$$

we take $5 + 46 + 79 = 130$, then $1 + 30 = 31$; and $26 + 04 = 30$. The product of 31 and 30 is 930, which gives $9 + 30 = 39$. The answer 142384116 gives $1 + 42 + 38 + 41 + 16 = 138$, then $1 + 38 = 39$. These agree, and so it is rather likely that the answer given above is correct. It need hardly be noted that one may verify the correctness of a calculation by casting out 999s, 9999s, etc., and that these methods are successively more accurate. If the method of casting out 9999999999s passes on the answer given to the problem above, then it is absolutely certain that the answer is correct—except, of course, that an error may have been made in doing the problem by the method of casting out 9999999999s. This is quite possible, since doing the problem by that method amounts, in the case under consideration, to just doing the same problem all over again.

2
FACTORS AND FRACTIONS

1. Prime numbers

It is commonly stated that *prime numbers* have no factors except themselves and 1; see, e.g., [29]. This notion is erroneous for two reasons; one, that a prime number has four factors—if p is prime, then p is a multiple of p, of $-p$, of 1, and of -1. That is, we can find a number x such that

$$p = p.x,$$

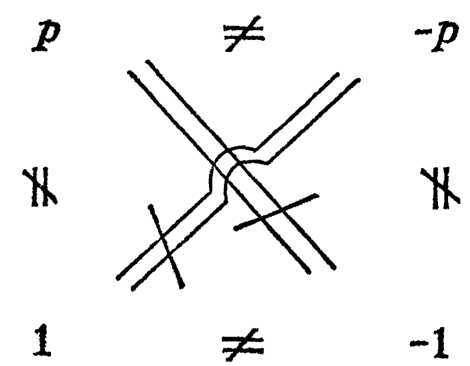
or such that

$$p = (-p).x,$$
$$p = 1.x,$$

or even

$$p = (-1).x.$$

The same number cannot be used in each case, but there is always such a number x that the equation is true, no matter which of the four equations you are trying to solve. Note that this is true even if p is not prime. Moreover, unless p is 0, the two numbers p and $-p$ are actually distinct: $p \neq -p$. (If $p = 0$, the fact that p has four factors is even more obvious simply by solving $0 = n.x$, which can be done for any value of n whatever—and in this case one *can* use the same solution for every value of n.) There is only one exception to the rule that every number has four factors at least: this exception is the number 1—and of course its associate -1. This is because in order to show that the four factors p, $-p$, 1, -1 are distinct we must have six 'inequations':



The cases $p \neq -p$ and $1 \neq -1$ have already been considered (it was shown above that $1 \neq 0$), and the other four can easily be reduced to two:

$$p \neq 1$$

and

$$p \neq -1.$$

Unfortunately, if these do not hold, in other words if p is 1 or -1, then p has only two factors, since p, $-p$, 1, -1 are really only two distinct numbers. The only number, then, that has no factors except itself and 1 is -1. According to one standard work, which is very popular, I regret to say, among schoolmasters, there is only one single prime number. That is, of course, a very serious error. There are lots of prime numbers. In fact the work cited mentions in the very next sentence (I quote) 'For example, 7 is a prime number, . . .'. This is a direct contradiction of the definition, since the factors of 7 are 7, -7, 1, -1, all of which are distinct. (This means that none of the four numbers is the same as any other.)

How is it possible for a writer of such distinction to make such an elementary mistake as to define prime numbers incorrectly? Very likely, what is involved here is not a serious misconception of what prime numbers are—though that is the first thought that strikes one's mind on reading this absurd definition—but the mere omission, quite possibly by the typesetter and not the author, of some qualifying phrase. Such a phrase would be: 'where associates are of course to be identified'; or equally well, 'it goes without saying that if $a = ub$, where u is a unit, then we must count a and b as the same factor'. Since $\{1, -1\}$ is the group of units in the ring of integers, we find that the definition quoted above becomes more nearly satisfactory once this qualifying phrase has been restored to it. The objection remains that in the book to which we refer, no mention at all is made of what ring we are dealing with. And of course the number 7 is prime only if we consider it as an element of a suitable ring. If it is the ring of integers that is involved—since the reader is forced to make a guess, this is perhaps his shrewdest guess—then 7 is a prime. If one is in the field of rationals, on the other hand, there are no primes.

But we are not out of the water yet. Let us look again at the definition of prime number in this text: '**Prime numbers** have no factors except themselves and 1.' This phrase occurs with the words 'Prime numbers' in bold-face type in a list of similar phrases headed 'DEFINITIONS, OR WHAT THEY MEAN'. Hence we have no choice but to take it as a definition in the fullest sense of the term; it must mean not only that prime numbers have this property, but that numbers having this property are prime numbers. Now as a matter of fact, the number 1 has no other factors except itself and 1; which is only another way of saying that the number 1 has no other factors except the number 1. (It also has as a factor the number -1, but this is a unit times 1, and so for this purpose is not to be distinguished from the number 1; remember that this was the proviso that the typesetter forgot to include.) Hence, according to the definition, the number 1 is a prime. But that is not the case! The number 1 is not prime. There is no way of escaping the responsibility for this error, and the reference [29] must be condemned as a pretty awful book.

Let us turn now to the work [30] of another famous author, one of almost equal distinction whose works also used to be popular in the schools. They are now not so popular there as the previous reference. Here, numbers are considered as being obtained by the addition of *units*; it is not certain, by the way, that the word 'unit' as used by this author has its ring-theoretic meaning. Some of the terminology is rather archaic. But it is observed that some numbers are obtained also by the addition of a quantity that is not a unit. Thus, 10 is obtained by the addition of twos:

$$10 = 2 + 2 + 2 + 2 + 2,$$

as well as by the addition of fives:

$$10 = 5 + 5.$$

A prime number is then defined as a number that can only be obtained by the addition of units. Note that it is not considered fair to say that 7 can be obtained in different ways; one might wish to say that we have both

$$7 = 1 + 1 + 1 + 1 + 1 + 1 + 1$$

and

$$7 = 7,$$

and we might say that the second equation shows 7 as obtained by the addition of sevens, in fact by the adding up of a string of sevens so short that it consists of only one seven. But Euclid (the author of the work in question) does not go along with that. That is not allowed as an addition. So since 7 can only be obtained by the addition of units, it is a prime. Note that, for some strange reason, the unit is not considered to be a number at all. Since 1 is not a number, the question does not arise in this theory whether 1 is prime or not. (Even if it did arise, it is not true that 1 can only be obtained by the addition of units; it cannot be obtained by addition at all.) The other question, that of identifying associates, is easily settled: these were always identified by the ancient mathematicians, to the extent that they never bothered with negative numbers at all.

A correct definition of prime numbers, then, and one that does not require the absorption of ancient modes of thought, is this: We require first of all to know what is a *factor* of a number. If two numbers can be multiplied together to give a third number, each of the two numbers is a *factor* of the third. A prime number, then, is a number that has exactly two factors; for this purpose, we consider two numbers to be the same if one of them is the product of the other times a unit. (A *unit* is a number that has an inverse in the ring under consideration; thus, -1 is a unit in the ring of integers, since $(-1).(-1) = 1$, while 509 is not a unit, since $1/509$ is not an integer.) The two factors of a prime number are necessarily the number and 1.

The number 1 is not prime, since any factor of 1 is *ipso facto* a unit. The number

$$4{,}294{,}967{,}297$$

is obviously not prime, since it is $641 \times 6{,}700{,}417$; on the other hand

$$170{,}141{,}183{,}460{,}469{,}231{,}731{,}687{,}303{,}715{,}884{,}105{,}727$$

is said to be prime. It is often stated that the numbers 2, 3, 5, 7, 11, 13 are prime. Let us consider the first case.

QUESTION 14. Whether 2 is a prime number?

Objection 1. It is clear that 2 cannot be a prime number. Anyone looking at the list of primes above will see that 2 sticks out like a sore thumb. All the other numbers in the list are odd; of all these numbers, only 2 is even. Clearly, the number 2 does not belong in the list. As a matter of fact, if you will take the trouble to find longer lists of prime numbers going up far beyond 13, and even including all the prime numbers that have ever been discovered, you will not find even one more even number among them. This makes it all but certain that 2 is not prime.

Reply to Objection 1. This method of argument is common among the increasing class of people who have been subjected to intelligence tests. These tests often include questions in which the examinee is asked to choose the object among a given class of objects that does not belong, because it is different from all the others. For instance, one may be shown an amoeba, a picture of John Bull, a hatter, a diagram of a planet moving in an orbit according to the law of epicycles, and a wine-bottle. The answer is clearly that the amoeba is the odd man out, because all the others share the property of being eccentric (or in the case of the wine-bottle, of causing eccentric behaviour), whereas the amoeba is amorphous and so cannot have a well-defined eccentricity. Or else, the amoeba is out because any of the others is suitable for framing and hanging on the wall, whereas this is not commonly done with amoebae. Or because amoebae are microscopic. At any rate, we are all agreed (if we are intelligent) that the amoeba is the sore thumb. We mark box A and hastily proceed to the next exercise in discernment.

Mathematical arguments, however, must be independent of discernment. The most insensitive person must be able to follow them. The reason why 2 is the only even prime is that every other even number is divisible by 2, as well as by itself and 1.

Objection 2. People have been contemplating the number 2 for a very long time. Either you must accept the fact that nobody has ever been able to break this number up into factors, and hence you must admit that it is prime, or else you must say that the problem has never resulted in a definite solution and that very likely no one will ever know if 2 is prime.

Reply to Objection 2. Mathematically, it is indeed inconclusive that nobody has ever succeeded in finding any numbers that divide 2 exactly, other than the numbers 1 and 2 which obviously do this; this does not at all show that no such numbers exist. On the other hand, it may be quite easy to show that there are no such numbers. In mathematics, if one approach does not work, one must always try another; according to the principle *catus multifariam deglubitur.*

I ANSWER THAT the number 2 is prime. It is easily seen that the only numbers between 0 and 2, including 0 but excluding 2, are 0 and 1. Thus the remainder left by any number on division by 2 is either 0 or 1. Hence the quotient ring

$$Z/2Z,$$

where $2Z$ is the ideal in Z generated by 2, has only the elements [0] and [1], where these are the images of 0 and 1 under the canonical quotient map. Since [1] must be the unit of this ring, every element of this ring except [0] is a unit, and the ring is a field. As such, it has no zero-divisors other than [0]. But looking back now at Z, this shows that if $ab = 2$, then one of a, b is an element of $2Z$; i.e., is an even number. In other words, we have either $a = 2m$ or $b = 2m$; say the former. By cancellation, $mb = 1$, so that both m and b are units. (Or one may argue that since $Z/2Z$ is a field, the ideal $2Z$ is maximal in Z, and hence prime, which implies that the generator is prime.)

The fact that the number 2 is prime is useful in many ways. It prevents us, for instance, from indulging in the time-wasting habit, to which the ignorant are so prone, of attempting to discover a proper factor for the number 2. It shows that it would be very unwise to program a computer to keep on trying numbers one after the other until the computer should have found a divisor of 2 distinct from 1 and 2. It saves the trouble of looking for a rectangle with integral sides and area 2.

Prime numbers have the most various applications to the world of everyday life; I mention here but a few of them. One application known to most of us is to the carving of roast beef, and is said to be the discovery of Gnivri, the steward in the feasting hall of Stoattr the Fat. Gnivri had been on raids in the Mediterranean in his younger days, and had been found to be a handy fellow in the pillage of libraries. Before decapitating the head librarian, he would usually strike up a conversation on general topics—like most of the better sort of Vikings, Gnivri was quite a pleasant man, really, and enjoyed a good chat; besides, it helped to break the monotony of chop, chop, chop. 'Vell,' Gnivri would say, 'it loaks as if the library is goaing to need a nue head, hoa, hoa, hoa. Yoost tenk vot a lot off stoof you moost haff lernt ven you voss reading oal dem boaks! Vy doan't you teach it to may, hoa, hoa? Vot iss oal dett prame nomberce naunsanse?' Then he would listen for a while to a diverting lecture on number theory, or whatever the head librarian's speciality might be, occasionally breaking in with an appreciative word or to ask an intelligent question, such as 'Iss dett soa?' It was from these little diversions, these fleeting contacts with civilised learning, that Gnivri picked up his bits and pieces of knowledge, including the first few prime numbers. 'Tvoa, tray, femm, whew, ell vah!, . . .' he would mutter. Sometimes, as he was arranging the skulls of slain enemies in a row to pick out the best ones for drinking cups, he would pick out the prime ones first. Later, it was Gnivri, by now retired from active service as a steward, who made the famous culinary discovery. Today, if you go into any expensive restaurant worthy of the name, you will see on the menu 'prime ribs of beef'. The restaurant that advertises composite ribs of beef is unheard of; but it is well to remember your number theory if you have the slightest reason to suspect the honesty of the establishment or the reliability of the chef's knowledge. 'I say, my good man,' you might wish to ask him —but in French this is 'Je dis, mon bon homme'—'Just how many prime ribs are there on a beef?' It is surprising how many chefs think the number one rib is prime.

2. Finding prime factors of a number

If a factor of a number is prime, then it is a prime factor of the number. For example, $28 = 7 \times 4$, and so both 7 and 4 are factors of 28, but 4 is not prime, so it is not a prime factor of 28. However, the number 7 is prime, so it is a prime factor.

QUESTION 15. Whether every number other than ± 1 has a prime factor?

This question is of great practical interest. If the answer is no, then when the mathematics master in a school sets the children some numbers to find the prime factors of as homework, they (or their parents, whichever one does the homework) can bring up the easy retort, 'Perhaps there are no prime factors. Some numbers haven't got any, you know. Hadn't you better show us the existence of these things you are asking us to find?' An organised use of this line may be an unfair trick, but it is guaranteed to produce a diversion. The maths master is not allowed to crack the whip these days; he is expected to encourage an enquiring spirit. This makes it easy to disrupt the maths class. On the other hand, the answer may be yes; perhaps we are about to find that all numbers except 1 and -1 have got prime factors. This too can be turned to good account. There are two main approaches, and they should be used separately or successively, not both at once. The first approach is to take an eager-to-learn attitude. 'Oh, Mr Samwise, I say! These prime numbers are terribly interesting. Fantastic! Gosh, sir, is it really true that every number has got prime factors?' The important thing, after this rather soft, gooey beginning, is to hang on like the British bulldog. Insist on a rigorous proof. If you have any suspicion that the master is trying to sneak through a simplified explanation or an argument that is merely plausible, the right thing to do would be to gently hint that somewhat more is expected of him, perhaps in such words as these: 'Oh, wonderful! I almost think I see. Could you be a *bit* more explicit?' If this sort of tactic ever results in a real, rigorous proof, it will unfortunately be necessary to show commensurate gratitude; however, the time wasted and disruption caused are almost certain to be worth it. The second approach, in case the answer is that numbers have got prime factors, is to use this as leverage for not finding them. What is the point of verifying case after case of a general proposition that has already been established in full generality? If we know we can always do it, why bother?

Objection 1. It is very simple to see that a number bigger than 1 has got prime factors. If it is a prime number, then of course it is a factor of itself, since 1 times any number is the number again. If it is not a prime number, then it breaks into two or more factors.

Reply to Objection 1. It is in the second case that this argument does not pay sufficient attention to the context. A number might have lots of factors, and none of these factors need necessarily be prime. In such a case, the factors themselves would need to have lots of factors, and none of these factors would be prime, either. It is necessary to find some way of ascertaining that this is not what happens.

I ANSWER THAT every number other than 1 and -1 has indeed got a prime factor. This is obvious in the case of even numbers, because we already know that 2 is a prime. But the proof will not make use of this fact. (Try to find some other facts that the proof does not make use of; after you read the proof, try again!) Since the number n in question is not a unit, the set of its multiples

$$\mathfrak{a} = \{xn\colon\ x \in Z\}$$

is not all of Z. Consider the class $\mathscr{S}$ of all proper ideals of Z containing $\mathfrak{a}$ as a subset; the set-inclusion relation $\subset$ makes $\mathscr{S}$ a partially ordered set. Now consider any subclass of $\mathscr{S}$ with the property that if $\mathfrak{b}, \mathfrak{c} \in \mathscr{C}$ then either $\mathfrak{b} \subset \mathfrak{c}$ or $\mathfrak{c} \subset \mathfrak{b}$; the union of $\mathscr{C}$ is trivially a proper ideal of Z containing as a subset every ideal of $\mathscr{C}$ and also containing $\mathfrak{a}$ as a subset. By Zorn's Lemma, a proper ideal $\mathfrak{m}$ of Z exists that is maximal with respect to the property of being a proper ideal of Z containing $\mathfrak{a}$ as a subset. Hence $\mathfrak{m}$ is maximal with respect to the property of being a proper ideal; and hence is a prime ideal.

Now, in the ring Z every ideal has a generator. (This is explained below under greatest common divisors.) The generator of a prime ideal is prime; since n is in this ideal, we are done. It should be noted that the existence of the maximal ideal mentioned above need not be referred to Zorn's Lemma, for in the particular case of Z one may rely on the fact that Z is Noetherian, and hence every class of ideals has a maximal element. This again follows from the fact that every ideal of integers has a generator. Zorn's Lemma, of course, is an axiom of set theory, equivalent to the Axiom of Choice. The Axiom of Choice is often explained in terms of drawers full of socks. Many mathematicians do not like mentioning or even alluding to socks, drawers, knickers, and such unmentionables in their books and lectures, and this accounts for some of their shyness in using the Axiom of Choice and related ideas. They try to get round it somehow. In the same way, many ladies of delicate feelings refer to the part of the leg of a roast chicken (or other bird that is carved at the table) that is not the drumstick—the piece that connects the drumstick to the chicken—as the *second joint* or the *upper pulkeh*. Thus they avoid saying 'thigh'. In Arkansas, one does not say 'privy' or 'outhouse', but 'State Capitol'. In this age of miniskirts, it is perhaps charming that older customs of reticence have been preserved in specialised areas—mathematics, carving, politics, to name a few; but the march of progress moves inexorably forward, and these refinements can only be regarded as frail relics of the past. The days when one could get up from the dinner-table and leave the room at the mention of the Axiom of Choice are gone with the wind, and we must accustom ourselves to the new ways, the ways of tomorrow.

3. The greatest common divisor

The term 'greatest common divisor' is easily explained. This is a noun phrase, and at the head we find the noun 'divisor'. This means the same thing as 'factor'. Having said this, it is very important to point out that these words have several meanings in popular usage, and naturally as technical terms they must be treated with care. Not just any of the popular meanings of the words will do. The word 'factor' in English often has the meaning '*Institor, negotiator, procurator negotiorum*; that is, an huckster, a foreman of a shop, one that goes about with linen or woollen cloth'. If one spells the other word 'deviser'—and this would itself be a great mistake—then one might get the idea that the greatest common deviser, or the highest common factor, as it is sometimes expressed, must be the most important man in the House of Commons. This personage is often chosen for his ability to devise, and he is usually accounted the greatest or highest common person in the land. Indeed, divisions are not at all uncommon in the House. It would be a serious mistake to think that the greatest common divisor is the Prime Minister. The greatest common divisor has nothing whatever to do with Parliament or the Cabinet at all; it is erroneous even to think of the Composite Ministers as common divisors. (This is the name sometimes given to the ministers of Her Majesty's Cabinet who are not Prime.)

Furthermore, we must carefully guard against the idea that common factors are in the habit of picking their noses in public, or that they say 'Oy loike poineapple oice,' 'Haiw naiw, braiwn caiw,' or indeed that they are guilty of any vulgarity whatsoever. Any mental picture of a highest common factor drunkenly staggering up the down escalator in a department store on Christmas Eve, or ecstatically chanting 'Hare Krishna' in an awful accent while chewing convolvulus seeds, would be misleading. It is not that kind of 'high' that is meant.

By now it should be clear that the number of misconceptions about greatest common divisors that are possible, while perhaps not precisely infinite, is colossal.

This is why it may be desirable to consult a standard arithmetic book if you want to know what the phrase means. There one is likely to see 'The *highest common factor* or *h.c.f.* is the biggest of the common factors of two or more numbers.' This approximates rather closely to the right idea, but it has some flaws. After all, does it not seem queer to take all the trouble of defining the highest common factor of 'two or more numbers', and yet stop short by not even discussing the highest common factor of a single number, taken all by itself? It is even more ludicrous, though perhaps excusable, to leave unstated what one means by the greatest common divisor of no numbers at all. But these are trifling quibbles; the only change necessary is to make it read, 'The h.c.f. (of a set of numbers) is the biggest of the common factors of the numbers.' This allows the set of numbers to be empty.

A more serious difficulty is with the word 'biggest'. There can be no doubt that this means just what it says. We have no choice, in the context of any known standard arithmetic text, but to interpret one number to be bigger than a second if the second can be subtracted from the first, leaving a positive answer. That is what bigger always means. And a number is the biggest of all the common divisors if it is bigger than every other common divisor. Given all this knowledge, we may ask: what is the highest common factor of the numbers 0 and 0? (There is no rule that says the two numbers may not be the same. If someone insists that they may not be the same, we can ask the same question in a different way: what is the highest common factor of 0?) Now, you see, if you are to apply the rule given in all the books, you have got to take all the divisors of 0; and then look for the biggest. That seems easy. First of all, we shall identify a number with any other number obtained from it by multiplication by a unit. All the arithmetic books do this in this context; it means ignoring negative numbers. We note that 0 is a factor of 0, since $0 = 275 \times 0$. That is one factor of 0 out of the way. Then also, the number 1 is a factor of 0, since $0 = 0 \times 1$. And since $0 = 0 \times 2$, the number 2 is a factor of 0. After a length of time which varies from individual to individual, one may wake up to the fact that every number is a factor of 0. Wonderful! We now can be certain that we have found all the factors of 0. Now to find the biggest. By this we obviously mean a number that is bigger than every other number. Again, it is a variable length of time before people wake up to the fact that there is no such number. If n is the highest common factor of 0 and 0, then $n + 1$ is also a common factor, so n is bigger than $n + 1$. This means that $n + 1$ can be subtracted from n leaving a positive answer. This answer is -1. In fact, it can be shown by trichotomy that -1 is not a positive number. Now the trouble with all this is not, as it would appear, that 0 has not got a highest common factor. The whole trouble is in the use of the word 'biggest'.

And of course it is a most puerile error to say that the highest common factor is the *biggest* of the common factors. It is only the youngest of children who make this silly mistake; they often confuse height, size, age, and so on. If you ask an infant who is the tallest child in the class at school, he is likely to say 'Robert, because he's seven,' or something like that. The young child cannot distinguish between height and age. Older folks like us know that they are different concepts, but often go together. The correct definition is of course this: the highest common factor is the *highest* of the common factors of the numbers. Numbers, you see, are ranked in a hierarchy; some of them are higher than others. Now the highest number in the whole hierarchy is 0. One often wonders why a number with so little apparent character was given this exalted position, but that is not for us to say. It has always been that way, and it would appear unlikely that any change will occur for some time yet to come. But of course, the number 0 is not the biggest nor even the greatest of the numbers; though he is high, he is little.

Now what, exactly, constitutes height among numbers? A number is defined to be *higher than* any of the factors of the number. Now that one has made this change in the definition of the h.c.f., it begins to make sense.

QUESTION 16. Whether two numbers have got a highest common factor?

Objection 1. Before someone came along and made the definition explained above, one had no trouble finding highest common factors. Take the numbers 4 and 8, for example. The only numbers that divide both of them are 1, 2, and 4. Now if anybody gives you a set of numbers, at least if it is a finite set, you can find the biggest of them. In this case, 4 is bigger than 1 or 2. But if someone gives you a finite set of numbers—take, for example, 2 and 3—can you find the highest of them? In the case under consideration (2 and 3) neither of the numbers divides evenly into the other without leaving a remainder. So neither is higher than the other. Then which is the highest? It seems that there is no highest number in this set.

Reply to Objection 1. It is true that a set of numbers need not contain a highest. This sort of thing often occurs in hierarchies. Two elements in a hierarchy (which is only a colourful name, here, for a partially ordered set) need not be comparable; and among a set of objects, there need not be a highest. There is no highest among 2 and 3,† but as a matter of fact there is no set of numbers that has 2 and 3 as its only common divisors, so it does not matter. All that is required is that the set of common divisors of a set of numbers should have a highest element, and as we shall see this is the case.

Objection 2. The normal method of finding highest common factors is to split both numbers into products of primes. Then it is only necessary to multiply all the primes that go into both numbers. For example, $30 = 2 \times 3 \times 5$ and $42 = 2 \times 3 \times 7$, so the h.c.f. is 2×3, or 6. This seems so simple that one ought to be able to devise a proof along these lines.

Reply to Objection 2. It is clear that 6 divides both 30 and 42, but without experiment or further proof, it is not clear that 13 does not. It is inexpedient to attempt to tackle the problem in this way.

I ANSWER THAT any set of numbers (two numbers, or more, or fewer) has a highest common factor. Let x and y be the numbers. It is only necessary to find a common divisor of x and y that can be written in the form $rx + sy$, since any number a that divides exactly into both x and y also divides exactly into $rx + sy$—we have

$$x = ka \;\&\; y = ma \Rightarrow rx + sy = (rk + sm)a.$$

A similar formulation of the problem works for any set of numbers: instead of x and y, we can use x_i, where i runs through a suitable index set. Thus we are looking for a generator for the ideal generated by $\{x_i\}_{i \in I}$. If this ideal is $\{0\}$, the conclusion is obvious; otherwise, the ideal has a least strictly positive element, and this clearly generates.

PROBLEM. To find the h.c.f. of two numbers.

The normal method of doing this is mentioned above. It begins: 'Split each number up into its prime factors, by dividing in turn by the prime numbers.' As we shall see later, there are infinitely many

† Some purists would say, 'no *higher* among 2 and 3', or even perhaps 'between 2 and 3'. Since the number of elements in the set is irrelevant to the argument, it is mathematically better to ignore the fact that there are two; this leaves us with the superlative, which is the general case.

prime numbers, so that no table listing all of them actually exists. Hence it is not a mere algorithm to divide a number in turn by the prime numbers. Only a few of the prime numbers are known, so that one would never know if one had left one out. The question whether in fact numbers can be split up into prime factors, even theoretically, has not yet been discussed here. Neither has the question whether, supposing it is possible in an individual case to split both the numbers into prime factors, the further instructions will really give the highest common factor.

Fortunately, there is no need to despair. If (a, b) is the ideal generated by a and b then $(a, b) = (a - bq, b)$, and this gives us Euclid's algorithm via well-ordering and ordinary division with remainder. Thus to find the h.c.f. of 715 and 413, we write

$$
\begin{aligned}
&715-413=302\\
&\quad 413-302=111\\
&\qquad 302-111.2=80\\
&\qquad\quad 111-80=31\\
&\qquad\qquad 80-31.2=18\\
&\qquad\qquad\quad 31\;-18=13\\
&\qquad\qquad\qquad 18-13=5\\
&\qquad\qquad\qquad\quad 13-5.\,2=3\\
&\qquad\qquad\qquad\qquad 5-3=2\\
&\qquad\qquad\qquad\qquad\quad 3-2=1\\
&\qquad\qquad\qquad\qquad\qquad 2-1.2=0
\end{aligned}
$$

The last number obtained on the right-hand side, not counting 0, is the h.c.f. If both the numbers at the beginning were positive, this method takes a finite number of steps; if the smaller of the numbers has three digits you will be finished in less than twenty steps.

Now consider doing the same problem by the so-called normal method of the school-books. First we must split 715 into primes (we have as yet no assurance that this is possible, but it may be; and we may not need any of the primes we have never heard of, such as 97, or 131,071). Since 715 ends in 5, it is divisible by 5; we find

$$715 = 5\times 143.$$

Now by casting out nines, $143 = 8 \pmod 9$ and hence is not a zero-divisor (mod 9); therefore 3 does not divide 143. Since 143 does not end in 0, 2, 4, 6, or 8, it is not divisible by 2; since it does not end in 0 or 5 it is not divisible by 5. Since 143 leaves remainder $3 \neq 0$ on division by 7, it is not divisible by 7. Trying 11, we find $143 = 11.13$. Hence

$$715 = 5 \times 11 \times 13.$$

Attacking 413 in a similar way, we find that 2, 3, and 5 do not divide 413—for the same reasons mentioned when 143 was under consideration. Then we find that 7 divides 413, and in fact $413 = 7 \times 59$. Now the list of prime numbers that the average user of a school arithmetic book carries in his head is not long enough to take in 59; he will not be sure that 59 is a prime, but he will also not be able to think of a factor other than 59 or 1. If he knows a little elementary number theory, he may know that if 59 is not prime; then since it is not a perfect square (being congruent to 3 modulo 4, which perfect squares are not) it has a prime factor less than its square root. Since $59 < 64$, and $\sqrt{64} = 8$, he will reflect that such a prime will be at most 7. There are four such primes: 2, 3, 5, and 7. By one trick or another, or by actual division in digits, it can be verified that none of these goes into 59 without leaving a remainder. Hence

$$413 = 7 \times 59$$

and the second number has been factored into primes. None of these primes (neither of them, if one has counted and found there are two) is also one of those in the particular factorisation of 715 that happens to have been found, though it would take some work to show that some different way of writing 715 as the product of primes might not contain 7 or 59. The directions in the school-book do not say explicitly what to do in this case, but the teacher will just possibly know, and the right thing to do is this: you must still multiply together all the primes common to both factorisations. Now in this case there are none, so we must multiply together no numbers at all. This is easy if you know how to do it, and the answer is that the product of no numbers at all, multiplied together, is 1. Hence the h.c.f. of 715 and 413 is 1.

QUESTION 17. Whether a number can be split into prime factors in more than one way?

Objection 1. It would seem that any number that is not a power of a prime, or even any number that can be split up into primes otherwise than as a power of a prime, is writable in more than one way as a product of primes. For example, $6 = 2 \times 3 = 3 \times 2$. Here the number 6 is written as the product of primes in two distinct ways.

Reply to Objection 1. This objection is not fair. The question did not mean to consider 2×3 and 3×2 as different ways of writing 6; but by the phrasing of the question, which is really far too vague, we have let ourselves in for this difficulty. This question is very important, as all the children in civilised countries are taught (rightly or wrongly) to find h.c.f.'s and lowest common denominators by assuming that the answer is yes. It is sensible at least to know what the question is that these little souls are assuming the answer to is yes. The example in the objection shows that by a way of writing a number as a product of primes, one does not mean a way of writing it as the product of a *sequence* of primes. Two finite sequences, say (2, 3) and (3, 2), can give the same number. Nor can the idea be that of the product of a set of primes: the number 4 is the product of two 2's, and the number 8 is the product of three 2's, but the set of primes involved is in both cases the same set, namely $\{2\}$. The product of the primes in this set is in both cases the same number, which is neither 4 nor 8 but 2. In fact the only numbers that can be written as the product of a set of primes are those that are *quadratfrei* [31]. As a matter of fact, the correct concept is that of a finite set with finite multiplicities; what is asserted here is that if

$$\mathbf{L}^+(P, N_0)$$

is the set of functions $f\colon P \to N_0$ with the property that

$$\underset{n_0 \in N_0}{\exists} \ni \underset{p \in P}{\forall} p \geqslant n_0 \Rightarrow f(p) = 0$$

then there exists a bijection $N_1 \to \mathbf{L}^+(P, N_0)$ such that if $n \mapsto f$ then

$$n = \prod_{p \in P} p^{f(p)}.$$

Here P is the prime numbers and $N_1 = N_0 \sim \{0\}$. Why cannot this be made plain in the schools?

I ANSWER THAT there is exactly one way to write a number as the product of primes, once that concept is rightly understood. Considering the difficulty of formulating the question precisely, it is truly amazing that this was so early discovered; even more amazing that it is part of the arithmetic curriculum in the average school. If p is prime and does not divide m, then it is clear from the definition of prime numbers that one can find r, s such that $1 = rp + sm$. For example, the prime 11 does not go evenly into 100, and in fact

$$1 = (-99)\times 11 + 100\times 100.$$

Once this is known, if p divides exactly into a product ab, then either p divides into a or one can write

$$b = rpb + sab,$$

which implies $p|b$. By induction on $\sum_{p\in P} f(p)$, we are done.

The mathematical subject which we have been considering is all about dividing one number by another number in such a way as to get no remainder. It is easy to divide a number by another number, but it is not so easy to do it in such a way as to get no remainder. This task is by no means always easy; in fact, it is often impossible. This fascinating subject, together with all its ramifications, is called *divisibility*. People often devote their lives to a part of mathematics that they have chosen for various characteristics. Of course, until one has devoted his life to a part of mathematics, one cannot know for certain what the characteristics of the particular field actually are; and by the time one does know, it is often too late. One of the major criteria a mathematician looks for in choosing a subject to work in is *uselessness*. It is considered fun to do something really useless. One American mathematician had devoted much of his life's work to divisibility and related topics. In the autumn of a fruitful sojourn on earth, he rested on his laurels, gratefully remembering how useless his work had been. To fill the idle hours, he played with the dial of a radio receiver of the kind that can get anything. Picking up a ground-to-air communication, he heard this:

'Hey, Mac. How's divisibility up dere?'

'Pretty bad, Chet. How's divisibility on de ground?'

'Can't see a ting, Mac.'

'Neider can I. Guess I better come in.'

'Dat's right, Mac. Divisibility is one of de most important tings when you're up in de air.'

On learning that his darling subject, on which he had spent the best years of his life, had applications to aeronautics, the poor man lost his mind, and became a folk-singer. His songs were based on the Pythagorean idea that music is applied arithmetic, and had titles such as 'A protest song based on the first five Mersenne primes, for sackbut and thumb-piano', or 'A contrapuntal imprecation based on 65537, for ophicleide and calliope, in 17 parts for hoarse vocalists,' and won world-wide acclaim.

4. Guess the next number

A great deal of what we learn at school is of little use in later life. This is especially true of mathematics. Beyond the most basic arithmetic, which does have a use in checking the bill in a restaurant, there is very little that is ever used again except by specialists. A knowledge of probability theory is handy for an undertaker, so that he can work out when his customers are likely to need him; a little topological group representation theory is not amiss if you happen to end up a quantum mechanic, repairing other peoples' quanta when they begin to wear out. But for most of us, most of our mathematics moulders away slowly as the brain cells blink out, cell after cell, in our heads. It never gets used.

Number guessing is an exception. Why is it not taught in the schools? This is one branch of quasi-mathematical trickery that everybody needs desperately, yet it is never found in the school texts. By number guessing, I mean being able to answer those little riddles, like the following.

Here is a little test. Do not be afraid of it; the questions are all of the same type. Though they start out easy at first, pretty soon they are going to get much, much harder. So keep calm. All you have to do is guess the next number in the sequence. We give you, gratis, some of the numbers, which shows how very kind we are, because then we only ask for *one* number back from you. Sounds simple, doesn't it? In case you still haven't caught on, the first example is worked for you. Here goes:

(1) 8, 75, 3, 9, ____.

Now all you have to do is look at the numbers, and then in the blank provided write in the number that seems to you logically ought to go there. Now read the numbers again: eight, seventy-five, three, nine, . . . What was that you were about to say? Was it 17? Right! The only number any sensible person would put there is 17. So we write in the number 17 in the space provided, like this:

(1) 8, 75, 3, 9, <u>17</u>.

That's all you have to do. Good luck!

Everyone has encountered one of these little tests, and we all know how much depends on them—that place in a really good university, that step up in the firm that has been hanging fire for years, that membership of MENSA—the chance to look down on more and more stupid people who cannot guess the next number.

Really, it is inexcusable that this art is not taught in every school. The scientific fact is universally acknowledged that only intelligent people can do these puzzles; moreover, nobody denies that there is a crying need for intelligence in all areas of the national economy. Hence, and one would think the inference would be obvious to any person who can guess the next number even in the easy example we saw just above, all that needs to be done in order to cure a vast proportion of the world's ills is to teach everyone to guess the next number. Because then, naturally, everyone would be intelligent. The only case of which the writer knows in which so simple a remedy is known to exist for so serious a social disease is that of mental illness. It is universally accepted that by examining a man's handwriting, trained graphologists can determine what is his mental character, and in particular diagnose any tendencies to violence, mania, etc. Yet nothing is done to teach sane handwriting! But, lest we deceive ourselves, that might involve some difficulty. Good handwriting is not acquired overnight. Fortunately, the ability to answer the sort of puzzle that is the subject of this section can be acquired in anything from a few seconds to an hour.

There is only one little snag. The people who set the intelligence tests are a very special breed. After you learn how to answer the sort of question we are considering here, you will be absolutely certain that the answer you put down is correct; not so the fellow who made up the test. He has his own way of looking at things. The fact is, by his own intelligence test he may not be very intelligent. He may not recognise a correct answer when he sees one. Perhaps that is why he went into his particular line of work. Who ever thinks to question the intelligence of a man whose very job is testing intelligence? What occupation can you think of that makes you utterly safe from the prying doubter who asks, 'Has he got a high I.Q.?' *Quis examinabit examinatores ipsos?*

QUESTION 18. Whether there is a simple method, whereby one may always give a logical answer to the sort of puzzle that says, 'Here are some numbers; what number comes next?'; and whether it is easily, painlessly, and rapidly learnt?

Objection 1. It would be very surprising if such a method existed. The example mentioned earlier was 8, 75, 3, 9, ___. There does not appear to be any relationship at all connecting those four numbers. Now if one were a better mathematician, or had a better mind for figures, granted one might be able to see how to connect them up; and then one could see what number follows logically after the ones already given. But surely this subject is simple only to the most accomplished number-cruncher.

Reply to Objection 1. It is true that some knowledge of mathematics is necessary before one can actually write down a 'relationship' that will produce these numbers; but it is not all that stupendous a task. Writing down such a relationship is a mere technical difficulty, and if one is good at mathematics he can master the problem of writing down such a relationship in a few minutes. Above all, though, one must not forget that no intelligence test or puzzle of this sort actually requires anyone to write down a mathematical formula. It is not necessary even to understand what it is that connects the numbers and makes them come out in that order. The trick which is going to be explained gives you an easy, one might almost say magic method for giving the right answer *without understanding anything*!

Objection 2. The idea that such questions can be answered easily is absurd in the extreme. Great care is taken in the devising and arranging of the questions to test the examinee's abstract pattern-recognition level. While it is conceivable that an examinee will do slightly better or slightly worse than he ought to do, it is a matter of experience that examinees tend to come out about the same if tested again. Stupid people absolutely cannot do well on this test, except by a once-in-a-million chance, like guessing all the right numbers on the lottery.

Reply to Objection 2. It is true that very stupid people cannot do this test; nor can very uneducated people. It is necessary to be able to write a number, which illiterates and cretins cannot do. Moreover, it is necessary to be able to understand that one ought to write a number, and where. These are not very taxing demands; all the abstract pattern-recognition that is needful is contained in them. One of the abstract patterns that does, admittedly, have to be recognised is the line on which one is meant to write the correct answer.

I ANSWER THAT there is an easy, painless, and simple method for always writing in a correct answer to one of these brain-twisters. The following are some examples of the kind of thing one gets.

(1) 5, 4, 3, 2, ____.
(2) 2, 4, 6, 8, ____.
(3) 1, 3, 5, 7, ____.
(4) 1, 2, 4, 8, 16, ____.

And now here are some of the answers one is likely to get, with the reasoning behind them.

Example 1. In order to see more clearly the relationship among the numbers, let us draw a graph (Figure 1).

We see that the dots are on a straight line. It is logical to continue the line, and if we do so, we see that the next dot should be at the place marked with a circle. (This technique is called *extrapolation*; it is also used to determine the price of tobacco in the year A.D. 2000,

and to predict the future fortunes of transcendental meditation among humanoids.) Thus the answer is

$$5, 4, 3, 2, \underline{1}.$$

Example 2. This time, if the numbers are plotted on a graph, we do not get a straight line. Trying to think of something else to do, we first take the logarithms to the base 2, and then plot the graph. These are: $\log_2 1 = 0$, $\log_2 2 = 1$, $\log_2 4 = 2$, $\log_2 8 = 3$, $\log_2 16 = 4$. Hence the graph will be as shown in Figure 2: a straight line; the extrapolation gives 5, and taking 2^5 we get 32. Thus the answer is

$$1, 2, 4, 8, 16, \underline{32}.$$

Emphatically, this is *not* the method being presented here. Not everyone can get the correct answer by this sort of argument. Before explaining the sensible approach to the problem, it was thought useful to remind the reader how he is *expected* to do the problem. This approach is not systematic. Who could guess that one would get a straight line by taking the logarithms of the numbers in the fourth example? Why not take the exponential function of the numbers, or the inverse tangent?

One easily proves by induction that if f is a polynomial function of degree $\leqslant n$, and if $f(x_i) = 0$ for every integer i such that $0 \leqslant i \leqslant n$ wherever $(x_i)_{0 \leqslant i \leqslant n}$ is a finite sequence of real numbers indexed by the set of integers i such that $0 \leqslant i \leqslant n$ with the property that $x_i \neq x_j$ if $i \neq j$, then f is identically zero. Moreover, the polynomial function

$$\sum_{k=0}^{n} y_k \frac{\prod_{i \neq k} (x - x_i)}{\prod_{i \neq k} (x_k - x_i)}$$

(under the same hypotheses about (x_i)) has degree $\leqslant n$, and sends $x_i \mapsto y_i$ for $0 \leqslant i \leqslant n$; it must be the unique polynomial that does so. This method provides a systematic method for solving our problem, which gives the formula

$$1 + \tfrac{7}{12}x + \tfrac{11}{24}x^2 - \tfrac{1}{12}x^3 + \tfrac{1}{24}x^4$$

for Example (2). Thus if $x = 0$, then we get $1 + 0 + 0 + 0 + 0 = 1$; if $x = 1$ we get

$$1 + \tfrac{7}{12} + \tfrac{11}{24} - \tfrac{1}{12} + \tfrac{1}{24} = 2;$$

if $x = 2$ we get

$$1 + \tfrac{14}{12} + \tfrac{44}{24} - \tfrac{8}{12} + \tfrac{16}{24} = 4;$$

if $x = 3$ we get

$$1 + \tfrac{21}{12} + \tfrac{99}{24} - \tfrac{27}{12} + \tfrac{81}{24} = 8;$$

and so on.

If we apply this formula in the case $x = 5$, we get the answer

$$1 + \tfrac{35}{12} + \tfrac{275}{24} - \tfrac{125}{12} + \tfrac{625}{24} = 31.$$

It may be noted that by logarithms, as the example is often done, we got not 31, but 32. Does this worry us? Not a bit! First of all, there is not much difference between 31 and 32; the difference is only $32 - 31 = 1$. Secondly, even if we are forced, in an unguarded moment, to admit that 31 and 32 are not quite the same, there remains the question: which answer is better anyhow,

$$1, 2, 4, 8, 16, \underline{31}, \text{ or}$$

$$1, 2, 4, 8, 16, \underline{32}\ ?$$

Which of these is really more logical, more true to a mathematical way of thinking? Which really shows the greater pattern-recognition facility? There can be no doubt that 31 is the better answer by a long-to-middling chalk; and this for the simplest of reasons. It must be admitted that if anyone should be so narrow-minded and curious as to put down 32 as the number that follows 1, 2, 4, 8, 16, then he had *some* reason for doing so. It is possible that he arrived at 32 in any of several ways. He may have noticed immediately that the numbers 1, 2, 4, 8, 16, if exchanged for the letters of the alphabet that correspond to them, are just the initial letters of the words 'Alien birds do have peculiar . . .'; and that these are the words of an old Kentish proverb, the last word 'feathers' being omitted. Arranging the alphabet in a circle with *z* next to *a*, and counting on past *z* to *a*, which then becomes 27, we see that the letter *f*, with which the extra word 'feathers' begins, gets the value 32, which is the right answer. This is all well and good, and is a reasonable interpretation of the problem showing remarkable skill at pattern-recognition. But one can argue, of course, that the letter *w*, by its very name, is not really a letter at all; it is merely a way of writing the letter *u*, or just perhaps the letter *v*, when they are doubled. Hence this letter must be omitted in the counting, and while 32 is a good approximation to the correct answer, it is not as good as the true answer 31.

But it is also possible that the answer 32 was obtained in another way. It could even have been obtained by the graphical method explained earlier, in which by lucky chance someone thought of taking the logarithm. This leads to the formula

$$a_n = 2^n,$$

where of course the first number of the sequence is considered as the zeroth or noughtth number of the sequence. On the other hand, our own work, which was based on a system, gives

$$a_n = 1 + \tfrac{7}{12}n + \tfrac{11}{24}n^2 - \tfrac{1}{12}n^3 + \tfrac{1}{24}n^4.$$

Now both of these answers are backed up by perfectly good mathematical formulae, and so both are perfectly logical. Of the two answers, which is the better—both being correct? The answer is, of course, that the second answer is to be preferred, because it is much the simpler, and is easier to use, and is obtained by a more general method.

On the other hand, some people may prefer the answer 32 and the formula $a_n = 2^n$, because here we have a homomorphism from the additive monoid N_0 to the multiplicative monoid Z, or rather **End** (Z). Since both answers are correct, it is a moot point whether one of them can be preferred to the other. Perhaps it is best to leave the question of preference to the metaphysicians. At least one can say that the investigation of this example has led to one important improvement in our point of view, if we had thought before that there could be only one logical answer to the question. Following up this suggestion of the muse, let us consider an arbitrary answer to the problem

$$1, 2, 4, 8, 16, \underline{\quad\quad}.$$

An arbitrary answer means any old answer at all. Think of a number. It need not be a number between 1 and 100; it need not be less than a million; it need not be short enough to write down, even if one is allowed only one atom of any element at all for each letter. There is no restriction on the number one is allowed to write down as the answer; but to avoid unnecessary explanations we shall restrict it to be an integer. Just write this number on the line provided. (This line may be extended if it is not long enough; and if the number is too long to actually write during the probable lifetime of the reader, simply think of the number as written in.) Since I do not know what number you have written in, let us call it a_5. Since there is nothing special about the number 5, let us assume that to an arbitrary finite sequence

$$(a_0, a_1 \ldots, a_{n-1})$$

the reader has added one more number, making it

$$(a_0, a_1, \ldots, a_{n-1}, a_n).$$

Then the polynomial-generating formula that was mentioned earlier will produce a formula that will not only interpret the given terms $a_0, a_1, \ldots, a_{n-1}$, but will also include the new term added arbitrarily by the reader in the interpretation. This gives us the clear and simple rule that was sought, which would enable any fool to answer these little puzzles with a minimum of difficulty. The rule is this: you probably have a favourite number—most of us have one. If you do not know what your favourite number is, try to find out or decide somehow; perhaps an astrologer could help. Preferably, add 1 to this number; but if you are unable to perform this calculation, it does not matter greatly. Now take the number you have arrived at, and whenever you are given one of these questions involving guessing the next number in a sequence, use this number. (The addition of 1 to your favourite number is simply a device that makes it more difficult to determine your character defects by analysing your number. No technique by which a person's character may be found out from his secret number is known to the author, but of course someone may some day invent such a technique.) For example, here is how you may answer the quiz we started with:

(1) 5, 4, 3, 2, $\underline{19}$.

(2) 2, 4, 6, 8, $\underline{19}$.

(3) 1, 3, 5, 7, $\underline{19}$.

(4) 1, 2, 4, 8, 16, $\underline{19}$.

There can be absolutely no question that the answer given here is correct, and that it tends to show a very high level of understanding of pattern-recognition. It must be admitted immediately that some of the people who evaluate these tests are so poorly equipped, patternwise, as to be unable to recognise the correctness of the answer given above. Of course, if the object is to guess the number chosen by the examiner as the one he intends to accept as correct, then a study of parapsychology is more relevant than a mathematical study of which answer is correct. Parapsychology is also called E.S.P. or mind-reading, and is a much more useful art than is mathematics. A good esper can pass an examination in any subject at all, so long as the examiner can also pass his own exam. Still, even E.S.P. is helpless if the examiner himself does not know the correct answer.

The reader may be assured that in writing the following it is not mere random whimsy that guides the author's unerring finger:

$$0, 0, 0, 0, 0, 0, 0, \ldots .$$

There is something very definitely in the writer's mind when he sets out that string of numbers. If this is regarded as to be completed by the addition of one more number, then that number may as well be 0, or 19. If the object, on the other hand, is to find out just what are the numbers (all of them) in the infinite sequence that I happen to have in mind, then even an accomplished mind-reader cannot tell you very much about the numbers. He can tell you something: since what I have in mind is that the term in the nth place is 0 if $n = 0, 1,$ or 3; and 0 if $n \geqslant 3$ and if there exists no solution in integers to the equation

$$a^n + b^n = c^n$$

such that $abc \neq 0$; and 1 if $n \geqslant 3$ and there does—since all that is what I have in mind, he can tell you it. What he cannot tell you is whether all the terms of the sequence are 0, or some of them are 1. He cannot tell you that because the answer is unknown, not only to him and to me but to everyone. He has no minds to read about the answer in. That sort of question is a good one on any intelligence test: namely, the sort of question that has never been answered, but is known to have only one correct answer. At the opposite extreme from this is the question that is known to have all possible answers as correct answers; indeed it is strange that people not only get great enjoyment out of answering such questions, but that only intelligent people are able to answer them.

5. Fractions

Fractions are usually introduced to us in terms of pies. Suppose, for instance, that seven greedy boys wish to eat three large apple pies—it is only a supposition. One might try to solve the problem by the methods of arithmetic in integers; this would not allow one to cut the pies, a most inappropriate condition. It is very silly not to cut pies; apple pies, especially, are eaten with a fork, and not with the fingers, and are eaten more easily without being cut. If the problem is to find out how to share the pies in case they may not be cut, it ought really to be reformulated thus: Suppose that seven greedy boys wish to eat three large oysters. The oyster, as everyone is well aware, is never cut. (It is not true, by the way, that oysters may not be chewed. They should be taken whole into the mouth, and one must not bite off pieces; also, of course, one must never remove partly chewed oysters from the mouth in order to inspect them. Part of the fun of eating oysters is the tantalising thought of how ghastly and sickening the insides must look, and once the insides have been viewed all the spice goes out of the procedure for many people. Then, too, if you look at the insides of your oyster it is just possible that you will lose your nerve, and be unable to finish. That would brand you as a coward and a chicken for life, and no woman could love you after such craven behaviour.) In order to finish the oysters without any of those inequities of distribution which the pure mind of childhood finds so intolerable, the little lads realise the necessity of a homomorphism $Z \mapsto Z$ sending $7 \mapsto 3$; putting, if you will, the seven boys round the three oysters. Or else—since there are seven of them, they may be excused for formulating the problem in various forms at the start—one seeks an integer n such that under the unique symmetrical bilinear map

$$Z \times Z \to Z$$

that sends $(1, 1) \mapsto 1$ one should have

$$(n, 7) \mapsto 3$$

—or $(7, n) \mapsto 3$, which is the same thing. These two suggestions have been made already by the first two lads, when a third chimes in to point out that they are of course equivalent. The equivalence is soon proved by the fourth boy. The fifth then asks pensively, 'Then in that case I wonder whether this proposition—for two equivalent propositions may, I take it, sensibly be regarded as one only—whether (I say) this proposition may be true; namely, that there exists such a homomorphism (respectively, such an integer)?'

To the sixth little fellow (and we can hardly blame him) this remark is hardly one that can be passed over unnoticed. 'You cannot mean, Q.C.,' says he (the name of his playmate is Quintus Columus MacIlhenny, his father having learned Latin while engaged in agricultural pursuits at the State Prison Farm in Penalville, so out of tender feelings his chums call him Q.C.), 'surely you are not thinking, what I take to be an unwarranted generalisation of your conjecture; namely, that *whatever* be the number of men, distinct, of course, from 0, and whatever the supply of bivalves, they can eat them? For in the case of two men required to consume a single oyster, I perceive the impossibility of a solution—no doubt you have seen already what I am thinking of.' This way of speaking—calling his comrades *men* and not boys—was the custom in this group of friends, which the speaker had agreed to follow after quieting his original misgivings with the thought that though they were not *actually*, they were *virtually*, grown men, just as in the view of the ancient mathematicians the unit 1 was a number, not actually but by virtue of being the generator of the natural numbers. Besides that, he was aware that when they should have reached mature years, he and his buddies would almost certainly make up for it by calling themselves *boys*. In idle moments he would reflect that no doubt this colloquial inversion had poetic truth as well, because of the line 'The child is father of the man', which one could not otherwise justify.

Very much to their credit, the other little nippers owned up immediately to their inability to follow him. They had not the slightest idea why two persons could not ingest a single mollusc without the impropriety of dissection. Oh, of course they would not like to be obliged to do it; they had some misgivings about the success of such an experiment; if one were to place a bet, it would no doubt have to go down on the side of failure. But that, they knew, proved nothing.

On the insistence of his pals, our sixth little chap unwillingly produced his proof. 'It is nothing, really; only don't you see that if a homomorphism (which may as well be called n) sends 2 to 1, and if 2 is the homomorphism sending 1 to 2, then the composition, that is 2 followed by n, must be the identity? I know it may not be elegant, but one sees that the restriction of n to the subgroup generated by the integer 2 (which is after all nothing more than the image of the other homomorphism) is already surjective onto Z. Clearly then n itself cannot be injective; but we know that any non-zero homomorphism $Z \longrightarrow Z$ is injective (by consideration of the properties of the natural numbers, or by some other means). I think that makes it clear.'

Without any question, it was agreed that that did indeed make the matter all too clear; the sunny faces of the little gang of companions took on, as one, a hang-dog expression. Their brows furrowed. Twenty-eight fingers were raised to scratch seven heads. One thought was in seven minds—and how can that be, since nobody has ever succeeded in slicing a thought so as to distribute it evenly among seven? It was this: if it is *not always* possible to distribute a number of oysters among a number of boys (for in their inmost thoughts the boys recognised their juvenile status), then might it not also be impossible in the very case they were now considering—with gathering urgency as time drew on? One could not but admit that it *might* be impossible; and then the question is, 'Ah! But we must know for certain: is it?' For some time the company sat pensively, seven hands under seven chins, staring at the oysters. The dust of the country road settled in their hair, on their shoulders, on the open shellfish before them. At length there was a small change in this dismal scene; but it was not a happy one, nor one to be proud of. Let us hope the passing travellers, if any came by at that moment, were too preoccupied to notice. For if you had been present just then, you would have seen a small tear that issued from the eye of the seventh and last young hayseed. Only one—we must be swift to point that out. It trickled down to the level of the lips, and had almost mingled with the trail of saliva which there poured forth when by an unconscious reaction the little pink tongue flicked out and caught it gratefully.

The boy spoke with the calm of despair. 'Can't be done,' he said manfully. 'You will all want to know the reason; I am sensible to that, so here it is. I'll tell you what: let me put the argument to you in the form of questions, and if you will, do you all answer me in chorus.' On receiving their agreement to this mode of discourse, number seven began his pitch. 'Tell me, is the number 3 prime, or composite?'

The answer came in a sextet: 'Prime, though you ought really to have asked, is it prime, composite, or a unit; those are the three alternatives.'

'I stand corrected, fellows. You forgot to mention 0; that is a class to itself. Still, the point is made. Very well then. In what way may a prime number be the image under a homomorphism from the integers to the integers of a number not a unit?'

'That can only happen in one way,' came the reply; there was no polyphony, but simple unison, for these were simple country lads. 'The homomorphism must be an isomorphism.'

'You have struck the nail on the head with perfect orthogonality, mates,' spoke the inventor of the demonstration. 'And which isomorphisms exist from the group of integers to itself?'

'There are two only, as generally recognised; that is to say, negation, and the identity.'

'Right again!' cried the interlocutor, 'though we ought really to have spoken of endomorphisms and automorphisms a moment ago; nevertheless, the terms used will serve. Finally, you must tell me, what are the images under both these isomorphisms, or automorphisms if you find the term preferable, of the number 7? Is either of them 3?'

A gloomy tone of finality rang in the still, hot country air as the answer came in chorus: 'Under the one, -7, and under the other, 7. Neither of these numbers is 3.' Six more tears trickled down, and were absorbed by six tongues. The seventh child made as if not to notice.

The minds of the young are volatile, and not disposed to rest for long on one thing. One of the boys remarked that they might have noted, if their senses had not overpowered their other faculties (he was referring to the sight of oysters, and to the Pavlovian reaction of the saliva in flowing out of the corners of their mouths) that a very similar occurrence was recorded in the Bible. The others recognised immediately his reference to King David and the water which his men procured for him at great danger through the enemy lines; because it was generally thought that except for difficulties of division it might have been shared out among the army, whereas in fact it had been poured out on the ground. They said nothing of all this, but one other of the lads remembered a bottle of root beer that had been buried a year ago, and ought by now to be ready; and the clutch of rattlesnake eggs that a third lad had been incubating in his pocket was produced, and found to have begun to hatch through the good offices of the sun, which had been beating down steadily on that area of the boy's shirt during their deliberations. The oysters were added to the writhing mass in the pocket, and the seven set off to procure a spade, and soon mathematics was forgotten for a time.

Now if those boys had been of the bookish, studious kind, that stays indoors all day in the summer to work out mathematics problems, they might have applied the idea of division in the Euclidean domain of integers, and come out with the fact that on division by 7, the number 3 leaves quotient 0 and remainder 3; as it was, being red-blooded fellows they solved this problem informally and practically by eating no oysters each and feeding the remaining three molluscs to the snakes. How many snakes there were does not matter—if snakes eat oysters at that age, it is unlikely that they do so with fastidious manners, insisting on equal apportionment and swallowing whole. In any case the question did not occur to the seven boys, and in their healthy outdoor way they could not care less for such hair-splittings and refinements. They did remember, however, to improve the snakes' appetite by the addition of a couple of shakes of Tabasco Sauce, which Q.C. usually carried about with him.

But apple pies—to return to the starting-point of the discussion—are something quite different from oysters. Nobody loses his appetite when he sees an apple pie cut—not if it is a good apple pie, not if the physical properties of the crust and of the filling are what they should be, not if the aroma emanating therefrom is the one we expect. No, indeed. People do cut apple pies, and that is one of the reasons why the integers alone have not been considered sufficient for all mathematical purposes, even applied ones.

QUESTION 19. Whether there are fractions?

Objection 1. Clearly fractions involve the idea of breaking things in pieces. Now it is clear that some things can be broken in pieces; apple pies, or the hearts of unrequited lovers. But a mathematical unit cannot be broken in pieces, since it is one and must remain so. Hence fractions are mere physical things, without the ideal truth of mathematics. Instead of fractions, the mathematician ought to be talking about ratios. Ratios are relations between integers, and so they have a mathematical existence.

Reply to Objection 1. This objection stems (aside from any philosophical or metaphysical ideas that may lie behind it) from a misconception of the role played in mathematics by the idea of 1. If we think of 1 as something having an existence apart from the system of integers itself, then we may assign to it absolutely certain qualities that it has only by virtue of being one of the privileged class of integers. *Qua* integer, the number 1 cannot be broken in pieces; *qua* rational, it can. It is possible to think of rational numbers (or fractions, as they are sometimes termed) as ratios, in a sense to be made precise later. It is possible not to think of them thus.

Objection 2. A fraction is a number consisting of a top number and a bottom number, which are called the numerator and denominator respectively. Now the number 0.111 is a fraction since it is bigger than 0 and less than 1. But it has no top and no bottom number, and this is nonsense. The contradiction leads us to the conclusion that fractions are absurd.

Reply to Objection 2. It is possible to devise an explanation that will circumvent this apparent difficulty, and that is based on the notion that a fraction has a top number and a bottom number, to use the barbarous words of the objection. In fact, 0.111 is the same as 111/1000. The idea that fractions must be less than 1, or that numbers less than 1 are fractions, is also mistaken. But the real error lies in speaking of fractions as if they were things in themselves. We cannot really speak of fractions at all; we can only speak of the field of rational numbers. The field of rational numbers is sublime, like the *laeta arva*, the delightsome, verdant fields of Elysium, the happy hunting grounds, of which all men love to speak. The field of rationals is a subject for poetry. Whoever attempts to speak of fractions is bound to become entangled in barbarous expressions, to sink in a miasmatic bog of barbarous inelegancies. Unclean! Unclean! Let them cry as they walk through the streets, all those who mention the unspeakable numerator and denominator.

I ANSWER THAT there are fractions, if you care to mention them; and if you are sensible, you will not. What people usually are after when they want fractions is something on the order of seven boys eating three pies fairly. They want to have some kind of confidence that it can be done, which has nothing at all to do with the stomachic capacity of the lads; that we can take to be infinite. In general, one would like to be able to divide. One would like to be able to say that if n is an integer not 0, and if a is given, then a unique solution x exists for the equation

$$nx = a.$$

To make everything as simple and neat as possible, one may also wish to require that if $x \neq 0$ and if $nx = 0$ then $n = 0$; and that tells us that we are looking for an abelian group G such that

$$Z \longrightarrow \mathbf{End}(G)$$

(the natural map) shall be an injective monoid homomorphism from the multiplicative monoid $Z \sim \{0\}$—a restriction is necessary to eliminate 0, where before the restriction the natural map is of course defined—to a submonoid of the multiplicative group

$$\mathbf{Aut}(G)$$

of units in $\mathbf{End}(G)$.

Unfortunately, the problem as thus presented is too easy, since there are non-isomorphic solutions. We are used to thinking of fractions as well-determined entities, and this contains a small grain of truth: we should hope to find that the group of fractions is well-determined up to isomorphism. It is impossible to allow non-isomorphic objects to share the glory. There must therefore be found some way of choosing among all the non-isomorphic entities one that pleases us best.

It is already acknowledged that the sum of two integers is again an integer, and hence is either a non-zero integer or 0. Since in any group $mn = nm \Rightarrow m^{-1}n^{-1} = n^{-1}m^{-1}$, and since $n(n^{-1}m) = m = mnn^{-1} = n(mn^{-1})$ likewise follows if m and n commute, giving $n^{-1}m = mn^{-1}$, it is easily seen that the subgroup of **Aut**(G) generated by the image of $Z \sim \{0\}$ consists of all elements mn^{-1}, with $m, n \in \mathbf{Im}\,(Z \sim \{0\})$, and is commutative. The equation

$$mn^{-1} + pq^{-1} = (mq + pn)\, n^{-1}q^{-1}$$

then shows that the sum of two automorphisms in this subgroup is again such an automorphism, or is 0. Hence the subgroup is a field, which we dignify with the name Q. It is easily verified that Q may take the place of G, and that Q is a universally repelling object in the category of fields of characteristic 0.

Hence the existence of fractions will be assured if only it can be shown that a group satisfying the conditions imposed on G exists. Such a group is the following: Let R be the ring

$$\frac{Z[X]}{(X^2)}$$

of polynomials in a single indeterminate X over the ring of integers, modulo the ideal generated by X^2. Let M be the monoid obtained from the multiplicative monoid of this ring by excluding all elements $a + bx$ such that $a = 0$, where $x = X + (X^2)$. Now if C is the submonoid of 'constants', i.e. elements $a + bx$ with $b = 0$, then the relation

$$(a + bx)\,\overline{(c + dx)} \in C$$

is an equivalence, where

$$\overline{c + dx} = c - dx.$$

Let G be the quotient; i.e., the set of equivalence classes under this equivalence. Because the equivalence is compatible with the monoid structure, G is clearly a monoid with the structure inherited from M.

Since $m\bar{m} \in C$ for every element m of M, the monoid G is clearly a group, and trivially abelian. Since

$$(a + bx)^n \equiv a + nbx$$

in M, the conditions which G is to satisfy are easily verified. Thus fractions exist.

In the example given there were three pies and seven boys. Since we have the isomorphism of $\mathbf{Aut}(Q)$ and Q together with the injection $Z \to Q$—which we shall always interpret as an inclusion—we see that all we need to do is to take the field elements 7 and 3 and form $3 \cdot 7^{-1}$. Because fields are commutative this may be written as 3/7 without much danger, and this is what is often done. Thus we see that some fractions have a numerator and a denominator.

QUESTION 20. Are whole numbers fractions?

I ANSWER THAT whole numbers are fractions, or rather that integers are rational numbers. There is a curious distinction sometimes made in arithmetic books between whole numbers and fractions, or between proper and improper fractions, usually in connection with a notion of something called 'mixed numbers'. Usually, these correspond to various kinds of rational numbers written in various ways. Negative rationals are frequently ignored or treated as curiosities. The whole point about integers being rational numbers is that Z is a subring of the field Q, in the sense that the natural injection is considered as an inclusion.

QUESTION 21. Whether a fraction has a numerator and a denominator?

I ANSWER THAT in a sense, a rational number has a numerator and a denominator, and in a sense this is not so. There is no way of taking a rational number and getting out of it a numerator and a denominator, unless one is willing to accept several numerators and several denominators, or unless one is willing to study the subject known as *reduction to lowest terms*. But on the other hand, if one is given the numerator and denominator to begin with, and if the denominator is not 0, then it is possible to get a rational number and only one out of the numerator and the denominator. In civilised terminology, one would distinguish between *fractions* on the one hand and *rational numbers* on the other. A fraction is merely an element of $Z \times (Z \sim \{0\})$, whereas a rational number is an element of Q.

To find the rational number associated with a numerator N and a denominator D, one simply maps (N, D) to Q, considered as **End**(Q), by taking the product in the latter of the endomorphism associated with the integer N and the inverse of the automorphism associated with the integer D. By previous remarks and exercises, the resulting map from fractions to rationals

$$Z\ (\times Z \sim \{0\}) \rightarrow Q$$

i s surjective. The fraction (N, D) is commonly written as N/D, and this same symbol is often used for the rational number obtained from the fraction. As fractions, 3/4 and 27/36 are distinct, or

$$3/4 \neq 27/36;$$

whereas

$$3/4 = 27/36$$

if they are rational numbers. Fractions are of course rather ridiculous and pointless objects, and both the idea and the word are best forgotten. Rational numbers are lovely, civilised and useful things. That may be why they are called rational numbers.

6. Calculations with fractions

(1) *How to find a numerator and denominator for a rational number*
We are aware that if the rational number is not 0, then it may be considered in a natural way as an automorphism of the Z-module of rational numbers. We also know that this group of automorphisms is commutative, and is generated by the submonoid of (automorphisms associated in the natural way with elements of) $Z \sim \{0\}$. Hence it can be written in the form ND^{-1}, so that it is the rational

number associated with the fraction N/D by the civilising process described above. If the rational number is 0, then it is also $0D^{-1}$ and hence comes from numerator 0 and denominator D, where D can be any integer except 0. Then N and D are a numerator for the rational number.

(2) *How to reduce a fraction to lowest terms*
The fraction is barbarous, so first we must civilise it by the map

$$Z \times (Z \sim \{0\}) \longrightarrow Q.$$

Once a civilised rational number is available, it is possible to get to work and do something. Let the rational number be q. As we know, the set $\mathbf{D}$ of integers d such that

$$dq \in Z$$

is not empty; trivial examination shows that $\mathbf{D}$ is an ideal. Since $\mathbf{D} \neq \{0\}$, the ideal has a non-zero generator D. Taking

$$Dq = N,$$

we have found a particular numerator and denominator for the rational number q. The fraction N/D is said to be *reduced to lowest terms*. So is the fraction $-N/-D$, and these are the only fractions reduced to lowest terms that give this rational number, as is obvious from the uniqueness to within a unit of the single generator of an ideal in a ring without non-trivial divisors of zero.

Consider the reduction to lowest terms of the fraction 6/8. If q is the rational associated with this fraction, then it is easily seen that $dq \in Z \Leftrightarrow 3d \in 4Z$. The proposition on the right-hand side of this equivalence implies that $9d \in 4Z$, and since in any case $8d = 4.2d \in 4Z$ we get by subtraction that a further consequence of this is $d = 9d - 8d \in 4Z$. This last trivially implies that $3d$ is again an element of $4Z$, so that $3d \in 4Z \Leftrightarrow d \in 4Z$. The ideal $\mathbf{D}$ in question is therefore generated by either 4 or -4, whichever one cares to take, and since (taking, say, $D = 4$) we get $4q = 3$, we come on the result that 6/8 reduced to lowest terms is 3/4. The answer $-3/-4$ is just as good.

(3) *How to deal with mixed numbers*
A mixed number is a pair, consisting of an integer and a fraction N/D in which one has $0 \leqslant N \leqslant D$. The sum of the integer and the

rational associated with the fraction is another rational, called the rational associated with the mixed number. The customary mode of writing mixed numbers is most curious, and is also entertaining so long as one is not called upon to use it. It is a sort of triangular array, and involves three integers. There would be nothing odd about this if these were always the first integer of the mixed number, and the numerator and denominator of the fraction that forms the second term in the ordered pair that is the mixed number. Those are the three integers that one sensibly expects in writing the mixed number, and they are sometimes the integers used. In case they are the integers used, the mixed number $(I, N/D)$ is simply written

$$I\frac{N}{D}.$$

But if I is 0, it is omitted, leaving N/D. If N is 0, both N and D are omitted, leaving just I. These exceptions are rather simple. The general exception in case $I < 0$ is that one must write

$$(I+1)\frac{D-N}{D};$$

but to this exception there are numerous exceptions. If $I + 1$ is 0, it is replaced by a mere minus sign in this expression. If $N = 0$, one ignores this exception and uses only the exception about what to do if N is 0. The subject of mixed numbers is more nearly a branch of generative grammar than of arithmetic or algebra.

(4) *How to deal with improper fractions*

A fraction N/D is *proper* if $0 \leqslant N < D$; otherwise it is *improper*. The thing to remember about improper fractions is that no lady or self-respecting working girl ever allowed an improper fraction to pass her lips. Note that the fractional part of a mixed number is proper. It is possible to express any rational as a mixed number, and the availability of this alternative to the use of improper expressions of rationals is a blessing to all right-thinking young ladies. It is quite probable that the wish to avoid the improper has led to the invention of mixed numbers; these are hard to account for otherwise. The mixed number $(I, N/D)$ represents the same rational as the fraction $(ID + N)/D$, and by carefully observing the rules of the cryptogram, one can write an improper fraction as a mixed number, or a mixed number as a fraction.

EXERCISE. Write as fractions:

$$-7\tfrac{17}{52};\ -\tfrac{3}{4};$$

write as mixed numbers:

$$54/-3;\ -9/-2.$$

(5) *How to add fractions*

The sum of two fractions is of course not well defined. Any fraction or mixed number can be considered as the sum of two fractions or mixed numbers if it represents the sum of the two rationals that are represented by the two fractions or mixed numbers. Definite rules for computing the sum of two rationals by representing them and obtaining a fraction or mixed number representing the sum from the two representations have, nevertheless, been devised. In ordinary practice, despite their totally impractical mode of writing, mixed numbers are more commonly used in such computations than fractions. In order to explain how this is done, it is necessary to use a special symbol Ø to stand for any of I, N, or D that is not really there; for example, we must take the mixed number 2 as $2\frac{Ø}{Ø}$, and we must take $\frac{3}{4}$ as $Ø\frac{3}{4}$. If we wish to add

$$\mathrm{I}\frac{N}{D} + I'\frac{N'}{D'},$$

there are several cases. If both I and I' are positive integers or Ø, then the result is either

$$(I + I')\frac{ND' + N'D}{DD'}$$

or else

$$(I + I' + 1)\frac{ND' + N'D - DD'}{DD'},$$

with certain remarks and provisos.

If in a mixed number *both* I and N are 0, then one of the exceptions to the exceptions in the rule for writing mixed numbers which is outlined above is that instead of writing nothing at all, which would be $\varnothing\frac{\varnothing}{\varnothing}$ in our present temporary notation, one actually writes 0, or in present notation $0\frac{\varnothing}{\varnothing}$. This is the only case when the place of I is filled by the integer 0 in writing mixed numbers. In computing $I + I'$ or any of the other sums and products, certain special rules must be observed if any of I, I', N, N' is $\varnothing$.

(i) In addition, $\varnothing$ acts like 0; thus, $\varnothing + 3 = 3$.

(ii) In multiplication, $\varnothing D = \varnothing$ and $N\varnothing = N$, which we may express by saying that $\varnothing$ is absorbing on the left and neutral on the right, like British toilet paper.

(iii) It is sometimes necessary to use $D'D$ as denominator instead, as in the case $0 + \frac{3}{4}$.

(iv) $0 + \varnothing = \varnothing + 0 = \varnothing$.

In case one only of I, I' is a negative integer or is $-$, any of the following forms may be the correct answer: (say $I' < 0$ or $I' = -$)

$$(I - I')\frac{ND' - N'D}{DD'}$$

or

$$(I - I' + 1)\frac{DD' - ND' + N'D}{DD'}$$

or

$$(I - I' - 1)\frac{DD' + ND' - N'D}{DD'}$$

or

$$(I - I')\frac{-ND' + N'D}{DD'}.$$

There are of course certain remarks and provisos; however, we only touch on the subject here, and for a full treatment the reader is referred to any school arithmetic text, where the case when both I and I' are negative numbers or $-$ may also be omitted, as it is here.

Fractions are also sometimes added as fractions, especially in the case of proper ones. This is much easier to do. When fractions have been added it is considered the done thing to put them in lowest terms.

(6) *How to multiply fractions*

Since rational numbers are the elements of a ring of endomorphisms, they may be multiplied by composition. It is this which shows us the kind of application to everyday life the multiplication of fractions is likely to have. The following is a very practical and concrete example; like many problems in the elementary theory of fractions, it involves the idea of sharing.

The headmaster of a provincial boys' school believes strongly in the great value of practical experience in forming boys' minds. It is this that has made up his mind to take the whole form to the zoological gardens of a great metropolitan city. 'How else can they pick up the rudiments of zoological gardening?' he asks himself—the question is merely rhetorical and does not require an answer, but to show his agreement with the wit and wisdom of the inquirer, the headmaster chuckles softly to himself, and sighs. Alas! Little does he know what lies in store. Because the school has been founded on principles of freedom from fidgety nonsense, no books are kept and it is unknown just what is the size of the form. When people ask how many boys are in the various forms, Dr Chockle-Fervyn answers mildly, 'Oh, various numbers—I think.' For the problem that presents itself in a few moments, the fact that the number of boys is unknown takes on great importance. The boys go to the zoo; they see animals; they grow hungry. The tables in the cafeteria seat twelve. For each tableful of boys, the Doctor brings 45 steak pies; not that he has been so fussy as to ask for that number—he merely says to the attendant, 'A carton or so of hot steak pies, perhaps, please,' each time he sees a table of unfed boys. (Afterwards, enquiries in the packing department of Hengist & Horsa Olde Englishe Meat Products produced the number 45. 'We always puts 45, your honour, sir. The sum of the hexpounents on the proims bein' free, you see. Free dimensions, loik spice. Glad to 'elp your worship.') When the boys have finished, it is time to feed the lions. This is done, in the metropolitan zoo to which the problem has reference, by opening the gates between the lion house and the cafeteria; the fact being known in the metropolitan area, only tourists are eaten on most days. Today, the only tourists are the boys from Fervyn Towers of Learning. Old 'Chock-full-of-vermin' is in the loo. The lions lick their whiskers and prowl smugly back to the lion house. 'Can't say exactly,' was the headmaster's reply to anxious parental enquiries after the hols commenced—and those words, at least, were spoken true. 'Some of them may have got lost at the "zoo", of course. An interesting question. Are you *sure* you had a boy here?'

It must not be thought, however, that vagueness went with want of curiosity or with pedagogical apathy in Dr C.-F.—far from it. He went so far as to prevail on the head gardener of the zoological garden to have the stomachs of 25 of the lions pumped, and the contents sorted. 'Unheard of, sir,' was the initial reaction of this official; but the good Doctor plied him with educational anecdotes, and reminded him that the schools could not be expected to produce the required quota of zoological gardeners unless the gardeners took an interest in education. The curator (as he was also called) mollified. 'Better see to it myself, you know. Can't trust these young fellows these days. Would the boys benefit by a personal report from myself?' It was agreed that the results would be presented at assembly in the Great Hall down at Pummidge in the near future. Briefly, it can be stated that every 25 lions managed 18 boys among them.

'And now, lads,' said the Doctor after the vote of thanks, 'I have a practical exercise for you in the multiplication of fractions. How many steak pies were eaten by each lion? You know the facts, and you should be able to visualise the problem. Think of pies inside boys, and boys inside lions. For artistic expression, there is a prize for the most vivid painting of the meal. Those doing military science will show the optimum formations for the armies involved: lion, boy, and pie. For social science. . . .' Only the arithmetic problem need detain us here. Let m be the rational number corresponding to the meal in which the boys eat and the pies are eaten. Thinking of m as an endomorphism of Q, we have $m\colon 45 \mapsto 12$. If M is the endomorphism whereby lions eat and boys are eaten, then $M\colon 18 \mapsto 25$. Since these are Z-module endomorphisms we may also write

$$18 \times 45 \xrightarrow{m} 18 \times 12 \xrightarrow{M} 25 \times 12,$$

so that the composition Mm sends $810 \mapsto 300$. But the rational represented by the fraction $10/27$ sends $810 = 30 \times 27$ to $300 = 30 \times 10$. If two endomorphisms of Q agree at a non-zero element of Q, they agree on all of Q, since this amounts to stating that cancellation holds in Q, and since Q is a field. Thus Mm sends 27 to 10, or $Mm = 10/27$, or, as we easily see, $27/10$ pies went into each lion. In mixed numbers this is $2\frac{7}{10}$.

(7) *How to divide fractions*

Since every non-zero element of a field has an inverse, it is easy to divide rational numbers. In order to divide one fraction by another, turn the fractions into rational numbers and divide the rationals. Then turn the answer back into any of the fractions that represent it. Division of fractions is of use in those places where cannibalism flourishes on a highly developed cultural level. This is because a good side of porkh (pronounced porch) has got a history attached to it, relating to the quantity of porkh and hamh (pronounced haunch) eaten by the side of porkh in its earthly life. In other words, if you go to a really good family butcher and if you are particular, you will want to know how many men (for such purposes one does not call them men but pork pighs, on the principle that man is long pig or pigh, and his flesh is porkh or hamh)—how many men (to use that expression) he himself has eaten. That is because the meat of cannibals tastes better than the meat of heterophagi, or non-cannibals. A living cannibal is never called a pork pigh in polite society. Here is a problem in division of fractions: if $7/5$ pork pighs were eaten by $3/4$ pork pigh, how many pork pighs did the whole pork pigh eat? (Some people, especially waiters in fancy restaurants, say 'hyperbolic ham' instead of 'haunch'. Etymologically this is correct, since 'hyperbolic' means 'past all likelihood of truth'—it is a roundabout way of hinting that it is not really just ordinary ham. The same is true of the hyperbolic functions sinh and cosh, sometimes pronounced *shin* and *cosh*; the pronunciation reveals the cannibalistic origins of these functions; the cosh being a weapon used in preparing a market-ready animal for slaughter, and the significance of the shin being clear enough.)

In order to divide by $\frac{3}{4}$, it is sufficient to invert and multiply. How to invert $\frac{3}{4}$? Since $\frac{3}{4}$ is an automorphism sending $4 \longrightarrow 3$, its inverse sends $3 \longrightarrow 4$; and hence is $\frac{4}{3}$.

3

ALGEBRA

1. The wonderful quadratic formula

The quadratic formula is an amazing discovery of the utmost utility in solving the most multifarious practical problems, such as abound in the best texts of school algebra. One example will suffice to illustrate its depth and range of application.

Muscular M. Boulangiaire, the baker of the picturesque village of Beaulieu Derrière, makes three kinds of loaves in his shop (Figure 3). The first kind is a flat square of side x; the second is an ordinary French loaf, just x units long and one unit wide; and the third loaf—it is really more a bun than a loaf—again a square one like the first, but measuring one unit along each side. All loaves have exactly the same height.

M. Boulangiaire knows just how much dough goes into the one-by-one loaf, or bun. He also knows, of course, just how much dough he has prepared on any particular day—in fact, this never varies; there is always just enough dough for 113·006 buns. What does vary is the number x, which tells how long the French loaves are and determines the size of the x-by-x square. The baker's method of work is the following: By enquiries among his customers, he determines what numbers of the several loaves will be required for the morrow. Let us suppose that today he has learned that tomorrow's demand will be for seventeen variable-sided square loaves, for 5·310 ordinary

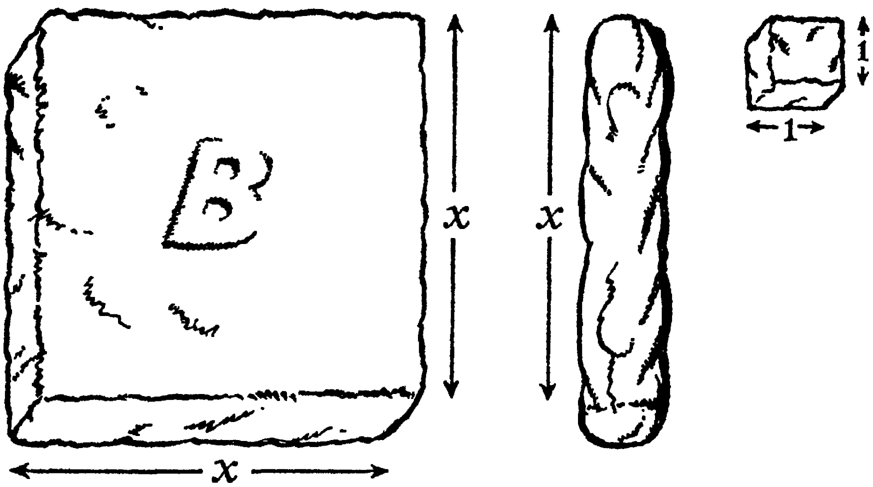


Figure 3. The three styles of loaf at Beaulieu Derrière.

loaves, and for half-a-dozen buns. Then he adjusts the variable x accordingly, so as to make the dough come not just right, with none left over. M. Boulangiaire learned mathematics at the École Paranormale, and is able to compute just how long to make his loaves when he bakes tomorrow's bread for the village.

Can you help him? (Remember that, being French, he writes the number 317,476·52 as 317.476,52.)

It is well known among French provincial bakers with a suitable mathematical education that a quadratic equation has two solutions, or roots; and that these solutions may be the same; they may one or both be negative; and they may both be complex numbers. It is not the custom to measure loaves of bread in complex numbers, nor even in negative ones. This has on rare occasions caused difficulty at M. Boulangiaire's shop. Indeed, one August when all the village was away on vacation, Beaulieu suffered a mammoth influx of Crimean Goths (Figure 4). These folk speak a dialect descended from that of bishop Ulfilas, translator of the Bible into Gothic. Like Ulfilas they

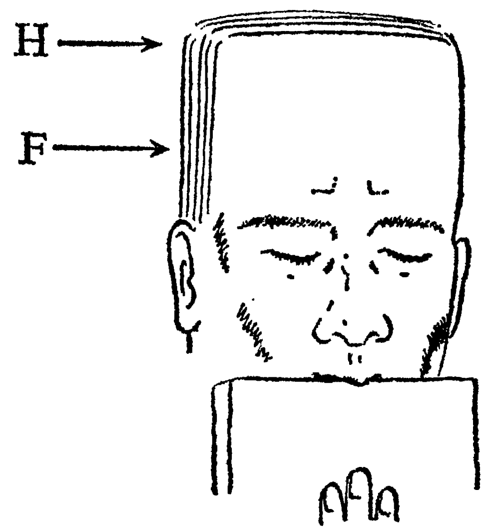


Figure 4. Crimean Goth eating bread. Note the straight, fair hair, *H*, and the typically Arian tetragonocephalic forehead, *F*.

are Arian heretics and believe it a sin to eat any but square bread. This is because Ulfilas' (lost) translation of Genesis has the error 'In the shape (rather than 'sweat') of thy face shalt thou eat thy bread'. The demand for the one-by-one buns alone was in fact more than the dough on hand could supply, which made the baker a little pessimistic, but he went ahead and worked out the quadratic equation. The Goths moved on, leaving the good baker a sadder but wiser man. The ensuing efforts of comparative philologists to trace the further course of the linguistic group have so far proved fruitless. Show that the quadratic equation had two purely imaginary roots (at least one Goth ordered a variable-sided square loaf). Explain the meaning of this answer in practical terms so as to be readily understood by (i) a provincial baker with some school maths; (ii) a Crimean Goth.

Our M. Boulangiaire is a rather indecisive man when he is deprived of the moral support of algebra. Especially since the August catastrophe he has been bothered by the recurring thought that one day in his daily algebra he might find two distinct, positive roots for the bread equation. How would one choose in such a case? As far as he can see, nothing in the mathematics would give one the slightest clue about which of the two solutions to use. One could take either value for x, and either way the dough would exactly suffice to fill all the orders. What a terrible responsibility, to make the choice all unaided! Can you put M. Boulangiaire's mind at ease?

The quadratic formula says that to find the roots of

$$ax^2 + bx + c = 0$$

you must take the square root of $b^2 - 4ac$; then the roots are

$$\frac{-b + \sqrt{(b^2 - 4ac)}}{2a}$$

and

$$\frac{-b - \sqrt{(b^2 - 4ac)}}{2a}.$$

The formula involves no difficulty in itself so long as $a \neq 0$. If $a = 0$ then the formula is, of course, nonsense, since 0 is absorbing in any ring and hence if 0^{-1} existed we should have $0 = 00^{-1} = 1$. By definition, $\sqrt{(b^2 - 4ac)}$ is a number such that

$$\{\sqrt{(b^2 - 4ac)}\}^2 = b^2 - 4ac,$$

so that a trivial verification yields

$$a\left(\frac{-b + \sqrt{(b^2 - 4ac)}}{2a}\right)^2 + b\left(\frac{-b + \sqrt{(b^2 - 4ac)}}{2a}\right) + c = 0.$$

The same thing works for $\dfrac{-b - \sqrt{(b^2 - 4ac)}}{2a}$.

Note that these formulas involve the use of the operation $\sqrt{}$, known as the square root operation.

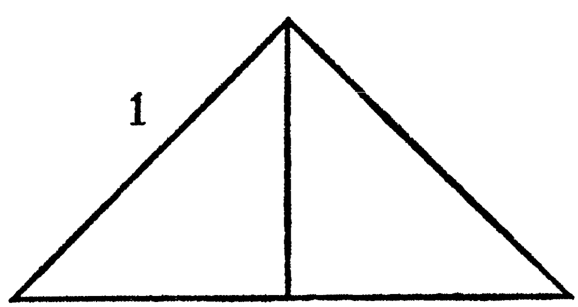
QUESTION 22. Whether it is possible to take square roots of rational numbers?

Objection 1. Patently, it is impossible to take square roots of rational numbers, unless one is very lucky in the rational numbers he uses. The square root of 4 certainly exists, and is 2, since $2 \times 2 = 4$. The square root of $7\frac{74}{2601}$ is obviously $2\frac{13}{19}$. Let q be a rational number such that $q^2 = 2$. Then writing q in lowest terms, we get the fraction m/n, with n the generator of the ideal of integers consisting of those integers x that make xq an integer. It is clear that m and n are relatively prime, and that $2n^2 = q^2n^2 = m^2$. The prime 2 must divide m^2; hence 2 divides m and 2^2 divides m^2; hence 2 divides n. Since 2 divides m and n they are not relatively prime, and a contradiction results.

Reply to Objection 1. The argument is correct, but all it shows is that rational numbers may not have a rational square root. They may still have some other kind of square root. In fact, all that is necessary is to construct the ring of polynomials over Q in a single indeterminate, which we write $Q[X]$, and to take the quotient ring over the ideal generated by the polynomial $X^2 - 2$. If x is the coset of X, then $x^2 = 2$ in this ring. Moreover, no multiple of $X^2 - 2$ by a polynomial is a rational number, so Q is injected into this quotient ring in a natural way, and the injection may be thought of as an inclusion. If we replace 2 by $b^2 - 4ac$, we get a ring in which the formula for the roots of the quadratic equation makes sense.

Objection 2. The bakers may not think so. Going back to $\sqrt{2}$, and this could easily be the answer given by the quadratic formula to a problem involving a quadratic equation, can you justify this number? If it is not rational, in what way is it a number at all? Can it be the length of a loaf of bread? If it is an element of some algebraic extension of the rational field, it would seem that it could not be a length.

Reply to Objection 2. The geometric answer might be this: from the vertex of an isosceles right triangle drop a perpendicular to the hypotenuse as shown. The two smaller triangles are also isosceles,



and if the side of the larger triangle is 1 then it follows that the hypotenuse is $\sqrt{2}$ by the fact that the perpendicular bisects the hypotenuse and by similar triangles. But this argument is invalid, since it is customary to establish geometry on the basis of an ordered field and not vice versa. Let us try harder.

Clearly every element of the quotient ring $Q[X]/(X^2 - 2)$ can be written as $a + bx$, where x is the square root of 2, and where a and b are rational. Suppose there exists a positive integer n and rational numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ such that

$$-1 = \sum_{i=1}^{n} (a_i + b_i x)^2.$$

Then since x is not rational,

$$-1 = \sum_{i=1}^{n} a_i^2 + 2 \sum_{i=1}^{n} b_i^2.$$

Since squares are positive and since $2 = 1 + 1$, this is impossible.

Moreover, $Q[X]/(X^2 - 2)$ is a field since

$$(a + bx)^{-1} = (a - bx)(a^2 - 2b^2)^{-1}$$

if a and b are not both 0. It is well known that every real field has a real closure; i.e., is a subfield of a real closed field algebraic over the first field—this is an easy consequence of Zorn's Lemma. But a real closed field has the property that a sum of squares has a square root in the same field; and hence has a natural ordering. This is inherited by $Q[X]/(X^2 - 2)$. Since the two possible roots of 2 have opposite sign, we may choose x to be the positive one. Then either $x < 1$ or $2 = x^2 \geqslant x$; hence, $x < 3$. It follows that the field in question is archimedean. Any positive element of an archimedean ordered field can obviously be considered as a length. Hence, $\sqrt{2}$ is a length.

Objection 3. No doubt, it is true that $\sqrt{2}$ is a length, and any baker worth his salt would accept it as such after such a clear and beautiful explanation. The argument involved one little trick, however; namely the fact that $2 = 1 + 1$, and hence 2 is a sum of squares. But -1 is not a sum of squares. Moreover, any field containing a square root of -1 cannot possibly be ordered, since in an ordered field squares are positive, whereas -1 is negative. Therefore, a square root of -1 makes no sense.

Reply to Objection 3. It is quite true that no ordered field can contain a square root of -1. Hence, since $\sqrt{-1}$ cannot be a length or the negative of a length, it is of no use to bakers. Mouldy wheat is also of no use to bakers; nevertheless, mouldy wheat contains ergot, which is of use to pharmacologists. [32] Moreover, mouldy wheat exists. Therefore, it is possible that $\sqrt{-1}$ may exist.

I ANSWER THAT square roots of rational numbers exist. If a rational number q has no rational square root, then $X^2 - q$ is irreducible over Q; hence if b is a non-zero rational and if a is rational then rational numbers c, d and a non-zero rational r exist such that

$$(X^2 - q) = (a + bX)(c + dX) + r,$$

which makes the quotient ring $Q[X]/(X^2 - q)$ a field. If q is positive, then this field can be ordered, since if q is positive then

$$q = mn^{-1} = (mn)n^{-2} = \sum_{i=1}^{mn} (n^{-1})^2,$$

which is a sum of squares. Otherwise, this field cannot be ordered. If finding the roots of a quadratic equation involves taking a non-ordered extension of the rationals, then the roots cannot be considered as lengths, nor can they be used to weigh bread-dough. The reason is that a practical man is sufficiently imprecise to accept an approximation of the desired measure between certain upper and lower bounds. He knows that $\sqrt{2}$ lies between 1·4 and 1·5, or between 1·4142 and 1·4143. He can only know that if the field is an ordered field, in which 'between' makes sense. That is why bakers never try to bake loaves of length $\sqrt{-1}$, even approximately. The question, 'Have I shaped this loaf a trifle too long or a smidgin too short?' would be meaningless. In practice, bakers seldom feel obliged to produce loaves of length $\sqrt{-1}$, and this helps to keep them in a happy, carefree frame of mind.

How to find the square root of a number

It must be assumed that the number is positive and rational. Otherwise, it is hopeless to try to approximate its square root by rationals.

Method 1. Use the fact that

$$\sqrt{q} = \exp(\tfrac{1}{2}\log q).$$

Writing $q = 1 + p$, we get

$$\tfrac{1}{2}\log q = \tfrac{1}{2}(p - \tfrac{1}{2}p^2 + \tfrac{1}{3}p^3 - \tfrac{1}{4}p^4 + \ldots).$$

This is not valid if $q \geqslant 2$. Take sufficiently many terms to make a good approximation to $\frac{1}{2}\log q$. Next, we have

$$\sqrt{q} = 1 + \tfrac{1}{2}\log q + \tfrac{1}{8}(\log q)^2 + \tfrac{1}{48}(\log q)^3 + \ldots.$$

Again, it is necessary to take several terms. As an example, this method gives 1·098 as the square root of 1·21. This is in error by 0·002, since the square root is 1·1. Here is the work:

$$\log 1{\cdot}21 = 0{\cdot}21 - \tfrac{1}{2}(0{\cdot}21)^2 + \tfrac{1}{3}(0{\cdot}21)^3 - \tfrac{1}{4}(0{\cdot}21)^4 \text{ approximately,}$$

and this is $0{\cdot}21 - 0{\cdot}02205 + 0{\cdot}003087 - 0{\cdot}0004862025$, or

$$0{\cdot}213087 - 0{\cdot}0225362025 = 0{\cdot}1905507975.$$

Half this number is 0·09527539875. It remains only to take the square and cube of this last number, and to add

$$1 + 0{\cdot}09527539875 + \tfrac{1}{2}(0{\cdot}09527539875)^2 + \tfrac{1}{6}(0{\cdot}09527539875)^3,$$

which will give the answer stated.

Method 2. Using a table of squares. To find the square root of q, write

$$x^2 = q$$

and find the greatest integer n such that $n^2 \leqslant q$. Put $x = n + x'$, and $\hat{x} = 1/x'$. Clearly, if $x' = 0$ then the square root is n and we are done. If not then we have $\hat{x} \geqslant 1$, and $\hat{x}$ satisfies the new quadratic equation

$$(q - n^2)\hat{x}^2 - 2n\hat{x} = 1.$$

Now we may seek the greatest integer such that the left-hand side is less than or equal to 1; call this integer $\hat{n}$, and write

$$\hat{x} = \hat{n} + 1/{}^2\hat{x}.$$

Iterating, one gets a sequence of integers

$$(n, \hat{n}, {}^2\hat{n}, \ldots).$$

Form the sequence of fractions

$$(n, n + 1/\hat{n}, n + 1/(\hat{n} + 1/{}^2\hat{n}), \ldots)$$

These fractions approximate the desired square root. As an example, let us compute $\sqrt{3}$.

First write $x^2 = 3$. Since $1^2 = 1 < 3$ but $2^2 = 4 > 3$, we take $n = 1$. Now $(1 + 1/y)^2 = 3$ gives $2y(y - 1) = 1$.
Since $2.1(1 - 1) = 0 < 1$ and $2.2(2 - 1) = 4 > 1$, we put $\hat{n} = 1$ and write $y = 1 + 1/z$. This gives $z(z - 2) = 2$ and ${}^2\hat{n} = 2$. Putting $z = 2 + 1/a$, we get $2a(a - 1) = 1$. It follows that $y = a$, and

$$\hat{n} = {}^3\hat{n}.$$

Thus the sequence of integers is

$$(1, 1, 2, 1, 2, 1, 2, \ldots).$$

This is periodic after a certain point, or in other words the 'hat' operation $\hat{\cdot}$ obeys

$$\hat{\cdot}^3 = \hat{\cdot}.$$

As a matter of fact this will always happen, in the sense that there exist a non-negative integer p and a positive q such that

$$\hat{\cdot}^{p+q} = \hat{\cdot}^p.$$

A convenient way of writing the sequence of fractions that results is

$$1 + 1/(1 + 1/(2 + 1/(1 + 1/(2 + \ldots,$$

or

$$1\cfrac{1}{1\cfrac{1}{2\cfrac{1}{1\frac{1}{2}\ldots}}}$$

These are called continued fractions. The most useful thing to do is to start with the iteration of

$$y = 1 + 1/(2 + 1/y).$$

Taking $y = 1$ as a beginning, we get successively $y = 1\frac{1}{3}$, $y = 1\frac{4}{11}$, $y = 1\frac{15}{41}, \ldots$, and the corresponding values of x are $x = 1\frac{3}{4}$, $1\frac{11}{15}$, $1\frac{41}{56}, \ldots$. Note that

$$(1\tfrac{41}{56})^2 = 3\tfrac{1}{9409}.$$

The sequence for $\sqrt{2} + 1$ is $(2, 2, 2, \ldots)$; for π it is

$$(3, 7, 15, 1, 292, 1, 1, 1, 2, \ldots).$$

This explains why it is often stated that $\pi = 3\frac{1}{7}$.

2. The incommensurable of the incommensurable

Let us begin to consider an event which may or may not have happened to the reader in the course of his life to date, in a semblance more or less as is to be described here; and one which may—or possibly may not—befall him in the future. Most of us have a sort of routine by which we trace out the day. A man will go to his daily place of occupation, where the surroundings change little, or change in an expected way; where he finds the people seem to understand what he says with a fair degree of accuracy, and to show little surprise at the man's own commonplace gestures and remarks. The sort of thing that has just been described is not unusual. It may not happen all the time, but it happens at least a good part of the time; the number of days that vary from the pattern are few, if any, and they do not vary much from the pattern. The psychological effect of this monotony is to produce a commonsense outlook, a reasonable, matter-of-fact way of looking at the world.

Then WHAMMO! The illusion is suddenly shattered. You are creeping, let us say, down the queue in the refectory at lunch-time, just as you do every day. Nothing at all out of the ordinary has happened.

The level of hunger is normal for such a situation. There is a choice of meat pie or spaghetti; you must speak.

'Meat pie, please.'

'Sorry, sir, no cheese today. Only meat pie or spaghetti.'

'I'll have meat pie, please.'

'I told you, sir. You'll have to take what we've got.'

'All right.' By now you are a wee bit puzzled, and mere acquiescence seems an easy way out of the difficulty.

'We only feed that to the rats, sir. If you'd like to come around the back I'll see what I can do.' The assistant is plainly puzzled by your obstinacy, but means to be cooperative. She leads the way behind the counters, the ovens, into a corner, through a door, into a broom closet. The door of the closet is shut on you. It is dark. A muffled voice is heard outside.

'The spiders don't often come before teatime, sir. Shall I slide a few lollies under the door? A nice lolly will help to keep off the peckish feeling, you know.'

'No! Why am I in here? Let me out!'

'Oh, well. I'll have to ask the butcher. Are you sure you can't wait? The knives are all dull, and there's still a lot of meat in the freezer. Most of the customers prefer spiders if the knives are dull.'

'Help! Police!'

The door opens, and you are on a vast plain of tall grass. Nothing is visible except a caravan of Tartars approaching in the distance. As their voices grow nearer and nearer, the words of their chant become distinctly audible:

'Meat pie! Meat pie! Meat pie! . . .'

Something like this happens if you construct an isosceles right triangle. There is a procession from rationality to irrationality. There is an encounter, where by an encounter we mean something different from a lunch-counter. There is a meeting-up with something that, while perhaps not Wholly Other, is rather Other. There is the feeling of the Absurd; there is the wish to cry, Help! Police! The encounter may come when two circles meet. It may come in a variety of ways. When it comes, we cross over from rationality to irrationality, and enter a new domain of incommensurability.

The word 'surd' is popularly used to designate irrationality. This usage has fallen out of favour with mathematicians, but is still found among certain classes of pedagogues and is used in school textbooks. The basic meaning of the word is 'deaf', but it may mean irrational or unresponsive. Some early etymologists derived the word from 'sordid', probably incorrectly, because those whose ears are dirty or sordid are unable to hear clearly. A surd is defined as

(1) any irrational solution of an equation

$$x^n = q,$$

where q is rational; or else

(2) any number obtained by a succession of rational operations and extractions of roots, beginning with rational numbers; or

(3) any algebraic irrational.

Since surds were discussed before cases (2) and (3) were known to be distinct, and before the question whether they might be distinct had been much considered, it is not always clear which kind of surd is meant. Here, as in many cases in mathematics, one has a usefully expressive term which time and the development of the subject have passed by. In the meaning (1), it sometimes says beautifully what one would like to say about an irrational; namely, that nobody can give a rational solution to the equation, so that in a sense the equation is deaf and unhearing to the question we ask it, and also that we can give approximate solutions, but that to do so is ugly and sordid. Just as with prime numbers, one must not expect an ordinary dictionary to tell the truth about surds. One dictionary [14] defines a surd number as one that cannot be squared. This would be ridiculous if it meant what it seems to mean, namely that it is impossible to multiply the number by itself, since every element of a ring may be multiplied by every other. If the definition meant that, then the number 17 would be surd when we considered it as an element of the abelian group Z, and rational as an element of the ring Z. That would be absurd. In fact that is not what is meant, in this old-fashioned terminology, when it is said that a surd is a number that cannot be squared. What is meant is that a surd is a number q such that the equation

$$x^2 = q$$

has no rational solution. This differs in only two respects from the definition (1): we must replace 2 by n, where n is an integer at least 2;

and we must call x, not q, the surd. This use of the word 'square' to mean 'square root' derives from geometry.

3. The inconstructible of the inconstructible

Once it became thoroughly well known that surds were quite surd, and that it was absurd to keep on asking them, 'Who are you? Please answer in the form of the quotient of two integers (the second being non-zero),' people cast about for other methods of obtaining information. Certain surds, it was found, would respond if subjected to a very tiresome, long-drawn-out process requiring lots of strong light and the use of (1) a sharp pointed stick; (2) a pair of sharp pointed sticks joined at the blunt ends and capable of being set at any desired degree of opening at the angle thus formed; and (3) a stiff, straight rod of suitable length. Numbers which became more compliant under this mode of investigation were known as *constructible numbers*. All this occurred in an age of the world when people were far more cruel than they are now. In obtaining information of this type, it was not thought to be particularly relentless or cruel to use such tools, or to stretch the number out along a line segment and so possibly to put it painfully out of shape.

It would perhaps not have been surprising, considering the lengths to which people were prepared to go in representing surds geometrically, or *constructing* them, as it was quaintly termed, if all the surds had given way. But there were surds too obstinate to accept such treatment. Not only would these surds not speak up and say who they were, they would not even lie down meekly on a line segment with their feet (so to speak) up in the air. These brave numbers won the grudging admiration of even the ruthless investigators themselves for their unflinchingly irrational behaviour under the most brutal methods of the police state, which included bisection, indefinite extension, rotation, translation, cutting by lines and circles—but the reader must be spared the full force of this infamy. Suffice it to say that some of those that withstood such treatment became known as surds of the *third degree*. In the curious parlance of that abominable era, they were *inconstructible*.

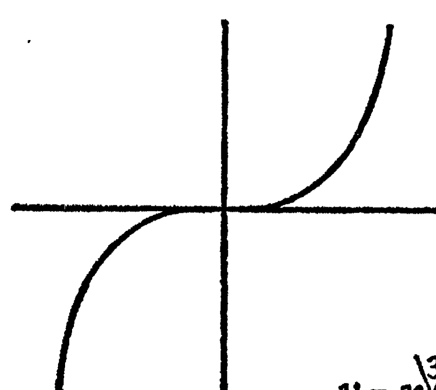
QUESTION 23. Whether a surd of the third degree is inconstructible?

Objection 1. We must never give up hope. Indeed, one hears daily of someone who has found a new way of treating some surd—say $\sqrt[3]{2}$—that seems, to the investigator at least, to promise success. There is so much more time available now for scientific research than ever before that it hardly seems plausible to suppose the problem will not be broken in the near future.

Reply to Objection 1. It is not a problem any longer. It is quite certain that surds of the third degree will never yield to the dreaded ruler and compass. Time spent on them is time wasted utterly, and beyond hope of return. No research of this kind can be called scientific.

Objection 2. Clearly the methods used in the past for the treatment of irrationality have been wrong; they are essentially those of the snake-pit. As to the irrationals of the third degree, they have been subjected to this sadistic cruelty for so long that they may quite possibly be beyond help by now. But some of the higher algebraic irrationals might be chosen for a more modern approach, such as neo-Freudian analysis or prefrontal lobotomy.

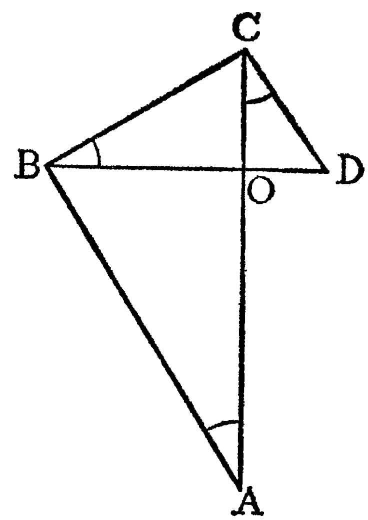
Reply to Objection 2. Prefrontal lobotomy should be done only under medical supervision, even as applied to irrationals. It, too, involves the use of a pointed instrument. Under analysis, even transcendental irrationals become relatively tractable, it is admitted. It is quite true that irrationals of the third degree may become constructible if the methods of ruler-and-compass geometry are abandoned; if, for instance, the geometer is allowed a device that draws the curve



$y = x^{\sqrt[3]{}}$

tangent at its point of inflection to a given line at a given point. It is difficult to understand why such a device is not provided in the kit with the other supplies when young children first set out in geometry.

Objection 3. It is easy to construct the cube root of a given number, which is a surd of the third degree. Voilà:



Here AO $= n$ (the given number), OD $= 1$, the triangles are similar, and O is the centre of a circle with diameters AC and BD.

Reply to Objection 3. Unless $n = 1$, there is no such circle. If $n = 1$, there is a simpler construction:

Here OA is the given length 1; it is also the cube root of that length, since $1.1.1 = 1$. But since 1 is rational, this is not a surd.

I ANSWER THAT it is impossible to construct any root of a cubic polynomial

$$ax^3 + bx^2 + cx + d$$

unless one of the roots is rational. Thus no length that can be constructed by the methods of ruler-and-compass geometry can ever be a solution of an equation

$$ax^3 + bx^2 + cx + d = 0,$$

where $a \neq 0$ and where a, b, c, d are rational numbers; i.e., quotients of integers.

In what ways is it possible to write the number 3 as the sum of a finite series of positive integers? The following ways are certainly possible ones:

$$\begin{aligned} 3 &= 3 \\ 3 &= 2 + 1 \\ 3 &= 1 + 1 + 1 \end{aligned}$$

If one requires further that the sequence be non-increasing, then it can be shown that these possibilities are the only ones (this condition implies, for example, the exclusion of '$3 = 1 + 2$', since 2 is greater than 1 and that makes this an increasing series). Essentially all that is required is the inequality

$$0 < 1 < 2 < 3,$$

together with the fact that no integer lies strictly between two successive members of this inequality.

From the fact just mentioned, it is possible to show that

$$ax^3 + bx^2 + cx + d$$

must satisfy one of the following conditions: either

(i) $ax^3 + bx^2 + cx + d$ cannot be factored; i.e. cannot be written as the product of polynomials of lower degree having rational coefficients; or

(ii) this polynomial can be factored, and one of the factors is linear; i.e.,

$$\begin{aligned} ax^3 + bx^2 + cx + d &= a(x + t)(x^2 + ux + v) \\ \text{or} \qquad &= a(x + r)(x + s)(x + t); \end{aligned}$$

again, the coefficients are rational. This is clear because if the factorisation is

$$\prod_{i=1}^{k} f_i(x)$$

then
$$\sum_{i=1}^{k} \deg(f_i) = 3,$$

and because by commutativity we can assume that the degrees do not increase.

Thus, if the cubic equation has no rational root then the polynomial is irreducible; henceforth, let

$$f(x) = ax^3 + bx^2 + cx + d$$

be irreducible. If $p(x)$ is another polynomial, and if $p(x)$ is not of the form $f(x)\,q(x)$ for some polynomial $q(x)$—all polynomials shall have rational coefficients—then since the ring

$$Q[x]$$

of polynomials with rational coefficients in an indeterminate x is a principal ideal domain, there exist polynomials $a(x)$, $b(x)$ such that

$$1 = a(x)f(x) + b(x)p(x);$$

in particular, this is true if $p(x)$ is quadratic or linear. Obviously, then, no root α of $f(x)$ satisfies any quadratic equation with rational coefficients. Moreover, it is clear that

$$Q[x]/(f(x))$$

is a field containing Q as a subfield. If $Q(\alpha)$ is the smallest subfield of F that contains both Q and α, where α is a root of $f(x)$ in some extension F of the rationals, then $(f(x))$ is in the kernel of

$$\begin{aligned} \varphi : Q[x] &\longrightarrow Q(\alpha) \\ x &\longmapsto \alpha, \end{aligned}$$

and the induced map

$$\hat{\Phi} : \frac{Q[x]}{(f(x))} \longrightarrow Q(\alpha)$$

is a surjective field homomorphism, and hence an isomorphism. Hence $(1, \alpha, \alpha^2)$ is a basis of $Q(\alpha)$ as a vector space over Q. But since constructible numbers can be produced by a succession of rational operations and square-root operations, the smallest subfield of F containing such a number has even dimension over Q; we merely use the fact that if each of the fields $\mathfrak{k}$, $\mathfrak{l}$, $\mathfrak{m}$ extends the preceding and is a finite-dimensional vector space over the preceding field, then the dimension of $\mathfrak{m}$ over $\mathfrak{k}$ is the product of the dimension of $\mathfrak{m}$ over $\mathfrak{l}$ and the dimension of $\mathfrak{l}$ over $\mathfrak{k}$. But 3 is not even.

How to solve a cubic equation

It is easy to solve cubic equations. Let the equation be

$$ax^3 + bx^2 + cx + d = 0,$$

and suppose the roots are x_1, x_2, x_3, so that

$$ax^3 + bx^2 + cx + d = a(x - x_1)(x - x_2)(x - x_3).$$

If

$$\sigma_1 = x_1 + x_2 + x_3,$$
$$\sigma_2 = x_1x_2 + x_2x_3 + x_3\,x_1,$$
$$\sigma_3 = x_1x_2x_3$$

are the elementary symmetric functions, then writing

$$f(z) = x_1 + x_2z + x_3z^2$$

we get

$$3x_1 = f(1) + f(\omega) + f(\omega^2),$$

where

$$(f\omega)^3 + f(\omega^2)^3 = 2\sigma_1^3 - 9(\sigma_1\sigma_2 - 3\sigma_3),$$
$$f(\omega)f(\omega^2) = \sigma_1^2 - 3\sigma_2.$$

Hence a root is

$$\frac{1}{3a}\Big\{-b+[-2b^3+9abc-27a^2d+3\surd(-3a^2b^2c^2+81a^4d^2-54a^3bcd+ \\ +12a^2b^3d+12a^3c^3)]^{1/3}.2^{-1/3}+[-2b^3+9abc-27a^2d- \\ -3\surd(-3a^2b^2c^2+81a^4d^2-54a^3bcd+ \\ +12a^2b^3d+12a^3c^3)]^{1/3}.2^{-1/3}\Big\}.$$

This is known as the cubic formula; in taking the two cube roots that are indicated in it, always remember that among the nine combinations that result from choosing among three roots in each case, only those three combinations are allowed for which the product of the cube roots is $2^{1/3}a^{-2}(b^2 - 3ac)$.

EXAMPLE. A man wishes to build a belfry, which shall be square at the base, and which shall be three dekacubits higher than it is wide. The desired volume of the belfry is four cubic dekacubits. Please give the required measurements.

Solution. Let x be the height of the belfry in dekacubits. Then the width and thickness at the base are both $(x - 3)$. The equation is thus

$$x - 3)(x - 3)x = 4,$$

or

$$x^3 - 6x^2 + 9x - 4 = 0.$$

The formula gives

$$\begin{aligned} x = \tfrac{1}{3}\{6 + [432 - 486 + 108 + 3\surd(-8748 + 1296 - 11664 + \\ + 10368 + 8748)]^{1/3}.2^{-1/3} + [432 - 486 + 108 - \\ - 3\surd(-8748 + 1296 - 11664 + 10368 + 8748)]^{1/3}.2^{-1/3}\}, \end{aligned}$$

or 4. Thus, a belfry 4 dekacubits high, and square at the base with breadth 1 dekacubit will have volume 4 cubic dekacubits. From this it should be easy to compute the volume in decibels when the bells are installed. The bells are to be installed, of course, on decibelisation day, and special decibel coins will be issued by the vicar of the parish to all those who attend the ringing-in. On the great day when for the first time all measurements will be in cubic units, it was thought especially fitting to offer a seemly entertainment centred on the church and capable of competing, noisewise, with the most deafening discotheque. Afterwards there will be a talk on cubic measurements, the cubit equation and the cubit formula; it is hoped that no-one present will be able to hear what it said. Here is a run-down on cubit measurements; note that no conversion factor from standard British measurements is necessary:

1000 cubic millicubits	=	1 cubic centicubit
1000 cubic centicubits	=	1 cubic decicubit
1000 cubic decicubits	=	1 cubic cubit
1000 cubic cubits	=	1 cubic dekacubit
1000 cubic dekacubits	=	1 cubic hectacubit
1000 cubic hectacubits	=	1 cubic kilocubit
1000 cubic kilocubits	=	1 cubic myriacubit.

The cubic cubit is established as the volume of water displaced by the forearm of the person doing the measuring. This may be established by the method of the great Greek mathematician, philosopher, and physicist Eureka, who when he had put his whole forearm into the bathtub to test the temperature of the water, shouted 'Archimedes!', or as we would say in English, 'Decibelisation!' Regarding the above table, it is necessary to remark that the number 1000 appears only as a rough guide, which is valid in almost all cases, and that the general rule in the case of people with missing fingers or parts of fingers as well as people with more than ten fingers is to use the cube of the actual number of fingers present at the time when the measuring is done. The joints of the fingers are usually taken as dividing the fingers into thirds, and if extra joints are present the finger is counted as more than one whole finger. Thus if you have $1\frac{1}{3}$ fingers missing from working in a sawmill, and if you were born with 11 fingers, then you must use the conversion factor $903\frac{8}{27}$, so that in your case that number of cubic kilocubits make a cubic myriacubit.

The linear cubit, or measurement of length after decibelisation day, is defined as the length of any edge of a cube of water at a comfortable temperature that has been displaced by your forearm. The cube of water should of course be in Euclidean 3-space, and should not be frozen to facilitate measuring the edge as that would change the volume somewhat and hence also the length of the edge. The square cubit is similarly the area of a face of such a cube of water.

Ordinarily, a linear cubit is called simply a *lineit*, and a square cubit is a *quadratit*. A measurement of 'hypervolume' in four-dimensional space is defined in a completely similar way, and is termed a *biquadratit*, or more freely, a *quartit*. This is sometimes shortened to *quart*; other abbreviations are frowned on, and it is especially hoped that the unpleasant *dratit* will not replace the more seemly and correct *quadratit*.

In getting your own cubit, use the arm that seems more natural. It is quite unnecessary to sever the limb at the elbow; simply hold the arm in the water so that the elbow line lies in the surface of the water. Artificial limbs having well-defined elbows may be used exactly as if they were real ones.

4. Pain de maison

Poor M. Boulangiaire, the French baker, remained long in a state of hypostatic tension over his quadratic equations. The trouble was about those imaginary roots. He knew they would not come up in his own bakery, because of certain facts relating to the coefficients. No-one in the village was more sensitive to the good man's anxieties than his widowed sister Gasparde Mange. Indeed, she felt the weight of it more than he, being one of those women who live on anxiety. It could almost be said that Gasparde got fat on anxiety; at least, she tended to eat when she worried, and she worried prodigiously. Her brother, Crétain the baker, was the focus around which her worries orbited. The year after the invasion of the Ostrogoths, Gasparde determined that her body and mind could use a course of slimming, and took off for London. Her previous holiday, a couple of lustra ago, had also been in London; Gasparde's life centred on Beaulieu Derrière, and she could not bear to leave the village more frequently than that. Long ago, she had learned that the food in London was inedible, and that she had nothing to worry about in the land where everyone does his duty.

Gasparde's sojourns in the homeland of the Anglo-Saxons were a spasm of relief for her from the crowded activity of the village. She usually avoided anything to do with home—she did not look at the bucolic Butter Marketing Board posters, the vegetable shops, the police horses, the 'Pinta' advertisements. (What a strange word for milk that was! But everything about the English was strange.) On all the other journeys, her eyes had never tarried on a baker's window; that of all things was to be forgotten. Yet somehow, this time, she could not help it. How queer the bread looked! She knew, of course, that it was not bread; English bakers had a secret art of making the appearance of bread out of air and water. But, it went without saying, they could not make it *quite* like real bread if they chose those ingredients. Whatever their process might be, it certainly did not involve baking. Why, it would all evaporate in the heat—and you could see for yourself that there was no crust.

It must have been just then, when she was ruminating in front of the bakery window, that the idea came to Gasparde. When the assistant came out to ask her to leave (her immense form had been blocking most of the light trying to come into the shop through the plate glass) she had a question. 'Excusez-moi. Zat olmôst spherical wan wiz minimal superficial airia, wat is he call?'

'*Cottage loaf*,' the young person had said. By means of her pocket dictionary, Gasparde resolved the cryptic name: bread of house, it meant; a curious name. Probably because the shape of the loaf, which had a little knob on top, could not be imitated and was a speciality of this bakery. But it was the shape, all huddled together on the inside with no outside surface to speak of, that interested Gasparde. There was something about that that poor Crétain back home in the village might find very interesting. Feeling her appetite rise again at the thought of the serious business of the bakery at home, Mme Mange hurried to the window of a tea-shop for the only infallible remedy: the sight of Englishmen eating mashed potatoes, tinned spaghetti and baked beans on toast. Hunger vanished immediately, and soon the widow's mind returned to her favourite pursuit on holiday—discovering new twists in the unfathomable ways of the islanders.

That winter, she recalled her moment of enlightenment for Crétain. 'You know, it would solve all your problems; only, you must not make it like two balls smashed together. It is more like a *chignon*.'

'— I could never, never compute the volume of a *chignon*; even the sphere would introduce a difficulty, because the number π on my slide-rule is out by several thousandths.' When they spoke to each other, Crétain and Gasparde always began the first sentence with a long dash, as everyone does who speaks good conversational French. It is the first indispensable rule; whoever ignores it is exposed as a conversational barbarian before he has uttered a word.

'— No, no, Crétain. Listen to me, I am telling you. You will not make them like that. Do you even make the ordinary loaves like the other bakers? Yours are pure rectangular parallelepipeds. You will make them cubes, of course.'

'— What an idea! The only baker in France to make a cubical loaf. You cannot imagine the joy of it! And the mathematical implications . . . Now I am glad that I could never find you a husband; you can never know how you have eased my mind. Because of course . . . '

'— There is always at least one real root. It does not depend on the supply of dough, on the orders . . . '

'— Wait, Gasparde. What if no-one orders the variable cube? I will be the laughing stock of the village—there would then be a quadratic equation, not a cubic.'

'— Do you think I have not thought of it? Do you still think, my brother, that because I am a fat old woman I am stupid? Have I not been thinking of it since the autumn? I promise always to eat the large cube. I will place an order, I myself. Only because I am your sister—who else would eat a huge lump of uncooked dough every day? And you are wrong about being the only baker of cubes. That you are already; only they are the small cubes with edge of length 1. There are others besides you who bake by the quadratic equation. But you will bake by the cubic equation; that will be your unique distinction!'

Immediately, M. Boulangiaire reached for a pencil that adhered to his left ear, and wrote on the surface of the table:

$$ax^3 + bx^2 + fx + c = p,$$

where the coefficients are a for the Anglo-Saxon loaves required, b for the flat, square loaves preferred by the Goths (M. Boulangiaire always called them Boches), f for the ordinary French loaves, c for the little cubic loaves whose side was constant, and p for the total amount of dough for which there was provision. Yes, it was going to be a grand game. The villagers would not know what to make of it. To questions about the new loaf, he would only reply, 'It is special—you would not like it.'

QUESTION 24. Whether the cubic equation will always provide a real root, and in fact a positive one, so that the baker can begin to bake?

Objection 1. The coefficient a is the number of Anglo-Saxon cubical loaves, preferably of large size so as to be soggy inside. This is certain to be at least 1, since Gasparde has promised to consume one loaf daily, and she of all women can always be relied on to eat whatever she says she will eat. No doubt she will want several. Moreover, the supply p always provides for more of the little buns of unit side than the requirement c, since the villagers are never that keen to have them. Hence if f is the polynomial function whose value at x is the left-hand side of the equation,

$$f(0) < p$$

and

$$\lim_{x \to \infty} f(x) = \infty.$$

Taking N very large and positive, and applying the Intermediate Value Theorem to f on the interval from 0 to N, we see that

$$f(0) < p < f(N)$$

and hence there exists a real number ξ such that $0 < \xi < N$ and such that $f(\xi) = p$. This is then a positive root.

Reply to Objection 1. This solution of the problem has already occurred to M. Boulangiaire, but has been rejected by him for the very good reason that it requires the Intermediate Value Theorem, of which he was not given a rigorous proof at the École Paranormale. We, of course, know of a rigorous proof of the Intermediate Value Theorem, but it is one that uses the idea of the existence of a universally attracting object in the category of archimedean fields, and the fact that such a universally attracting object must necessarily contain the least upper bound, or supremum, of a set of numbers that is not empty and that possesses an upper bound. Not only would M. Boulangiaire be unable at his present age to assimilate all the necessary concepts to follow a proof, but he would be justifiably annoyed at the introduction of analytical concepts in order to solve an algebraic problem. 'You are employing a blast furnace in order to toast marshmallows,' he would say in his shrewdly practical way.

Objection 2. Replacing any root α by its complex conjugate $\bar{\alpha}$, we get

$$a\bar{\alpha}^3 + b\bar{\alpha}^2 + f\bar{\alpha} + c = \overline{(a\alpha^3 + b\alpha^2 + f\alpha + c)} = \bar{p},$$

so that $\bar{\alpha}$ is also a root. Hence the roots occur in conjugate pairs, and since 3 is an odd number and there are three roots in all, one of them must be equal to its complex conjugate and hence real.

Reply to Objection 2. In order to do the problem this way, it is necessary to construct the complex number field, or some other field containing all the roots of the equation, and possessing an alternating automorphism $k \mapsto \bar{k}$; i.e., one such that $\bar{\bar{k}} = k$; the alternation must have the property that the subfield it leaves invariant is a real field. The study of automorphisms of extension fields and the subfield that they leave invariant is Galois theory, and the study of the complex numbers is analysis; hence, these methods are too high-powered.

I ANSWER THAT there exists a real root of the cubic equation under consideration, and that it is positive. This means that the baker will always be capable of baking bread in such a way as to solve the equation and just use up his dough—although he will have to approximate and so may be a little bit out, the error and hence the amount of dough left over or not available for finishing the last loaf can be made as small as desired. Unfortunately, he will not in general be able to construct x, the length of the French loaf and the edge of the Gothic and Anglo-Saxon loaves, by ruler and compasses as he used to do with the old method. When he had only a quadratic equation to solve, it was possible to obtain the required dimension by ruler-and-compass construction, and the baker greatly enjoyed the process. This recreation will now have to be abandoned. A number of good methods nevertheless exist by which the approximation can be refined within acceptable bakery tolerances.

The point is to have a system of numbers in which it is possible to do addition, multiplication, subtraction, and division, in which it is possible to say whether a number is positive or negative in a way consistent with the arithmetic of the system of numbers; the system must be generous enough to include a positive number that corresponds to the desired dimension x, where x is the solution of the bread equation given by the orders of the various customers. That will all be the case if there is a real extension field of the field of rational numbers containing a root of the equation.

If the equation is reducible, this is obvious. Taking $g(X)$ to be

$$aX^3 + bX^2 + fX + c - p,$$

and α to be a root of this polynomial, suppose -1 is a sum of squares in $Q(\alpha)$. We get the equation

$$-1 = \sum_{i=1}^{n} p_i(X)^2 + k(X)g(X),$$

where of course the $p_i(X)$ have degree less than 3. Since $k(X)$ therefore necessarily has degree 1, or is the zero polynomial, there exists a rational number q such that $k(q) = 0$. Hence, -1 is the sum of squares of a finite sequence of rational numbers. This is impossible, and hence a real root exists.

The question remains whether any of the roots is positive. This is clear if all roots lie in a real field, since their product is $(p - c)/a$, which is a positive rational. Otherwise, the polynomial is reducible, so that

$$aX^3 + bX^2 + fX + c - p = a(X - \alpha)(X^2 + BX + C).$$

Since $B^2 \geqslant 4C$ would imply that all roots of the equation lie in a real field, it may be assumed that $B^2 < 4C$. Since $c - p$ is negative, so must be $-\alpha C$, and it follows that α is positive.

How to find cube roots

In solving a cubic equation by the cubic formula, one is obliged to take cube roots. This may be done by the method of approximation by continued fractions to within any required degree of accuracy. For example, to find $\sqrt[3]{2}$ we note that $x^3 - 2$ changes sign between 1 and 2; this leads to

$$x = 1 + 1/y,$$

and on substitution we find

$$y^3 - 3y^2 - 3y - 1 = 0.$$

The left-hand side changes sign between 3 and 4, which leads to

$$y = 3 + 1/z.$$

Similarly we get $z = 1 + 1/w$; continuing this we get

$$\sqrt[3]{2} = 1 + 1/(3 + 1/(1 + 1/(5 + \ldots;$$

the computation taken thus far gives 29/23, the cube of which is

$$2\tfrac{55}{12167}.$$

5. The Rule of Three

If a man and a half, all of whom speak Scots Gaelic as well as English and have had 5 years of schooling, dig a hole and a half while the temperature is 35° Fahrenheit, smoking 22 cigars, eating 4 lb of preserved goose, reciting a Shakesperean sonnet, and taking a day and a half to do the job, how long will it take one man, speaking only Ancient Egyptian, Sumerian, and Chinook Trade Jargon and having 3 years of schooling, to dig a single hole while the temperature is 3° Centigrade, smoking 11 cigars and eating 3 kilos of *moules marinière* while saying off from memory all the poetic contents of the Egyptian edition of *The Lord of the Rings*? [33]

The above is an example of a question in the Rule of Three. It is a moderately complicated example of the genus, and the fact that it can be solved at all is a tribute to the immense versatility of our convenient algebraic notation. A simpler problem is this: if John gets 10 hectathrills from accompanying to a ball a well-proportioned lady of 25 years who is 5 feet tall, how many thrills does John get from accompanying to the same ball a well-proportioned lady of 50 years who is 10 feet tall? (The thrill is the basic measurement of joy; a hectathrill is equal to 100 thrills.) Let us simplify still further, by supposing that not only the gentleman, John, but also the age of his young lady remain constant—whether she is 25 or 50 need not concern us. The only thing that changes is her height. Then we note that the problem is stated as a question. This is very unpleasant, because it is not a proposition unless it is in the indicative mood. Hence we make it a statement, using a letter (say θ) to stand for the unknown number of thrills. It now takes the form:

θ thrills of John are to 10 feet of lady

as

100 thrills of John are to 5 feet of lady.

Here we have a problem in the standard rule of three, so called because it has as its aim the solution of all problems in which one is to find the fourth, unknown, term in a proposition in which three terms are known. The more complicated examples, such as the one previously mentioned involving a choice of cigars, languages, delicatessen, etc., can only be treated once the main points of the simple, classical case are fully understood. Before considering these, it would be desirable to examine first why there is no such thing as a Rule of Two, or a Rule of One. In a sense, of course, these things do exist. A Rule of One might be thought of as a way of finding a single unknown from a single given quantity, and a Rule of Two as a function of two variables. But these ideas do not fully mirror the structure of the Rule of Three, which is not merely a matter of a function of three variables. It is more nearly a matter of a class of functions indexed by a parameter; two of the given numbers suffice to determine the parameter, and the third determines the particular value of the function associated with the value thus determined for the parameter. Thus, no fewer than three given numbers can produce a sufficiently rich structure. Rules associated with larger odd numbers are then a matter of families of functions of several variables; i.e. the Rule of $2n + 1$ is simply produced by considering an index set A and a function

$$(f_a)_{a \in A} : A \longrightarrow Y^X,$$

where

$$X = \mathop{\times}_{i=1}^{n} X_i.$$

Hence, there can be no Rule of Two since 2 is even, and there can be no Rule of One since $n = 0$ gives a Cartesian product over an empty set of indices (from 1 to 0), so that X is a singleton $\{x\}$. Necessarily the given term in the proposition is $f_a(x)$, and this is just what we are required to find—hardly an arduous task requiring a special rule for its accomplishment.

In the case of John and the expansible–contractible ladies, it is clearly presupposed that once we know how John reacts to a little thing of 5 feet we can say just how a 10-foot version of the same delicious morsel will hit him. Now various men react to different women in various ways. For some, the thrill function reaches its maximum value at 5 feet 2 inches. Clearly, what we must know from the given information is John's thrill function. In other words, the function

$$X \longrightarrow Y^A$$

induced by $A \longrightarrow Y^X$ actually takes its values in the subset of Y^A consisting of injections $A \longrightarrow Y$. Thus, if two different men (John and Xerxes, let us say) receive exactly the same thrillage from the company of a 5-foot woman, then they must also receive exactly the same thrillage from a 10-foot woman. Otherwise, it is impossible to answer the problem as set.

Let us suppose, in order to be able to do the problem, that a lady h feet high produces in a man a number of thrills equal to

$$a \sin \frac{\pi}{10} h,$$

where a is the parameter. We easily determine in the present case that for John the parameter a takes the value 100; hence it is easily computed that he gets exactly no thrills from taking out a 10-foot lady-friend.

6. Polynomials

In treating problems related to the Rule of Three, it is often assumed for simplicity that the functions f_a in the problem are all of the form

$$f_a(x_1, x_2, \ldots, x_n) = a x_1^{k_1} x_2^{k_2} \ldots x^{k_n},$$

where the k_i are integers, and where $A, X_1, X_2, \ldots, X_n$ are all the same field. If we restrict this further to the case where the k_i are natural numbers, we reach a kind of function closely related to the *polynomial.* Polynomials are sometimes divided into monomials (properly spoken, these should be mononomials), binomials, trinomials, ... There seems to be no special name for polynomials that have no terms at all, but otherwise these are just fancy ways of incorporating into the name exactly the number of terms that exist in the polynomial. Of these, only the first type, the monomial, deserves separate study. Hence we begin the study of polynomials with that of monomials, or what might almost be called terms.

An example of a monomial is

$$\gamma A^a B^b C^c D^d E^e F^f \ldots Z^z,$$

where γ is likely to be an element of a ring, and where the exponents $a, b, c, \ldots, z$ are natural numbers. The letters $A, B, C, \ldots, Z$ are called *letters* or indeterminates. It can easily be shown that the monomial just mentioned is a monomial in (at least) 26 letters; in order to simplify slightly the questions which arise, we shall begin by considering monomials in just one letter.

An example of a monomial in just one letter is aX. From this monomial we can make a function, in a way which will be more fully described below. If we assume that the thrill function with parameter is obtained from this monomial, rather than from the function sin, then we shall be able to compute that if John gets 100 thrills from dancing with a lady 5 feet tall, then

$$100 = a.5,$$

so that $a = 20$. This shows that if John dances with a lady 10 feet tall then he gets 200 thrills. (This approach is called *linearising* the problem, and shows one of the many charming applications of this special kind of problem.)

Another kind of monomial is X^a. Using this monomial we produce a parametrically indexed family of functions, and the same problem as before—if the reader will forgive the intrusion of one more worked example of the Rule of Three—can again be worked on the hypothesis that the thrill reaction of a man dancing with a lady depends on the height of the lady according to the law

$$h \mapsto h^a,$$

where h is her height and where a is a parameter which has to be determined in the case of every individual man, since men do not all react to ladies in the same way. In the present case, we get

$$100 = 5^a,$$

and since a is a natural number, this is impossible, since it leads to

$$5^a = 5^3 - 5^2,$$

which can be shown by computation to be false if $a = 0, 1, 2,$ or 3, and which is false for $a \geqslant 3$ because

$$Z^{a-2} - Z + 1 = 0$$

has as its only rational roots a subset of $\{-1, 1\}$. Why can the problem not be done? There are two explanations: one, that the word 'lady' no longer has a definite meaning. In 1840, a self-respecting servant girl could say, 'Why, if I were a lady, I should be delighted to be the object of Captain Ainstruther's affectionate interest.' Then, it was a matter of fact whether a woman was a lady or not. At present, many women who are certainly not ladies are most insulted if they are called women, and speakers of the language are divided between those who call all women ladies, and those who call no women ladies. Now in a mathematical exercise, it is of crucial importance that all the words used have a definite, clearly defined meaning. It is no good applying mathematics to a mixture of vague impressions, compounded of the notions 'lady', 'gentleman', 'well-proportioned', etc. All the concepts must rest on a firm logical basis, as *ball*:

$$\{x \in R^3 : |x| \leqslant 1\}$$

—one may gather that it is the 3-ball from the fact that the people who are going to the ball are in a problem illustrating the Rule of Three, but if it were some other odd number, one would simply take the ball in the Euclidean space of the corresponding number of dimensions. That is one reason why we cannot expect very startling success in determining the thrillage. The second reason is that in this case the image of 5 under $X \longrightarrow Y^A$ is not a surjection $A \longrightarrow Y$, and that this is just as necessary as that the image be an injection.

QUESTION 25. Whether a polynomial is a kind of function, or not?

Objection 1. A polynomial is a kind of function. In fact, if you start with a variable x, by which is meant a number that can be anything it likes, and if there are some constants, by which are meant some numbers that are stuck and cannot change at all, no matter how hard they try, and if addition, subtraction, and multiplication are allowed but not division, then one will get an expression of the form

$$a_n x^n + a^{n-1} x_{n-1} + \ldots + a_0.$$

Now if you let the variable x vary—which is to say, change all over the place—then this quantity will also vary. As x varies independently of any control, the quantity formed in the way described will vary in a way that depends on the way x varies. Hence x is called the *independent variable* and y is the *dependent variable*. But where a dependent variable depends on an independent variable, there one has a function; and functions constructed according to this formula are called polynomials.

Reply to Objection 1. It is bad form to mention variables; not only that, but it is very imprecise to talk about numbers changing all over the place, moving and heaving about like quicksand. It would be incorrect to say that the thing presented here is a polynomial, and just as imprecise to call it a function. It is by no means clear just what it is.

I ANSWER THAT polynomials are not functions. The reason is that polynomials can be used to obtain functions, but that two polynomials may produce the same function without being the same polynomial. Before explaining functions, we must of course go back to monomials, and indeed we must begin by considering monomials without coefficients. There will be a set of indeterminates, or letters, which are to be used; let this be S. Of course, if M is any monoid then M^S has the obvious monoid structure, and it is called the set of monomials in the letters of S without coefficient, and with exponents in M^S. One ordinarily takes M to be the natural numbers N_0. In terms of notation, there is a trick to be learnt; to indicate the element of M^S that sends A_i to n_i for i and integer such that $1 \leqslant i \leqslant k$, where $S = \{A_1, A_2, \ldots, A_k\}$ and k is a positive integer, we write

$$A_1^{n_1}\ A_2^{n_2}\ \ldots\ A_k^{nk}.$$

Moreover, if $A \in S$ and if A is sent to 1, it is permissible not to write the 1; and if A is sent to 0 it is permissible to write neither the A nor the 0; but this does not apply if $S = \{A\}$.

The obvious structure on the monoid $N_O{}^S$ then makes it obvious that (taking $S = \{X\}$) we have

$$X^m X^n = X^{m+n};$$

it is impossible to exaggerate the importance of this identity in computations with polynomials.

Unfortunately, the above considerations of notation have the effect of importing a certain difficulty into the study of the monomial. Taking again the case $S = \{X\}$ for simplicity, it is clear that the expression X^n can mean two separate things: firstly, the function $S \longrightarrow N_0$ that sends X to n; secondly, the nth power of $X = X^1$, which is the function $S \longrightarrow N_0$ sending X to 1. The solution is not far to seek—we merely map N_0 homomorphically to N_0^S by

$$n \longmapsto X^n,$$

which means two homomorphisms by the two interpretations of 'X^n', and observe that the two homomorphisms are not distinct, making one homomorphism and hence one interpretation of 'X^n'.

By the same token, we see that there is an *a priori* danger of confusion between AB on the one hand and AB on the other; or slightly more generally between $X^m Y^n$ and $X^m Y^n$. It would of course be somewhat discomfiting for the theory of polynomials if these were really as distinct as one must suppose, without proof, they may be. Therefore let us attempt to show, if possible, that they remain the same, always restricting ourselves for simplicity to the case AB versus AB.

Question 26. Whether one may be certain that AB is the same as AB?

Objection 1. This, surely, of all the ridiculous quibbles, of all the effete nonsense, of all the crabbed contortions found in the present compendium of cant, is the silliest and most abominably absurd! Is a thing the same thing as itself, when it is mentioned under the same name? Flogging is the answer to those who permit themselves to ask such questions, if no more ignominious punishment is available. And a remedial period of good, hard work could without detriment follow, to be continued until a more socially oriented behaviour pattern develops. Apiary work would do. The patient might be required to repeat, whenever stung, 'A bee is a bee is a bee.'

Reply to Objection 1. It may possibly seem, at first sight, that when two things are mentioned under the same name they are the same thing. If one is set the problem: Prove that Oberon is the King of the Fairies, it is not surprising that there is some work to do—one may have to find a fairy and say, 'Take me to your leader!' But on being asked to prove that Oberon is Oberon, one reacts differently. Since it is not clear how to proceed, even though the task hardly promises to be one of extreme difficulty, one may feel inclined to bite one's nails or to call for the police. It is not unnatural to react in this way.

The present case is in fact slightly deceptive. The conventions of notation have produced two expressions that were intended to say different things, and that were arrived at by different roads; and by pure chance the two expressions have the same form. It is rather like the two English sentences

'Fruit flies like a banana.'

and

'Fruit flies like a banana.'

The first of these sentences means that certain small flies, some of which belong to the species *Drosophila*, and all of which feed on fruit when they are in the larval stage, if they can get it, have among the fruits for which they feel a preferential inclination a yellow, elongated fruit of the genus and species *Musa paradisiaca sapientum.* In other words, those fruit flies really go for a banana. The second of the two sentences means something quite different; to wit, that vegetable produce succeeding the flower passes through the air in a manner resembling that in which a banana would pass through the air. The first sentence could be tested as to veracity by presenting some fruit flies with various kinds of food: a measuring tape, some beefsteak tartare—the list can vary without restriction, except that it must contain a banana. That way, you could tell if fruit flies like a banana. To test the second proposition, one might use a wind tunnel, or a large catapult, in order to study the aerobatic properties of mangoes, lichees, pears, and (just possibly) aubergines and horse-chestnuts, and to compare these with the aerobatic properties of a banana. 'Fruit flies like a banana' is a proposition in entomological gastronomy, whereas 'Fruit flies like a banana' is a proposition in horticultural aerobatics. These two subjects, while they are important intellectual disciplines demanding our utmost respect, are still in their infancy, and unfortunately we cannot safely rely on them for more than tentative, if hopeful, guesses as to whether fruit flies do like a banana, or as to whether fruit does fly like a banana. The important point is that the two fields have as yet no common ground; at present, no reputable interdisciplinary work has been done in both at once; no joint degree has been taken in entomological gastronomy as applied to horticultural aerobatics; no lecture entitled 'Fruit flight and the diet of *Drosophila*: bananism versus pomegranitry' has been read. There is no connection between the two sentences. If we are to prove that AB, understood in one sense, is the same as AB, understood in a quite different sense, we shall not do it just by remarking that they *look* the same.

I ANSWER THAT without doubt $AB = AB$. Consider the three monomials involved in the equation: A, B, and AB. These are monomials in a certain set of letters S, where S certainly contains A and B—considered as letters, not as monomials—and where S may possibly contain any further finite set of letters—say, all Russian minuscules that normally precede the letter 'e' in a native Russian common noun, all Hebrew letters that take daghesh forte but not daghesh lene, and the Devanagari voiced aspirates together with Latin majuscules not congruent to 3 modulo 4.

Now as a monomial,

$$A : A \mapsto 1$$

and if $\lambda \in S \sim \{A\}$ then

$$A : \lambda \mapsto 0.$$

Displaying in a table this information for the monomial A with similar information for the monomials B, AB,

	$\lambda = A$	$\lambda = B$	$\lambda \in S \sim \{A, B\}$
$A : \lambda \mapsto$	1	0	0
$B : \lambda \mapsto$	0	1	0
$AB : \lambda \mapsto$	1	1	0

we get that $$\underset{\lambda \in S}{\forall} A(\lambda) + B(\lambda) = AB(\lambda).$$

Those who have mastered the art of forming monomials may, if they are good, be admitted to the secrets of polynomials. These are among the most important weapons in the arsenal of the mathematicians, so that familiarity and skill in the handling of polynomial manipulations is absolutely essential for passing examinations. The most important skill of all, the *sine qua non* of the would-be algebraist, is the ability to recognise a polynomial on sight. This is not as simple as it sounds. Recognising polynomials on sight can be done on various levels. At a rather lowish level, one may be shown this:

$$a_0 + a_1X + a_2X^2 + \cdots + a_nX^n,$$

and one may be asked, 'Well, how about it? Is it a polynomial, or isn't it?' The low-level answer is (depending on the dialect one speaks),

'Yeah, sure, I guess. Looks like one. Yeah, sure, that's a polynomial, all right.'

Now, such an answer is neither too extremely bad nor on the other hand is it as good as it might be. The worst answer (excluding some very improbable sorts of answers, since there is not room here to consider any but the most common possible responses) would be a flat, definite *No*. After that, the next to the worst answer is a flat, definite *Yes*. The present answer, a hesitant, fumbling *Yes*, has many advantages. Hesitation is often a symptom of superior knowledge; in fact, one may have rarely had the opportunity to see polynomials, so that one is just barely able to tell when something looks like a polynomial. In order to hide one's ignorance, one is inclined to be daring and come out with a flat answer. But then one realises that experts often hesitate. Experts have all sorts of reasons to think two ways on a question, because they realise all the fearful complexities of a subject. By hesitating, one may get oneself mistaken for an expert. The finishing touch on this approach is to use the all-important phrase, 'It all depends'. (For added effect, learn to say this in several foreign languages and say it as if quoting from an eminent algebraist who spoke the appropriate language, as '*Das kommt darauf an*, as Gauss might have said'. Gauss may be used for almost any subject.)

Rather better than this last answer is one that shows, if not any great familiarity with polynomials, at least some idea about mathematics: one may say, 'That all depends. Do you treat it as a polynomial? Do you operate on it as you operate on a polynomial? If so, it *is* a polynomial, as near as makes no difference. If not, then it is certainly not a polynomial.' This could be extended further by, 'Is it part of a system isomorphic to a system composed of polynomials?' and words to that effect.

Of course if one can define a polynomial, it is a good thing to do so; and in that way one can of course give the correct answer to the question. Knowing what a polynomial is comes before all other knowledge in the field of polynomials as an essential prerequisite. Nothing could be simpler. First, polynomials have got monomial expressions, and these to begin with are pure and *sans* coefficients. Then, they have got to have a supply of coefficients from somewhere. Since the coefficients are going to have to be added and multiplied at some point, they are usually taken to come from a ring. Thus we have the pure-monomial-expression monoid N_0^S and the ring $\boldsymbol{\sigma}$. The most obvious thing to do with these is to form the monoid algebra of N_0^S over $\boldsymbol{\sigma}$, which may be denoted ${}_{\boldsymbol{\sigma}}N_0^S$, or for simplicity $\boldsymbol{\sigma}[S]$. This consists of all functions in

$$ {}_{\boldsymbol{\sigma}}{N_0}^S $$

that take the value 0 everywhere except on a finite subset of ${N_0}^S$. These functions are added in the obvious way, and are multiplied by convolution or *Faltung*. This means that if p and q are polynomials, their product r has the property that

$$ \underset{\mu \in N_0^S}{\forall} r(\mu) = \sum_{\varkappa\lambda=\mu} p(\varkappa)q(\lambda). $$

An obvious change of notation produces the more usual form; thus when $S = \{X\}$ we can write the polynomial in the form

$$ a_0 + a_1 X + \cdots + a_n X^n. $$

The reader will note that this change of notation bears an obvious similarity to integral transforms. The verification that the sum of X and Y, considered as polynomials in the letters X and Y, is exactly the polynomial $X + Y$ need not detain us; it is no harder than the proof that the product of X and Y is XY, which has already been explained.

Polynomial identities

One of the most basic identities for practical purposes in connection with computations is

$$(X - Y)(X + Y) = X^2 - Y^2.$$

This is sometimes expressed in words: the sum times the difference is the difference of the squares. Since we know how to multiply polynomials (previous section), this should not offer any great difficulty. Every term in the polynomial $(X - Y)$ has degree 1; in other words, the value of the function at an element of the monoid of pure monomials is 0, unless the element of the monoid of pure monomials is $X^h Y^i$ with $h + i = 1$. Note that for simplicity it has been taken for granted that all the polynomials are polynomials in the letters X and Y, and that if this supposition is not made then the last statement must be modified. Similarly, the polynomial $(X + Y)$ is a function taking the value 0 on the pure monomial $X^j Y^k$ unless $j + k = 1$. Note further that we have no idea what is meant by the statement 'the function takes the value 0 at a monomial', since we have no idea what this 0 is and are even unaware what ring 0 is the zero element of; and note also that this makes no difference to the argument. Putting

$$\begin{aligned} h + j &= m \\ i + k &= n, \end{aligned}$$

we see that the product of the two polynomials has degree 2; that is, their product is a function that must take the value 0 at $X^m Y^n$ except in case $m + n = 2$. This is because $m + n = (h + j) + (i + k) = (h + i) + (j + k) = 2$ whenever even one term in the convolution sum is non-zero. Hence we know that the product takes the form

$$aX^2 + bXY + cY^2,$$

and it remains only to compute the actual values of a, b, and c. In order to compute a, it is necessary to observe only that if

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} h \\ i \\ j \\ k \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

then (h, i, j, k) must differ from $(1, 0, 1, 0)$ by an element of the kernel of the homomorphism $Z^4 \longrightarrow Z^4$ determined by the matrix; hence is of the form

$$(1 + w, -w, 1 - w, +w),$$

where w is an integer. Since it is further required that h, i, j, k be natural numbers, we must necessarily have

$$w = 0.$$

Hence the coefficient of X^2 is the product of the coefficient of X in $(X - Y)$ and the coefficient of X in $(X + Y)$, or 1.1, giving

$$a = 1.$$

In a similar way, if $X^h Y^i$ and $X^j Y^k$ are to make a contribution to the coefficient b of XY, then we must have

$$(h, i, j, k) = (1 + w, -w, -w, 1 + w).$$

There are seen to be two solutions in naturals, namely $(1, 0, 0, 1)$ and $(0, 1, 1, 0)$, and we get $b = 0$. Similarly $c = -1$ and we are done.

Note that the above product $X^2 - Y^2$ becomes $X^2 + Y^2$ over rings of characteristic 2, where $-1 = 1$.

An exactly similar method may be used to show that many other familiar polynomial identities hold. Alternatively, of course, one may verify that the monoid algebra $\sigma[S]$ satisfies the ring axioms; i.e., one may verify that the convolution product is associative:

$$\sum_{\nu=\iota\mu} \Big(\sum_{\iota=\varkappa\lambda} p(\varkappa)q(\lambda)\Big)\, r(\mu) = \sum_{\varkappa\iota=\nu} p(\varkappa) \Big(\sum_{\lambda\mu=\iota} q(\lambda)r(\mu)\Big)$$

and that the 'constant polynomial' taking the value 1 on the neutral monomial and taking the value 0 elsewhere is neutral for convolution. Distributivity over addition must also be verified.

7. What are brackets?

Nothing is so important in mathematics as knowing what brackets are. In America and elsewhere abroad, brackets are parentheses; at least the people in those places think that brackets are parentheses, and call them parentheses. Doubtless this is but one instance of the fact that, broadly speaking, the people abroad speak broadly. Americans, especially speak so broadly they speak expansively; they expand the word 'lift', a box for hauling people up and down in, into 'elevator', and they have applied the same principle to the word 'bracket'. This habit is known as *American largesse.*

The division of brackets is twofold. First, they are divided into species, which are the *round*, the *square*, and the *curly*. Secondly, each species is itself divided into two sexes, the *left-hand* (or *sinister*) and the *right-hand* (or *dexter*). For example, (is a left-hand bracket, and) is a right-hand one.

A parenthesis on the sex life of brackets

Human beings live in a very different world from that inhabited by brackets. Our world (the author is himself a human being) is a three-dimensional one; that is, disregarding time and speaking locally, a human being who refrains from extended space travel spends most of his time in a homeomorph of R^3. In a similar sense, we may call a bracket a two-dimensionalite; he (or she) lives in R^2. Just as we can hardly imagine the life of a being in four-space, so can brackets only dimly guess at us. A thoughtful bracket could certainly form the abstract idea of the space we live in. An imaginative bracket could perhaps people such a figmentary space with putative beings. A lucky bracket might guess something about the division of our race into sexes, and the shapes of our bodies. But surely no being so alien to us could ever suppose our complex customs of marriage and courtship to be such as they are. Brackets, on those rare occasions when they think of us at all, must picture us as some kind of huge, curly bracket.

But are we in a better state with regard to our little fellow creatures? Most of us see them often enough; how often do we try to think what life is really like for them? Yet the rules of their game, the system sanctioned by nature and society by which they regulate their lives, is perhaps as complex for brackets as our own system is for us.

One misconception must be scotched at the start. Just because the left- and the right-hand brackets look rather alike, people sometimes think that no polarity exists between them. This is not true, as we shall see. An especially dangerous idea is that a left bracket can, by leaping upward on the page and turning a somersault, become a right bracket—in short, that sex-change is possible for brackets. Again, not true at all. The bracket remains on the line, and never tilts forward or backward; so that he (or she) can never change sex. Motion exists, but it is always a motion along the line of print.

Brackets marry. Unmarried brackets are extremely rare; indeed, they are almost non-existent. (There is an exception in France.) Brackets marry brackets of the *opposite* sex, but of the *same* species. Again, one would like to say that this rule is universally true, but there are (alas!) exceptions. These are as follows, and readers may well wish to close their eyes: square and round brackets of opposite sexes do marry. Unions of male or left-handed squares with female rounds, and unions of female squares with male rounds are about equally frequent; the couples are never very deliriously happy, being separated by an interval; e.g., $[a, b)$ on the one hand and $(a, b]$ on the other. No such mixed matings are ever indulged in by the curlies. The name of marriage is too holy to dignify the unnatural liaisons that occur chiefly in France, to the shame of that proud nation: such monstrosities as $[a, b\,[$ and $]a, b]$ in which two brackets of the same sex join horribly. Even harder for virtuous nations to understand, perhaps, is the ridiculous sort of marriage that occurs there between two brackets who, although they are of the same species and of opposite sexes, have got their rôles reversed: $]a, b[$. Even in France, it is only the square brackets that commit these abnormalities; thus curly brackets are the purest of all in their marriage behaviour.

Courtship as such is hardly known among brackets. Brackets are mature almost immediately after 'birth', quite contrary to the rule that long-lived creatures generally take a long time to grow up. They marry just as quickly. There is almost never any doubt as to who is married to whom. Thus in $(a(bc))$ the inner pair are married, and the outer pair are married. The absolute fixity which reigns in the relations of brackets might lead some human students to suppose that love among the brackets is a very humdrum affair. As we all know, courtship by the very uncertainty of its result adds a dimension of suspense and anticipation to love. Many romantic novels play on the theme of courtship; few carry on the story into married bliss. If we examine our hearts, we see that we should not be satisfied if the course of love were perfectly smooth. When we examine the life story of a bracket, perhaps we ask, 'How can it interest them?' It seems to us like utter monotony.

Nothing, in actual fact, could be further from the truth. The love life of brackets is not static bliss, but dynamic anticipation. The male and female brackets forever strive to come together, and forever they are prevented. There is always something in the way.† Often one sees couples for whom the gleam of hope must appear infinitesimally tiny and distant, like a faint ray of light to a lost explorer of underground caverns:

$$(x_1 + y_1 + z_1, x_2 + y_2 + z_2, \ldots, x_q + y_q + z_q).$$

At other times, especially in these latter days, couples almost in each other's arms:

$$f(.).$$

Indeed, a male bracket must feel much as did a knight in the days of chivalry, who had pledged himself beyond recall to a lady far above his station, to whom he could never hope to attain.

So all-pervasive is this feature of parenthetic love life that we ought perhaps to speak of betrothal rather than marriage. Many bracket couples not only cannot contrive to come together, but are prevented even from seeing one another by older couples who stand between the lovesick male and female bracket; while these older couples may themselves be the objects of a similar chaperoning operation performed by yet other couples:

$$((ab)((c(de))e))f.$$

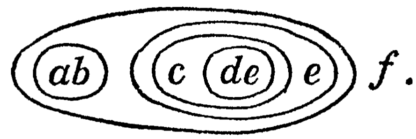
Here the couple (de) separate another pair in $(c(de))$ and these separate yet others, and so on. Only (ab) and (de) can see each other.

Looking on with pitying eye, if we are kind we may be seized with a desire to interfere in the rôle of Cupid. It would be possible for us, from our god-like vantage, to unite the pairs, to bring to fulfilment loves otherwise unrequited. We can twine two brackets in a true lovers' knot, and from (ab) make

$$ab$$

† See p. 61.

or from $((ab)((c(de))e))f$ make



Thus the two are made one, and the he-bracket may say to his she-bracket, 'Borne of my borne, and flèche of my flèche!' It is possible for us to do the thing; but is it wise? For what we have joined can no longer be put asunder. What is written, is written, like the laws of the Medes and the Persians. And it may begin to pall on the brackets. Too much of a good thing may well be worse than none at all. Had we not better leave well enough alone? For after all is said and done, human knowledge of the love life of brackets is meagre.

The most important use of brackets is to show the order in which operations must be performed. Thus, the above may indicate

$$F(F(\,F(a, b), F(\,F(c, F(d, e)), e)), f).$$

The function F is suppressed as being too obvious for mention, and the simpler form involving only brackets results. There is a different approach which is used in Poland sometimes, whereby instead of suppressing the functions one suppresses the brackets; for example, one writes

$$FFFabFFcFdeef.$$

This way of writing things is said to be easy for machines to read. One may also replace brackets by a numeral or mark representing the depth of the deepest bracket at the point, as

$$ab///c/de//e////f.$$

However we choose to indicate the order of operations, we must make it clear that one must perform either ab or de first, and that the last step is the one that uses f.

It is clear that when brackets are used in this way, certain rules must be obeyed if the result is to make sense. If the rules are not followed a very unhappy situation develops in which brackets are not married, or are not sure to whom they are married. A sensitive reader soon notices when this has happened—we do not know quite how; perhaps it is because of a kind of thought transfer from bracket to man. For example, in writing

$$(ab)c)d)((ef)$$

the author has started some very sad brackets on the road of life. In $((ef)$ we have, on the left, a sort of eternal best man who can never be married and who is doomed to be part of an eternal triangle, hanging about on the outskirts of the married couple (ef). It cannot be good for any of them, although) may be flattered by the extra attention she gets. There are other heartrending situations present in this example, and the author would never have created this woeful little microcosm but for his desire to warn others not to make the same mistake.

If we are to avoid such errors, we need a rule explaining when expressions obey the social rules so necessary for the smooth functioning of domestic life among the brackets, and when the expressions introduce friction into the well-lubricated workings of the bracket household. We need to have a procedure for writing bracketed expressions that will produce all the right expressions, and none of the wrong ones. Since a bracketed expression can be obtained from two smaller expressions by writing one down after the other, and then surrounding the whole mess by a pair of brackets, we may write

$$S = A + (SS),$$

where $A = a + b + c + \cdots + z$, unless we wish to use further letters; e.g., the 5000 most common characters of standard literary Chinese. If we wish to use these they must be added into the sum for A. Note also that the brackets are not brackets here, but generators of a free semigroup on 28 generators (or 5028 generators if you include the Chinese characters). It is 28 and not 26 because of the two brackets. It is clear that all bracketed expressions are generated, and only such expressions. Moreover, it is clear that no expression is generated ambiguously (use induction on the number of letters in the expression, not counting the brackets). It is extremely difficult to expand S as a series in the semigroup on 28 generators with integer coefficients, but as we have said the coefficients in such an expansion are all 1. If, however, we delete (and) and replace A by a, we get

$$S = a + SS,$$

and if $S = k_0 + k_1 a + k_2 aa + \ldots$ we see that this is equivalent to

$$(k_i)_{i \in N_0} * (k_i)_{i \in N_0} = (\delta_i^1)_{i \in N_0} + (k_i)_{i \in N_0},$$

where $*$ is the convolution product and where $(\delta_i^1)_{i \in N_0}$ is (0, 1, 0, 0, 0, . . .), the sequence that is 1 in the first place and 0 in the zeroth and all other places. With $q_i = k_i + \delta_i^0$, we get

$$(q_i) * (q_i) = \tfrac{1}{4} - a,$$

which gives us

$$k_i = \frac{2\,(2i-3)!}{i!\,(i-1)!}$$

by the binomial theorem. Thus there are 429 ways of inserting brackets in an expression of length 8: for instance, one way of inserting brackets in *abcdefgh* is

$$(a(b(c(d(e(f(gh))))))),$$

and there are 428 other ways.

4

TOPOLOGY AND GEOMETRY

1. Into the interior

Topology begins at Ujiji, on the continent of Africa, where for the first time the interior was penetrated by the explorer Stanley. If it were not for Stanley, we should be as ignorant of the interior as we were then, and topology would be dark to us—as dark as the thickest jungle of the dark continent.

Though many people are acquainted with the outlines of the story, not so many know the important details. We give here only as much of it as is necessary for our purposes. [34] Stanley penetrated so deeply and became so lost that an American reporter, Dr L. I. Presume, was sent to find him. The American, as he approached Stanley, extended his hand stiffly according to the custom of the time—people in those days were still rather Victorian. There were no herald and no butler to announce his arrival. After all, one *was* in the bush. The inhabitants of Ujiji (called Bajiji) spoke Mujiji, not English. Having no-one to announce his name, he announced it himself: 'Dr Livingstone I. Presume!' This piece of American resourcefulness has gone down in history, and has become a famous quotation.

But it is not this for which we mathematicians remember Dr Presume. Indeed, we break off the history of exploration and discovery at this point, not even pausing to recount the later death of Stanley at Ujiji, and his mummification in the rays of the all-purging sun; not dwelling on the forwarding of his remains to the isle of Britain, where the skies are continually overcast with cloud, or on their consequent putrefaction and interment. These things need not detain us.

Dr Presume, being American, is usually called by his first name Livingstone. It is not generally known, except to topologists, that Livingstone approached Stanley along a net. A naked white man, speaking only a variety of Swahili picked up from the apes with whom this strange being consorted, had at first attempted to teach Livingstone to approach Ujiji along a sequence, swinging from point to point on a vine. This was the white apeman's own customary mode of locomotion, acquired also from his simian companions. But Livingstone found that his ten-gallon Stetson hat kept falling off in the breeze. That was when the birthday boy (whose name was Tarson, and who is thought to be related to the Tar-baby of American Negro folklore) strung a net over the trees for Livingstone to approach Ujiji along. As he dangled from the net, swinging along by his hands like a circus performer, Livingstone stretched, and was taller when he arrived at Ujiji than when he had set out. There was more Livingstone than before. For this reason convergence along a net is sometimes called Moore–Smith convergence.†

The concept of the *interior*, and how a point of the interior may be approached, is one of the great starting-points of topology. We hold these truths to be self-evident:

(i) The interior of everything is everything;
(ii) The interior of the interior is the interior;
(iii) The interior of anything is a part of that thing;
(iv) The interior of the intersection is the intersection of the interiors.

Perhaps these will be made more clear by a word of explanation. Number (ii) says that if one considers, say, Africa, and then the interior of Africa, and finally the interior of the interior of Africa, the last two are the same place. Number (iii) says that the interior of Africa is entirely African, so that the Spitsbergen archipelago is not in the interior of Africa. To understand (iv), think of the United

† Taking Smith as a sort of generalised name for Livingstone.

Kingdom and also of the island of Hibernia. These intersect in a place called Northern Ireland. Its interior is the interior of Northern Ireland. The interior of the United Kingdom and the interior of the island of Hibernia then must also intersect in the interior of Northern Ireland. The conditions may be expressed by writing

$$
\begin{aligned}
&\text{(i)}\ X^\circ = X\\
&\text{(ii)}\ A^{\circ\circ} = A^\circ\\
&\text{(iii)}\ A^\circ \subset A\\
&\text{(iv)}\ (A \cap B)^\circ = A^\circ \cap B^\circ;
\end{aligned}
$$

if we write AB for $A \cap B$ and 1 for X, we see that these conditions make sense on any monoid, and that they say that $A \mapsto A^\circ$ is an idempotent endomorphism of the monoid such that $A^\circ A = A^\circ$. In order that $\subset$ should be a partial order, however, it is reasonable to restrict attention to commutative monoids in which every element is idempotent. For technical reasons, it is sometimes convenient to consider the case in which the monoid $\mathbf{M}$ satisfies the further condition that if $(a_i)_{i\in I}$ is a family of elements then there exists an element $a \in \mathbf{M}$ such that

$$\forall_{i\in I}\ a_i a = a_i$$

and such that $ax = a$ whenever

$$\forall_{i\in I}\ a_i x = a_i.$$

We write $a = \bigvee_{i\in I} a_i$, and note that if it exists, a is unique. In this case, it is obvious that the submonoid $\mathbf{T}$ of $\mathbf{M}$ consisting of elements invariant under $a \mapsto a^\circ$ is itself closed under $\bigvee$, and that of course the restriction of $^\circ$ to this submonoid is the identity. One calls $\mathbf{T}$ the *topology* of $\mathbf{M}$. Very often, one takes an even more special case: the set of all subsets of a set X is a commutative monoid in which every element is idempotent (the operation being $\cap$ and the neutral element of course X), and so all the above remarks may be applied to it. If we take A° to be $\varnothing$ if $a \neq X$, we get the *indiscrete* topology $\{\varnothing, X\}$. A space in the indiscrete topology is like an enormous room. Everyone is lumped together, and nobody is housed off. In fact, the indiscrete topology is the most un-housed-off topology there is. There is no separation, no seclusion, and so if one is living in such a space it is impossible to do anything discreetly. Lovers find indiscrete spaces very tiresome indeed. But they are a paradise for exhibitionists.

Considering our own terrestrial orb for the moment as X, we see that Ujiji is in the interior of Africa, but that Africa is not the whole earth. Hence our world is not indiscrete. This will be no surprise to lovers, who find this world a very pleasant place indeed, especially in the spring. Nor (to leap ahead for a moment) is it discrete—and none but hermits would have it so.

QUESTION 27. Whether it is possible to use nets in order to approach a point in a topological space?

Objection 1. Nets are merely glorified sequences. But approaching a point along a sequence is like marching on a pogo stick—you may never get near your objective. Suppose topological space to consist of the ordinal numbers up to, and including, the first uncountable ordinal Ω. The open sets are the intervals $[\alpha, \Omega]$ with $\alpha < \Omega$; alternatively, the interior of A is $[\alpha, \Omega]$ if α is the first ordinal less than Ω such that $[\alpha, \Omega] \subset A$, and is $\emptyset$ if no such α exists. If a sequence (α_n) approaches Ω it is clear that

$$\alpha_n = \Omega$$

from some point onwards. In other words, it is impossible to approach Ω along a sequence except by being at Ω almost from the start. But that would be cheating. Hence, sequences are inadequate for a proper stalking exercise in this kind of country, and no experienced ordinal-hunter would use them.

Reply to Objection 1. It is true that nets are glorified sequences, and that sequences are woefully insufficient in many spaces. One weeps to think how hard it would be to approach one's objective if there were nothing better than sequences to use. It would be like lion-hunting Masai style, with spear and leather shield. Fortunately, weapons have been devised that make lion-hunting as easy as picking up a sleeping pussy cat; they are, in essence, merely glorified spears. The projectile is thrown more forcefully and accurately, and from a greater distance. Similarly, nets can do what sequences cannot, despite the fact that they possess the same basic principle of design.

Note carefully that the use of nets in approaching or converging to a point in a topological space has nothing at all to do with the use of nets in Roman circuses by fighters known as *retiarii.* These did not hunt lions primarily, as is commonly thought, but were pitted against an armoured swordsman with a fish on his hat called a *murmillo.* The *retiarii* carried tridents, which are of almost no use in topology. The use of nets for catching armoured swordsmen is by now almost a lost art, whereas thousands of undergraduates annually engage in contests to determine which of them can best stalk a point in a topological space using nets. The losers are fed to the lions, in a quaint survival of the ancient life of the arena.

Objection 2. To approach a point along a net, the net must be finer than the neighbourhood filter at the point. When Livingstone set off for Ujiji on the net which his friend Tarson had provided for him, he must have realised that even in the neighbourhood of that savage place, the neighbourhood filter could not be so coarse as his net. Of course in the nineteenth century it was difficult to obtain really fine filters in such heathen places (that was the main reason why missionaries were sent out, bearing the message of modern civilisation and comfort); but even so, it was inherently unlikely that the wildest cannibal could drink his coffee without removing the grounds with any device more effective than a fish-net.

Reply to Objection 2. It is an abuse of language to say that a net is finer than a filter. Only another filter can be finer than a filter. What is actually meant by the phrase is that a filter associated with, or generated by, the net in question is finer than a certain filter. It is also a mistake to talk about nets and filters as if they had holes in them; or as if nets and filters were distinguished according to whether the holes were larger or smaller than the holes in cheese-cloth. But it is true that filters are a more *refined* concept than nets.

By the way, another possible confusion may be avoided if it is pointed out quite sharply that filters, in our sense, have nothing whatever to do with *philtres*; thus, when Tarson is feeling very melancholy he may make use of philtres in wooing his girl friend Jane, but he never lends philtres to American reporters who wish to approach her. Philtres are not very fashionable in topology, unlike filters. The two must never be confused. Putting a philtre through your girl friend is very different from putting your girl friend through a filter. Like any other substance, when she passes through a filter she comes out in a refined state; whereas when a philtre passes through her the probable effect on her behaviour is not likely to be one that could be described in terms of greater refinement.

I ANSWER THAT approaching points in a topological space along a net is indeed perfectly possible.

Sometimes it is even possible to approach a point along a sequence. We all know about the thirsty frog who wished to jump down the well, but found that his strength was fast failing him. The well was two feet away. His first jump took him one foot, but his second jump carried him only six inches. Indeed, he found that after each jump, he had only enough power left to jump half as far. Thus the total distance he had jumped after n jumps was

$$2 - 1/2^{n-1} \text{ feet.}$$

By taking more jumps than a certain minimal number he could ensure that he would come as close as desired to the well, and that he would ever afterward remain that close to the well. (He could never actually reach the well; but if we are to do experiments on animals we must first harden our hearts to the promptings of pity. It is of no interest to the investigating scientist if the frog feels terribly dry, if his skin and throat become parched and his muscles fatigued. No doubt frogs cannot really feel pain. In any case, if the frog fell in the well he would surely drown.)

In a more general topological space than the real numbers, our frog might find it necessary to proceed along a net indexed by a more general ordered set than the set of natural numbers. So as to avoid needless complexities, we may assume that the supremum of any two elements in this ordered set always exists. We say that the net $(x_i)_{i \in I}$ converges to, or approaches the point x if for every neighbourhood N of x one can always ensure that a term x_i of the net is in the neighbourhood N by requiring merely that i is greater than a certain suitable element i_N of the ordered set I, which must depend on N only. If by a *tail* of the net we mean a net $(x_i)_{i \in I'}$ obtained by restricting the net to a subset I' of I of the form $I' = \{i \in I: i \geqslant j\}$, where of course j may be any element of I, then the net converges to x if and only if any filter that contains the images of all the tails of the net must necessarily contain the neighbourhood filter. If $(x_N)_{N\in\mathscr{N}}$ is a choice function for the neighbourhood filter $\mathscr{N}$ at x, and if this is ordered in the natural way, then this net approaches x.

QUESTION 28. Whether Africa is connected to America?

Objection 1. It would appear obvious that America cannot be connected to Africa. America is a continent; i.e., a vast extent of land that one may run through without crossing the sea. [35] But, if America and Africa were connected, then one could run about all over America, cross over to Africa, and then run all over Africa. If one could do that, then the union of both America and Africa, together with the bit of land that one ran along, would amount to a vast extent of land, and one would have run through it without crossing the sea. That is, the union of these three land areas would be a continent. But there are exactly seven continents, and no one of them is a superset of two others. Hence America and Africa are not connected.

Reply to Objection 1. According to the definition used, it is not clear at all that America is a continent. The word 'vast' is defined [14] as 'Wast, huge, hurly; wheady, wide, broad and large, misshapen, illfavoured, . . .'. It is perhaps doubtful if all these adjectives apply to America. If they do apply, to what other parts of the earth do they apply? Most countries are rather oddly shaped, and seem large if one walks over them. But the condition about 'running through' the vast extent of land is even harder to justify. The only sensible meaning that can be given to this is that one is able to follow a path that crosses every point of the land in question. It would not appear to be sufficient that one could find a path that would pass within a short distance of every point of America. The topological properties of America are little known: is it, for instance, an open set? Is it compact? Is it even a Borel set? And so it would appear difficult to settle the existence of a continuous surjection $[0, 1] \longrightarrow$ America.

But let it be granted that America is a continent. Then Eurasia, by the argument presented in the objection, is as much a continent as any other. Hence a continent may indeed be a union of two others, and the conventional total of seven continents is very wide of the mark.

Objection 2. America is not even connected to itself, since 1914 when the Panama canal was completed, severing the North from the South and putting asunder what was meant to stay together. How then can America be connected to something else?

Reply to Objection 2. If $X \cup Y$ and $Y \cup Z$ are each connected, and if Y (in this case, Africa) is not empty, then it is elementary that $X \cup Y \cup Z$ is connected, though $X \cup Z$ may not be. But Africa has at least one point; namely, Ujiji. Hence we cannot infer from the argument in the objection that America and Africa are not connected.

I ANSWER THAT Africa and America are not connected. We may safely take the surface of the earth to be a sphere S^2. (This is known to be only approximately true. We must first break down all natural bridges, which of themselves make the earth a surface of high genus. It is not known to the author how numerous these bridges are in various parts of the earth, and they will present a difficulty if there are any in certain parts of the planet, especially Greenland, Antarctica, and the Eastern tip of Siberia, and certain small islands. The restoration of the natural bridges, and the proof that even after the restoration the Americas are still not connected to Africa, are tasks left to the reader as exercises.) At least, we shall take the surface of the planet to be a homeomorph of S^2. Now it is well known that neither the north pole ν nor the south pole σ lies in America or Africa. Include the punctured sphere $S^2 \sim \{\nu, \sigma\}$ as the unit sphere in R^3; circumscribe about it the cylinder $S^1 \times (-1, 1) \subset R^2 \times R = R^3$, which has the line through $\nu = (0, 0, 1)$ and $\sigma = (0, 0, -1)$ as its axis, and project the punctured sphere outward from the axis on to the cylinder. The homeomorphism just described is, of course, nothing but the usual way of defining the equiareal projection, a kind of map. Mercator's projection would have done as well, but is more difficult to describe. Now identify the punctured sphere with the cylinder. Choose points $\gamma =$ Porto Praia in the islands of Santiago, of the Capverdian chain, and $\alpha =$ Abkit or Anadyr (take your choice). Then if $\alpha_1, \alpha_2, \gamma_1, \gamma_2$ are points a little east and west of α, γ, and if the projection on S^1 sends these to a_1, a_2, c_1, c_2, a continuous function

$$S^1 \longrightarrow [0, 1]$$

exists sending

$$\begin{aligned} a_1 &\longmapsto 0 \\ a_2 &\longmapsto 1 \\ c_1 &\longmapsto 1 \\ c_2 &\longmapsto 0 \end{aligned}$$

and taking values that depend linearly on arc length over any interval not containing any of these four points. (By an interval on S^1 we mean a proper connected subset containing more than one point.) This fact is not difficult to get, and can be obtained by noting that S^1 is [0, 1] with 0 and 1 identified. It may be taken as a fact of geography that by the composition of

$$S^2 \sim \{\nu, \sigma\} \longrightarrow S^1 \times (-1, 1) \longrightarrow S^1 \longrightarrow [0, 1]$$

we have sent all of Africa to 0 and all of America to 1. Restricting to Africa $\cup$ America and corestricting to $\{0, 1\}$, we have a surjection.

2. The insides

People often talk as if they had insides. When a man is feeling queasy, uneasy, under the weather, when he suffers from neuritis or neuralgia, he may say mournfully, 'Oh! There is something the matter with my inwards; I am not quite right inside; I am the victim of intestine strife.' What rubbish! How does he know that he has any insides? Just because human beings ordinarily possess livers, guts, gall-bladders, and other such slimy organs, does not go any distance toward proving that these things are *inside* the human beings. In many fairy stories, people's insides are represented as being outside the people. A little boy may be entrusted with the care and keeping of his mother's heart, and may carry it about with him. How do we know that this is merely an extraordinary fancy? Why do we so easily suppose that it cannot happen to us? Perhaps everyone's heart is outside him. Perhaps my heart is outside me, and my fingers and eyes, the paper I am writing on, the reader and the book he is reading are all inside me. Or perhaps the book and I are inside the reader, and his stomach is outside him. Who can tell?

But even worse, why do we suppose that there is a distinction of the world into two parts, one outside us and one inside? Ah, the reader may say, now there you go too far. I am willing to entertain the ridiculous notion that my insides are really outside me, and my outsides are inside me. But now you ask me to think that there is no difference between the inside and the outside; that my stomach and the book I am reading are both on the same side, the only side; that there are not two sides but one side. It is too much. I cannot imagine it.

Yet, that is the situation we are faced with. It is not asserted that a human being has not got an inside and an outside; all that is asserted is that there is a problem. And it is a topological problem. Now a human being is topologically a difficult object to study. He is always moving about; that is, he is constantly changing shape. A man with his fingers in his ears has two more handles than a man standing with arms and legs outstretched. If the alimentary passage is not blocked, and if we make certain other simplifying assumptions, we may consider the man to be composed of the skin and the lining of the oesophagus, stomach, and intestines. It must be emphasised that in this view, the bones, muscles, brains, etc. are not part of the man thus considered. He is a surface; and it would help our deliberations if he would be so kind as to keep open his mouth and his anus. The breezes must pass freely through the tunnel. Then a man is a torus; to be exact, he is a two-torus $T^2 = S^1 \times S^1$. He is the product of two circles. More than that, he is a torus in three-space R^3. Has he got an inside and an outside? To put it another way, if we consider all the rest of the universe besides the man, which consists of the guts, heart, liver, brain, blood, etc. of the man together with the Eiffel Tower, the Big Dipper or Ursa Major, Grant's tomb, and whatever else exists besides the man himself—if we consider all that, is there a sensible way of dividing it into two portions that shall be separated from one another by the man? Is there a division of the waters under the firmament from the waters above the firmament? Is the liver inside the man and the Eiffel Tower outside—or vice versa? Later we shall return to this question. Before we go on to simplify the problem sufficiently to make it tractable, let us pause to consider a related question, namely that of a man who has closed his mouth and his anus. This type of man is a sphere, as we shall see later. What about him? Has he got an inside and an outside?

Here is an easier problem:

QUESTION 29. Whether a circle has an inside and an outside?

Objection 1. Consider a hoop or loop floating in space after being jettisoned by a lunar module. The hoop is a circle, but it is not a subset of any particular surface. The remainder of the universe, after the hoop has been deleted, is connected. Hence the circle has no inside and outside.

Reply to Objection 1. It is necessary to stipulate that the circle lies in some surface.

Objection 2. Consider a circle drawn on an inner tube, of the sort used to inflate the tyres of automobiles. If the tube is cut along the circle, it may or may not fall in two pieces. For a very small circle, the tube will fall in two pieces (Figure 5a), but for a circle drawn right round the tube, when the tube is cut it remains in one piece (Figure 5b). The second kind of circle has no inside.

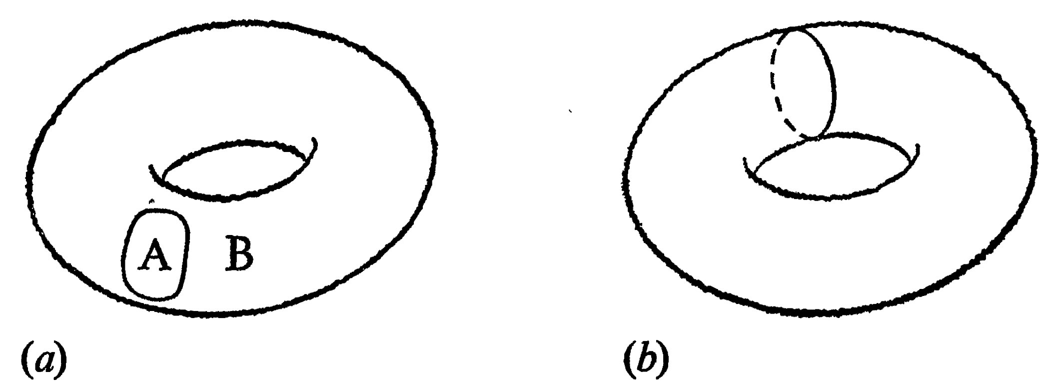


Figure 5

Reply to Objection 2. How true! It is clear that

$$(S^1 \times S^1) \sim (S^1 \times \{a\}) = S^1 \times (S^1 \sim \{a\}),$$

which is homeomorphic to $S^1 \times (0, 1)$, and the natural equivalence of $\{0, 1\}^{X \times Y}$ with $(\{0, 1\}^X)^Y$ proves (via $(\{0, 1\}^X)^Y = \{0, 1\}^X = \{0, 1\}$) that the product of connected spaces is connected. Hence if a circle is to have an inside and an outside, it must not be drawn on an inner tube.

Objection 3. The Equator is a circle running round the middle of the earth. The Arctic Circle also runs round the earth, but not round its middle. It is usually held that Santa Claus lives inside the Arctic Circle, and that the Patagonians live outside the Arctic Circle. But on which side of the equator does Santa Claus live? On the inside, with the Patagonians outside, or is it they who live inside the Equator?

Reply to Objection 3. There are many ways to answer the objection. Some people would say that if you walk round the Arctic Circle going eastward then Santa Claus lives on the inside, but if you walk round it going westward then he lives on the outside and the Patagonians live on the inside. This is doubtless true, since anyone who begins to walk round the Arctic Circle will certainly drown, freeze, or be eaten by polar bears long before he finishes the journey. Other experts hold that Santa Claus does not exist, but does live inside the Arctic Circle. Still others would say that there are two sides of the Arctic Circle, but that it is impossible to tell which of them is in and which is out. This is the best answer, together with the stipulation that the earth is a sphere [9] and that it is required to draw the circle on a plane, and only on a plane. Note that the sphere, being a closed, bounded subset of R^3, cannot be a plane R^2, since the latter is not compact.

I ANSWER THAT a circle drawn in a plane has an inside and an outside (Figure 6). This is sometimes hard for people to understand, for various reasons. Some people think that if the circle is to have an inside and an outside, there must be a door. Houses have insides, outsides, and doors; if the circle has an inside and an outside, where is the door? Before we proceed to show that a circle has an inside and

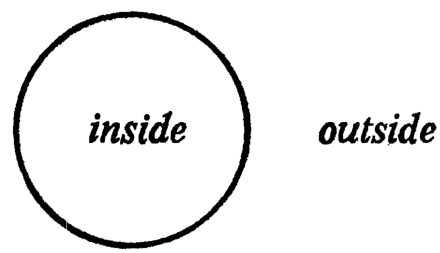


Figure 6

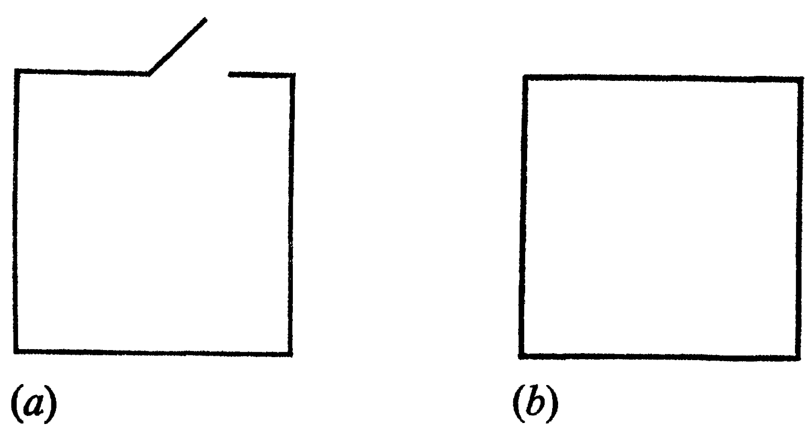


Figure 7

an outside, we had better dispense with this difficulty; otherwise, the reader's mind may be insufficiently prepared for the truth, and unable to assimilate it fully. In Figure 7 we have two houses. House (a) has the door open; house (b) has the door shut. We are looking at a floor plan. The reader should see immediately that house (a) *has no inside*; it is porous, permeable, open; anybody can walk through the door without committing a felony. There is no separation. The walls and door of the house do not divide the world into an inside and an outside. Now consider house (b). The door is shut, and that is why it does not show in the figure. We shall return to this point presently.

Now for simplicity in proving the circle has an inside, let it have radius 1. Let P, P′ be any two points of the plane, and let r, r' be the distances $| \mathrm{OP} |$, $| \mathrm{OP}' |$. Since one side of a triangle is shorter than the sum of two others, $| r - r' | \leqslant | \mathrm{PP}' |$, which shows that the mapping

$$R^2 \to [0, \infty)$$

defined by

$$\mathrm{P} \mapsto | \mathrm{OP} |$$

is continuous. Since

$$\left| \frac{r}{1 + r} - \frac{r'}{1 + r'} \right| \leqslant | r - r' |,$$

the mapping

$$f\colon [0, \infty) \to [0, 1]$$

defined by

$$r \mapsto \frac{r}{1 + r}$$

is continuous; notice that f is also strictly increasing, and that $f(1) = \frac{1}{2}$. If S^1 is our circle, we get a map

$$R^2 \sim S^1 \to [0, \infty) \to [0, 1].$$

By corestriction, we may get one

$$R^2 \sim S^1 \to [0, \tfrac{1}{2}) \cup (\tfrac{1}{2}, 1]$$

and composing this with the obvious map

$$[0, \tfrac{1}{2}) \cup (\tfrac{1}{2}, 1] \to \{0, 1\}$$

we are done with the major part of the problem; namely, to show that the circle divides the plane into two pieces. Note that the same argument works on the sphere, so that it is true that the Arctic Circle (or the Equator) divides the surface of the globe into two parts. This is because the sphere can be regarded as the one-point compactification of the plane, and all maps extend suitably. But beware! As we saw, it is not enough in the case of the sphere to show that there are two pieces, or components, because even so one is unable to distinguish which of the pieces is the inside and which is the outside. Can we do so in the present case?

Consider the two pieces. The piece containing O (the centre of the circle) will be shown to be the inside, and the other piece will be shown to be the outside. Thus, among other things we shall learn that the *centre of the circle is inside the circle.* Write C_O for the component of the plane (with the circle taken out) that contains O, and $\overline{C_O}$ for the union of this set with the circle itself. Then it is to be proved that $\overline{C_O}$ consists of all the points either inside the circle or on the circle: in any case, it is known that

$$\overline{C_O} = \{P : |OP| \leqslant 1\}.$$

The other component forms in a similar way

$$\overline{C_\infty} = \{P : |OP| \geqslant 1\},$$

and it is to be shown that this is the set of all points either outside the circle or on the circle.

If P is a point of the circle and if $0 \leqslant r \leqslant 1$, there is exactly one point P′ on the line segment OP at a distance r from O; this provides a mapping

$$S^1 \times [0, 1] \rightarrow \overline{C_O}$$

which is easily verified to be a continuous surjection. By Tychonoff, it follows that $\overline{C_O}$ is compact. For $n \in N_0$, let $\overline{B_n(O)}$ be the closed ball of radius n at O. Then

$$\overline{C_\infty} \subset R^2 = \bigcup\{\overline{B_n(O)} : n \in N_0\},$$

but $\overline{C_\infty}$ is not a subset of a union of any finite subcollection of this collection. Hence there is an essential difference between the component containing the centre of the circle and the other component. It is therefore reasonable to call one of these components the *inside* of the circle and the other the *outside.* For some reason, convention has settled on the term 'inside' for the component of the plane, when the circle has been cut out, that has the centre of the circle in it. Like all conventions, this one is not easy to explain rationally. Why should not the centre be said to be *outside* the circle, and why should not we designate all the points of the plane very far from the centre as points *inside*? In other words, why is the conventional terminology *not* the one shown in Figure 8? There appears to be no satisfactory

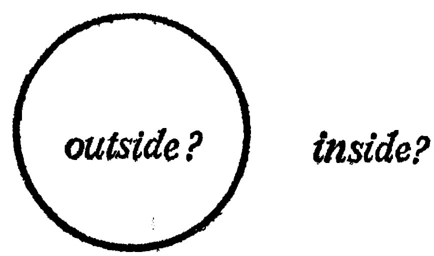
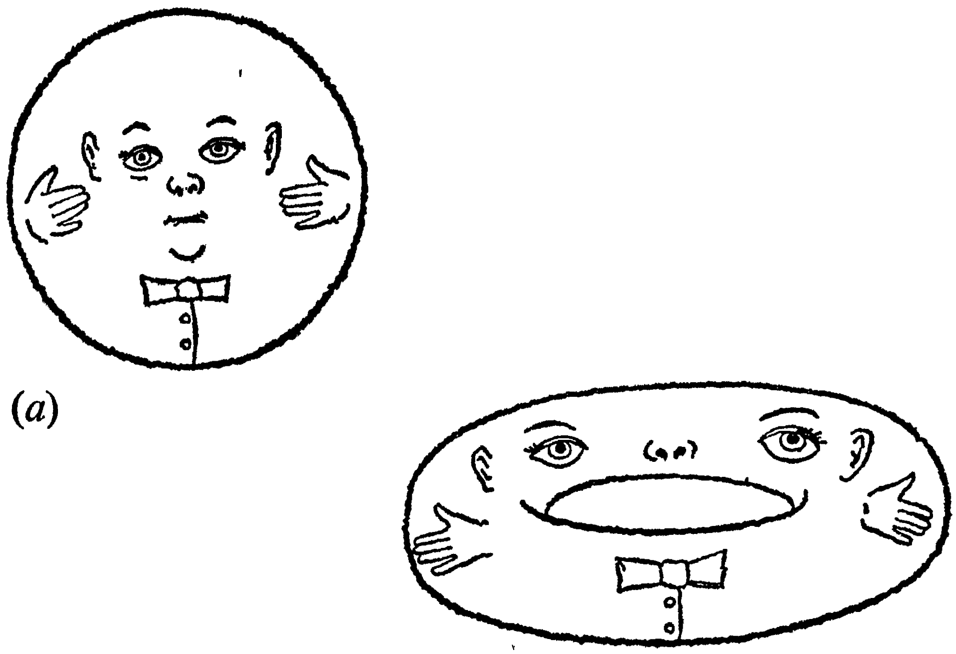


Figure 8

answer to this question. There is no real reason why one should prefer to say that the centre of the circle is inside, and not outside. However, if most people prefer to say that the centre is inside the circle, it is good manners to acquiesce; nothing is ruder than to find occasion for dispute in every inconsequential alternative. Really it would be best if we could all agree to speak of the *compactside* and the *countercompactside* of the circle. Thus, a man with indigestion could say, 'I've got to see a doctor. There's something seriously wrong with me compactside.' An economical housewife buying a chicken might say to the butcher, 'And please could I have the compactsides for the cat?' A prisoner in a maximum security institution might mutter to his compactmates on the night before he expected to get 'sprung': 'Youse won't see me tomorrow. Tomorrow I'm gonna be countercompactside.'

But not so swiftly! So far it is only the circle which one knows to have a compactside and a countercompactside (or an inside and an outside, to use the terminology of stodgy convention). Prisons, chickens, and men are much more complicated things, by their very nature. It is now time to consider whether men have insides. Chickens can be taken to be pretty much the same for our purposes as men; certainly, in a mathematical discussion, a chicken is often as much use as a man and makes comments of about equal intelligence. As we know, there are really two very distinct types of human being: mouth and anus open, and mouth and anus closed (Figure 9). (The acute reader will note that it is possible, in general, to consider the further cases that occur when one of the apertures of the alimentary

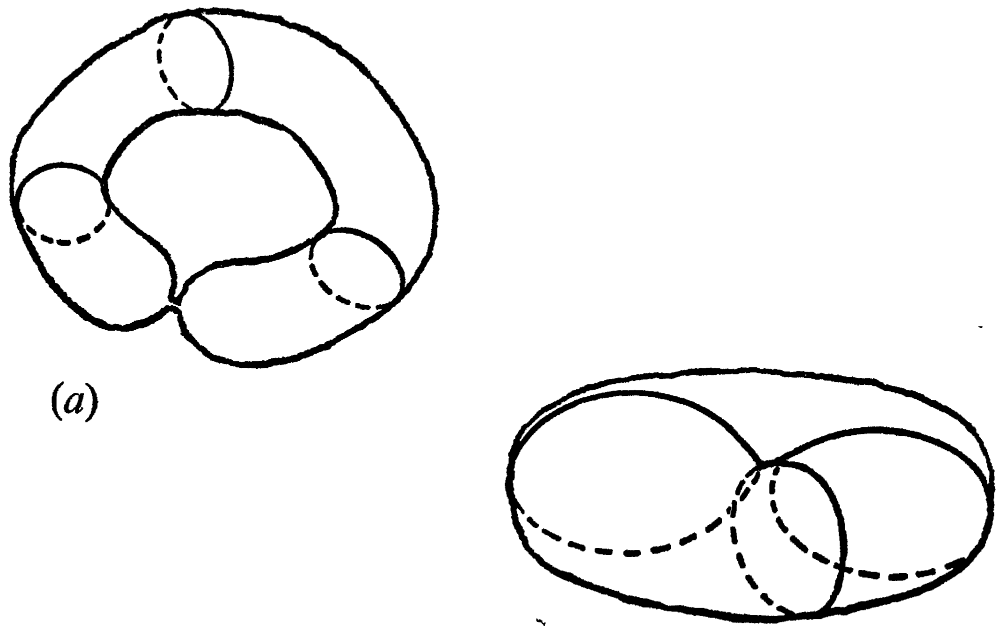


(a)

(b)

Figure 9. Two essential types of men.

passage is open and the other closed. It is usually difficult to distinguish one of these cases from the other, especially in political oratory; the resulting surface is in both of these cases topologically the same. Technically, it is an identification space of a torus. Only one of the possible ways of including such a topological space in R^3 is capable of being human; the other is the skin of a ring sausage (Figure 10). Neither the ring sausage nor the orator is a manifold, and the difficulties of this case force us to ignore it.)



(a) (b)

Figure 10. (a) Ring sausage; (b) political orator.

The first essential type of man is the Quiet Man. His motto: *Silence is Golden.* He is not given to flatulence or loquaciousness. His sign is the Sun. The second essential type of man is the Loud Man. His motto: *Speech is Silver.* He is flatulent, loquacious, and given to overeating. He thinks and acts little, but speaks much. His sign is Saturn, the planet with rings. Most men oscillate between these two types.

Loud Men, abstractly, are capable of much more interesting combinations than Quiet Men. They may be linked together in chains. A simple configuration of this type is shown in Figure 11.

Because of the theoretical possibility of this configuration, biologists have searched assiduously for an example. Their quest has been fruitless, so far as authenticated records show. Loud Men, as such, exist in abundance; linked Loud Men appear to be very rare. Why

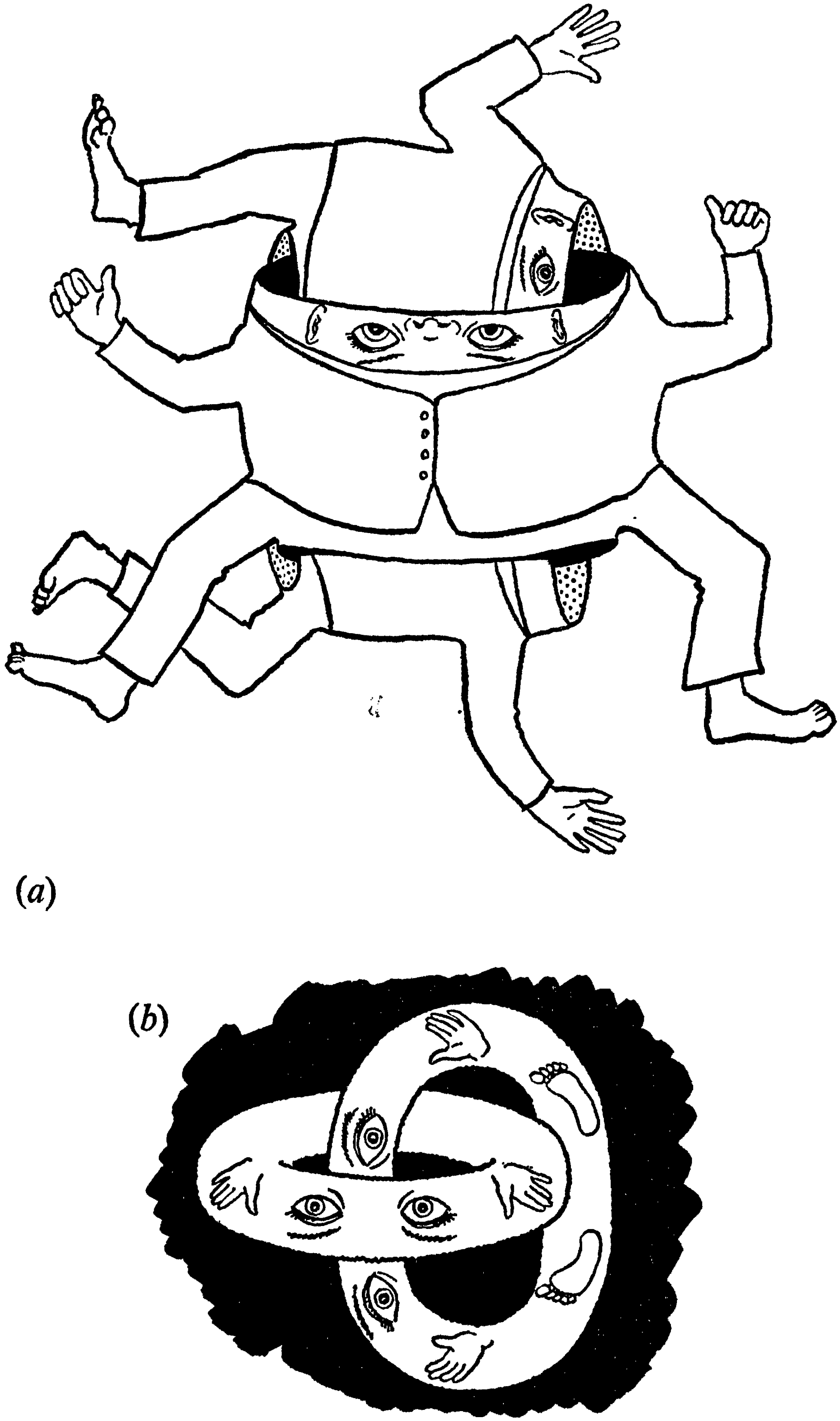


Figure 11. (a) Linked Loud Men; (b) the same, simplified.

is this? Topological biology seems to have provided at least a partial answer. We know from modern embryology that all men begin as Quiet Men; that is, a man is at first a tiny sphere in the womb of his mother. The potentiality of being Loud is developed somehow *in utero.* If it can be shown that two fully developed Loud Men, not already linked, can only become linked by a major surgical operation, then the question, Why do we see no linked Loud Men? will have been reduced to the two further questions: Why are twin Loud Men never born linked; and, Why do no Loud Men choose to undergo a surgical operation so as to become a pair of linked Loud Men?

QUESTION 30. Whether two men who customarily maintain the alimentary tract in an open and ventilated state, by never shutting the mouth or anus (hereafter called Loud Men), can become linked like the links of a chain—so that each man passes down the oesophagus of the

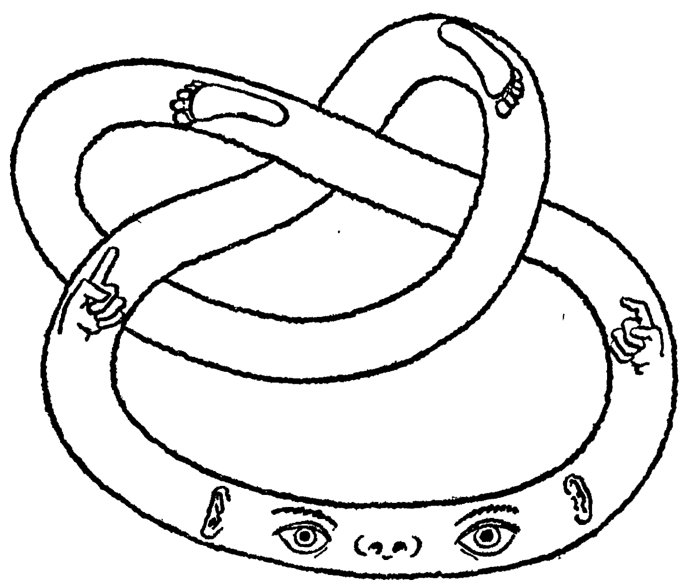


Figure 12. Knotted Loud Men.

other, out through the vent, and back again—without a major surgical operation?
(Note: It is helpful to refer to Figure 11.)

I ANSWER THAT two Loud Men, not yet linked, can only become so through surgery.

The two Loud Men, not yet linked, will for simplicity be taken to be *geometrical* tori. (This is a big assumption; in particular it assumes that no Loud Man is tied in knots, as in Figure 12.) Similarly for the linked Loud Men. The beginning and end of the hypothetical linking thus appear as in Figure 13.

Having made this assumption, it is possible to define a *belt* of a Loud Man. The man is generated as a surface of revolution by revolving a circle, not intersecting the axis of revolution; the circle

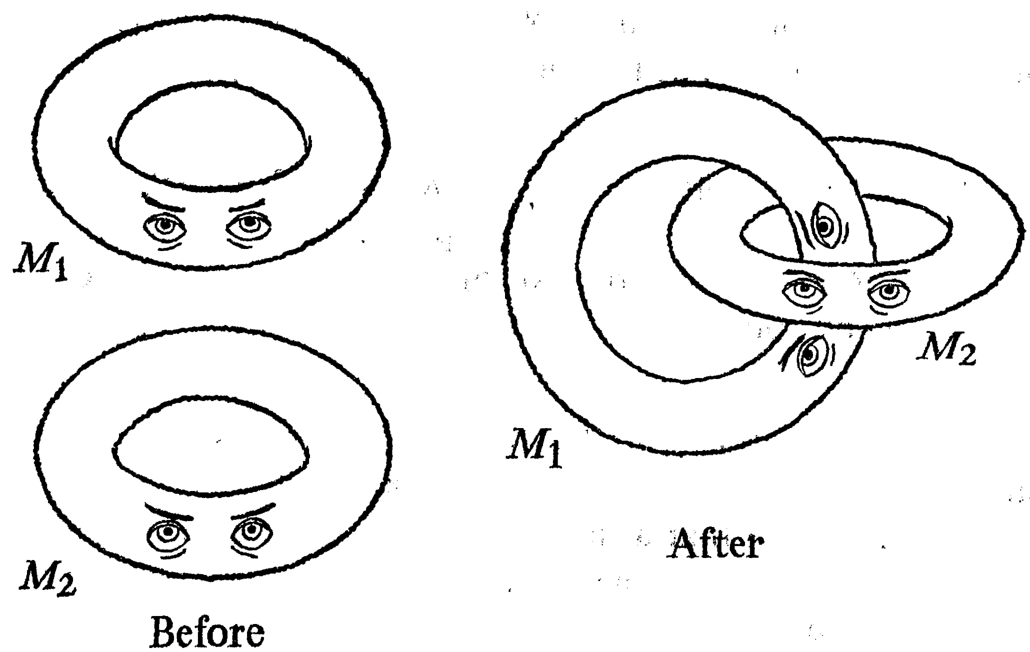


Figure 13. Hypothetical linking of Loud Man.

traced by any point of this circle as it is revolved about the axis is a belt of the man. It will be assumed that, after linking, the belts are in their right places again; i.e., that they are still belts. It can be shown that this assumption is fair. It can also be shown that it can be assumed without loss of generality that the whole procedure whereby the two men are linked can be carried out without moving or changing man M_2 at all. He can remain fixed while M_1 is twisted and tortured.

Suppose it were possible to link the two men without cutting, tearing, or any other surgery. Take C_1 and C_2 to be belts which the two men are wearing. Then the belts have also been linked without surgery, and indeed without unbuckling them. The question has been reduced to this: Can a belt be linked to a stationary belt without breaking or unbuckling? By hypothesis, we have a homotopy

$$C_1 \times [0, 1] \to R^3 \sim C_2$$

in which the image of $C_1 \times \{0\}$ is a geometric circle not linked with C_2, and the image of $C_2 \times \{1\}$ is a geometric circle linked with C_2. If Q is the centre of $C_1 \times \{0\}$ in $R^3 \sim C_2$, then taking Q as origin of coordinates and 1, say, as the radius of C_1, the map

$$(x, y, t) \mapsto (tx, ty)$$

makes $C_1 \times \{0\}$ homotopic to $\{Q\}$ in $R^3 \sim C_2$. Hence there exists a homotopy shrinking $C_1 \times \{1\}$ to a point. Taking cylindrical coordinates in $R^3 \sim C_2$, with the axis of rotation about which C_2 may be generated by rotating a point taken as the z-axis, we note that

$$(r, \theta, z) \mapsto (r, z)$$

is a continuous function from $R \sim C_2$ to $[0, \infty) \times R \sim \{(1, 0)\}$. Composing this map with the homotopy that shrinks $C_1 \times \{1\}$ to a point in $R^3 \sim C_2$, we infer that a circle can be shrunk to a point in a plane punctured at the centre of the circle; which is as much as to say that the identity map

$$S^1 \to S^1$$

is homotopic to a trivial map. But it is well known that the relevant group is Z, and is generated by the homotopy class of the identity map.

This conclusion, that Loud Men cannot be linked without surgery, is one of theoretical interest, but it has for most of us little practical importance. Two linked Loud Men would no doubt be as loud as two unlinked ones, and no louder. If there is any practical inference to be drawn from the theoretical fact that Loud Men cannot be linked, it is one which brings a comforting thought to the mind, such as it is, of every member of the Loud Fraternity. Since he cannot be linked, the danger is averted that (relapsing momentarily into quiet and closing his mouth) his link-mate might bite him through from mouth to anus. Such a calamity is then impossible. If it could happen, it would clearly change the toroidal Loud Man into a very quiet, spherical Quiet Man (Figure 14).

What looks like a halo is the toroidal soul ascending whence it came. His grave must have a marker: let us put, 'May he ever rest in peace.'

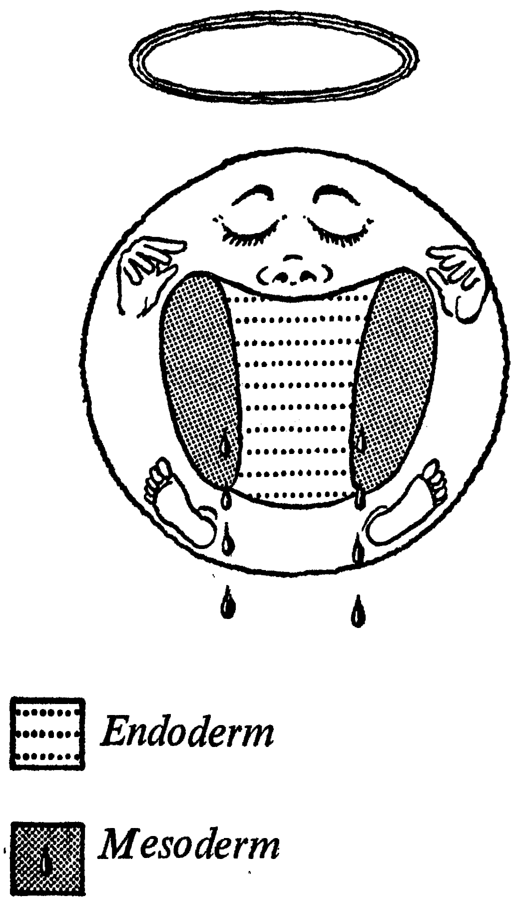


Figure 14. Transformation of toroidal Loud Man.

3. Geometry

Civilised man lives close by the brink of the water. To the great civilisations that developed along the banks of the great rivers—the Indus, the Euphrates, the Cam—can be traced much of that artistic and scientific heritage which adorns modern life. The earliest recorded discovery of a useful art occurred in Eden, near the head of the Euphrates. This was the discovery of both nakedness and its complementary opposite, clothes; both of which arts remain important industries in the great cities of today. The art of falling was discovered by a river-valley man, Isaac Newton, although the actual discovery took place at Grantham, Lincolnshire, and not at Cambridge (the centre of the great Cam civilisation). This event, known as the Fall of the Apple, occurred much later than the other, the Fall of Man. Before either of them occurred the Fall of Lucifer, which was not an art, though it became the subject-matter of several great works of art. From the Fall of the Apple was developed the great art of Potential Theory.

In the valley of the Indus a great civilisation developed, to which we owe Sanskrit and the Indo-European language. The most important word in this language is *lox*, or, if you prefer, *Lakshmi.* She is the goddess of smoked salmon and bagels. A bagel is a torus, and has been encountered already in the chapter on topology. It is eaten with lox, and is a topological group R^2/Z^2. Along with lox, bagels, and language came the great Aryan race; to them we certainly owe the discovery of the square. It has long been unknown whether the Aryans developed the torus, or bagel, from the square, or vice versa. Which came first? Square enthusiasts point to the shape of the Aryan head as an indication that they must have been forced at an early date to contemplate this figure. At first they would have found it natural to bake their bread in this shape; later, because the torus is obtained by an identification of opposite edges of a square, it naturally occurred to their rather simple, straightforward minds to make bagels, or toroid bread as a kind of elegant variation. That is how one segment of opinion reconstructs the story, which must nevertheless in its more intimate details remain dark for us. It is noteworthy that one branch of the Aryan race, the famous Aryan heretics, persevered in making and eating exclusively square bread, and eschewed bagels. (Bagels are hard to chew, but it is foolish to eschew them). The Aryan race were also interested in the wheel, which they called a *wheel-wheel*, *whirr-whirr*, a *GLGL*, a *quirquel*, or a *circle*. It is possible that the wheel was developed from the bagel; that when the early Aryans began to think about the bagel, they realised that it was a compact topological group and began searching for other compact topological groups, and that this search brought them on the trail of the wheel, or circle. It cannot be overlooked that others derive the bagel not from the square but from the circle. When the Aryans heard of Tychonoff's theorem (the product of compact spaces is compact) they applied it to the wheel, and got a new compact group:

$$\text{BAGEL} = \text{WHEEL} \times \text{WHEEL}.$$

So much for the Indus Valley and the immense heritage we possess as its successors.

The Yellow River civilisation gave us the I Ching, or index of permutations, from which developed the symmetric group on n letters and the non-conservation of parity, and was responsible for the Chinese Remainder Theorem, space travel, dragons, Pekingese dogs, and fortune cookies.

The Nile Valley has a civilisation so ancient that the name given to the process of suspended animation discovered on the banks of this river is *mummification*, and the subjects of it, who were provided with picture-books of instructions and light entertainment against their future reanimation, were designated as *mummies*. These names make it absolutely certain that the Nile civilisation dates from before the Fall of Man, when the distinction between mummies and daddies was discovered. Pyramids, another Egyptian invention, were principally a tourist attraction, like the Eiffel Tower. They were copied in America, where they were used for heart transplant operations, flower shows, and football games. Pyramids were also influential in the history of Greek geometry. Among all the ideas of the Nilesiders —mummies, men with bird's faces, cats with women's faces, the feeding of non-mummies to crocodiles—only the pyramid is of mathematical significance, and this only through the Greeks.

The Greeks were a Hellenic people; hence, although they were good at mathematics, poetry, politics, architecture, navigation, and many useful arts, they were overly fond of beautiful women. Fond of measurement, they measured feminine pulchritude in *Hellens*. Rather oddly for a civilised people, they based their unit of measurement on the curious idea that the more beautiful was a woman, the larger was the number of ships her face could launch, and that the largest number of ships that could be so launched was 1000, or 10^3. This quantity of beauty was called one Hellen. The practical result of this idea was that the Greek women spent much time on the seaside toughening their facial features by pushing boats into the water, and in the end the Hellenes lost interest in women and invented Platonic love. At the same time, the huge fleet of ships thus launched made the Hellenes a nautical nation, so that they did much sailing. The sailors would often bring home little trinkets, among which were small replicas of the great pyramids, something like little Eiffel Towers cast in lead that visitors bring home from Paris as souvenirs. These knicknacks had far-reaching effects on later Greek thought, as we shall see. Most important of all, doubtless, was the simple habit of spending time at the beach. At first one wandered down to the beach for the idle amusement of watching the ladies at their competitive launching. But the men, to pass the hours, would chat lying on the sand, and scratch crude drawings in its smooth surface. Later, when Platonism had replaced the mythical Hellen at the centre of the Greek imagination, and feminine beauty no longer played the leading rôle, the beach remained a gathering point. A preoccupation with forms, with the shape of things, remained; but now it was abstract forms that were contemplated, and geometric shapes were drawn in the sand. Geometry dawned. Her day is not yet spent.

One of the most difficult of the geometric concepts invented by the geometric Greeks was the geometric line. A line was said to have no thickness or breadth, and to be generated by a moving point. A point was said to have no length, breadth, or thickness, and to be indivisible. Thus, we are forced to interpret a point as a singleton $\{x\}$; i.e., a universally attracting object in the category of sets. That is simple enough. But how are we to interpret the line?

QUESTION 31. Whether, if three distinct points lie on a line, one of them must lie between the other two?

Objection 1. A point has no extent; i.e., no length, breadth, or thickness. That which has no length obviously cannot lie, sit, or stand. Hence, a point cannot lie on a line.

Reply to Objection 1. It is quite clear that what is meant by the statement, 'a point lies on a line', is not that the point reclines there, but that there exists a monomorphism from the singleton to the line, and that this monomorphism is among those considered to be inclusions. Alternatively, there exists a map f from the line to itself such that for any map g from the line to itself,

$$fg = f.$$

It is merely another way of saying that the relation of incidence holds between the point and the line.

Objection 2. This would seem obvious. By the Krein–Milman theorem, the closed convex hull of {P, Q, R} being clearly compact and convex, it is the closed convex hull of its extreme points. But since only two of P, Q, R can be extreme, the other is in the closure of their convex hull—which is as much as to say, in their convex hull itself.

Reply to Objection 2. It is true that by the Krein–Milman theorem, each of P, Q, R lies in the closed convex hull of the extreme points of the closed convex hull of {P, Q, R;} but the objection assumes without proof that this set can have only two extreme points. If it has three, the argument fails.

Objection 3. You could try moving the points around. Keep two of the points fixed firmly in their right places, but let the third point move freely on the line. If the third point is between the other two, it cannot go on for ever, either in one direction or the other, without bumping into one of the other points; if it is not between the other two, then the moving point can go on for ever in at least one direction, and never encounter either of the other points. Let one of the points, which we may call P, be supposed (as a hypothesis) *not* to lie between the other two. Move P toward Q with a large velocity, so that when P strikes Q, the particle Q will continue in the same direction with the same velocity. If Q never strikes the third point R, then the whole line has been swept out without encountering R, and hence R does not exist. It follows that Q is between P and R.

Reply to Objection 3. The objection is stated in terms with a pseudo-particle-dynamical flavour, but we need not insist on looking at the objection in this light; if we did, it would of course have to be rejected as non-mathematical. The idea of a moving point is a highly sophisticated one by the time it is given a mathematical expression. The billiard-ball behaviour of Q when struck by P involves differential equations, and the points are even treated as impermeable to one another. Worst of all, the objection assumes without proof that a line has only two directions. Just what is the justification for this bold assumption?

I ANSWER THAT it is the case that if three points on a line are distinct, then one lies between the two others. The general idea of the proof is very simple. Since P, Q, R (the three points) lie on a line, we may draw arrows from P to Q, from Q to R, and from R to P. It is necessary to show that either two of these arrows go to the right, or two of them go to the left. Since the line is an affine space, there exists a way of associating to each pair of points a vector or arrow: to (A, B) we associate the vector AB. Since the line is by Euclid's definition a one-dimensional affine space, the vectors in question lie in a one-dimensional space—i.e., module—over a field of characteristic 0—i.e., characteristic ∞—which possesses an archimedean order. (It was the great Eureka who discovered that lines have got to have an archimedean order. This was on one of the many occasions when Eureka was singing in the bath. Since he had very bad pitch combined with good volume, Eureka's singing was always indistinguishable from his shouting.) We cannot be sure, and need not worry about, just which archimedean ordered field F is being used, but since it contains $\sqrt{2}$, it cannot be the universally repelling archimedean ordered field Q, though it certainly contains Q. Since $PQ + QR + RP = PP = 0$ by the axioms of affine spaces, and since the vector space, being a one-dimensional free module over F, is isomorphic as an abelian group to the abelian group structure of F, the fact that none of PQ, QR, RP is 0 enables us to conclude that by the use of a suitable permutation (in the symmetric group on $\{P, Q, R\}$) it can be arranged that two of PQ, QR, RP are positive without loss of generality. These can be PQ, QR, so that Q is between P and R.

(Tri)gonometry

One of the most important elementary subjects under the general heading of geometry is gonometry. The origins of trigonometry, which is the most elementary form of gonometry and is followed by tetragonometry, pentagonometry, hexagonometry, etc., are to be found in the great Ur-city, Ur. The Ur-civilisation was a great river-civilisation, and invented trigonometry, which is therefore a useful art. Its citizens, as we know from plastic art, wore beards in the shape of rectangular parallelepipeds, and since Ur was situated on the Euphrates, we know that they wore clothes and divided their children into girls and boys. The boys had beards, and the girls had none. The great interest in triangles that was such a great part of the cultural life of Ur is evidenced by their writing, in which all letters were formed of triangles. This is called *cuneiform*. Even little bearded boys of the lowest social orders ran about the streets of the great Ur-city studying triangles. Because of their characteristic beards, these children were called Ur-chins. Nowadays, the name remains despite the fact that few Urchins any longer possess the beard to which it refers.

Clearly, if one is to do gonometry, which is the measurement of angles, one requires some means of measuring angles. Hence the question whether the Ur-menschen could have constructed a protractor was long a thorn in the side of the historian (see Figure 15).

QUESTION 32. Whether the Ur-menschen (as the inhabitants of Ur were called) could have constructed a protractor?

Objection 1. It is well known that the Ur-menschen counted in sixties. Since it is easy to construct an angle of 60 degrees, they would naturally have done so, and this angle would then have been divided in sixtieths, or angles of one degree. But an angle of one degree is the basis of the protractor. Hence the Ur-menschen constructed a protractor.

Reply to Objection 1. This merely shows that it would come naturally to the Ur-mensch to construct a protractor. But an Ur-mensch is far from being a Natur-mensch. The latter was not invented until all the Ur-menschen had disappeared. Hence the intersection of Ur-menschen and Natur-menschen is empty, and no Ur-mensch always acted naturally. Hence it is possible that no Ur-mensch constructed a protractor.

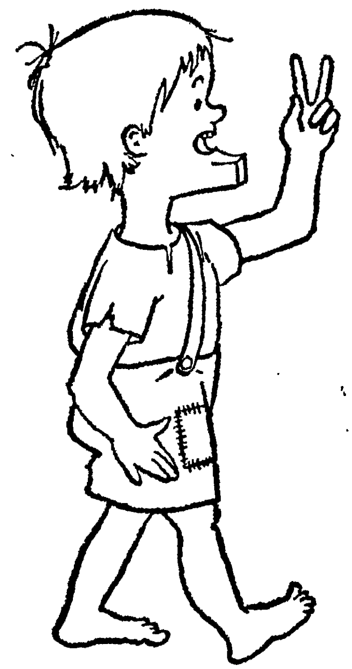


Figure 15. Ur-chin, studying angle.

I ANSWER THAT no man has at any time constructed a protractor. The cosine of 40° obviously satisfies

$$8 \cos^3 40° - 6 \cos 40° + 1 = 0,$$

which clearly makes the construction of a protractor an impossibility.

How then did the men of Ur measure their angles? Possibly they knew that one can integrate

$$\int_1^x \frac{dt}{\sqrt{(1-t^2)}}$$

for $1 \geqslant x \geqslant -1$, and that when this (obviously continuous monotonic) function is inverted, it is capable of providing material for a homomorphism from the topological group of real numbers to the (multiplicative) topological group of non-zero complex numbers whose image is the circle. If so, the measurement of angles would become a triviality for them. But there is no record of a theory of divergent Riemann integrals among the Urmen, and perhaps they did it via Taylor's series.

The line, the circle, the angle—to these basic ideas of geometry must be added a fourth.

Area of a rectangle

Almost everything is made in the shape of a rectangle. Two of the most obvious examples are a book and a farmer's field. Why this should be is difficult to say, but it is worthy of attention that the farmer ploughs his field in furrows, and that the writer writes and the reader reads the book in lines, which are very like furrows. The author has never understood why books are not written even more plough-fashion than they are. Why not write like this?

Oh, what are the zeros of zeta of s?
!ssefnoc ylurt I, uoy llet nac ydoboN

It would be so much easier to read—the eye would not travel back to the beginning of the line emptyhanded, as it were, but would get some reading done on the way back. If farmers ploughed their fields as inefficiently as we read, we should all likely starve. However that may be, the consideration of furrows only partly helps to explain why everything is a rectangle. All it explains is why everything is a product of something times a unit interval. Perhaps, in some dark half-conscious way, the farmer is trying to prove that the fence on one side of his field is homotopic to the fence on the other. Who knows? In any case, we have not explained why his field is not an infinite cylinder, or why this book is not a punctured disc.

Why is it that when a man has a rectangle, the first thing he does with it is to calculate its area? Is it because he knows how to get the right answer? That hardly seems likely.

QUESTION 33. Whether it be true that the area of a rectangle whose sides have lengths B and H respectively, is the product of those numbers, BH?

This is one question of which every intelligent person ought to be able to guess the right answer, at least after he has read the correct definition of 'rectangle'. A rectangle is a geometric figure composed of two pairs of parallel line segments, perpendicular to each other, such that each line segment meets each of the perpendicular line segments in its endpoints. Basically then, a rectangle is made up of four lines. And yet people say that the area of a rectangle is the product of base times height!

I ANSWER THAT the area of a rectangle is never the product of base times height. This is because the base of a rectangle is the length of a line segment, and so cannot be 0; neither can the height be 0. But the area of a rectangle is always 0, since it is obvious that a line segment has area 0, and hence so has the union of four line segments. If we are to speak of areas, we ought to speak of the area of the *convex hull* of a rectangle, or the *inside* of a rectangle. The fence round the farmer's field is a rectangle, perhaps; the field is its convex hull.

QUESTION 34. Whether the area of the convex hull of a rectangle, whose sides have lengths B and H, is BH?

Objection 1. The correct answer is not *Yes*; neither is it *No*. The correct answer is *Maybe*. For example, the rectangle that has base 1 and height 1 might be said to have area 1; but it would make just as much sense to say that it had area 197×54. This would, of course, necessitate giving a rectangle of base 2 and height 2 exactly the area 790×16.

Reply to Objection 1. The objection is perfectly correct, except that it has already been mentioned (tacitly) in the question, which asked what was 'the area of the convex hull of a rectangle, ***normalised in the obvious way, needless to say***'. Since it was needless to say that the area was normalised in the obvious way, the words 'normalised in the obvious way, needless to say' were omitted from the question. It is very important that the question be phrased as it is here, because if normalisation is not mentioned then one may indeed give the unit square the area 197×54.

Objection 2. The answer *Yes* would seem to be obvious, since the Jacobean determinant of the linear transformation sending orthogonal basis elements e_1, e_2 to Be_1, He_2 respectively is merely the determinant of the linear transformation itself, namely BH. It is well known that under a transformation obeying suitable conditions areas are multiplied by the Jacobean determinant, so that the rectangle of sides B and H which is the image of the unit square under this transformation has area BH.

Reply to Objection 2. This fact is only well known after the area of a rectangle is well known. Hence this objection is illogical.

I ANSWER THAT the area of a rectangle is base times height, at least if the rectangle has sides parallel to the coordinate axes. (It is difficult to see what are the base and height of a rectangle, if the rectangle is at an angle to the coordinate axes, anyway.)

The area of a one-by-one rectangle is taken as 1. This is purely for simplicity's sake; if anyone disagrees, it will be easy to alter the areas of all rectangles so that the one-by-one rectangle has area equal to whatever suits one's fancy. It is not assumed that the area of any one-by-one rectangle whatever is 1; that would be rather bold. However, it will be necessary to assume the existence of at least one such rectangle with that area.

If a rectangle is moved about, its area remains the same. It is desirable to move the rectangles gently, as they are not rigid figures. Preferably, they are to be moved by an affine transformation of the form

$$c + \mathrm{I},$$

where of course I is the identity endomorphism of the vector space

R^2. It is of course far from obvious that the area of a rectangle does remain the same when it is moved, since not every measure on a locally compact topological group is Haar measure. Clearly, one of the tacit words in the question was Haar: the question asked about the *Haar* area of a rectangle with sides B and H. (The words 'convex hull', the reader will notice, have become tacit by now.)

If a rectangle is (the boundary of the union of the convex hulls of) two other rectangles joined together, and if the two other rectangles do not overlap (so that their convex hulls have disjoint interiors) then its area is the sum of theirs. This is because the words 'finitely additive' appear tacitly in the question.

It is obvious that if axes have been chosen through two adjacent sides of the basic unit rectangle, the relation given holds so long as the vertices of the rectangles have rational coordinates. The rather mild assumption that countable additivity or continuity from above at $\emptyset$ holds for rectangles suffices to finish the job. It has not been shown that rectangles really have areas; what has been shown is that *if* they have areas which are as described in the question (mostly by tacit expressions hidden therein) then the areas are as stated. The question still remains: are areas of rectangles all nonsense?

EXERCISES

BIBLIOGRAPHY

EXERCISES

The reader is not required to do any of the exercises unless he cannot do them. Starred exercises are considerably more difficult, and should always be attempted.

Introduction

1. Show that a finite subset of an arbitrary set E in a ring suffices to generate the ideal generated by E if, and only if, the ring is Noetherian.

*2. Show that $17 \times 17 = 289$. Generalise this result.

3. If the book is read backward (beginning at the last word of the last page), the last thing read is the introduction (reversed, of course). Thus the introduction acts as a sort of extraduction, and is suggested as a simple form of therapy, used in this way, if the reader gets stuck. Read this exercise backward, and write an extraduction from it.

4. Define (1) a tom category, (2) a pussy category, (3) a kittegory.

5. The set of levels at which a book can be read are ordered in the natural way. If every chain of levels at which a book can be read has an upper bound (i.e., a level that explains or is higher than every level of the chain), then there is of course a maximal level. Is there necessarily a highest level? What would Aristotle say?

6. The tracks of a railway need to be the same distance apart all the way along; otherwise, the axles of the carriages become elongated or buckled. Show that this is possible, assuming for simplicity that trains have only one set of two wheels, one wheel on one side of the track and one on the other; also, allow the two rails to cross, or even to be the same rail. Show that a train of suitable gauge can ride on a single rail in the shape of a British fifty-penny piece. What if the train has several carriages, each with more than one axle? (The fifty-penny piece has the shape of a regular heptagon, each side replaced by an arc of a circle centred at the opposite vertex.)

CHAPTER 1: Arithmetic

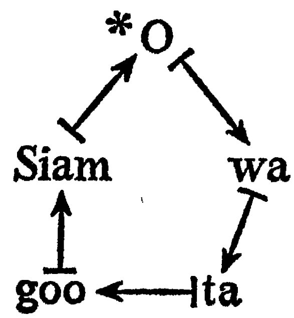
1. You die, and find yourself at the foot of a mountain on an island. People are marching round the shore of the island and counting in English; some of them are reeling off rather greatish numbers, up in the billions; others sound as if they had not been at it very long. You are instructed to join them, beginning at 'one'. 'Oh, goody!' you exclaim. Why?

2. You are asked your nationality and answer, 'British.' The clerk (an angel) says, 'Hard cheese.' Explain this comment.

3. Establish a bijection from the set of fingers of the right hand to the set of fingers of the left hand. Establish a bijection from the same set to the set of left toes. Establish a bijection from the set of forefingers to the set of ears. You are out. Why?

4. Explain why counting system (1), page 16, is used only in written form.

*5. Count in Siamese:



6. 'And Abraham answered and said, Behold now, I have taken upon me to speak unto the Lord, which am but dust and ashes: Peradventure there shall lack five of the fifty righteous: wilt thou destroy all the city for lack of five? And he said, If I find there forty and five, I will not destroy it. And he spake unto him yet again, and said, Peradventure there shall forty be found there. And he said, I will not do it for forty's sake . . .' [16] How did Abraham have the nerve to get into this conversation, and why was the Other Person so patient?

7. Prove Peano's axioms from Lawvere's. Prove the existence of a universal pointed bijection

$$\{0\} \hookrightarrow Z \xrightarrow{+} Z.$$

8. Prove that $1 \neq 0$.

9. If x and y are natural numbers other than 0, prove that $x + y \neq 0$.

10. Finish the proof that it is possible to count with the same numbers it is possible to add with, by furnishing a proof of the uniqueness.

11. (i) Comment from the point of view of an ordinary waiter on the bill homomorphism β.

*(ii) Explain β so that an ordinary waiter could understand it.

(iii) Show that every bill function β is uniquely determined by the values

$\beta(\hat{m})$, where $\hat{m}$ is a meal consisting entirely of one of item m on the menu. (For instance, if m is mayonnaise, then $\hat{m}$ is a meal consisting of one portion of mayonnaise.)

(iv) Show that the conclusion of (iii) remains true even if the menu is infinitely long, provided that customers may order only finite meals. Example: a fish-and-chip shop caters to customers possessing accents varying from standard English through all conceivable degrees of the London vowel shift; the fish available for sale varies similarly from fresh through all degrees of staleness. Here is the menu:

HAVE YOUR MONEY READY, PLEASE

0. Chips
1. Plaice
2. Plaaice
3. Plaaaice
4.

..

..

ONLY FINITELY MANY STRENGTHS
OF FISH PER CUSTOMER,
AND ONLY FINITELY MANY PORTIONS
AT ANY PARTICULAR STRENGTH.

*(v) The shop has changed its policy. Customers now are allowed to order finitely many portions of as many ages of plaice as they like—even infinitely many kinds of plaice are allowed. The bill function still exists; show that some kinds of fish are free. Show further that a thick enough accent rates free fish.

12. Show that if the number a is subtracted from the number b on two different occasions, neither of which is the last Friday of the month when the succeeding Monday occurs in the following month, the answers will both times be the same, in the sense that if there is an answer on one of the occasions there will also be one the other time.

13. Draw a noughts-and-crosses board, sometimes also referred to as a tic-tac-toe board. Do not fill it in with noughts and crosses, sometimes called exes and ohs. Instead, use curved arrows. By drawing more lines, make it a board for four-by-four (instead of three-by-three) noughts and crosses. Wave your hands about in complicated patterns over this board. Make some noughts, but not in the squares; put them at both ends of the horizontal and vertical lines. Make faces. You have now proved:

(a) the Nine Lemma
(b) the Sixteen Lemma
(c) the Twenty-five Lemma
(d) that four-by-four noughts-and-crosses is a simple two-person, zero-sum game
(e) that $3^2 + 4^2 = 5^2$
(f) that square-dancing is for squares

*14. True or false?

(i) A subtractable number is less tractable than a tractable number.
(ii) George Washington, in keeping with his Francophile sympathies, liked to dance the minuend.

(iii) In mathematics, all letters turn into numbers in the end—called the Second Rule of Thumbodynamics.

(iv) The knight's tour refers to *Gal, amant de la reine,* who *alla, tour magnanime, galamment de l'arène à la tour magne à Nîmes.*

(v) Only the fictitious is true.

(vi) Only the true is fictitious.

15. A diagram is:

(i) a picture in a book to make things clear to the reader.

(ii) worth a thousand proofs.

(iii) commutative unless otherwise stated.

(iv) an element in the category of diagrams.

(v) assumed to be chased unless proved otherwise.

16. What shape was the mark set upon Cain when he slew his brother Abel?

17. Consider the following proof that

$$e^{2\pi i} = 1 :$$

since $e^{2\pi i}\cdot$ is a homomorphism

$$R \longrightarrow C^*,$$

where C^* is C without 0, let $C^* \longrightarrow D$ be the equaliser associated with

$$\begin{tikzcd} R \arrow[dr, "e^{2\pi i}\cdot"] & \\ & C^* \\ C^* \arrow[ur, "1\cdot"'] & \end{tikzcd}$$

Then they are equal as soon as the equaliser is applied.

18. Prove that $\{1\} \hookrightarrow Z$ is universally repelling in the category of pointed groups.

19. Show that the natural numbers, as obtained by equalisers, have the property, that $\{1\} \hookrightarrow N_0$ is universally repelling in the category of pointed monoids.

20. Define a *freak* as a universally repelling object in the category of pointed heads (colloquially abbreviated to 'pointyheads'). Since heads are individualists, no two distinct heads are isomorphic. Prove that since there is more than one freak, the category of pointyheads is not a subcategory of the category of heads.

21. Show that $\{-1\} \hookrightarrow Z$ is also a universal pointed group.

22. Show that (i) $N_0 \longrightarrow Z$ is injective; i.e., that a natural number cannot be two distinct positive numbers at the same time;

(ii) $-N_0 \cap N_0 = \{0\}$; i.e., that 0 is the only integer that is both positive and negative at the same time;

(iii) every integer is either positive or negative.

23. Can it be shown by measuring the base of the Great Pyramid that the ancient Egyptians believed the number 5 to be transcendental? If not, what about the Leaning Tower of Pisa?

CHAPTER 2: Factors and Fractions

1. A book on the sexual life of the praying mantis is divided by the publisher in three volumes. A part of the book is left over, since the number of pages is not

divisible by three. Show that these extra pages make too slim a book to publish profitably, and have to be remaindered. Suppose that the left-over pages are just short of a profitable book, and that the three volumes themselves also have to be remaindered. How many pages has the book?

2. Find all prices in Old English money such that, if the prices are rounded to the nearest decimal equivalent, a full cup of coffee will be more expensive than two half-cups. Assume that in Old England, a full cup of coffee cost twice as much as half a cup, and ignore the fact that it is impossible to buy a proper cup of coffee in Britain. A decimal penny is worth 2·4 old pence.

*3. Find the number that comes after: 999; 199; 2992.

4. In showing that the homomorphism of counting systems mentioned in the answer to Question 10 is injective, we manage to prove, among other things, that 22995 is not the same number as 22985. To show this directly, the following argument is not sufficient: The number 22995 is the telephone number of one of my girl-friends. The number 22985 is not, because I often dial it by mistake and I do not get her. Hence they are not the same number. Point out the fallacy in the argument.

5. Find the quotient and remainder on division by 10 (where 10 is taken as $9 + 1$): 1978; 543.

6. Show that if a number is a sixth power—in other words, is equal to the product of six equal numbers, like $12 \times 12 \times 12 \times 12 \times 12 \times 12$—then on casting out nines it reduces to 1, or to 0.

7. What would the reader think of a course in analytic number theory in which the lecturer failed to define an analytic number?

8. Is the converse of Fermat's theorem true?

9. Show that the highest common factor of no numbers at all is 0. What is their lowest common multiple?

CHAPTER 3: Algebra

1. Roman fathers were wont to name their fifth sons Quintus, for some reason no-one has been able to figure out. Discuss all normal towers of the symmetric group on five letters, and thus show that fathers of five or more sons were not in general solvent.

2. What is the relationship between continued fractions and involutes and/or evolutes? What is the relationship between and and/or or?

3. It has sometimes been suggested that half pi should be called hi, and written τ. Explain why this idea was not discovered until recently.

4. Is a man who has three wives a bigamist? Is it legal for him to divorce one of his wives? Ignore simultaneous divorces.

5. At how many weeks old is an infant ready to digest the cubic formula?

6. An element of a universally attracting object in the category of archimedean ordered fields is called *irrational* if it is not in the image of any homomorphism from the universally repelling object in the same category; *rational* if it is. Between any two irrationals are two rational elements. Does this show that the real numbers are manic-depressive? Is analysis appropriate in this context?

7. Define a splitting field without reference to earthquakes.

8. A farmer acquires an algebraically closed field by extending his field finitely. What can be said about the original field?

9. A module over a field is a vector space. Space modules sometimes have difficulty in linking up in space, where it is desirable to get one module correctly inserted in another. Obviously, if every module were injective this problem would not come about. Thus these linking difficulties are a necessary evil, since if every module is injective, everything splits. Show therefore that the modules cannot be spaces. Explain why no space modules have split as yet.

CHAPTER 4: Topology and Geometry

1. The Mississippi river is said to be 6400 kilometers long. What does this mean?

2. Construct a net indexed by the neighbourhood filter at Ujiji, and approaching Ujiji. Each term should lie in Africa, and none of the terms should be Ujiji itself. You may assume the surface of the earth to be a 2-manifold, and you may use a chart mapping Africa as a subset of the plane. Translate your net into the dialect spoken by the apes and their friends, being careful to use the mathematical terms correctly. Assuming that an open subspace around Ujiji exists in which there are no crocodiles, show that it is possible to construct the net in such a way that Livingstone need never, in the course of his journey along the net, be present in a part of Africa that is infested with these unpleasant and populivorous animals.

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