Discrete Math Binomial Theorem

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1 Introduction

The binomial theorem is a result about expanding the powers of binomials, or sums of two terms. The coefficients of the terms in the expansion are the binomial coefficients $\binom{N}{k}$. The theorem and its generalizations can be used to prove results and solve problems in combinatorics, algebra, calculus, and many other areas of mathematics.

2 Questions

1. 5 people became friends at a Summer Camp. On the last day, they shook each other's hands to say farewell. How many handshakes were there?

   Answer: 10

   Solution 1: Let's count it slowly, by per person. The first person has 4 other hands to shake. Now let's ignore the first person. The second person has 3 other hands to shake (excluding the first person). Now let's ignore the second person. The third person has 2 other hands to shake (excluding the first and second person). Now let's ignore the third person. The fourth person has 1 other hands to shake (excluding the first, second and third person). This is all that remains.

   Hence, there are $4 + 3 + 2 + 1 = 10$ handshakes that occurred.

   Solution 2: We can draw a graph to represent this. There are 5 vertices representing each of the students. Between each vertex, we draw an edge to represent a handshake. We can count that there are 10 edges, hence there are 10 handshakes that occurred.

2. Properties of Binomial Coefficients:

   What positive integer n satisfies

   $$\binom{n+2}{7} = 41 \cdot \binom{n}{5}?$$

   - 108

- 41
- 40
- 0

Answer: 40

Observe that the equation can be rewritten as

$$\binom{n+2}{7} = 41 \cdot \binom{n}{5} \frac{(n+2)(n+1)n(n-1)}{7!} = 41 \times \frac{n(n-1)}{5!}.$$

Since n is positive, dividing both sides of this by $n(n-1)$ gives

$$(n+2)(n+1) = 42 \cdot 41.$$

This is satisfied for $n = 40$.

3. Binomial Theorem Expansions

   For what value of N, is the following an algebraic identity:

$$(x+y)^N = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4?$$

- 1
- 2
- 3
- 4

Answer: 4

By the binomial theorem, we know that

$$(x+y)^4 == x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

4. Binomial Theorem challenge:

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r}^{-1}$$

If n is an an odd positive integer, find the value of this sum.

- -1
- 0
- 1
- Infinity

Answer: 0

Recall that $\binom{n}{r} = \binom{n}{n-r}$. Since n is odd, $n - r$ has the opposite parity as r, so

$$(-1)^r \binom{n}{r}^{-1} = -(-1)^{n-r} \binom{n}{n-r}^{-1}.$$

Thus, when we pair the terms in this way, each pair sums to 0, so the summation is 0.

5. Binomial Theorem Challenge 2:

$$\sum_{k=0}^{n} \left[(-1)^k \binom{n}{k} (n-k)^n \right] = ?$$

- n!
- 1
- (n-1)!
- 0

Answer: n!

In a bijective, number of elements in domain = number of elements of co-domain = (n) and range is equal to co-domain.

In a bijective, No. of one-one functions is equal to number of onto functions.

No.of. onto functions = $\sum_{k=0}^{n} (-1)^k \binom{n}{n-k}^m$.

Here, m = n.

So, $\sum_{k=0}^{n} (-1)^k \binom{n}{n-k}^n$ = number of one-one functions from n to n = n!.

6. Binomial Theorem Challenge 3:

Find the number of odd coefficients in the expansion of $(a+b)^{2015}$.

- 512
- 1024
- 2048
- 2047

Answer: 1024

We will make use of a consequence of Lucas' Theorem, whereby a binomial coefficient $\binom{m}{n}$ is odd if and only if none of the digits in the binary expansion of n is greater than the corresponding digit in the binary expansion of m.

In this case we have $m = 2015$, which has a binary of 11111011111.

So all those binomial coefficients $\binom{2015}{n}$ in the expansion of $(a+b)^{2015}$ for which n does not have a 1 in the 6th from the right digit in their binary expansion will be odd.

Since all numbers from 2016 to 2047 do have a 1 in this position, we can then conclude that the number of values n from 0 to 2015 inclusive that do not have a 1 in this position will be $\frac{2048}{2} = 1024$.