Section 3.3 "Diagonalization and Eigenvalues"

Plan:

- Eigenvalues and Eigenvectors
- Properties of eigenvectors and eigenvectors
- Diagonalization and eigenvectors and eigenvectors
- Algebraic and geometric multiplicity

Eigenvalues and Eigenvectors

Eigen $\rightarrow$ characteristic

Def^n (Eigenvalues and Eigenvectors)

If A is an $n \times n$ matrix, a number λ is called an **eigenvalue** of A if

$$A\mathbf{x} = \lambda \mathbf{x} \text{ for some column } \mathbf{x} \neq \mathbf{0} \text{ in } \mathbb{R}^n$$

In this case, $\mathbf{x}$ is called an **eigenvector** of A corresponding to the eigenvalue λ, or a λ-**eigenvector** for short.

Example: For $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ is an eigen vector corresponding to eigen value $\lambda = 4$.

Sol^n)

How to find eigenvalues and eigenvectors?

Def^n (characteristic Polynomial)

If A is an $n \times n$ matrix, the **characteristic polynomial** $c_A(x)$ of A is defined by

$$c_A(x) = \det(xI - A)$$

Def^n (Finding eigenvalues and eigenvectors)

Let A be an $n \times n$ matrix.

1. The eigenvalues λ of A are the roots of the characteristic polynomial $c_A(x)$ of A.
2. The λ-eigenvectors $\mathbf{x}$ are the nonzero solutions to the homogeneous system

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

of linear equations with $\lambda I - A$ as coefficient matrix.

Example: Find eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$

Def^n) Algebraic Multiplicity of an Eigen Value

An eigen value λ is said to have multiplicity 'n' if it occurs n-times as a root of characteristic polynomial.

Question: Is $c_A(\lambda) = c_{A^T}(\lambda)$?

Sol^n)

Question: Suppose A has a zero eigenvalue, what can you say about invertibility of A?

Sol^n)

Recall: A is invertible $\Leftrightarrow$ $A\vec{x} = 0$ has only trivial sol^n

Fact: $\det(A) =$

Question) Suppose λ is an eigenvalue of A with corresponding eigen vector $\vec{x}$. Prove $A^k \vec{x} = \lambda^k \vec{x}$, $k = 0, 1, \ldots$

Sol^n) Let's do it for $k=2$.

To prove: $A^2 \vec{x} = \lambda^2 \vec{x}$ --- (1)

I know: $A\vec{x} = \lambda \vec{x}$

Pf)

What happens when $k = -1$?

when $k=-1$, we are looking at eigenvalues of A^{-1}

Then, $\frac{1}{\lambda}$ will be an eigen value of A^{-1} ($\because \lambda \neq 0$)

Question) Suppose $\vec{x}$ and $\vec{y}$ are eigenvectors corresponding to λ. Prove that linear combination of $\vec{x}$ and $\vec{y}$ is also an eigenvector corresponding to λ.

Sol^n) Given: $A\vec{x} = \lambda\vec{x}$ and $A\vec{y} = \lambda\vec{y}$

To prove: $A(s\vec{x} + t\vec{y}) = \lambda(s\vec{x} + t\vec{y})$

Try it out!

What happens when $\vec{x}$ and $\vec{y}$ correspond to different eigen values λ and β? Is $\vec{x} + \vec{y}$ an eigen vector corresponding to either λ or β?

Sol^n) Not always true.