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Analytical modelling of driven pendulum with small angular displacement using Undetermined Coefficients Method

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Abstract

Many researchers have worked on dynamical systems in recent times because of its relevance in Engineering and Applied Physics. This paper attempts to model a damped driven simple pendulum as dynamical system and solve the model using Undetermined Coefficients Method. The model is a second order differential equation. The results obtained show that the motion of the pendulum is affected, significantly, by the driving force and the torque of the pendulum. Specifically, as driving force increases the response maximum amplitude of the pendulum decreases.

Keywords: Analytical modelling; Driven Pendulum; Angular Displacement; Undetermined Coefficients Method.

1. Introduction

When a pendulum is acted on, both by a periodic driving force, it can display both ordered and chaotic behaviours, for certain ranges of parameters [1]. The pendulum is a dynamical system. Undetermined Coefficients is one of the methods that can be used to find a particular solution to a nonhomogeneous differential equation of the form

$$y'' + P(t)y' + q(t)y = g(t)$$

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One of the main advantages of this method is that it reduces the problem down to an algebra problem. The algebra can get messy on occasion, but for most of the problems it will not be terribly difficult. Another nice thing about this method is that the complementary solution will not be explicitly required, although as we will see knowledge of the complementary solution will be needed in some cases. There are two disadvantages to this method. First, it will only work for a fairly small class of *g(t)*'s. The class of *g(t)*'s for which the method works, does include some of the more common functions, however, there are many functions out there for which undetermined coefficients simply will not work. Second, it is generally only useful for constant coefficient differential equations [2]. The method is quite simple. All that is needed to do is look at *g(t)* and make a guess as to the form of $Y_P(t)$ leaving the coefficient(s) undetermined (and hence the name of the method). Plug the guess into the differential equation and see if the values of the coefficients can be determined. If the values for the coefficients can be determined, then the guess is done correctly, if values for the coefficients could not be found then the guess is incorrect [2, 3]. The objective of this paper is to solve driven pendulum with small angular displacement with the aid of undetermined coefficients method.

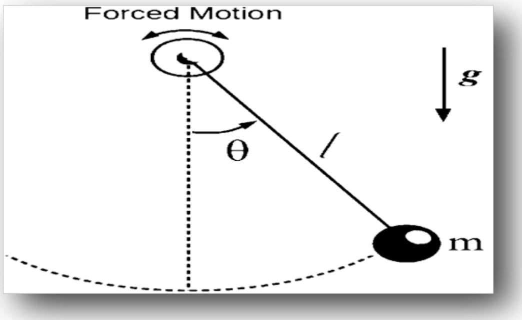

Figure 1: Driven Pendulum [4]

2. Problem Formulation

Our investigation of pendulum dynamics begins with Newton's second law of motion; which states that the relationship between an object's mass *m*, its acceleration *a*, and the applied force *F*, is [5]:

$$F = ma. \tag{1}$$

In terms of rotation, equation (1) becomes

$$R = Ia \tag{2}$$

Where I is the moment of inertia of pendulum, 'a' is the angular acceleration and R is the torque acting on the pendulum. The torque is indirectly proportional to the angle of rotation θ [6]:

$$R = - c \sin \theta \tag{3}$$

Where c is the torque of the pendulum at $\theta = 90^0$.
But, for simple pendulum [6,7],

$$c = mgd \tag{4}$$

where, mg is the normal force and d is the centre of mass.
Angular acceleration 'a' is the change in angular velocity:

$$a(t) = \frac{d}{dt}(v) = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) \tag{5}$$

$$Ia(t) = I\frac{d^2\theta(t)}{dt^2} \tag{6}$$

$$\Rightarrow R = I\frac{d^2\theta(t)}{dt^2}, from\ (1) \tag{7}$$

Substituting equation (3) into (7) gives:

$$-c\sin\theta = I\frac{d^2\theta(t)}{dt^2} \tag{8}$$

Now, when θ is small,

$$\frac{d^2\theta(t)}{dt^2} = \frac{-c}{I} \tag{9}$$

This can be written as;

$$\ddot{\theta} + \frac{c}{I}\theta = 0 \tag{10}$$

But when θ is not 0 and relatively large, $\sin\theta \neq 0$.
Equation (8) can be written as:

$$\ddot{\theta} + \frac{c}{I}\sin\theta = 0 \tag{11}$$

If the pendulum is driven, then a new term representing the driving force will come into the equation as follows:

$$\ddot{\theta} + \frac{c}{I}\sin\theta = k\cos(w_D t) \tag{12}$$

Where,
k is the amplitude of the driving force.
w_D is the angular driving force.
t is the time.
$w_D t$ is the phase of the driving force term.
For the pendulum under consideration,

$$I = ml^2 \tag{13}$$

Where

m and l are the mass of the pendulum bob and length respectively.
Substituting equation (13) into (12) gives;

$$\ddot{\theta}+\frac{c}{ml^2}\sin\theta = k\cos(w_D t) \tag{14}$$

This is a nonhomogeneous second order linear ordinary differential equation to be solved. Equation (14) takes the form;

$$\ddot{\theta}+p(t)\dot{\theta}+q(t)\theta = g(t) \tag{15}$$

Where g(t) is a non-zero function.
The associated homogeneous differential equation to equation (15) is

$$\ddot{\theta}+p(t)\dot{\theta}+q(t)\theta = 0 \tag{16}$$

2.1 Theorem

Suppose that θ_1 and θ_2 are two solutions to equation (15) and that q_1 and q_2 are a fundamental set of solutions to the associated homogenous differential equation (16) then, $\theta_1(t)-\theta_2(t)$ is a solution to equation (16) and it can be written as

$$\theta_1(t)-\theta_2(t) = c_1 q_1(t)+c_2 q_2(t) \tag{17}.$$

Now, because θ_1 and θ_2 are solutions to equation (15), it is known th

$$\ddot{\theta}_1+p(t)\dot{\theta}_1+q(t)\theta_1 = g(t) \tag{18}$$

$$\ddot{\theta}_2+p(t)\dot{\theta}_2+q(t)\theta_2 = g(t) \tag{19}$$

Since q_1 and q_2 are a fundamental set of solutions to equation (16), they form a general solution and so any solution to equation (16) can be written in the form;

$$q(t) = c_1 q_1(t)+c_2 q_2(t) \tag{20}$$

Therefore, since $\theta_1(t)-\theta_2(t)$ is a solution to equation (16), it can be written as

$$\theta_1(t)-\theta_2(t) = c_1 q_1(t)+c_2 q_2(t) \tag{21}.$$

The above theorem can used to write down the form of the general solution to equation (15). To solve the nonhomogeneous differential equation, the associated homogenous equation needs to be solved, which for constant coefficients equations is pretty easy to do, and a solution is needed for equation (15). There are two common methods for finding particular solution: the undetermined coefficients and variation of parameters. For this paper the undetermined coefficients method is used.

3. Problem Solution

Equation (14) is solved analytically using the undetermined Coefficient methods as follows:
Let
The amplitude of driving force (k) = 1
The angular driving force (W_D) = 2
The mass of the bob (m) = 1/16
Length of the string (l) =2
Torque of the pendulum (c) = -1
Equation (14) becomes

$$\ddot{\theta}+0\dot{\theta}-4\theta=\cos(2t) \tag{22}$$

The associated homogenous differential equation of equation (22) is

$$\ddot{\theta}+0\dot{\theta}-4\theta=0 \tag{23}$$

Characteristic equation for equation (23) and its roots are

$$r^2+0r-4=(r-2)(r+2)=0 \tag{24}$$

$$\Rightarrow r_1=2, r_2=-2 \tag{25}$$

The complementary solution then is

$$q_c(t)=c_1\cos(2t)+c_2\cos(-2t) \tag{26}$$

Since the differential equation, under consideration, has a cosine so let a guess of the particular solution be

$$\theta_p(t)=A\cos(2t) \tag{27}$$

Differentiating and plugging into the differential equation (22) gives;

$$-4A\cos(2t)+0(2A\sin(2t))-4A\cos(2t))=\cos(2t) \tag{28}$$

Collecting like terms yields,

$$-8A\cos(2t)=\cos(2t) \tag{29}$$

‘A’ needs to be picked so that the same function can be gotten on both sides of equal signs.

$$-8A=1 \tag{30}$$

$$\Rightarrow A=-\frac{1}{8} \tag{31}$$

A particular solution to the differential equation is therefore

$$\theta_p(t)=-\frac{1}{8}\cos(2t) \tag{32}$$

Now considering an increase in the driving force, starting with driving force (W_D) = 3:
A guess of the particular solution in this case is

$$\theta_p(t)=A\cos(3t) \tag{33}$$

$$\Rightarrow -9A\cos(3t)+0(3A\sin(3t))-4(A\cos(3t))$$
$$=\cos 3t \tag{34}$$

$$-13A\cos(3t)=\cos(3t) \tag{35}$$

$$\Rightarrow A=-\frac{1}{13} \tag{36}$$

$$\therefore \theta_p(t)=-\frac{1}{13}\cos(3t) \tag{37}$$

When the driving force (W_D) = 4, the particular solution is as follows:
The guess is

$$\theta_p(t) = A\cos(4t) \quad (38)$$

$$\Rightarrow -16A\cos(4t) + 0(4A\sin(4t)) - 4(A\cos(4t)) = \cos 4t \quad (39)$$

$$-20A\cos(3t) = \cos(4t) \quad (40)$$

$$\Rightarrow A = -\frac{1}{20} \quad (41)$$

$$\therefore \theta_p(t) = -\frac{1}{20}\cos(4t) \quad (42)$$

When the torque of the pendulum (c) = -4, driving force W_D = 2 and other parameters remaining the same, equation (14) becomes,

$$\ddot{\theta} + 0\dot{\theta} - 16\theta = \cos(2t) \quad (43)$$

The characteristic equation and the roots of equation (38) are given as;

$$r^2 + 0r - 16 = (r-4)(r+4) = 0 \quad (44)$$

$$\Rightarrow r = 4, r = -4 \quad (45)$$

The complementary solution is in the form

$$q_c(t) = c_1\cos(4t) + c_2\cos(-4t) \quad (46)$$

Let a guess of the particular solution be

$$\theta_p(t) = A\cos(2t) \quad (47)$$

Differentiating and plugging into the differential equation (22) gives;

$$-4A\cos(2t) + 0(2A\sin(2t)) - 16A\cos(2t)) = \cos(2t) \quad (48)$$

Collecting like terms yields,

'A' needs to be picked so that the same function can be gotten on both sides of equal signs.

$$-20A = 1 \quad (49)$$

$$\Rightarrow A = -\frac{1}{20} \quad (50)$$

A particular solution to the differential equation is therefore

$$\theta_p(t) = -\frac{1}{20}cos(2t) \quad (51)$$

4. Result Discussion

From the figures 1,2,3 and 4, it shows that as the driving force increases, with time, the frequency increases. Also the response maximum amplitude increases as the torque 'c' of the pendulum at $\theta = 90^0$c increases, as shown in figure 1 and figure 5. It was also noticed that as the driving force increases the response maximum amplitude of the pendulum decreases. The dynamic response of the driven pendulum, dynamical system is periodic.

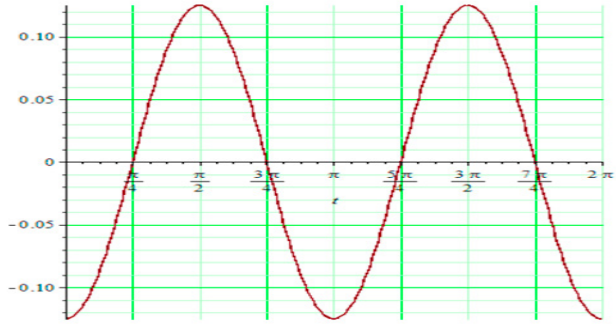

Figure 2: Shows angle of rotation when driving force (W_D)= 2 and c = -1 as time increases

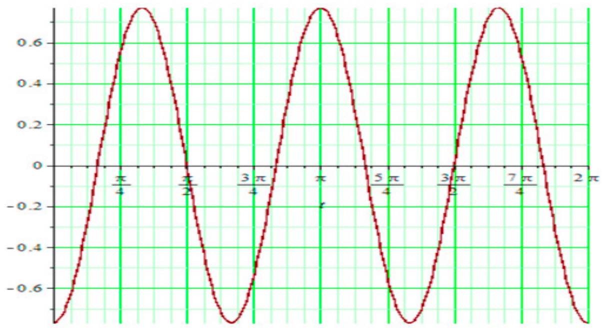

Figure 3: Shows angle of rotation when driving force (W_D) = 3 and c = -1as time increase

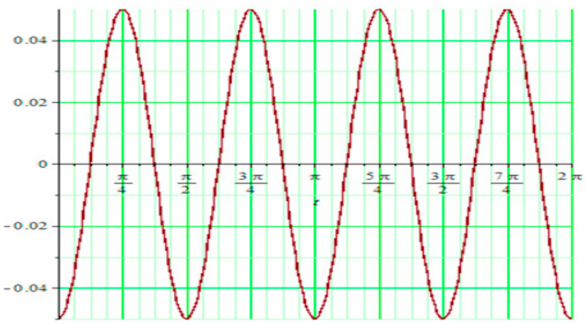

Figure 4: Shows angle of rotation when driving force (W_D) = 4 and c = -1 as time increases

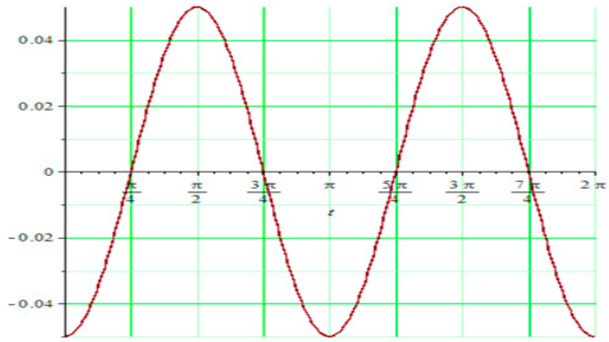

Figure 5: Shows angle of rotation when $W_D = 2$ and c = - 4 units as time (t) increases.

5. Conclusion

This paper attempted to analyze the dynamics of a driven pendulum. The modelled equation of motion of the dynamical system was solved using undetermined coefficients method. From the results obtained, it was observed that both the driving force and the torque of the pendulum affect the dynamic response of the driven pendulum. The undetermined coefficients method used was easy to apply.

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