Canonical Quantization Procedure and Time-Energy Uncertainty

A Toy Model of Quantum Gravity

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The Time-Energy Uncertainty Relation

First postulated by Heisenberg [1], the Time-Energy Uncertainty is:

$$\underbrace{\Delta E}_{\text{Hamiltonian } \hat{H} \text{ eigenvalue}} \cdot \underbrace{\Delta t}_{?} \geq \hbar/2 \tag{1}$$

Problem 1: No self-adjoint time operator

No axiomatic derivation similar $\Delta\hat{x} \cdot \Delta\hat{p} \geq \hbar/2$ based on their commutation relation $[\hat{x}, \hat{p}] = i\hbar$, since a time operator $\hat{t}$ does not exist.

Problem 2: Different interpretations of time uncertainty

1. Energy-dispersion (ΔE) of a state and its lifetime (Δt)
2. Energy exchange (ΔE) and exchange time-frame (Δt)
3. Energy measurement (ΔE) and measurement/preparation time (Δt)

Notation and Klein-Gordon equation

Notation:

1. four vectors: $x^\mu = (t, \vec{x})$ and $p^\mu = (E, \vec{p})$
2. Minkowski metric: $\eta_{\mu\nu} = (-,+,+,+)$
3. Einstein summation convention: $p_\mu p^\mu = p_\mu p_\nu \eta^{\mu\nu} = -p_0^2 + \vec{p}^{\,2}$

Def: A constraint on a system is a dynamical relation that the system must obey, of the form:

$$\phi(x^\mu, p_\mu) = 0 \tag{2}$$

e.g. the mass-shell constraint:

$$p_\mu p^\mu + m^2 c^2 = 0 \tag{3}$$

For $\hat{p}_\mu = -i\hbar\partial_\mu$, quantizing the above yields:

$$\left(\hat{p}_\mu \hat{p}^\mu + m^2 c^2\right)|\psi\rangle = (-\hbar^2 \partial_\mu \partial^\mu + m^2 c^2)\,|\psi\rangle = 0 \tag{4}$$

i.e. the Klein-Gordon equation.

Free Relativistic Bosonic Particle

The Hamiltonian after quantizing the simplest example is:

$$\hat{H}(x^{\mu}, p_{\mu}) |\psi\rangle = \frac{1}{2}(\hat{p}_{\mu}\hat{p}^{\mu} + m^2c^2) |\psi\rangle = 0 \tag{5}$$

This is called a Hamiltonian Constraint.

Problem of Time in Quantum Gravity

$$i\hbar\frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle = 0 \tag{6}$$

Proposed solution: $4 + 1$ formalism

Introduce an extra parameter time λ giving the system's evolution as:

$$i\hbar\frac{\partial}{\partial \lambda} |\psi\rangle = \hat{H}_0 |\psi\rangle \tag{7}$$

World Line Parametrization

An observer moves in space-time as parametrized by the time evolution parameter λ. [2]

e.g. identify the parameter with proper time:

$$\lambda = \alpha\tau \tag{8}$$

Just a specific gauge choice, in general, $\lambda = f(\tau)$.

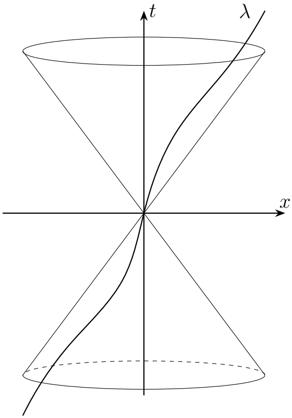

Figure 1: World line parametrization by λ

Free Relativistic Bosonic Particle (Again)

The Hamiltonian constraint after quantizing the simplest example is:

$$\hat{H}\left|\psi\right\rangle_m = \frac{1}{2}\Big(\underbrace{\hat{p}_\mu \hat{p}^\mu}_{\text{operator}} + \underbrace{m^2 c^2}_{\text{scalar value}}\Big)\left|\psi\right\rangle_m = 0 \tag{9}$$

Reinterpret as parameter time independent equations:

$$\hat{H}_0\left|\psi\right\rangle_m = \hat{p}_\mu \hat{p}^\mu \left|\psi\right\rangle_m = i\hbar\frac{\partial}{\partial\lambda}\left|\psi\right\rangle_m \overset{!}{=} -m^2 c^2 \left|\psi\right\rangle_m \tag{10}$$

solved by physical wave functions $\left|\psi\right\rangle_m \in \mathcal{H}_m \subset \mathcal{H}$.
But for arbitrary (not necessary on mass-shell) wave functions:

Parameter time evolution

$$\hat{H}_0\left|\psi\right\rangle = \hat{p}_\mu \hat{p}^\mu \left|\psi\right\rangle = i\hbar\frac{\partial\left|\psi\right\rangle}{\partial\lambda} \tag{11}$$

Extension to Four Coordinate Dimensions

The wave functions $|\psi\rangle$ live in a four dimensional Hilbert Space, taken to be $\mathcal{H} = L^2(\mathbb{R}^4)$.

$$\langle\psi_1|\psi_2\rangle = \int_{\mathbb{R}^4} \mathrm{d}^4x\, \psi_1^*(x^\mu)\psi_2(x^\mu) \quad \text{with} \quad \|\psi\| = \sqrt{\langle\psi|\psi\rangle} \leq \infty \qquad (12)$$

Problem: How to normalize wave functions in time?

Take $U(t) = \exp\{-iEt\}$, then any wave function would be non-normalizable, as:

$$\langle\psi|\psi\rangle = \int_{\mathbb{R}} dt U^*(t)U(t) \int_{\mathbb{R}^3} \mathrm{d}^3x\, \psi^*(\vec{x})\psi(\vec{x}) = \int_{\mathbb{R}} dt \to \infty \qquad (13)$$

Time Gaussian Wave Packets!

Time-localized Wave Functions

Particles localized in time

Wave packets evolve in parameter time λ and propagate in the space-time.

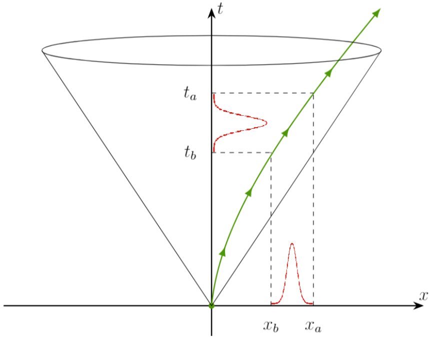

Figure 2: Space-time probabilities

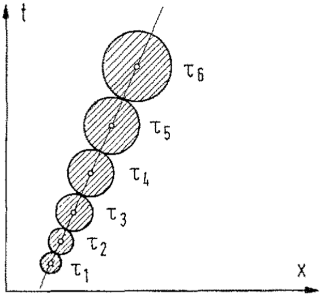

Figure 3: Gaussian Spreading [2]

Probability Interpretation

The Born probability interpretation [3] holds! By construction, the probability density $\rho = \psi^* \psi$ is positive and moreover:

Continuity equation

$$\partial_\lambda \rho + \partial_\mu j^\mu = 0 \tag{14}$$

where:

$$j^\mu = -i(\psi^* \partial^\mu \psi - \psi \partial^\mu \psi^*) \tag{15}$$

Time-Energy Uncertainty

Uncertainty relations [4] for operators $\hat{A}$ and $\hat{B}$:

$$(\Delta \hat{A})^2 (\Delta \hat{B})^2 \geq \frac{1}{2} \left| \langle [\hat{A}, \hat{B}] \rangle \right|^2 \tag{16}$$

From the position-momentum commutation relation $[\hat{x}^\mu, \hat{p}_\nu] = i\hbar \delta^\mu_\nu$:

$$\Delta \hat{x}^\mu \Delta \hat{p}_\nu \geq \frac{1}{2} \left| \langle \psi | [\hat{x}^\mu, \hat{p}_\nu] | \psi \rangle \right| = \frac{\hbar}{2} \delta^\mu_\nu \tag{17}$$

The zeroth component is:

Main result: The Time-Energy Uncertainty

$$\Delta x^0 \Delta p_0 = \Delta t \Delta E \geq \frac{\hbar}{2} \tag{18}$$

Summary

1. Introduced another time evolution parameter λ to implement the Hamiltonian constraint as an eigenvalue problem.
2. Promoted coordinate time $t = x^0$ to an operator.
3. Reformulated Relativistic Quantum Mechanics for the Bosonic Free Particle accordingly: $\psi \in L^2(\mathbb{R}^4)$, probability interpretation, etc.
4. Axiomatically obtained the Time-Energy Uncertainty Relation via operator commutation relations.

References

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