Little's Law

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OR 647

Lecture 1b

Little's Law

Little's Law, Notation

- Let $A^{(1)}$, $A^{(2)}$, $A^{(3)}$, … be a sequence with

- Let $D^{(1)}$, $D^{(2)}$, $D^{(3)}$, … be a sequence with

- Let $W^{(k)} \equiv$

- Let $Q(t)$ = # of indices k such that

Little's Law, Assumptions

- $$\frac{A^{(k)}}{k} \rightarrow$$
- $$\frac{\sum_{j=1}^{k} W^{(j)}}{k} \rightarrow$$

Little's Law

Then

- $$\frac{\int_0^t Q(s)ds}{t} \to L$$

and

-

Observations

- The hypotheses requires a convergence of average system behavior.
  - Precludes…
  - Arrival rate…
- Little's law does not mention a "queue". Minimum physics is:
- Little's law does not <u>not</u> require:
  - Poisson arrivals, independent arrivals, first-come-first-served, exponential service, etc.

Key Questions

- What is the "system"?
  - Different definitions give different "versions" of Little's law.
  - Little's law is more like a "principle"
- Based on definition of "system"
  - What is L, average # in system?
  - What is W, average time in system?

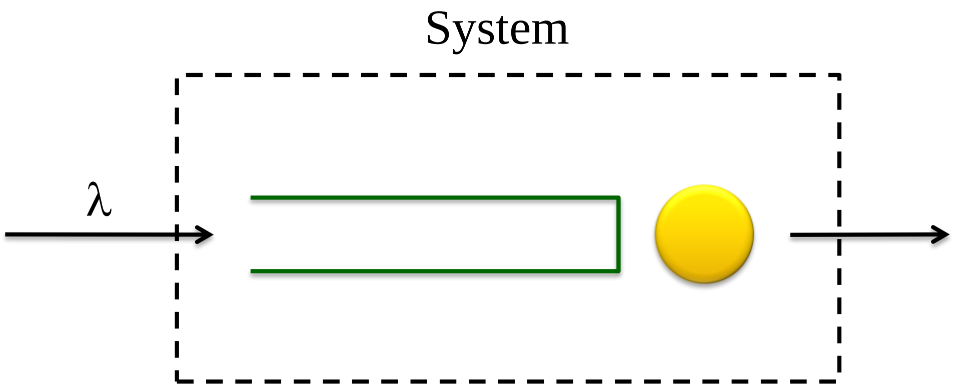
Common Notation

- # of customers in system / queue
- average time in system / queue
- average arrival rate
- average service rate, where S = random service time
- offered load
- # of servers
- utilization or traffic intensity

Notes

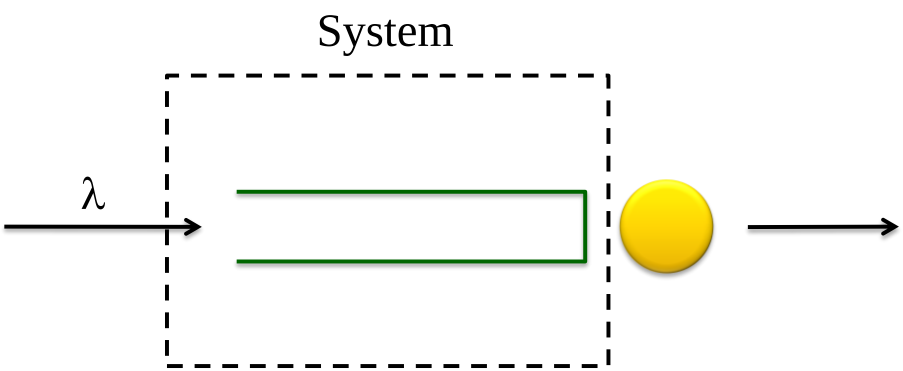
- The offered load *r* is the expected number of customers in service assuming an infinite-server system.
  - This is in contrast to the carried load, which is the expected number of customers in service in the actual system.
  - This difference matters for loss systems, but not for loss-less systems.
- For single server systems, *r* and ρ are interchangeable.

Example #1



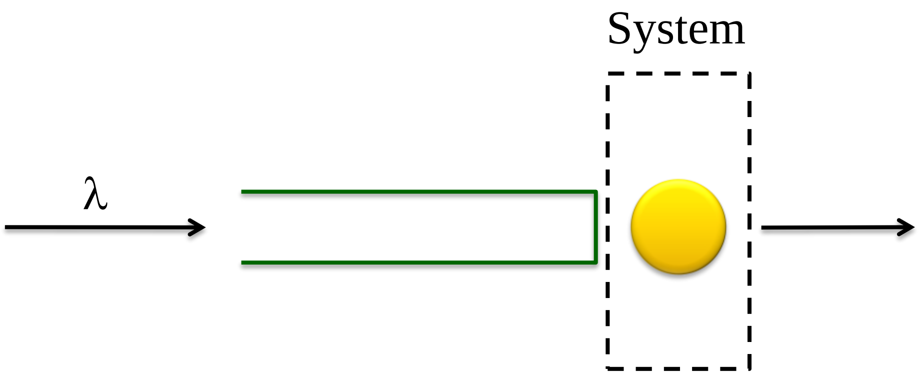
- Average # in system
- Average time in system
- Little's Law:

Example #2



- Average # in "system"
- Average time in "system"
- Little's Law:

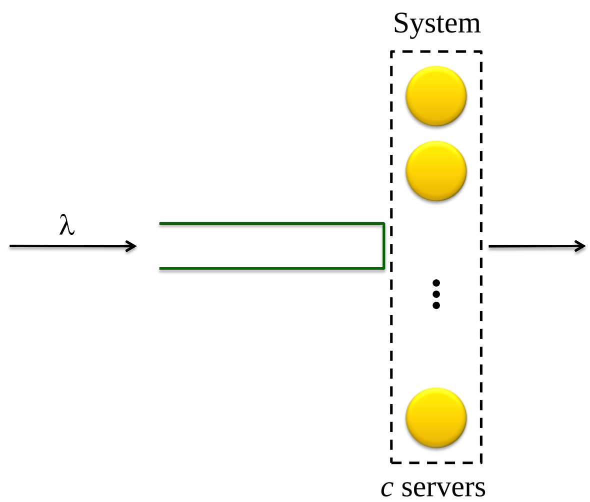
Example #3: Single-Server



- "L":
- "W":
- Little's Law:

Note: This result is valid for any <u>loss-less</u> <u>*single-server*</u> system (provided steady-state exists).

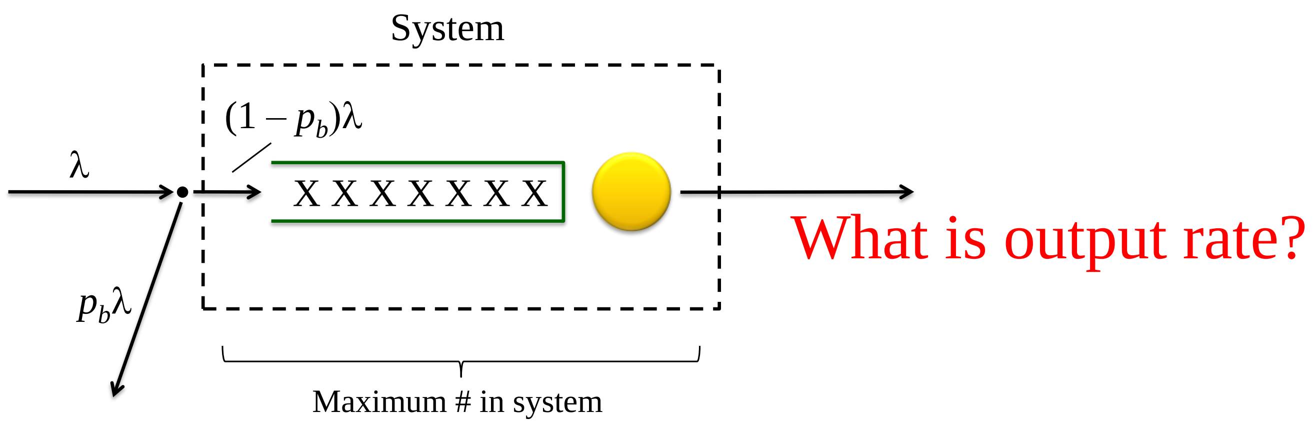
Example #4: Multiple-Server



- "L":
- "W":
- Little's Law:

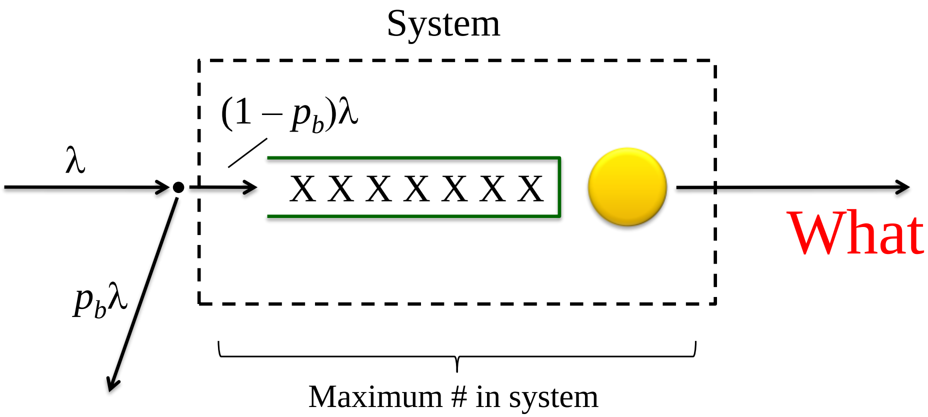
Notes

Example #5: Loss System



- "L":
- "λ":
- "W":
- Little's Law:

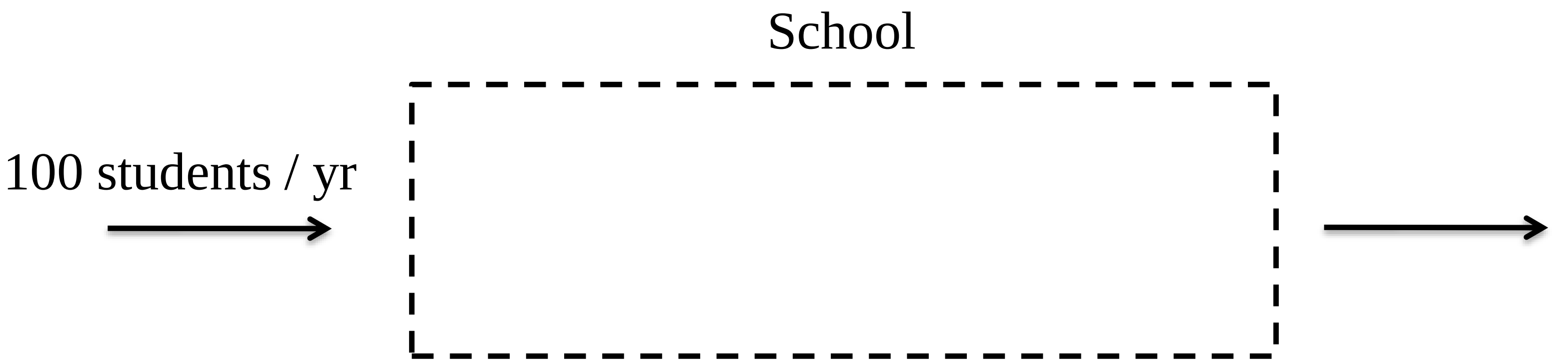
Example #5: Loss System



What is output rate?

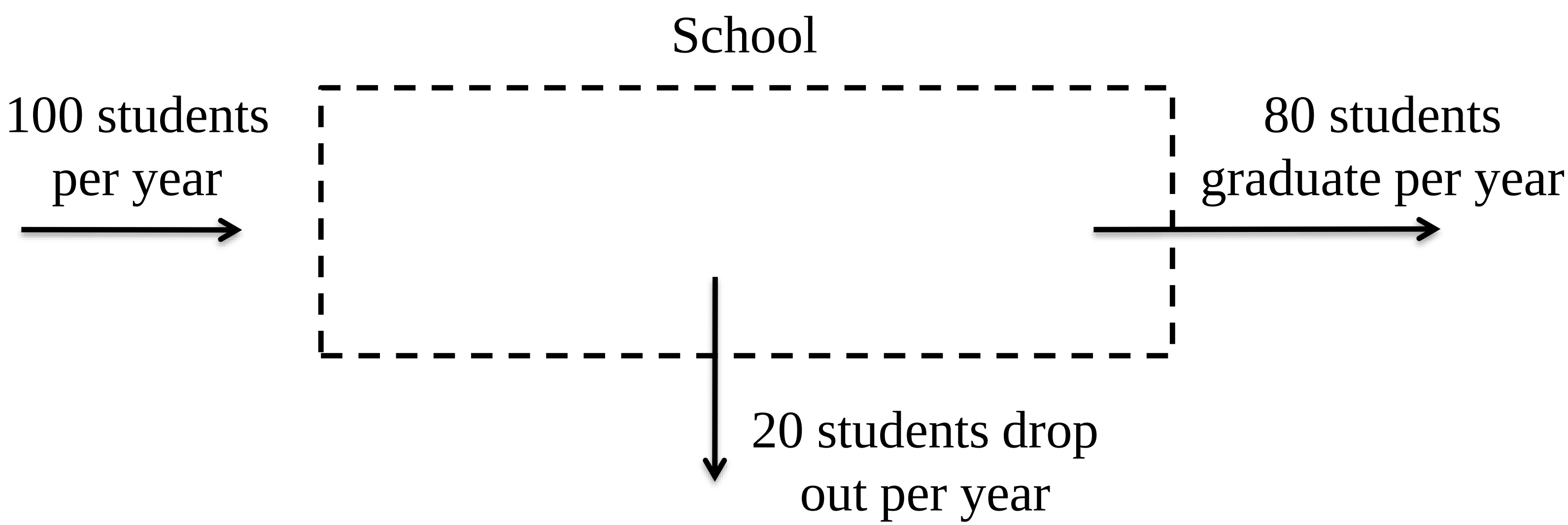
- What if you interpret "W" as the average time spent in system for all customers (including blocked customers who count as zero against the average)?

Example #6: Non-Queueing



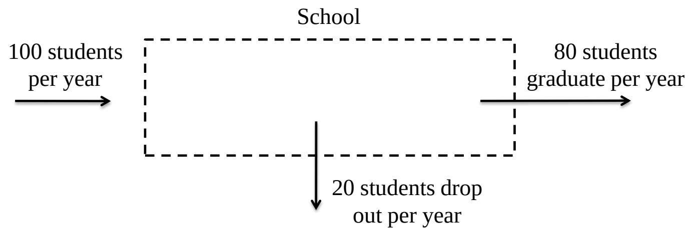
- Suppose average time in school is 4 years.
- What is average # of students enrolled?

Example #6b: Variation



- Suppose average # in school is 400.
- What is average time spent in school?

Example #6b: Variation



Graphical View of Little's Law

Definitions

- $A(t)$ = number of arrivals by time t
- $D(t)$ = number of departures by time t.
- $L(t)$ = number in the system at time t [= $A(t) - D(t)$].
- $W^{(n)}$ = time in system of nth customer [= $D^{(n)} - A^{(n)}$].

Customer #	Arrival Time	Departure Time
1	0	1
2	2	5
3	3	11
4	6	13
5	7	14
6	8	15
7	12	16
8	14	18
9	19	24
10	20	25
11	24	26
12	26	28

We are not making any assumptions about queueing. The system is FCFS in this example since the departure order is the same as the arrival order. We could mix up the departure times without changing the overall result. However, assuming FCFS makes the graphical interpretation below easier.

Graphical View

Customer #	Arrival Time	Departure Time
1	0	1
2	2	5
3	3	11
4	6	13
5	7	14
6	8	15
7	12	16
8	14	18
9	19	24
10	20	25
11	24	26
12	26	28

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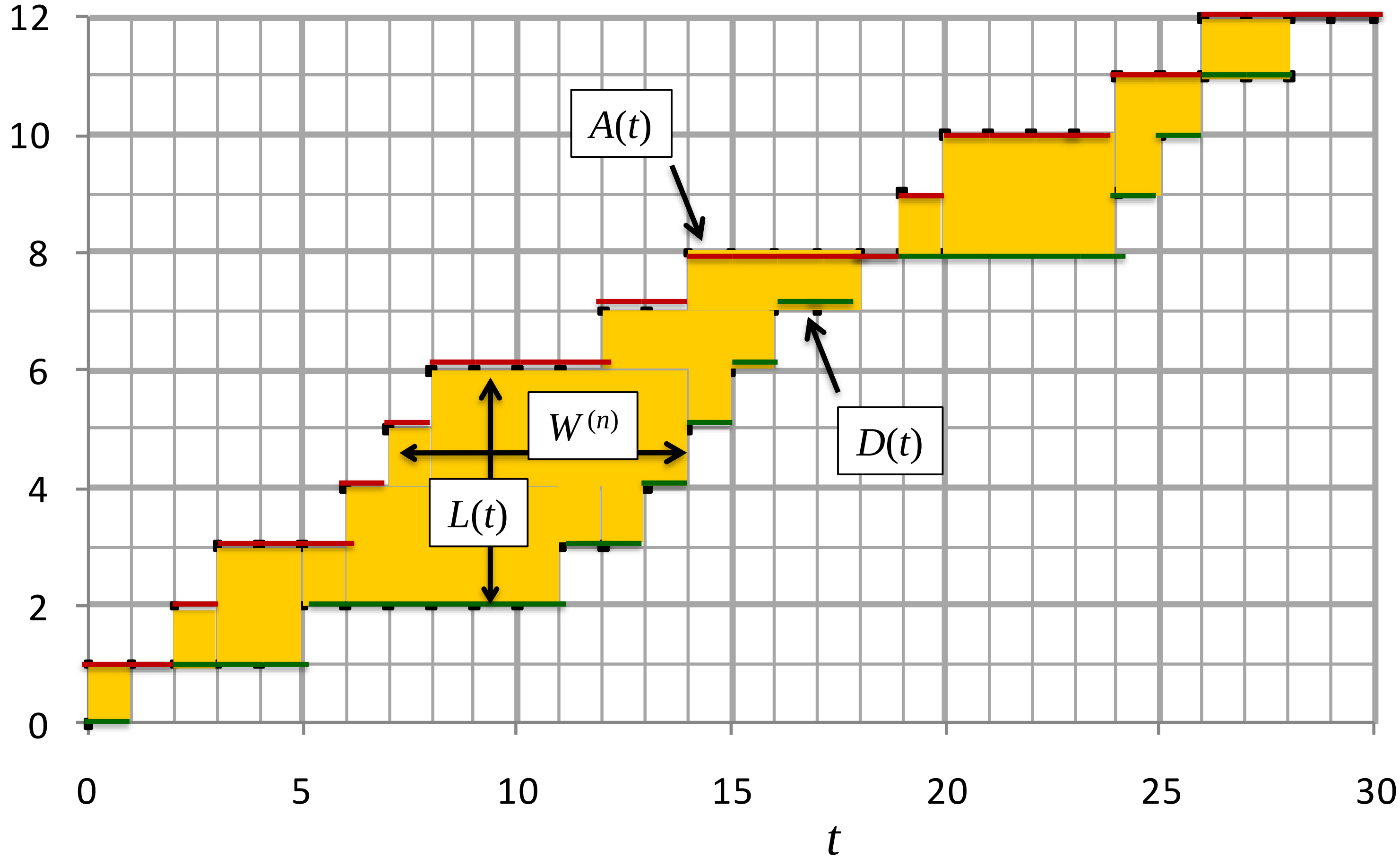
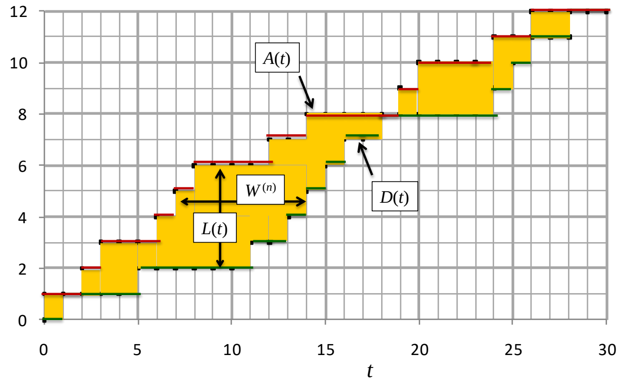
Graphical Interpretation



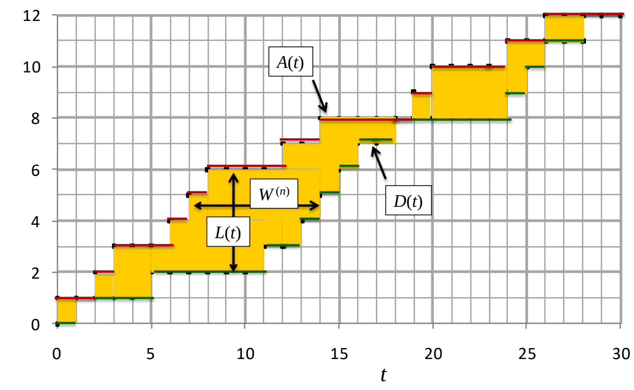
"Proof" of Little's Law

- Integrate along *x*-axis



"Proof" of Little's Law

- Integrate along *y*-axis



"Proof" of Little's Law

- Set areas equal to each other:

$$\int_0^T L(t)dt = \sum_{i=1}^{n} W^{(n)}$$

$$\frac{\int_0^T L(t)dt}{T} = \frac{n}{T} \cdot \frac{\sum_{i=1}^{n} W^{(n)}}{n}$$

Use of Little's Law

- In a queueing system, if you know one of L, L_q, W, W_q, you can get the other 3.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
L \arrow[r, leftrightarrow, "L = \lambda W"] \arrow[d, leftrightarrow, "L = L_q + r"'] & W \arrow[d, leftrightarrow, "W = W_q + 1/\mu"] \\
L_q \arrow[r, leftrightarrow, "L_q = \lambda W_q"] & W_q
\end{tikzcd}$$