HOMEWORK 2
OR647: QUEUEING THEORY, SPRING 2021

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1

A cloud computing service is modeled as a fluid queue with time-dependent arrival rate $\lambda(t)$ and constant service rate μ. Consider a situation where the arrival process has a burst. Initially, the arrival rate is $\lambda_1 < \mu$. Then, for a period of length T, the arrival process is $\lambda_2 > \mu$, after which it returns back to λ_1.

a. What is the maximum queue length?

Solution. Let the burst begin at time $t_0 = 0$ and end at $t_T = T$. For times $t < t_0$, the cumulative number of arrivals, $A(t)$, equals the cumulative number of departures, $D(t)$, since $\lambda(t) = \lambda_1 < \mu$. Therefore the queue length L is zero. For times $t_0 \leq t < t_T$ however, $\lambda(t) = t_2 > \mu$ and cumulative arrivals are greater than cumulative departures. We also know that the queue length at time t, $L(t)$, is the difference $A(t) - D(t)$. Furthermore, the moment when the queue length is greatest is t_T. This follows from the fact that the queue length is increasing from t_0 to t_T and begins decreasing at time t_T. So then,

$$\begin{aligned}
L_{\text{max}} &= A(t_T) - D(t_T) \\
&= \int_{t_0}^{t_T} \lambda(t)dt - \int_{t_0}^{t_T} \mu dt + A(t_0) - D(t_0) \\
&= T(\lambda_2 - \mu).
\end{aligned}$$

b. What is the length of the busy period (i.e., the time from the start of the burst until the queue empties)?

Solution. Let T_e be the length of time from t_T to the end of the busy period, t_e. We know the queue length at $t = t_T$ and we know the queue empties at rate $(\mu - \lambda_1)$.

$$\begin{aligned}
T_e(\mu - \lambda_1) &= T(\lambda_2 - \mu) \\
T_e &= \frac{T(\lambda_2 - \mu)}{\mu - \lambda_1}.
\end{aligned}$$

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So the total busy period, T_b, is

$$\begin{aligned} T_b &= T + T_e \\ &= T + \frac{T(\lambda_2 - \mu)}{\mu - \lambda_1} \\ &= \frac{T(\mu - \lambda_1)}{\mu - \lambda_1} + \frac{T(\lambda_2 - \mu)}{\mu - \lambda_1} \\ &= \frac{T(\lambda_2 - \lambda_1)}{\mu - \lambda_1} \end{aligned}$$

c. What is the total delay (among all jobs) due to the burst?

Solution. Find $\int_T A(t) - D(t) + \int_{T_e} A(t) - D(t)$ geometrically.

$$\frac{1}{2}T^2(\lambda_2 - \mu)\left(\frac{\lambda_2 - \lambda_1}{\mu - \lambda_1}\right)$$

2

A road segment is modeled as a fluid queue. The arrival rate is shown in the graph below, with a morning rush hour. After the morning rush hour, the transportation department begins maintenance work on the road. At 9 a.m., one of the three lanes on the road is closed, reducing the capacity of the road from 15 cars per minute to 10 cars per minute. The maintenance period ends at noon.

d. Determine the average delay per car (over all cars from 6 a.m.to 1 p.m.).

Solution. Calculate the area of $A(t) - D(t)$ (193,500) geometrically and divide by the total number of vehicle arrivals in the time period (2400) to find the average delay of 41.61 minutes.

time, t (min)	60	120	180	360	390(f)
$\lambda(t)$ (veh/min)	10	20	30	5	5
$D'(t)$ (veh/min)	10	15	15	10	15
$A(t)$ (veh)	600	1800	3600	4500	4650
$D(t)$ (veh)	600	1500	2400(e)	4200	4650
Queue length (veh)	0	300	1200	300	0
Total delay (veh·min)	0	9000	54000	189000	193500
Average delay (min)	0.0	5.0	15.0	42.0	41.61(d)

e. What is the maximum length of the queue over this period?

Solution. Calculate the $A(t) - D(t)$ iteratively from $t = 0$ and choose the largest value (1200 vehicles).

f. At what time does the queue length return to 0?

Solution. Find the intersection of $A(t)$ and $D(t)$ geometrically to find 390 minutes elapesed time (12:30 p.m.).

3. PROBLEM 8.12

Airplanes arrive at an airport at a rate of 30 per hour. The arrival capacity of the airport is 40 airplanes per hour during good weather. During periods of fog, the arrival capacity drops to 20 airplanes per hour. Suppose that the airport experiences fog from 8 am to 10 am. Using a fluid approximation, and considering only arrivals between 8 am and noon, what is the average delay for each airplane?

Solution. Solve geometrically as before. 10 minutes.

4. PROBLEM 8.17

A traffic signal has a left-hand turn lane. The signal alternates from a red light to a green left-hand turn arrow (cars can turn left) to a solid green light (cars can turn left but must yield to oncoming traffic). Cars flow through the signal at a rate of 15 per minute when there is a left-hand arrow, but only 10 per minute when there is a regular green light. The signal spends 2 min in red, 1 min as a green left-hand arrow, and 1 min as a green light. The arrival rate of cars who want to turn left is 6 per minute. Under a fluid approximation, what is the average time spent waiting to turn left?

Solution. Solve geometrically as before. 23 minutes

5. PROBLEM 8.18

The following diagram shows a freeway on-ramp. The maximum rate that cars can pass through point A is 60 per minute. Due to a stop light at B, the in-flow of cars to the freeway comes in waves. The stop light is “off” for 2 min (during which no cars enter the freeway) and “on” for 1 min, during which cars enter the freeway at a rate of 30 per minute. During periods of congestion at A, assume that up to half of the flow-rate is applied to cars arriving from B (i.e., cars merge on a one-to-one basis).

a. What is the maximum flow rate of cars through point C that maintains stability of the system?

Solution. The average flow from B in one cycle is $30/(1 + 2) = 10$ veh/min. So the maximum stable flow from A is $60 - 10 = 50$ veh/min.

b. Assume that the flow rate of cars through C is 45 per minute. Using a fluid approximation, what is the average delay per car on the freeway due to the on-ramp?

Solution. During the “on” phase the departure rate is 30 veh/min (60/2) otherwise it may be as high as 60 veh/min. We assume congestion is cleared which means we will consider the one-minute “on” phase first to accumulate congestion. Solving geometrically as before. 5 seconds.

c. What is the average delay per car on the on-ramp?

Solution. This time, we begin with the “off” phase. Assume the arrival rate is constant (10 veh/min). Solving geometrically as before. 20 seconds.

6. PROBLEM 2.12

A certain software company has a technical support line. Requests for technical support arrive according to a Poisson process with rate $\lambda = 20$ per hour hour. What is the probability that:

a. No calls arrive during 1 hour?

Solution.

$$\Pr\{N(1) = 0\} = e^{-20\cdot 1}\frac{(20\cdot 1)^0}{0!} = e^{-20} \approx 2.061 \times 10^{-9}$$

b. Exactly 5 calls arrive during 1 hour?

Solution.

$$\Pr\{N(1) = 5\} = e^{-20\cdot 1}\frac{(20\cdot 1)^5}{5!} \approx 5.496 \times 10^{-5}$$

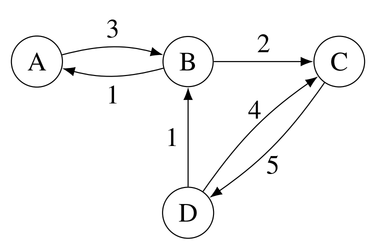
c. 5 or more calls arrive during 1 hour?

Solution.

$$\begin{aligned}\Pr\{N(1) \geq 5\} = 1 - \Pr\{N(1) \leq 4\} &= 1 - e^{-20\cdot 1}\sum_{i=0}^{4}\frac{(20\cdot 1)^i}{i!} \\ &\approx 0.99998\end{aligned}$$

7. Problem 2.13

The following diagram represents a continuous-time Markov chain (where the numbers represent transition rates q_{ij}). Find the fraction of time the chain spends in each state.



Solution.

$$\{q_{ij}\} = \begin{bmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 1 & 4 & 0 \end{bmatrix}$$

$$\{v_i\} = \{q_{ij}\}J_{j,1} = \begin{bmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 1 & 4 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 5 \\ 5 \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} -3 & 3 & 0 & 0 \\ 1 & -3 & 2 & 0 \\ 0 & 0 & -5 & 5 \\ 0 & 1 & 4 & -5 \end{bmatrix}$$

The fraction of time spent in each state are the stationary probabilities.

$$\mathbf{0} = \vec{p}\mathbf{Q} \wedge \sum_i p_i = 1$$

$$\begin{bmatrix} [\mathbf{Q}]^\mathsf{T} \\ J_{1,j} \end{bmatrix} \vec{p} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}$$

$$\vec{p} = \begin{bmatrix} \frac{1}{16} & \frac{3}{16} & \frac{3}{8} & \frac{3}{8} \end{bmatrix}$$

8. Problem 2.15

Customers arrive at a shuttle stop according to a Poisson process with rate 3 per hour. Shuttles arrive at the stop according to a Poisson process with rate 1.5 per hour. Suppose that each shuttle can hold at most 2 customers. Suppose that at most 4 people wait for the shuttle (subsequently arriving customers are turned away).

a. Model this process as a continuous-time Markov chain. Give the rate transition matrix $\mathbf{Q}$.

Solution. Assume arriving shuttles are empty.

$$\{q_{ij}\} = \begin{bmatrix} 0 & 3 & 0 & 0 & 0 \\ 1.5 & 0 & 3 & 0 & 0 \\ 1.5 & 0 & 0 & 3 & 0 \\ 0 & 1.5 & 0 & 0 & 3 \\ 0 & 0 & 1.5 & 0 & 0 \end{bmatrix}$$

$$\{v_i\} = \{q_{ij}\}J_{j,1} = \begin{bmatrix} 3 \\ 4.5 \\ 4.5 \\ 4.5 \\ 1.5 \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} -3 & 3 & 0 & 0 & 0 \\ 1.5 & -4.5 & 3 & 0 & 0 \\ 1.5 & 0 & -4.5 & 3 & 0 \\ 0 & 1.5 & 0 & -4.5 & 3 \\ 0 & 0 & 1.5 & 0 & -1.5 \end{bmatrix}$$

b. Give the probability transition matrix $\mathbf{P}$ of the embedded discrete-time Markov chain.

Solution.

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

c. Solve for the stationary probabilities p_i (of the CTMC) and π_i (of the embedded DTMC).

Solution. For the CTMC

$$\mathbf{0} = \vec{p}\mathbf{Q} \wedge \sum_i p_i = 1$$

$$\begin{bmatrix} [\mathbf{Q}]^{\mathsf{T}} \\ J_{1,j} \end{bmatrix} \vec{p} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}$$

$$\vec{p} = \begin{bmatrix} \frac{11}{57} & \frac{10}{57} & \frac{4}{19} & \frac{8}{57} & \frac{16}{57} \end{bmatrix}$$

For the embedded DTMC

$$\mathbf{0} = \vec{\pi}(\mathbf{P} - \mathbf{I}) \wedge \sum_i \pi_i = 1$$

$$\begin{bmatrix} [\mathbf{P} - \mathbf{I}]^{\mathsf{T}} \\ J_{1,j} \end{bmatrix} \vec{\pi} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}$$

$$\vec{\pi} = \begin{bmatrix} \frac{11}{64} & \frac{15}{64} & \frac{9}{32} & \frac{3}{16} & \frac{1}{8} \end{bmatrix}$$

d. What is the average number of customers who enter a shuttle?

Solution. Calculate the average number of customers weighted by probabilities $\vec{p}$.

$$\frac{11}{57} \cdot 0 \text{ passengers} + \frac{10}{57} \cdot 1 \text{ passenger} + \left(\frac{4}{19} + \frac{8}{57} + \frac{16}{57}\right) \cdot 2 \text{ passengers}$$
$$= \frac{82}{57} \text{ passengers} \approx 1.44 \text{ passengers}$$

9. Problem 3.10

The following is a graph of the arrival rate of customers (over a 2-hour period). You decide to model this as an M/M/1 queue with service rate 30 per hr. Compute the average wait in queue if you:

a. Assume that the arrival rate is constant over the 2-hour period (dashed line)

Solution. Let $\lambda = 15/\text{h}$ and $\mu = 30/\text{hr}$, then $\rho = \frac{1}{2}$ and $W_q = \frac{\rho}{\mu - \lambda} = \text{h}/30 = 2$ min.

b. Assume that the arrival rate is constant over specific intervals (solid line). Assume the queue is in steady state over each interval, compute the average wait in queue for each interval and then compute the overall average wait in queue.

Solution. For $0 < t \leq 40$ and $80 < t \leq 120$, let $\lambda = 10/\text{h}$ and $\mu = 30/\text{h}$, then $\rho = \frac{1}{3}$ and $W_q = \frac{\rho}{\mu - \lambda} = \text{h}/60 = 1$ min. For $40 < t \leq 80$, let $\lambda = 25/\text{h}$ and $\mu = 30/\text{h}$, then $\rho = \frac{5}{6}$ and $W_q = \frac{\rho}{\mu - \lambda} = \text{h}/6 = 10$ min. The average wait time in the queue for the two-hour period is $\frac{2}{3} \cdot 1 \text{ min} + \frac{1}{3} \cdot 10 \text{ min} = 4$ min.

c. Does the simpler model in (a) overestimate or underestimate the congestion compared with using a nonstationary model?

Solution. As shown in parts (a) and (b) the simpler model underestimates the congestion as compared with the nonstationary model.

10. Problem 3.11

A car rental agency has a kiosk at the airport with a single customer service agent. Arrivals to the kiosk are Poisson with a rate of 8 per hour. The time to process a customer is exponential with a mean time of 5 minutes.

a. What is the average time that a customer spends at the kiosk (wait in queue plus service time)?

Solution. Let $\lambda = 8/\text{h} = \frac{2}{15}/\text{min}$ and $\mu = \frac{1}{5}/\text{min}$, then

$$\text{E}[T] = \frac{1}{\mu - \lambda} = 15 \text{ min.}$$

b. Suppose that an arriving customer will be late for a wedding if the time spent at the kiosk is more than 20 minutes. What is the probability the customer makes it to the wedding on time?

Solution. $W(t) = 1 - e^{-(\mu-\lambda)t}$

$$\begin{aligned} \text{Pr}\{T < 20\} &= W(20) \\ &= 1 - e^{-(\frac{1}{5} - \frac{2}{15})20} \\ &= 1 - e^{-(\frac{1}{15})20} \\ &\approx 0.736 \end{aligned}$$

c. What is the probability that the customer is late to the wedding but not less than 5 minutes late (i.e., the time spent at the kiosk is between 20 and 25 minutes)?

Solution.

$$\begin{aligned} \text{Pr}\{20 < T < 25\} &= W(25) - W(20) \\ &= 1 - e^{-(\frac{1}{15})25} - W(20) \\ &\approx 0.075 \end{aligned}$$

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