OR 647: Queueing Theory, Spring 2021
Homework Solution 3
Due Wed. Mar. 3, 2021

1. Problem 1.17

The following spreadsheet is constructed to solve the problem:

Customer	Interarrival Time	Service Time	Arrival	Start Svc	End Svc	Queue Wait	Time in Syst
1	1	3	0	0	3	0	3
2	9	7	9	9	16	0	7
3	6	9	15	16	25	1	10
4	4	9	19	25	34	6	15
5	7	10	26	34	44	8	18
6	9	4	35	44	48	9	13
7	5	8	40	48	56	8	16
8	8	5	48	56	61	8	13
9	4	5	52	61	66	9	14
10	10	3	62	66	69	4	7
11	6	6	68	69	75	1	7
12	12	3	80	80	83	0	3
13	6	5	86	86	91	0	5
14	8	4	94	94	98	0	4
15	9	9	103	103	112	0	9
16	5	9	108	112	121	4	13
17	7	8	115	121	129	6	14
18	8	6	123	129	135	6	12
19	8	8	131	135	143	4	12
20	7	3	138	143	146	5	8
	Average	6.2			Average	3.95	10.15

Part a: $W_q = 3.95$, $W = 10.15$
Part b: The average system waiting time of those customers who had to wait for service is the average of the numbers in the right-hand column among those rows for which the queue wait is bigger than zero, i.e., the average of 10, 15, 18, 13, 16, 13, 14, 7, 7, 13, 14, 12, 12, 8, which is 12.29.

Using a time horizon of [0, 146], the arrival rate is $\lambda = 20 / 146$. Since the system starts and ends in an empty state, we have $L_q = \lambda W_q = .541$ and $L = \lambda W = 1.39$.

The fraction of time the server is idle is $1 - \rho = 1 - (20/146)\, E[S] = 0.151$.

Alternate solution: The solution above uses 9 as the first interarrival time. It is OK to use 1 as the first interarrival time. The problem statement may be ambiguous, so either way is valid.

2. Problem 1.18

The following table shows the arrival times and departure times (given values). The remaining columns are computed as follows:

- The time in system for each customer = departure time minus arrival time (assuming FCFS).
- The time a customer starts service = max(arrival time, previous customer departure time)
- Service time = departure time minus time customer starts service
- The time in queue = time a customer starts service minus arrival time
- Idle time (time system was idle before customer arrival) = max(arrival time of current customer minus departure time of last customer, 0)

Arrival Time	Departure Time	Time in System	Start Svc	Service Time	Time in Q	Idle
5	7	2	5	2	0	5
10	17	7	10	7	0	3
15	23	8	17	6	2	0
20	29	9	23	6	3	0
25	35	10	29	6	4	0
30	38	8	35	3	5	0
35	39	4	38	1	3	0
40	44	4	40	4	0	1
45	46	1	45	1	0	1
50	60	10	50	10	0	4
	Average:	6.3		4.6		

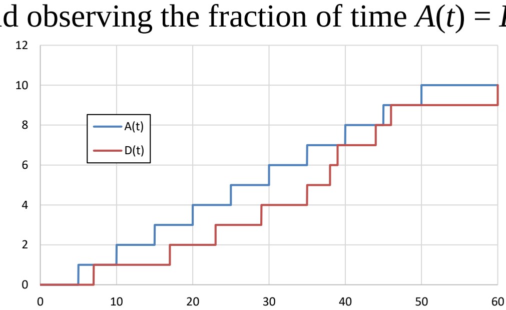
The average time in the system is $W = 6.3$. The arrival rate over the time interval [0, 60] is $\lambda = 10 / 60 = 1/6$ per minute. Since the system starts and ends in an empty state, we can use Little's law on the sample values: $L = \lambda W = (1/6)\cdot 6.3 \boxed{=1.05}$

The total time the system is idle is 14. Thus the percent idle time is $\boxed{14/60 = 7/30}$. Since the server starts and ends in an empty state, this could also be computed using Little's law applied to the server:

Average # in service = average time server busy = $\lambda E[S] = (10/60) \cdot 4.6 = 46 / 60 = 23 / 30$. Thus,

$\boxed{1-\rho = 7/30}$

One could also answer this question by plotting $A(t)$ and $D(t)$ and computing the time average of $A(t) - D(t)$ and observing the fraction of time $A(t) = D(t)$.



3. Problem 1.19

The following spreadsheet is used to obtain the solution:

Arrival Time	Time Start Svc	Time Exit	# in System	# in Queue	Time in Queue
1	1.00	3.22	0	0	0.00
2	3.22	4.98	1	0	1.22
3	4.98	7.11	2	1	1.98
4	7.11	7.25	2	1	3.11
5	7.25	8.01	2	1	2.25
6	8.01	8.71	3	2	2.01
7	8.71	9.18	4	3	1.71
8	9.18	9.40	3	2	1.18
9	9.40	9.58	2	1	0.40
10	10.00	12.41	0	0	0.00
11	12.41	12.82	1	0	1.41
12	12.82	13.28	2	1	0.82
13	13.28	14.65	1	0	0.28
14	14.65	14.92	1	0	0.65
15	15.00	15.27	0	0	0.00

- The 3rd column (exit time) is the 2nd column plus the service time.
- The 2nd column (time a customer starts service) is the previous entry of the 3rd column.
- The 4th column (# in system as seen by an arriving customer) is the number of previously arriving customers whose exit time (3rd column) is after the arrival time of the customer in question.
- The 5th column (# in queue as seen by an arriving customer) is 4th column minus 1, but at least 0.
- The last column is the 2nd column minus the 1st column

The average of the 5th column is $\boxed{L_q^{(A)} = 12/15 = 0.8}$

L_q is the total person hours spent in the queue divided by the total time interval. The number of person hours spent in the queue is sum of the last column. The total of this column divided by total time (15.27) is the answer: $\boxed{L_q = 17.02/15.27 = 1.1146}$

4. The table below gives the arrival and service times of a sequence of customers to an airline ticket counter. The counter has one line and customers are served in a first-come-first-served manner. Note: Use the same time horizon for parts (a) and (b) in computing the average number in queue.
   a. Assuming one server, compute the average wait in queue and the average number in queue.
   b. Assuming there are two servers who are working at half speed (i.e., double the service times), compute the average wait in queue and the average number in queue.
   c. Compare the answers in (a) and (b) and discuss why there is a difference.

Customer #	Arrival time (min)	Service time (min)
1	0	6
2	6	4
3	9	6
4	10	1
5	15	2
6	17	1
7	19	3
8	23	5
9	29	8
10	35	6

Part a: See the spreadsheet solution below. The average wait in queue is 1.4. The average number in queue (using a time horizon of [0, 47]) is .30.

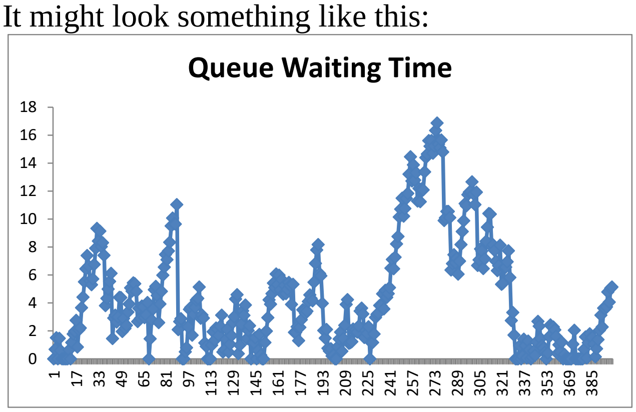
Part b: See the spreadsheet solution below. The average wait in queue is 1.5. The average number in queue (using a time horizon of [0, 47]) is .32. Note that the 2-server solution requires some manual bookkeeping regarding which server becomes free after a departure and the corresponding start time of a given customer. When there are two servers, it is possible for one customer to "pass" another, so it is not possible to directly use Lindley's equation as before.

Part c: In this problem, having one server that is twice as fast is better than having two servers (with respect to average wait in queue).

			1 Server			2 Servers			
Cust. #	Arr. Tm. (min)	Svc. Tm. (min)	Enter Svc	Depart	Wait in Q	Enter Svc	Server	Depart	Wait in Q
1	0	6	0	6	0	0	1	12	0
2	6	4	6	10	0	6	2	14	0
3	9	6	10	16	1	12	1	24	3
4	10	1	16	17	6	14	2	16	4
5	15	2	17	19	2	16	2	20	1
6	17	1	19	20	2	20	2	22	3
7	19	3	20	23	1	22	2	28	3
8	23	5	23	28	0	24	1	34	1
9	29	8	29	37	0	29	2	45	0
10	35	6	37	43	2	35	1	47	0
				Wq	1.4			Wq	1.5
				Lq	0.297872			Lq	0.319149

5. [10 points] Reconstruct the Excel spreadsheet given in class for airplane arrivals and departures to/from a single runway. Specifically, create columns containing the following elements: Operation #, inter-arrival time (min), actual arrival time (min), arrival or departure (A or D), runway hold time, queue waiting time, departure time, arrival count (1 for arrivals 0 for departures), departure count (0 for arrivals 1 for departures). The spreadsheet should also contain the following global parameters: % arrivals (0.5), arrival runway hold time (1 min), departure runway hold time (1.5 min), operation arrival rate (40 / hr), average service time, rho.
    a. Graph the queue-waiting time for a given sequence of 400 simulated operations.
    b. Vary λ from 5, 10, …, 45, 50. Calculate the average queue-waiting time W_q for each of these cases (based on simulation of 400 operations).
    c. Calculate the average queue-waiting time for each of these cases for an M/M/1 queue with the same λ and μ. Plot both values of W_q as a function of λ and compare the results. How good is the M/M/1 queue as an approximation you're your simulation?
    d. Repeat part (b) when 75% of operations are departures.

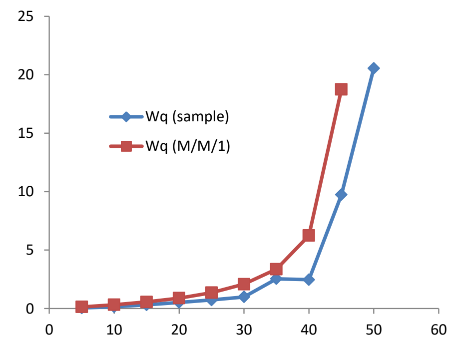
Part a: It might look something like this:



Part b: You might get something like the first two columns in the table below (W_q is in minutes).

Part c:

lambda (per hr)	Wq (sample)	rho	Wq (M/M/1)
5	0.047521886	10%	0.145348837
10	0.150529472	21%	0.328947368
15	0.316095377	31%	0.568181818
20	0.522588505	42%	0.892857143
25	0.733518357	52%	1.358695652
30	0.996278527	63%	2.083333333
35	2.540654804	73%	3.365384615
40	2.464444979	83%	6.25
45	9.735736343	94%	18.75
50	20.54245858	104%	-31.25



The fit is not too bad for low ρ, but it gets worse as ρ gets larger. Of course, the negative result when ρ > 1 is not valid.

Part d: Note that the average service time goes up since there are more departures. Also, λ = 45 and λ = 50 both correspond to ρ > 1. Below are some sample results.

lambda (per hr)	Wq (sample)
5	0.10066353
10	0.183570681
15	0.373709408
20	0.734485353
25	1.036148697
30	1.389883048
35	2.276526173
40	7.541172658
45	30.72356596
50	44.03113108

6. A hardware store has a "merchandise pickup" window where customers arrive to load previously purchased items into their car. Arrivals to the pickup window follow a Poisson process with rate 45 per hour. The time for a store employee to load merchandise into a customer's car is exponentially distributed with a mean of 3 minutes. There are 3 employees. What is the average time it takes for a customer to get his/her merchandise loaded into the car (wait in queue plus service time)?

This is an M/M/3 queue with λ = 45, μ = 20, and c = 3, so r = 2.25 and ρ = .75. The average number in queue is

$$p_0 = \left[\frac{r^c}{c!(1-\rho)} + \sum_{n=0}^{c-1}\frac{r^n}{n!}\right]^{-1} = \left[\frac{2.25^3}{3!(.25)} + 1 + 2.25 + \frac{2.25^2}{2}\right]^{-1} = \frac{1}{13.375}$$

$$L_q = \frac{r^c}{c!}\frac{\rho}{(1-\rho)^2}p_0 = \frac{2.25^3(.75)}{6(.25)^2}\frac{1}{13.375} \doteq 1.7033$$

$$W_q = \frac{L_q}{45} \doteq .03785$$

$$\boxed{W = W_q + \frac{1}{\mu} \doteq .08785 \text{ hr} = 5.271 \text{ min}}$$

7. A local polling location has 5 voting machines. People arrive to cast their votes according to a Poisson process with rate λ = 75 per hour. The time to cast a ballot is exponential with a mean of 2 minutes.
   a. What is the fraction of time that a given voting machine is being utilized?
   b. What is the fraction of time that 3 or fewer machines are in use?
   c. What is the average line length?

Part a: This is an M/M/c queue with λ = 75, μ = 30, and c = 5, so r = 2.5 and ρ = 0.5. The fraction of time a voting machine is utilized is the same as ρ, so the answer is 0.5.

Part b

$$p_0 = \left[\frac{r^c}{c!(1-\rho)} + \sum_{n=0}^{c-1} \frac{r^n}{n!} \right]^{-1} = \left[\frac{2^5}{5!(.5)} + 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} \right]^{-1} \doteq .0801$$

$$p_1 = \frac{\lambda}{\mu} p_0 \doteq .20025$$

$$p_2 = \frac{\lambda}{2\mu} p_1 \doteq .25031$$

$$p_3 = \frac{\lambda}{2\mu} p_1 \doteq .20859$$

$$\boxed{p_0 + p_1 + p_2 + p_3 \doteq .73926}$$

Part c

$$\boxed{L_q = \frac{r^c \rho}{c!(1-\rho)^2} p_0 = \frac{2^5(.5)}{5!(.5)^2} \cdot .0801 = .13037}$$

8. Arrivals to a queueing system follow a Poisson process with rate 18 per hour. Service times are exponential with rate 20 per hour. Each server is paid $40 per hour while working and $10 per hour while idle. The company determines that it loses $2 per customer per hour spent waiting in the queue, due to "ill-will" or lost customer loyalty.
    a. What is the hourly cost to the company if 1 server is employed?
    b. What is the hourly cost to the company if 2 servers are employed?
    c. If the company adds a 3rd server, would the cost go up or down compared with 2 servers? (Give a one- or two-sentence explanation without working the numbers out fully).

Part a: The hourly cost due to "ill will" is $\$2 \cdot L_q$.
The hourly cost per server is $\$40 \cdot \rho + \$10(1-\rho)$, since ρ is the fraction of time a server is busy.

Thus, the total hourly cost is $2L_q + c(30\rho + 10)$, where c is the number of servers.

For $c = 1$, we have $\rho = 18/20 = .9$. And,

$$L_q = \frac{\rho^2}{1-\rho} = \frac{.81}{.1} = 8.1$$

The total hourly cost is:

$$2L_q + c(30\rho + 10) = 16.2 + (30 \cdot 0.9 + 10) \boxed{= \$53.2}$$

Part b: For $c = 2$, we have $r = 18/20 = 0.9$ and $\rho = 18/40 = .45$.

$$p_0 = \left[\frac{r^c}{c!(1-\rho)} + \sum_{n=0}^{c-1} \frac{r^n}{n!}\right]^{-1} = \left[\frac{.9^2}{2(.55)} + 1 + .9\right]^{-1} \approx .37931$$

$$L_q = p_0 \frac{r^c}{c!} \frac{\rho}{(1-\rho)^2} \approx .37931 \frac{.9^2(.45)}{2(.55)^2} \approx .22853$$

The total hourly cost is:

$$2L_q + c(30\rho + 10) = .45705 + 2(30 \cdot 0.45 + 10) \boxed{= \$47.457}$$

Note: You can also compute the server cost as $\$10p_0 + \$50p_1 + \$80(1 - p_0 - p_1)$.

Part c: It will go up. The total work is conserved so the total non-idle costs ($40 per hour) will be the same no matter how many servers. Adding a 3rd server will basically add an additional cost of $10 per hour for idle time. The potential benefit comes from a reduction in the cost associated with L_q. But $2L_q$ is about 0.9 .45705, so the maximum possible reduction in this cost is about $0.90, significantly below $10.

9. Problem 3.32

Postponed to Homework #4

10. Problem 3.37

Postponed to Homework #4