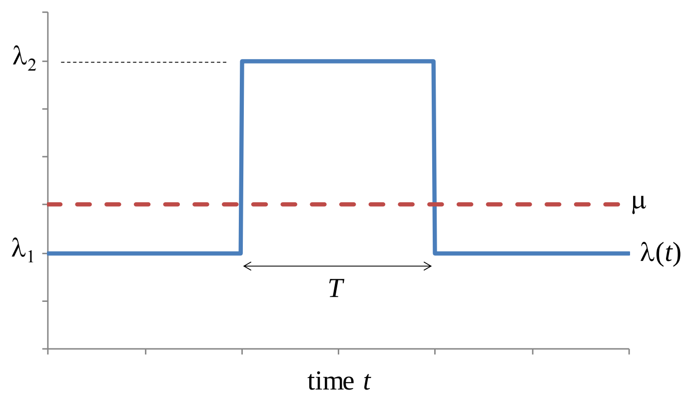
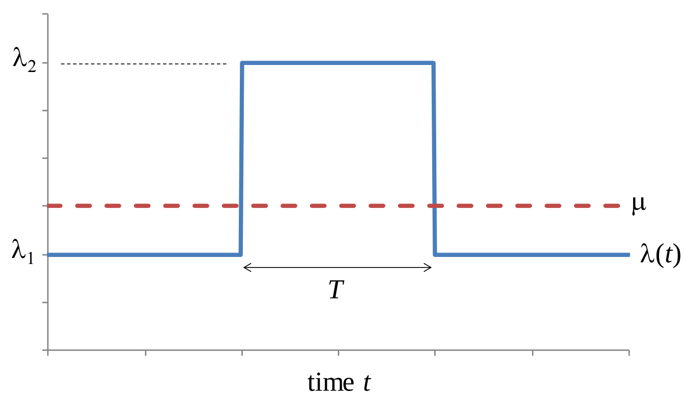
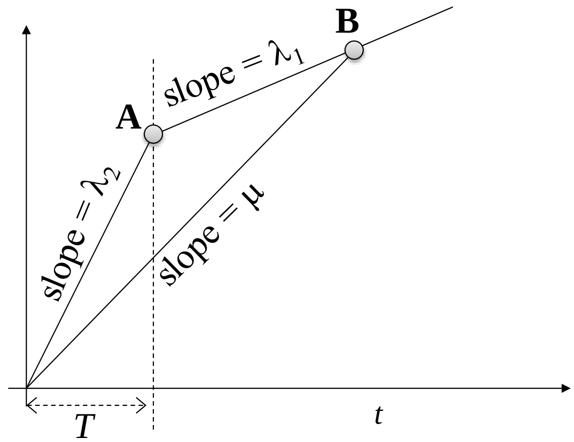
OR 647: Queueing Theory, Spring 2021
Homework Solution 2
Due Wed. Feb. 17, 2021

1. A cloud computing service is modeled as a fluid queue with time-dependent arrival rate $\lambda(t)$ and constant service rate μ. Consider a situation where the arrival process has a burst. Initially, the arrival rate is $\lambda_1 < \mu$. Then, for a period of length *T*, the arrival process is $\lambda_2 > \mu$, after which it returns back to λ_1.
   a. What is the maximum queue length?
   b. What is the length of the busy period (i.e., the time from the start of the burst until the queue empties)?
   c. What is the total delay (among all jobs) due to the burst?





The following diagram shows the arrival and departure curves.



Part a: The maximum queue length occurs at $t = T$. The queue length $A(t) - D(t)$ at this point is $\boxed{\lambda_2 T - \mu T = (\lambda_2 - \mu)T}$

Part b: The coordinate of point A is $(T, \lambda_2 T)$.

Point B is the intersection of the following lines:

$$y = \mu t$$

$$y = \lambda_1 (t - T) + \lambda_2 T$$

Solving gives:

$$\mu t = \lambda_1 (t - T) + \lambda_2 T$$

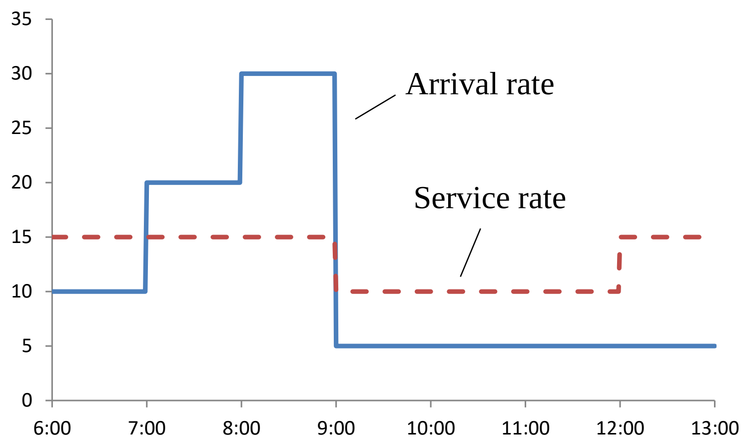
$$(\mu - \lambda_1) t = -\lambda_1 T + \lambda_2 T$$

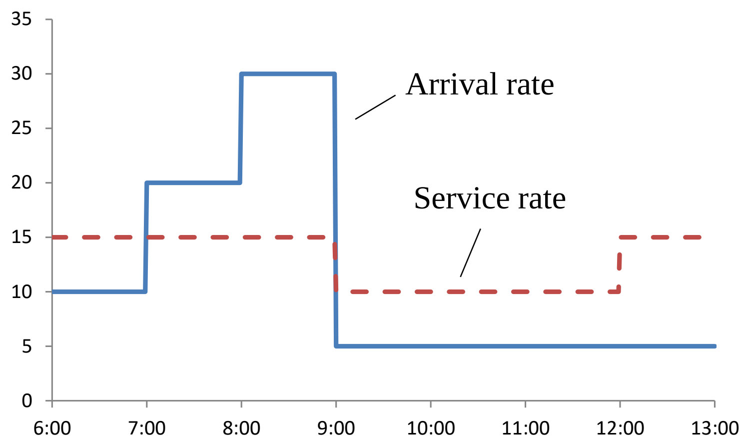
$$t = \frac{\lambda_2 - \lambda_1}{\mu - \lambda_1} T$$

Thus, the busy period ends at time $\boxed{t = \frac{\lambda_2 - \lambda_1}{\mu - \lambda_1} T}$

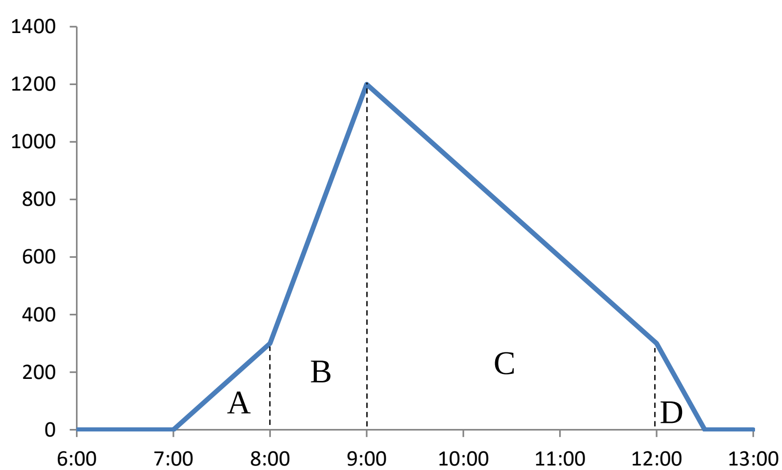
Part c: The area of the triangle is $\boxed{\frac{1}{2} bh = \frac{1}{2} \cdot \frac{\lambda_2 - \lambda_1}{\mu - \lambda_1} T \cdot (\lambda_2 - \mu) T = \frac{(\lambda_2 - \lambda_1)(\lambda_2 - \mu)}{2(\mu - \lambda_1)} T^2}$

2. A road segment is modeled as a fluid queue. The arrival rate is shown in the graph below, with a morning rush hour. After the morning rush hour, the transportation department begins maintenance work on the road. At 9am, one of the three lanes on the road is closed, reducing the capacity of the road from 15 cars per minute to 10 cars per minute. The maintenance period ends at noon.
   d. Determine the average delay per car (over all cars from 6am to 1pm).
   e. What is the maximum length of the queue over this period?
   f. At what time does the queue length return to 0?





Part a: The following is a plot of queue length as a function of time. The queue increases at a rate of 5·60 = 300 per hour from 7 to 8. It increases at a rate of 15·60 = 900 per hour from 8 to 9. It decreases at a rate of 5·60 = 300 per hour from 9 to 12. It decreases at a rate of 10·60 = 600 per hour from 12 to 12:30.



The area of triangle A is 300/2 = 150 (in hours)
The area of trapezoid B is 1500/2 = 750 (in hours)
The area of trapezoid C is 1500/2*3 = 2250 (in hours)

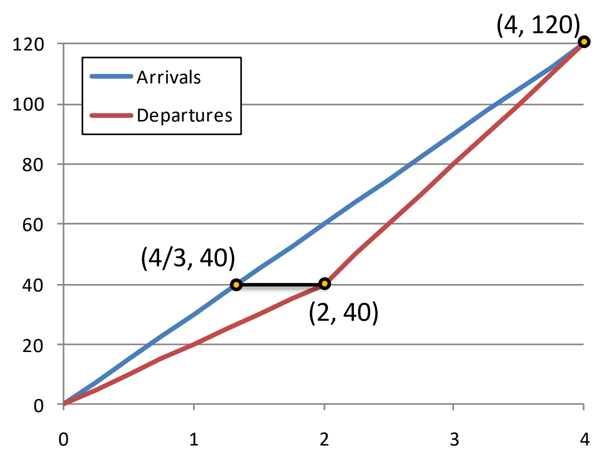
The area of triangle D is 300/4 = 75 (in hours)
The total area is 3,225 (in hours)

The total number of arrivals from 6am to 1pm is 60(10 + 20 + 30 + 5 + 5 + 5 + 5) = 4,800. The average delay per car is: $\boxed{3225/4800 = .67188 \text{ hr}}$

Part b: The maximum length of the queue is $\boxed{1200}$

Part c: The queue ends at time $\boxed{12:30}$

3. Problem 8.12



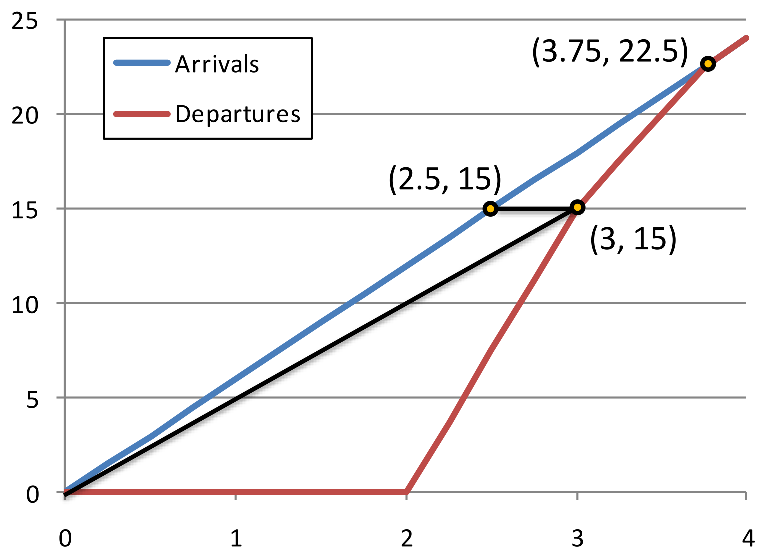
Summing up the area of the triangles gives:

$$A = \frac{1}{2}\left(\frac{2}{3}\cdot 40\right) + \frac{1}{2}\left(\frac{2}{3}\cdot 80\right) = 40$$

The average wait is

$$\boxed{W = \frac{40}{120}\text{hr} = 20\text{ min}}$$

4. Problem 8.17



Summing up the areas of the triangles gives:

$$A = \frac{1}{2}(2 \cdot 15) + \frac{1}{2}(.5 \cdot 15) + \frac{1}{2}(.5 \cdot 7.5) = 20.625$$

The average wait is

$$\boxed{W = \frac{20.625}{24} = .859 \text{ min}}$$

5. Problem 8.18.

<u>Part a</u>: The on-ramp contributes 10 cars per minute on average. Thus the maximum flow through point C is 50 cars per minute on average.

<u>Part b</u>: During the first minute (when the light is green), the freeway cars get a service rate of 30 / minute. During this time, there is no queue in the on-ramp (since the on-ramp arrival rate is 30 / minute which equals the service rate of 30 / min). During minutes 2-3, the service rate is 60 until point B. See the diagram below.

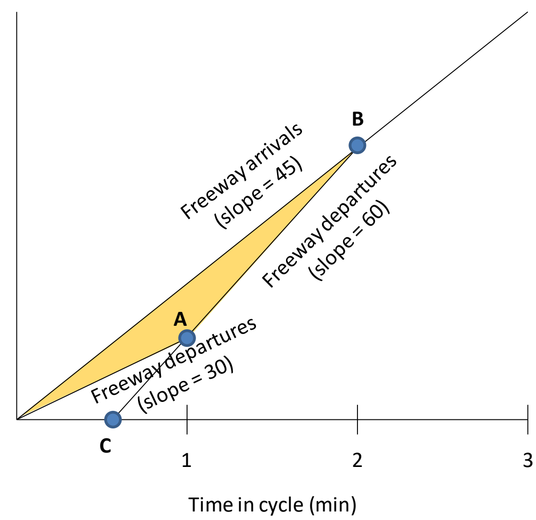
Point A = (1, 30)
Point B = (2, 90)
Point C = (1/2, 0)
To get point B, solve

$$45t = 30 + 60(t-1)$$
$$15t = 30$$
$$t = 2$$

The area of the triangle is

$$\frac{1}{2}(0.5 \cdot 90) - \frac{1}{2}(0.5 \cdot 30) = 15$$

Thus, the average delay is $\boxed{15/135 = 1/9 \text{ min}}$



Part c: Under a fluid approximation, the on-ramp cars experience no delays since the max flow rate equals the max service rate (30 / min).

6. Problem 2.12

Part a: The number of arrivals in 1 hour is a Poisson random variable with mean 20. The probability that no calls arrive during 1 hour: $\boxed{e^{-20}}$

Part b: The probability that exactly 5 calls arrive during 1 hour: $\boxed{e^{-20}\dfrac{20^5}{5!}}$

Part c: The probability that 5 or more calls arrive during 1 hour:

$$\boxed{1-\sum_{k=0}^{4} e^{-20}\frac{20^k}{k!} = 1-e^{-20}[1+20+20^2/2+20^3/6+20^4/24] \approx .999983}$$

7. Problem 2.13

The transition rate matrix is $\mathbf{Q} = \begin{bmatrix} -3 & 3 & 0 & 0 \\ 1 & -3 & 2 & 0 \\ 0 & 0 & -5 & 5 \\ 0 & 1 & 4 & -5 \end{bmatrix}$.

Then, $\vec{p}\mathbf{Q} = 0$ gives

$$\begin{aligned} 3p_A &= p_B \\ 3p_B &= 3p_A + p_D \\ 5p_C &= 2p_B + 4p_D \\ 5p_C &= 5p_D \end{aligned}$$

Solving these equations with the added condition that the probabilities sum to 1 gives:

$$\boxed{[p_A, p_B, p_C, p_D] = [1/16, 3/16, 6/16, 6/16]}$$

8. Problem 2.15

Part a: Let $X(t)$ denote the number of people at the stop at time t. The transition-rate matrix is:

$$\boxed{\mathbf{Q} = \begin{bmatrix} -3 & 3 & 0 & 0 & 0 \\ 1.5 & -4.5 & 3 & 0 & 0 \\ 1.5 & 0 & -4.5 & 3 & 0 \\ 0 & 1.5 & 0 & -4.5 & 3 \\ 0 & 0 & 1.5 & 0 & -1.5 \end{bmatrix}}$$

Part b: $\boxed{\mathbf{P}=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 & 0 \\ 1/3 & 0 & 0 & 2/3 & 0 \\ 0 & 1/3 & 0 & 0 & 2/3 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}}$

Part c: $\vec{p}\mathbf{Q}=0$ gives

$$3P_0 = 1.5P_1 + 1.5P_2$$
$$4.5P_1 = 3P_0 + 1.5P_3$$
$$4.5P_2 = 3P_1 + 1.5P_4$$
$$4.5P_3 = 3P_2$$
$$1.5P_4 = 3P_3$$

From the last two equations, we have: $P_3 = (1/2)P_4$ and $P_2 = 1.5P_3 = (3/4)P_4$.

Plugging into third equation: $4.5(3/4)P_4 = 3P_1 + 1.5P_4$, so $P_1 = (5/8)P_4$.

Plugging into second equation: $4.5(5/8)P_4 = 3P_0 + 1.5(1/2)P_4$, so $P_0 = (11/16)P_4$

Normalizing: $1 = P_0 + \ldots + P_4 = [(11/16)+(5/8)+(3/4)+(1/2)+1]P_4$, so $P_4 = 16/57$

$\boxed{P_0 = 11/57, P_1 = 10/57, P_2 = 12/57, P_3 = 8/57, P_4 = 16/57}$

Solving $\pi = \pi\mathbf{P}$:

$$\pi_0 = (1/3)\pi_1 + (1/3)\pi_2$$
$$\pi_1 = \pi_0 + (1/3)\pi_3$$
$$\pi_2 = (2/3)\pi_1 + \pi_4$$
$$\pi_3 = (2/3)\pi_2$$
$$\pi_4 = (2/3)\pi_3$$

From the last two equations: $\pi_4 = (4/9)\pi_2$

Plugging this and 4th equation into 2nd and 3rd equations:

$$\pi_1 = \pi_0 + (2/9)\pi_2$$
$$\pi_2 = (2/3)\pi_1 + (4/9)\pi_2$$

Solving the 2nd equation for π_2: $\pi_2 = (6/5)\pi_1$

Plugging into the 1st: $\pi_1 = \pi_0 + (2/9)(6/5)\pi_1 = \pi_0 + (4/15)\pi_1$ or $\pi_1 = (15/11)\pi_0$.

Combining together:

$$\pi_1 = (15/11)\pi_0,\ \pi_2 = (18/11)\pi_0,\ \pi_3 = (12/11)\pi_0,\ \pi_4 = (8/11)\pi_0.$$

Normalizing:

$$1 = \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 = [1 + (15/11) + (18/11) + (12/11) + (8/11)]\pi_0$$

So, $\boxed{\pi_0 = \frac{11}{64}, \pi_1 = \frac{15}{64}, \pi_2 = \frac{18}{64}, \pi_3 = \frac{12}{64}, \pi_4 = \frac{8}{64}}$

Part d: An arriving shuttle sees n customers waiting with probability P_n. With probability $P_2 + P_3 + P_4$, the shuttle picks up 2 customers. With probability P_1 the shuttle picks up 1 customer. With probability P_0, the shuttle picks up no customers. Thus, the average number of passengers on a shuttle is: $\boxed{2P_4 + 2P_3 + 2P_2 + P_1 = 82/57}$

9. Problem 3.10

Part a: $\lambda = 15, \mu = 30, \rho = 1/2$

$$\boxed{W_q = \frac{1}{\mu}\frac{\rho}{1-\rho} = \frac{1}{30}}$$

Part b: For the first and third intervals, $\lambda = 10, \mu = 30, \rho = 1/3$

$$W_q = \frac{1}{\mu}\frac{\rho}{1-\rho} = \frac{1}{60}$$

For the second interval, $\lambda = 25, \mu = 30, \rho = 5/6$

$$W_q = \frac{1}{\mu}\frac{\rho}{1-\rho} = \frac{5}{30}$$

The fraction of customers arriving during the 2^{nd} interval is 25 / (10 + 25 + 10) = 5/9. The fraction of customers arriving during the 1^{st} + 3^{rd} intervals is (10 + 10) / (10 + 25 + 10) = 4/9.

$$\boxed{W_q = \frac{5}{9}\frac{5}{30} + \frac{4}{9}\frac{1}{60} = \frac{54}{540} = \frac{1}{10}}$$

Part c: The simpler model under-estimates the congestion. This is essentially due to the non-linear nature of delays. The actual congestion during the busiest period will more than offset the lack of congestion during the least busy periods.

10. Problem 3.11

Part a: $\lambda = 8$, $\mu = 12$. So $\boxed{W = \frac{1}{\mu - \lambda} = \frac{1}{4}\text{hr}}$

Part b: Let T be the random time in the system

$$\Pr\{T \le t\} = 1 - e^{-(\mu-\lambda)t}$$

$$\Pr\{T \le 1/3\} = 1 - e^{-(12-8)(1/3)} \approx .736$$

Part c

$$\Pr\{20/60 \le T \le 25/60\} = \Pr\{T \le 25/60\} - \Pr\{T \le 20/60\}$$

$$= 1 - e^{-4(25/60)} - (1 - e^{-4(20/60)}) \approx .0747$$