hw3

March 3, 2021

[1]:

```
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
from scipy.stats import norm
from scipy.optimize import newton
from math import factorial
from math import sqrt
```

[2]:

```
%pprint
```

```
Pretty printing has been turned OFF
```

1 Problem 1 (1.17)

1.1 1 (a)

[3]:

```
T = [1,9,6,4,7,9,5,8,4,10,6,12,6,8,9,5,7,8,8,7]
print(T)
```

```
[1, 9, 6, 4, 7, 9, 5, 8, 4, 10, 6, 12, 6, 8, 9, 5, 7, 8, 8, 7]
```

[4]:

```
S = [3,7,9,9,10,4,8,5,5,3,6,3,5,4,9,9,8,6,8,3]
print(S)
```

```
[3, 7, 9, 9, 10, 4, 8, 5, 5, 3, 6, 3, 5, 4, 9, 9, 8, 6, 8, 3]
```

[5]:

```
A = [0] + list(np.cumsum(T)[:-1])
print(A)
```

```
[0, 1, 10, 16, 20, 27, 36, 41, 49, 53, 63, 69, 81, 87, 95, 104, 109, 116, 124,
132]
```

[6]:

```
U = [0 for i in range(len(A))]
print(U)
```

```
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
```

[7]:

```
D = U.copy()
print(D)
```

```
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
```

[8]:

```
for i in range(len(A)):
    D[i] = U[i] + S[i]
    if i < len(A)-1:
        U[i+1] = max(D[i], A[i+1])
print(U)
```

```
[0, 3, 10, 19, 28, 38, 42, 50, 55, 60, 63, 69, 81, 87, 95, 104, 113, 121, 127,
135]
```

[9]:

```
print(D)
```

```
[3, 10, 19, 28, 38, 42, 50, 55, 60, 63, 69, 72, 86, 91, 104, 113, 121, 127, 135,
138]
```

[10]:

```
W_q = [u-a for u,a in zip(U,A)]
print(W_q)
```

```
[0, 2, 0, 3, 8, 11, 6, 9, 6, 7, 0, 0, 0, 0, 0, 0, 4, 5, 3, 3]
```

[11]:

```
W = [w+s for w,s in zip(W_q,S)]
print(W)
```

```
[3, 9, 9, 12, 18, 15, 14, 14, 11, 10, 6, 3, 5, 4, 9, 9, 12, 11, 11, 6]
```

[12]:

```
W_ave = sum(W)/len(W)
print(W_ave)
```

```
9.55
```

[13]:

```
W_q_ave = sum(W_q)/len(W_q)
print(W_q_ave)
```

```
3.35
```

[14]:

```
S_ave = sum(S)/len(S)
print(S_ave)
```

```
6.2
```

1.2 1 (b)

[15]:
```
rate = len(D)/D[-1]
print(len(D), D[-1], rate)
```

```
20 138 0.14492753623188406
```

[16]:
```
L_q = rate*W_q_ave
print(L_q)
```

```
0.48550724637681164
```

[17]:
```
L = rate*W_ave
print(L)
```

```
1.384057971014493
```

[18]:
```
W_q_ave_wait = sum(W_q)/sum([w>0 for w in W_q])
print(W_q_ave_wait)
```

```
5.583333333333333
```

[19]:
```
print(sum(S), D[-1], sum(S)/D[-1], 1-sum(S)/D[-1])
```

```
124 138 0.8985507246376812 0.10144927536231885
```

[20]:
```
print(A)
print(U)
```

```
[0, 1, 10, 16, 20, 27, 36, 41, 49, 53, 63, 69, 81, 87, 95, 104, 109, 116, 124,
132]
[0, 3, 10, 19, 28, 38, 42, 50, 55, 60, 63, 69, 81, 87, 95, 104, 113, 121, 127,
135]
```

[21]:
```
Ti = list(set(A+U))
Ti.sort()
print(Ti)
```

```
[0, 1, 3, 10, 16, 19, 20, 27, 28, 36, 38, 41, 42, 49, 50, 53, 55, 60, 63, 69,
81, 87, 95, 104, 109, 113, 116, 121, 124, 127, 132, 135]
```

[22]:
```
N_q=[sum([a <= t for a in A])-sum([u <= t for u in U]) for t in Ti]
print(N_q)
```

```
[0, 1, 0, 0, 1, 0, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0,
1, 0, 1, 0, 1, 0]
```

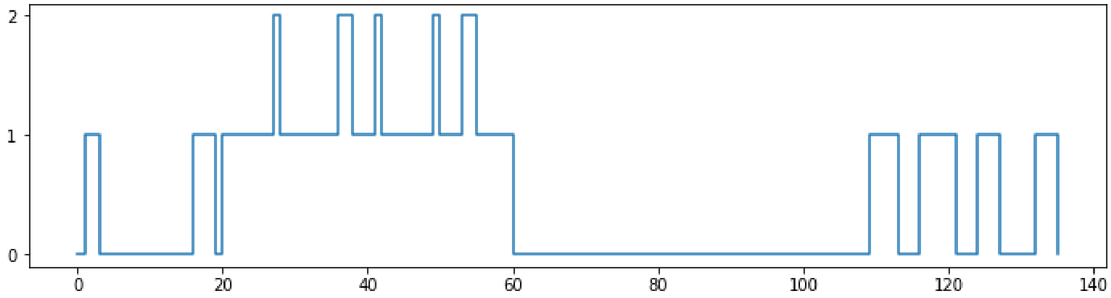
[23]:
```
N=[sum([a <= t for a in A])-sum([d <= t for d in D]) for t in Ti]
print(N)
```

```
[1, 2, 1, 1, 2, 1, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, 1, 1, 1, 1, 1, 1, 1, 2, 1,
2, 1, 2, 1, 2, 1]
```

[24]:
```
max(N_q)
```

[24]:
```
2
```

[25]:
```
plt.figure(figsize=[12,3])
plt.plot(Ti,N_q,drawstyle='steps-post')
plt.yticks(range(max(N_q)+1))
plt.show()
```



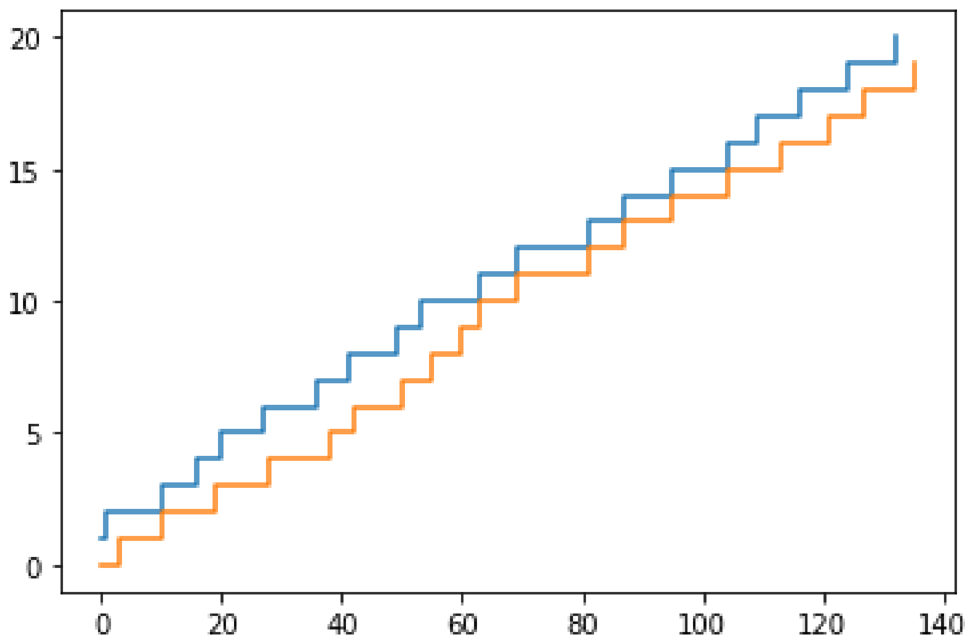
[26]:
```
sum([n*t for n,t in zip(N_q,np.diff(Ti))])/D[-1]
```

[26]:
```
0.4855072463768116
```

[27]:
```
L_q
```

[27]:
```
0.48550724637681164
```

[28]:
```
plt.plot(A,[i for i in range(1,len(A)+1)],drawstyle='steps-post')
plt.plot(U,range(len(U)),drawstyle='steps-post')
#plt.plot(D,range(len(D)),drawstyle='steps-post')
#plt.plot(Ti,N,drawstyle='steps-post')
#plt.plot(Ti,N_q,drawstyle='steps-post')
plt.yticks(list(set([i*(i%5==0) for i in range(len(A)+1)])))
plt.show()
```



```
[29]: print(sum(S), sum(W_q), sum(W))
```

```
124 67 191
```

```
[30]: print(sum(S)/20, sum(W_q)/20, sum(W)/20)
```

```
6.2 3.35 9.55
```

```
[31]: df = pd.DataFrame(dict(Customer=[i for i in␣
       ↪range(1,21)],T=T,S=S,A=A,U=U,D=D,W_q=W_q,W=W))
      print(df.to_latex(index=False))
```

```
\begin{tabular}{rrrrrrrr}
\toprule
 Customer &  T &  S &  A &  U &  D & W\_q &  W \\
\midrule
        1 &  1 &  3 &  0 &  0 &  3 &   0 &  3 \\
        2 &  9 &  7 &  1 &  3 & 10 &   2 &  9 \\
        3 &  6 &  9 & 10 & 10 & 19 &   0 &  9 \\
        4 &  4 &  9 & 16 & 19 & 28 &   3 & 12 \\
        5 &  7 & 10 & 20 & 28 & 38 &   8 & 18 \\
        6 &  9 &  4 & 27 & 38 & 42 &  11 & 15 \\
        7 &  5 &  8 & 36 & 42 & 50 &   6 & 14 \\
        8 &  8 &  5 & 41 & 50 & 55 &   9 & 14 \\
        9 &  4 &  5 & 49 & 55 & 60 &   6 & 11 \\
       10 & 10 &  3 & 53 & 60 & 63 &   7 & 10 \\
       11 &   6 &   6 &  63 &  63 &  69 &   0 &   6 \\
       12 &  12 &   3 &  69 &  69 &  72 &   0 &   3 \\
       13 &   6 &   5 &  81 &  81 &  86 &   0 &   5 \\
       14 &   8 &   4 &  87 &  87 &  91 &   0 &   4 \\
       15 &   9 &   9 &  95 &  95 & 104 &   0 &   9 \\
       16 &   5 &   9 & 104 & 104 & 113 &   0 &   9 \\
       17 &   7 &   8 & 109 & 113 & 121 &   4 &  12 \\
       18 &   8 &   6 & 116 & 121 & 127 &   5 &  11 \\
       19 &   8 &   8 & 124 & 127 & 135 &   3 &  11 \\
       20 &   7 &   3 & 132 & 135 & 138 &   3 &   6 \\
\bottomrule
\end{tabular}
```

2 Problem 2 (1.18)

[32]:

```
A = [5*i for i in range(1,60//5+1)]
print(A)
```

```
[5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60]
```

[33]:

```
D = [7, 17, 23, 29, 35, 38, 39, 44, 46, 60]
print(D)
```

```
[7, 17, 23, 29, 35, 38, 39, 44, 46, 60]
```

[34]:

```
Ti = list(set(A+D))
Ti.sort()
Ti = [0] + Ti
print(Ti)
```

```
[0, 5, 7, 10, 15, 17, 20, 23, 25, 29, 30, 35, 38, 39, 40, 44, 45, 46, 50, 55,
60]
```

[35]:

```
N=[sum([a <= t for a in A])-sum([d <= t for d in D]) for t in Ti]
print(N)
```

```
[0, 1, 0, 1, 2, 1, 2, 1, 2, 1, 2, 2, 1, 0, 1, 0, 1, 0, 1, 2, 2]
```

[36]:

```
L = sum([n*t for n,t in zip(N[:-1],np.diff(Ti))])/D[-1]
print(list(np.diff(Ti)))
print(sum([n*t for n,t in zip(N[:-1],np.diff(Ti))]))
print(D[-1])
print(L)
```

```
[5, 2, 3, 5, 2, 3, 3, 2, 4, 1, 5, 3, 1, 1, 4, 1, 1, 4, 5, 5]
68
```

```
60
1.1333333333333333
```

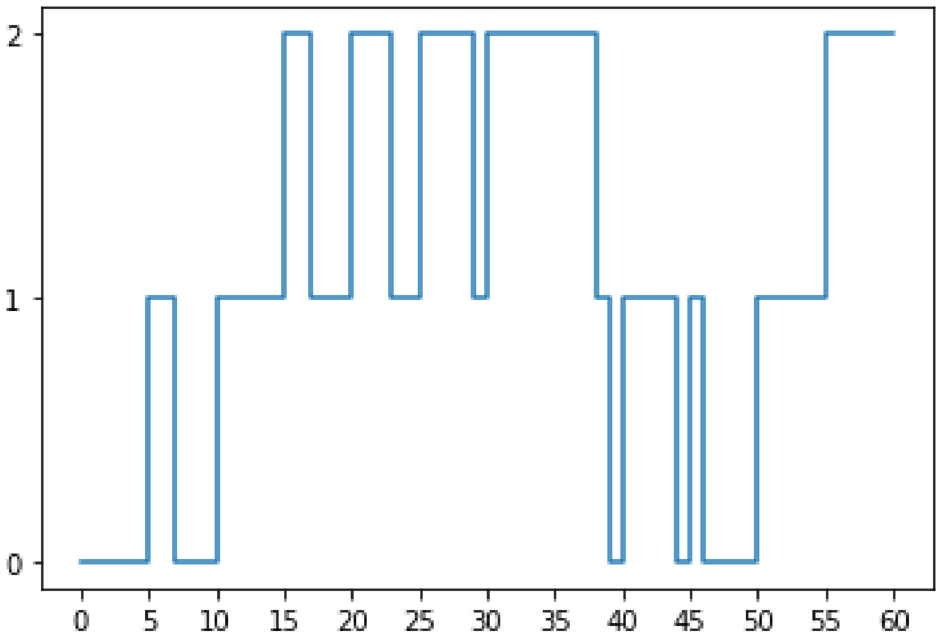
[37]:

```
W_i = sum([n*t for n,t in zip([n==0 for n in N],np.diff(Ti))])/D[-1]
print([n*t for n,t in zip([n==0 for n in N],np.diff(Ti))])
print(list(np.diff(Ti)))
print(sum([n*t for n,t in zip([n==0 for n in N],np.diff(Ti))]))
print(W_i)
```

```
[5, 0, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 4, 0, 0]
[5, 2, 3, 5, 2, 3, 3, 2, 4, 1, 5, 3, 1, 1, 4, 1, 1, 4, 5, 5]
14
0.23333333333333334
```

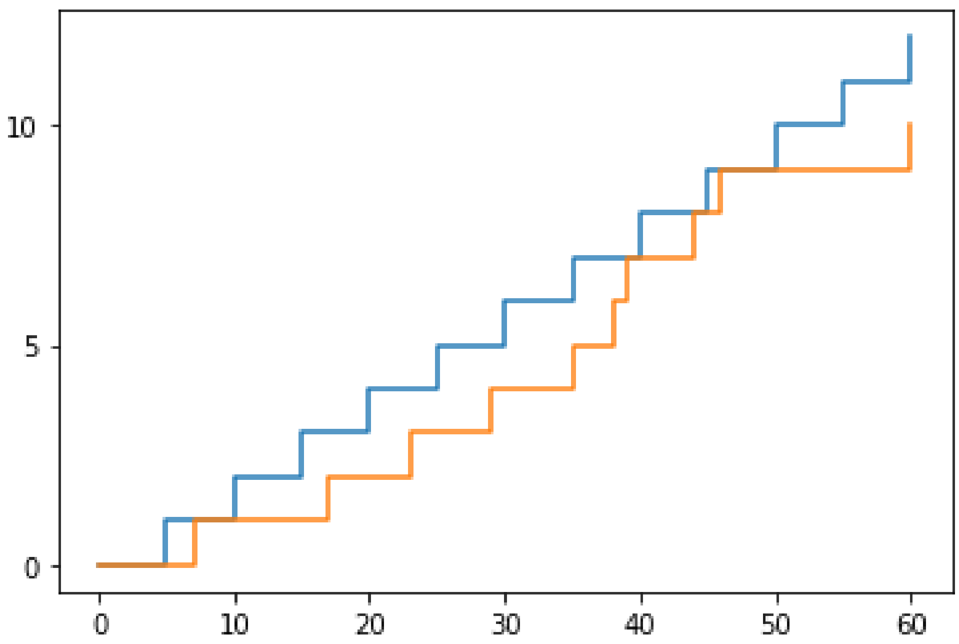
[38]:

```
plt.plot(Ti,N,drawstyle='steps-post')
plt.yticks(range(max(N)+1))
plt.xticks(list(set([i*(i%5==0) for i in range(max(Ti)+1)])))
plt.show()
```



[39]:

```
plt.plot([0]+A,range(len(A)+1),drawstyle='steps-post')
plt.plot([0]+D,range(len(D)+1),drawstyle='steps-post')
#plt.plot([0]+D,range(len(D)+1),drawstyle='steps-post')
#plt.plot(Ti,N,drawstyle='steps-post')
#plt.plot(Ti,N_q,drawstyle='steps-post')
plt.yticks(list(set([i*(i%5==0) for i in range(len(A)+1)])))
plt.show()
```



3 Problem 3 (1.19)

[40]:
```
A = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]
print(A)
```

```
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]
```

[41]:
```
S = [2.22,1.76,2.13,0.14,0.76,0.70,0.47,0.22,0.18,2.41,0.41,0.46,1.37,0.27,0.27]
print(S)
```

```
[2.22, 1.76, 2.13, 0.14, 0.76, 0.7, 0.47, 0.22, 0.18, 2.41, 0.41, 0.46, 1.37,
0.27, 0.27]
```

[42]:
```
D = [0 for i in range(len(A))]
U = D.copy()
U[0] = A[0]
for i in range(len(A)):
    D[i] = U[i] + S[i]
    if i < len(A)-1:
        U[i+1] = max(D[i], A[i+1])
print(U)
print(D)
```

```
[1, 3.22, 4.98, 7.11, 7.25, 8.01, 8.709999999999999, 9.18, 9.4, 10, 12.41,
12.82, 13.280000000000001, 14.650000000000002, 15]
```

```
[3.22, 4.98, 7.11, 7.25, 8.01, 8.709999999999999, 9.18, 9.4, 9.58, 12.41, 12.82,
13.280000000000001, 14.650000000000002, 14.920000000000002, 15.27]
```

[43]:

```
Ti = list(set(A+U))
Ti.sort()
print(Ti)
```

```
[1, 2, 3, 3.22, 4, 4.98, 5, 6, 7, 7.11, 7.25, 8, 8.01, 8.709999999999999, 9,
9.18, 9.4, 10, 11, 12, 12.41, 12.82, 13, 13.280000000000001, 14,
14.650000000000002, 15]
```

[44]:

```
N_q=[sum([a <= t for a in A])-sum([u <= t for u in U]) for t in Ti]
print(N_q)
```

```
[0, 1, 2, 1, 2, 1, 2, 3, 4, 3, 2, 3, 2, 1, 2, 1, 0, 0, 1, 2, 1, 0, 1, 0, 1, 0,
0]
```

[45]:

```
L_q = sum([n*t for n,t in zip(N_q,np.diff(Ti))])/D[-1]
print([round(t,4) for t in np.diff(Ti)])
print(sum([n*t for n,t in zip(N_q,np.diff(Ti))]))
print(L_q)
```

```
[1.0, 1.0, 0.22, 0.78, 0.98, 0.02, 1.0, 1.0, 0.11, 0.14, 0.75, 0.01, 0.7, 0.29,
0.18, 0.22, 0.6, 1.0, 1.0, 0.41, 0.41, 0.18, 0.28, 0.72, 0.65, 0.35]
17.020000000000007
1.1146037982973154
```

[46]:

```
N_q_A = [N_q[i+1]-1 if N_q[i+1]>N_q[i] else 0 for i in range(len(N_q[:-1]))]
print(N_q_A)
```

```
[0, 1, 0, 1, 0, 1, 2, 3, 0, 0, 2, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0]
```

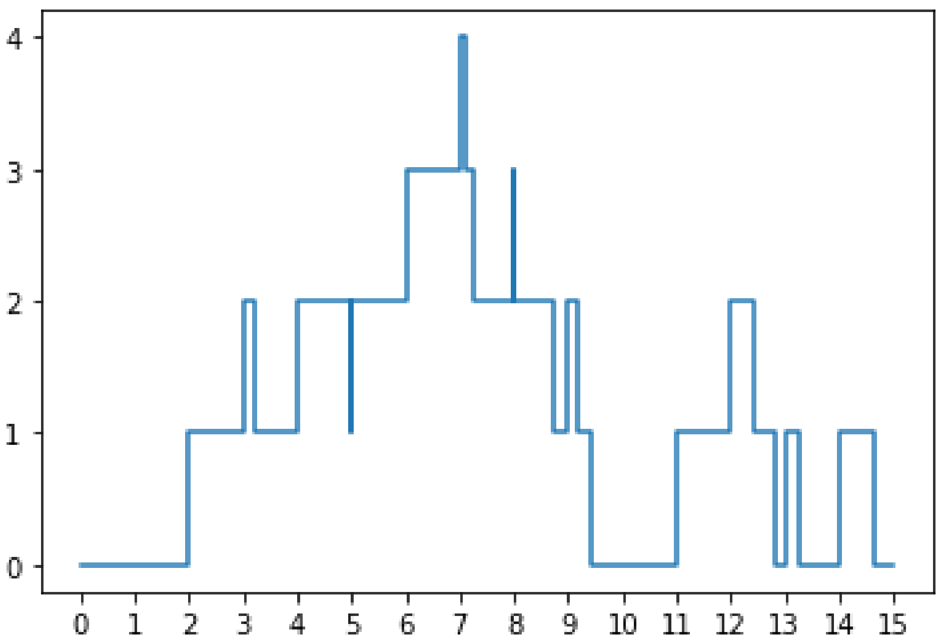
[47]:

```
L_q_A = sum(N_q_A)/sum([n>0 for n in N_q_A])
print(sum(N_q_A))
print(sum([n>0 for n in N_q_A]))
print(L_q_A)
```

```
12
8
1.5
```

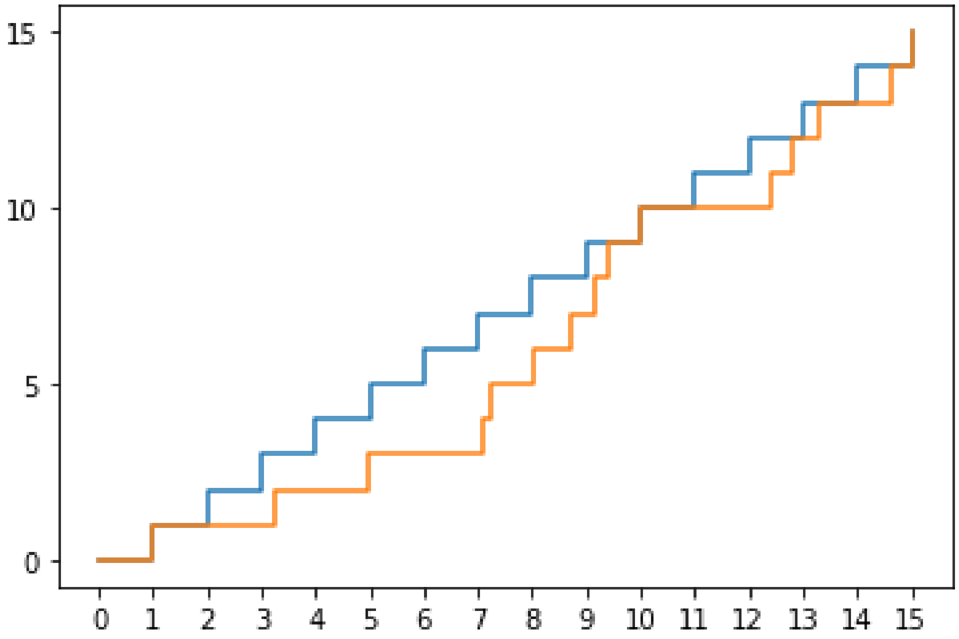
[48]:

```
plt.plot([0]+Ti,[0]+N_q,drawstyle='steps-post')
plt.yticks(range(max(N_q)+1))
plt.xticks(range(16))
plt.show()
```



[49]:

```
plt.plot([0]+A,range(len(A)+1),drawstyle='steps-post')
plt.plot([0]+U,range(len(U)+1),drawstyle='steps-post')
#plt.plot([0]+D,range(len(D)+1),drawstyle='steps-post')
#plt.plot(Ti,N,drawstyle='steps-post')
#plt.plot(Ti,N_q,drawstyle='steps-post')
plt.yticks(list(set([i*(i%5==0) for i in range(len(A)+1)])))
plt.xticks(range(16))
plt.show()
```



[50]:

```
df = pd.DataFrame(dict(A=A,S=S,U=U,D=D))
print(df.to_latex(index=False))
```

```
\begin{tabular}{rrrr}
\toprule
  A &     S &      U &      D \\
\midrule
  1 &  2.22 &   1.00 &   3.22 \\
  2 &  1.76 &   3.22 &   4.98 \\
  3 &  2.13 &   4.98 &   7.11 \\
  4 &  0.14 &   7.11 &   7.25 \\
  5 &  0.76 &   7.25 &   8.01 \\
  6 &  0.70 &   8.01 &   8.71 \\
  7 &  0.47 &   8.71 &   9.18 \\
  8 &  0.22 &   9.18 &   9.40 \\
  9 &  0.18 &   9.40 &   9.58 \\
 10 &  2.41 &  10.00 &  12.41 \\
 11 &  0.41 &  12.41 &  12.82 \\
 12 &  0.46 &  12.82 &  13.28 \\
 13 &  1.37 &  13.28 &  14.65 \\
 14 &  0.27 &  14.65 &  14.92 \\
 15 &  0.27 &  15.00 &  15.27 \\
\bottomrule
\end{tabular}
```

4 Problem 4

4.1 4 (a)

[51]:

```
A=[0,6,9,10,15,17,19,23,29,35]
S=[6,4,6,1,2,1,3,5,8,6]
```

[52]:

```
D = [0 for i in range(len(A))]
U = D.copy()
U[0] = A[0]
for i in range(len(A)):
    D[i] = U[i] + S[i]
    if i < len(A)-1:
        U[i+1] = max(D[i], A[i+1])
print(U)
print(D)
```

```
[0, 6, 10, 16, 17, 19, 20, 23, 29, 37]
[6, 10, 16, 17, 19, 20, 23, 28, 37, 43]
```

[53]:

```
D_4a = D[-1]
print(D_4a)
```

```
43
```

[54]:

```
Ti = list(set(A+U))
Ti.sort()
print(Ti)
```

```
[0, 6, 9, 10, 15, 16, 17, 19, 20, 23, 29, 35, 37]
```

[55]:

```
N_q=[sum([a <= t for a in A])-sum([u <= t for u in U]) for t in Ti]
print(N_q)
```

```
[0, 0, 1, 1, 2, 1, 1, 1, 0, 0, 0, 1, 0]
```

[56]:

```
W_q = [u-a for u,a in zip(U,A)]
W_q_ave = sum(W_q)/len(W_q)
print(W_q)
print(W_q_ave)
```

```
[0, 0, 1, 6, 2, 2, 1, 0, 0, 2]
1.4
```

[57]:

```
L_q = sum([n*t for n,t in zip(N_q,np.diff(Ti))])/D_4a
print(L_q)
```

```
0.32558139534883723
```

[58]:

```
df = pd.DataFrame(dict(A=A,S=S,U=U,D=D,W_q=W_q))
print(df.to_latex(index=False))
```

```
\begin{tabular}{rrrrr}
\toprule
 A &  S &  U &  D &  W\_q \\
\midrule
 0 &  6 &  0 &  6 &    0 \\
 6 &  4 &  6 & 10 &    0 \\
 9 &  6 & 10 & 16 &    1 \\
10 &  1 & 16 & 17 &    6 \\
15 &  2 & 17 & 19 &    2 \\
17 &  1 & 19 & 20 &    2 \\
19 &  3 & 20 & 23 &    1 \\
23 &  5 & 23 & 28 &    0 \\
29 &  8 & 29 & 37 &    0 \\
35 &  6 & 37 & 43 &    2 \\
\bottomrule
\end{tabular}
```

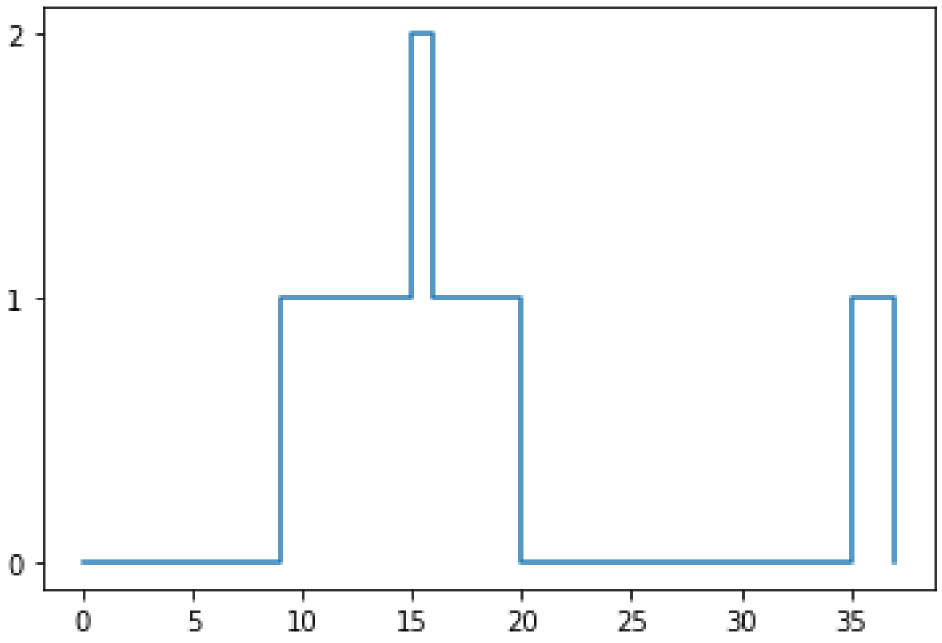
[59]:

```
W_q_ave = sum(W_q)/len(W_q)
print(W_q_ave)
```

```
1.4
```

[60]:

```
plt.plot([0]+Ti,[0]+N_q,drawstyle='steps-post')
plt.yticks(range(max(N_q)+1))
plt.xticks(list(set([i*(i%5==0) for i in range(max(Ti)+1)])))
plt.show()
```



4.2 4 (b)

[61]:

```
A=[0,6,9,10,15,17,19,23,29,35]
S=[6,4,6,1,2,1,3,5,8,6] # From 4 (a)
S=[2*s for s in S]
```

[62]:

```
D = [0 for i in range(len(A))]
U = D.copy()
U[0] = A[0]
U[1] = A[1]
D[0] = U[0] + S[0]
for i in range(1, len(A)):
    D[i] = U[i] + S[i]
    if i < len(A)-1:
        U[i+1] = max(min(max(D), max(D[:D.index(max(D))]+D[D.index(max(D))+1:
↪])), A[i+1])
print(U)
print(D)
```

```
[0, 6, 12, 14, 16, 20, 22, 24, 29, 35]
[12, 14, 24, 16, 20, 22, 28, 34, 45, 47]
```

[63]:

```
Ti = list(set(A+U))
Ti.sort()
print(Ti)
```

```
[0, 6, 9, 10, 12, 14, 15, 16, 17, 19, 20, 22, 23, 24, 29, 35]
```

```
[64]: N_q=[sum([a <= t for a in A])-sum([u <= t for u in U]) for t in Ti]
      print(N_q)
```

```
[0, 0, 1, 2, 1, 0, 1, 0, 1, 2, 1, 0, 1, 0, 0, 0]
```

```
[65]: L_q = sum([n*t for n,t in zip(N_q,np.diff(Ti))])/D_4a
      print(L_q)
```

```
0.3488372093023256
```

```
[66]: W_q = [u-a for u,a in zip(U,A)]
      print(W_q)
```

```
[0, 0, 3, 4, 1, 3, 3, 1, 0, 0]
```

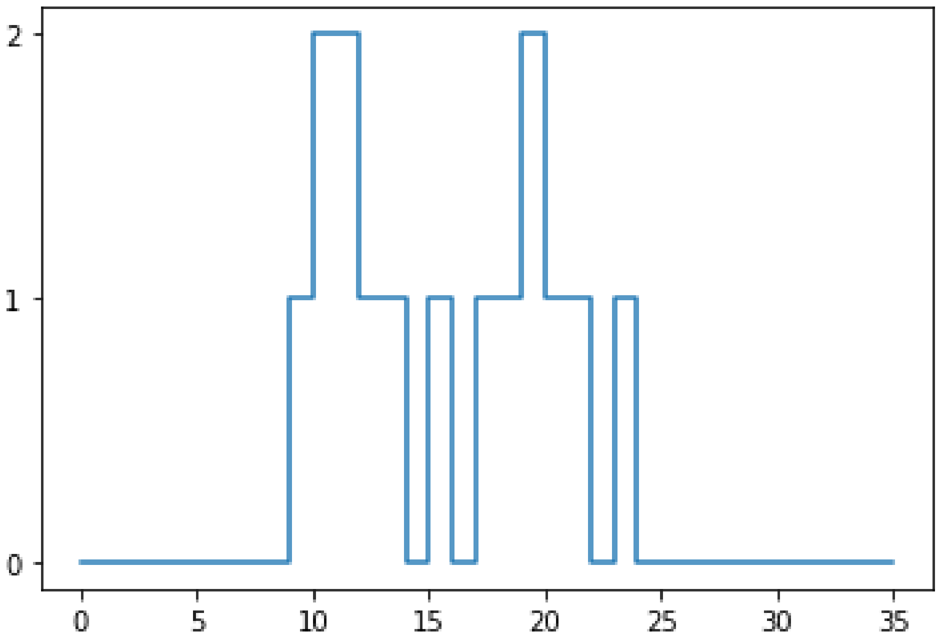
```
[67]: df = pd.DataFrame(dict(A=A,S=S,U=U,D=D,W_q=W_q))
      print(df.to_latex(index=False))
```

```
\begin{tabular}{rrrrr}
\toprule
  A &   S &   U &   D &  W\_q \\
\midrule
  0 &  12 &   0 &  12 &    0 \\
  6 &   8 &   6 &  14 &    0 \\
  9 &  12 &  12 &  24 &    3 \\
 10 &   2 &  14 &  16 &    4 \\
 15 &   4 &  16 &  20 &    1 \\
 17 &   2 &  20 &  22 &    3 \\
 19 &   6 &  22 &  28 &    3 \\
 23 &  10 &  24 &  34 &    1 \\
 29 &  16 &  29 &  45 &    0 \\
 35 &  12 &  35 &  47 &    0 \\
\bottomrule
\end{tabular}
```

```
[68]: W_q_ave = sum(W_q)/len(W_q)
      print(W_q_ave)
```

```
1.5
```

```
[69]: plt.plot([0]+Ti,[0]+N_q,drawstyle='steps-post')
      plt.yticks(range(max(N_q)+1))
      plt.xticks(list(set([i*(i%5==0) for i in range(max(Ti)+1)])))
      plt.show()
```



5 Problem 5

[70]:

```
arr_frac = 0.5
dep_frac = 1 - arr_frac
arr_hold_mean = 1
dep_hold_mean = 1.5
rate_hr = 50
rate = rate_hr/60
service_mean = arr_frac * arr_hold_mean + dep_frac * dep_hold_mean
rho = rate*service_mean
print(rate, service_mean)
```

```
0.8333333333333334 1.25
```

[71]:

```
num_ops = 400
N = 10000
```

[72]:

```
def sim(arr_frac, arr_hold_mean, dep_hold_mean, rate_hr, num_ops, exp_service =␣
 ↪True):
    rate = rate_hr/60
    T = np.random.exponential(1/rate,num_ops)
    op_type = np.random.binomial(1, arr_frac, num_ops)
    if exp_service:
        S = [np.random.exponential(arr_hold_mean) if b==1
            else np.random.exponential(dep_hold_mean) for b in op_type]
    else:
        S = [arr_hold_mean if b==1 else dep_hold_mean for b in op_type]
    A = [0] + list(np.cumsum(T[:-1]))
    D = [0 for i in range(len(A))]
    D[0] = S[0] + A[0]
    for i in range(1,len(A)):
        D[i] = S[i] + max(D[i-1], A[i])
    W_q_n = [d-s-a for d,s,a in zip(D,S,A)]
    return sum(W_q_n)/num_ops
```

[73]:

```
rates = [i*5 for i in range(1,11)]
```

[74]:

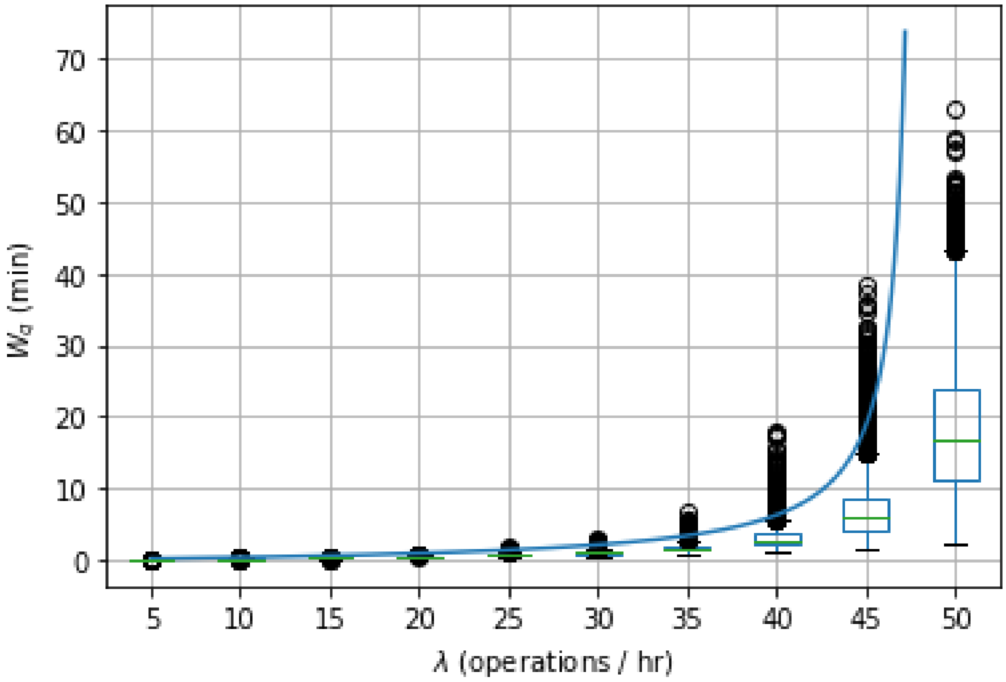
```
data = [[sim(arr_frac, arr_hold_mean, dep_hold_mean, r, num_ops, False) for i␣
 ↪in range(N)] for r in rates]
```

[75]:

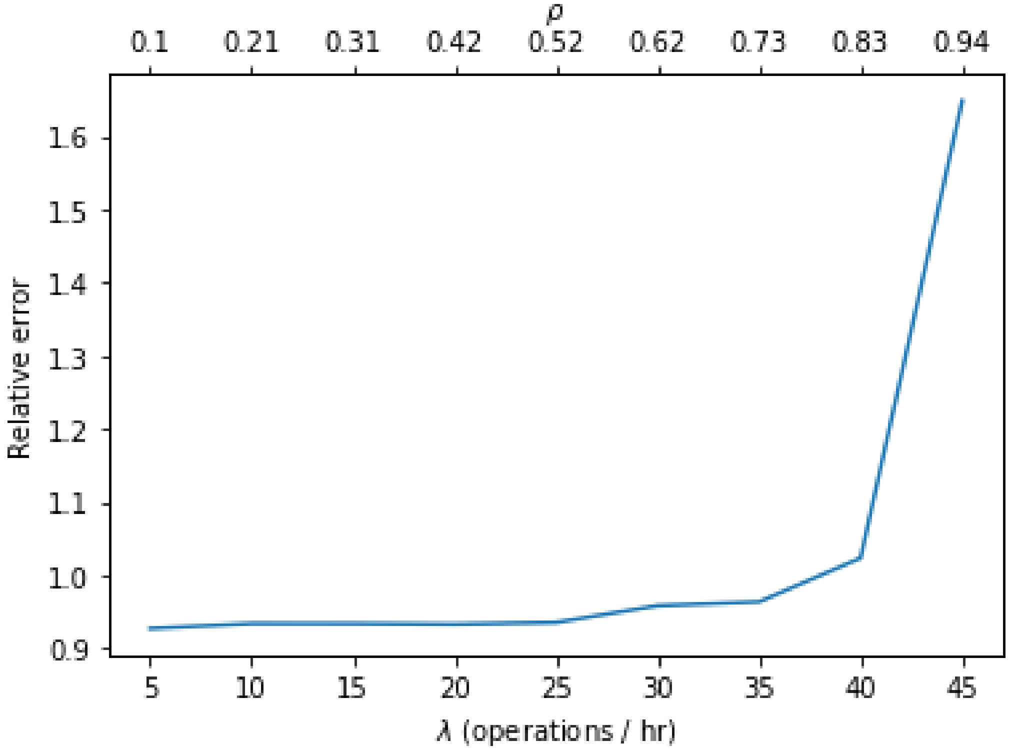
```
df = pd.DataFrame(data)
df = df.transpose()
df = df.rename(columns=dict(zip([i for i in range(10)],rates)))
```

[76]:

```
plt.figure()
df.boxplot()
x = np.arange(1, 9.45, 0.01)
y = [i*5*service_mean/60/(1/service_mean-i*5/60) for i in x]
plt.plot(x,y)
#plt.title('Average queue-waiting time')
plt.xlabel(r'$\lambda$'+' (operations / hr)')
plt.ylabel(r'$W_q$'+' (min)')
plt.savefig('W_q1.pdf')
plt.show()
```



```
[77]: m = [rate*service_mean/60/(1/service_mean-rate/60) for rate in rates][:-1]
      d = np.array(df.mean())[:-1]
      rho = np.array(rates)*service_mean/60
      err = abs((d-m)/d)
      #plt.title('Relative error by arrival rate')
      plt.ylabel('Relative error')
      plt.plot(rates[:-1],err)
      plt.xlabel(r'$\lambda$'+' (operations / hr)')
      plt.twiny()
      plt.xlabel(r'$\rho$')
      plt.plot(rates[:-1],err, alpha=0)
      plt.gca().set_xticklabels([0]+[round(r,2) for r in rho][:-1])
      plt.savefig('er1.pdf')
      plt.show()
```



```
[78]: arr_frac = 0.25
      dep_frac = 1 - arr_frac
      arr_hold_mean = 1
      dep_hold_mean = 1.5
      rate_hr = 50
      rate = rate_hr/60
      service_mean = arr_frac * arr_hold_mean + dep_frac * dep_hold_mean
      rho = rate*service_mean
      print(rate, service_mean)
```

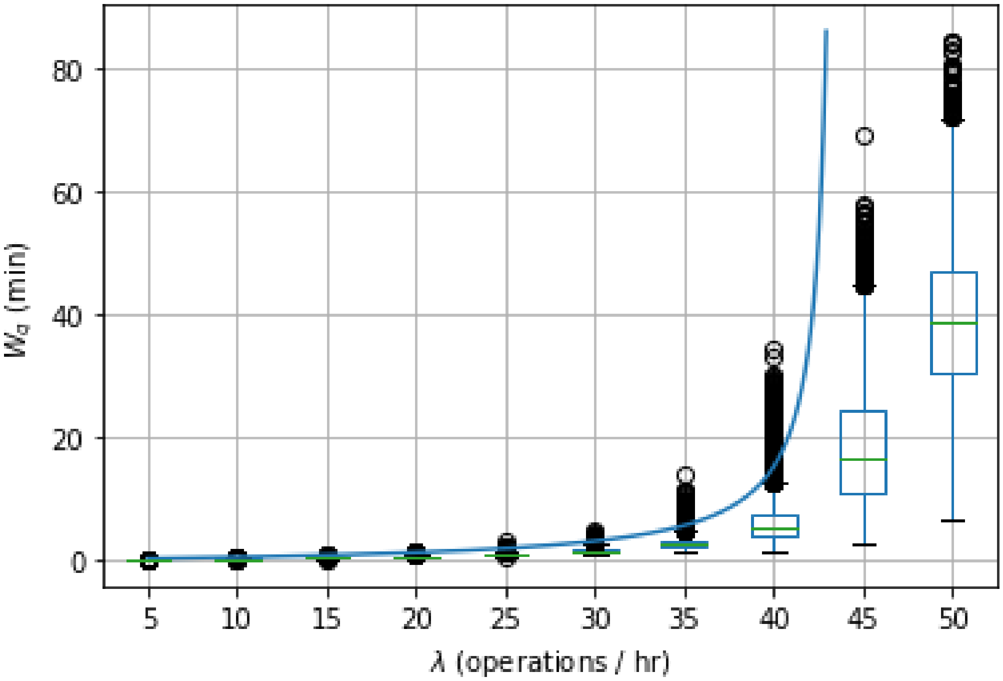
```
0.8333333333333334 1.375
```

```
[79]: data = [[sim(arr_frac, arr_hold_mean, dep_hold_mean, r, num_ops, False) for i␣
       ↪in range(N)] for r in rates]
```

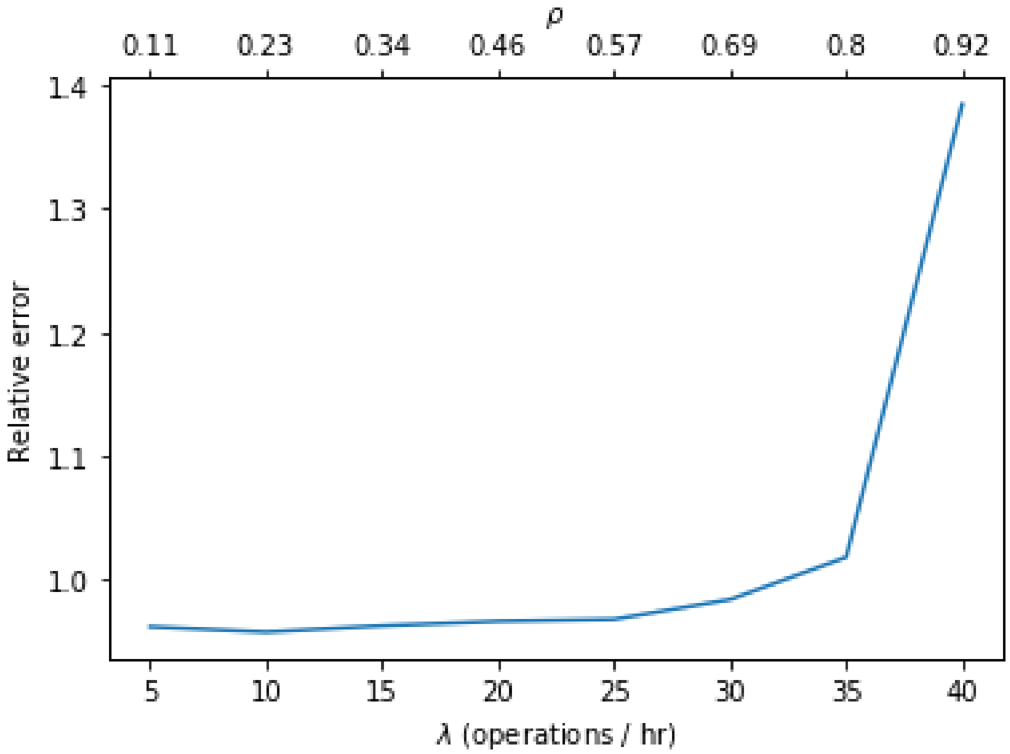
```
[80]: df = pd.DataFrame(data)
      df = df.transpose()
      df = df.rename(columns=dict(zip([i for i in range(10)],rates)))
```

```
[81]: df.boxplot()
      x = np.arange(1, 8.6, 0.01)
      y = [i*5*service_mean/60/(1/service_mean-i*5/60) for i in x]
```

```
plt.plot(x,y)
#plt.title('Average queue-waiting time')
plt.xlabel(r'$\lambda$'+' (operations / hr)')
plt.ylabel(r'$W_q$'+' (min)')
plt.savefig('W_q2.pdf')
plt.show()
```



[82]:
```
m = [rate*service_mean/60/(1/service_mean-rate/60) for rate in rates][:-2]
d = np.array(df.mean())[:-2]
rho = np.array(rates)*service_mean/60
err = abs((d-m)/d)
#plt.title('Relative error by arrival rate')
plt.ylabel('Relative error')
plt.plot(rates[:-2],err)
plt.xlabel(r'$\lambda$'+' (operations / hr)')
plt.twiny()
plt.xlabel(r'$\rho$')
plt.plot(rates[:-2],err, alpha=0)
plt.gca().set_xticklabels([0]+[round(r,2) for r in rho])
plt.savefig('er2.pdf')
plt.show()
```



[83]:

```
arr_frac = 0.5
dep_frac = 1 - arr_frac
arr_hold_mean = 1
dep_hold_mean = 1.5
rate_hr = 50
rate = rate_hr/60
service_mean = arr_frac * arr_hold_mean + dep_frac * dep_hold_mean
rho = rate*service_mean
print(rate, service_mean)
```

```
0.8333333333333334 1.25
```

[84]:

```
num_ops = 400
N = 1000
```

[85]:

```
def sim(arr_frac, arr_hold_mean, dep_hold_mean, rate_hr, num_ops, exp_service =␣
 ↪True):
    rate = rate_hr/60
    T = np.random.exponential(1/rate,num_ops)
    op_type = np.random.binomial(1, arr_frac, num_ops)
    if exp_service:
        S = [np.random.exponential(arr_hold_mean) if b==1
```

```
            else np.random.exponential(dep_hold_mean) for b in op_type]
    else:
        S = [arr_hold_mean if b==1 else dep_hold_mean for b in op_type]
    A = [0] + list(np.cumsum(T[:-1]))
    D = [0 for i in range(len(A))]
    D[0] = S[0] + A[0]
    for i in range(1,len(A)):
        D[i] = S[i] + max(D[i-1], A[i])
    W_q_n = [d-s-a for d,s,a in zip(D,S,A)]
    return sum(W_q_n)/num_ops
```

[86]:

```
rates = [i*5 for i in range(1,11)]
```

[87]:

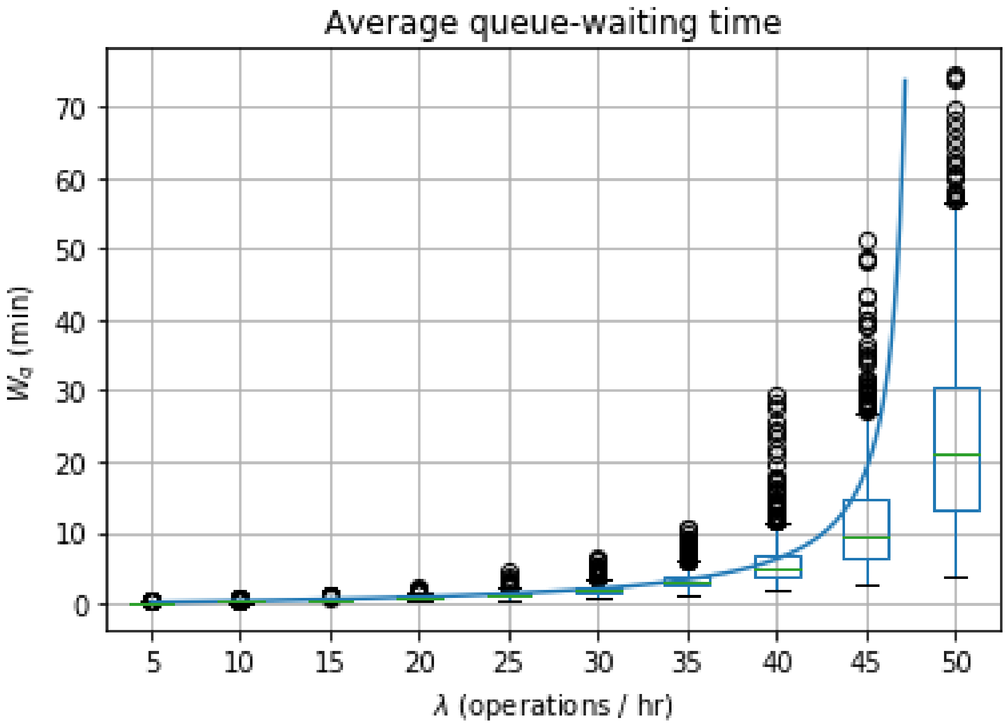
```
data = [[sim(arr_frac, arr_hold_mean, dep_hold_mean, r, num_ops, True) for i in␣
 ↪range(N)] for r in rates]
```

[88]:

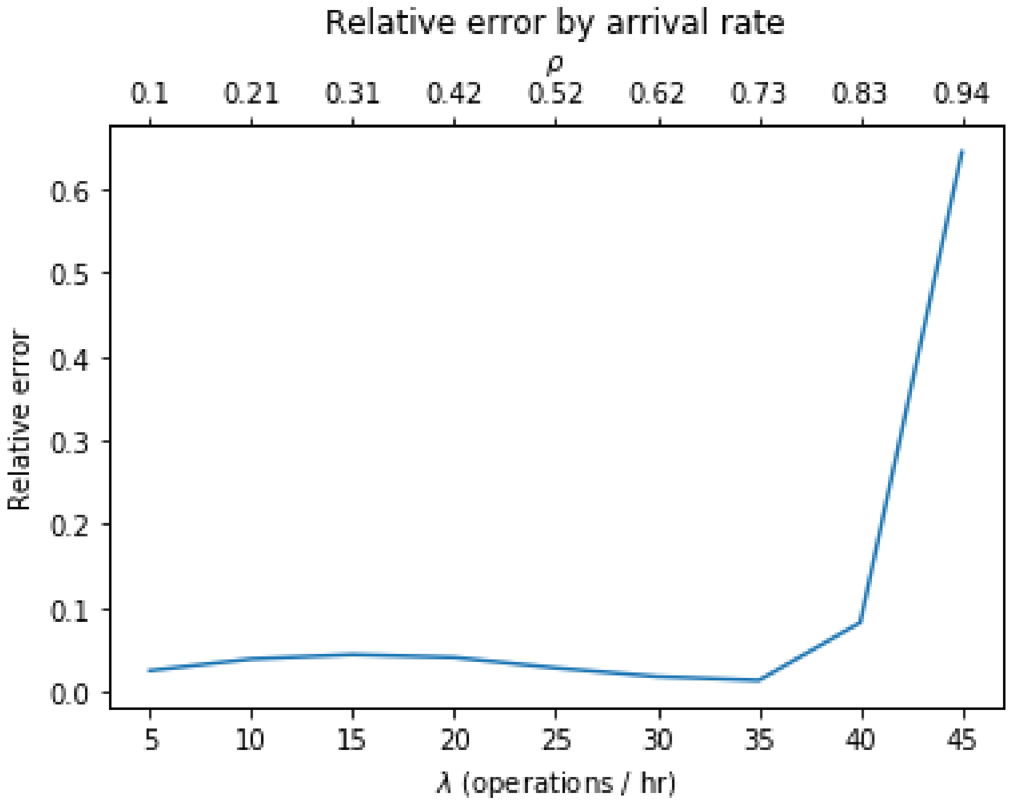
```
df = pd.DataFrame(data)
df = df.transpose()
df = df.rename(columns=dict(zip([i for i in range(10)],rates)))
```

[89]:

```
plt.figure()
df.boxplot()
x = np.arange(1, 9.45, 0.01)
y = [i*5*service_mean/60/(1/service_mean-i*5/60) for i in x]
plt.plot(x,y)
plt.title('Average queue-waiting time')
plt.xlabel(r'$\lambda$'+' (operations / hr)')
plt.ylabel(r'$W_q$'+' (min)')
plt.show()
```



```
[90]: m = [rate*service_mean/60/(1/service_mean-rate/60) for rate in rates][:-1]
      d = np.array(df.mean())[:-1]
      rho = np.array(rates)*service_mean/60
      err = abs((d-m)/d)
      plt.title('Relative error by arrival rate')
      plt.ylabel('Relative error')
      plt.plot(rates[:-1],err)
      plt.xlabel(r'$\lambda$'+' (operations / hr)')
      plt.twiny()
      plt.xlabel(r'$\rho$')
      plt.plot(rates[:-1],err, alpha=0)
      plt.gca().set_xticklabels([0]+[round(r,2) for r in rho][:-1])
      plt.show()
```



[91]:

```
arr_frac = 0.25
dep_frac = 1 - arr_frac
arr_hold_mean = 1
dep_hold_mean = 1.5
rate_hr = 50
rate = rate_hr/60
service_mean = arr_frac * arr_hold_mean + dep_frac * dep_hold_mean
rho = rate*service_mean
print(rate, service_mean)
```

```
0.8333333333333334 1.375
```

[92]:

```
data = [[sim(arr_frac, arr_hold_mean, dep_hold_mean, r, num_ops, True) for i in␣
 ↪range(N)] for r in rates]
```

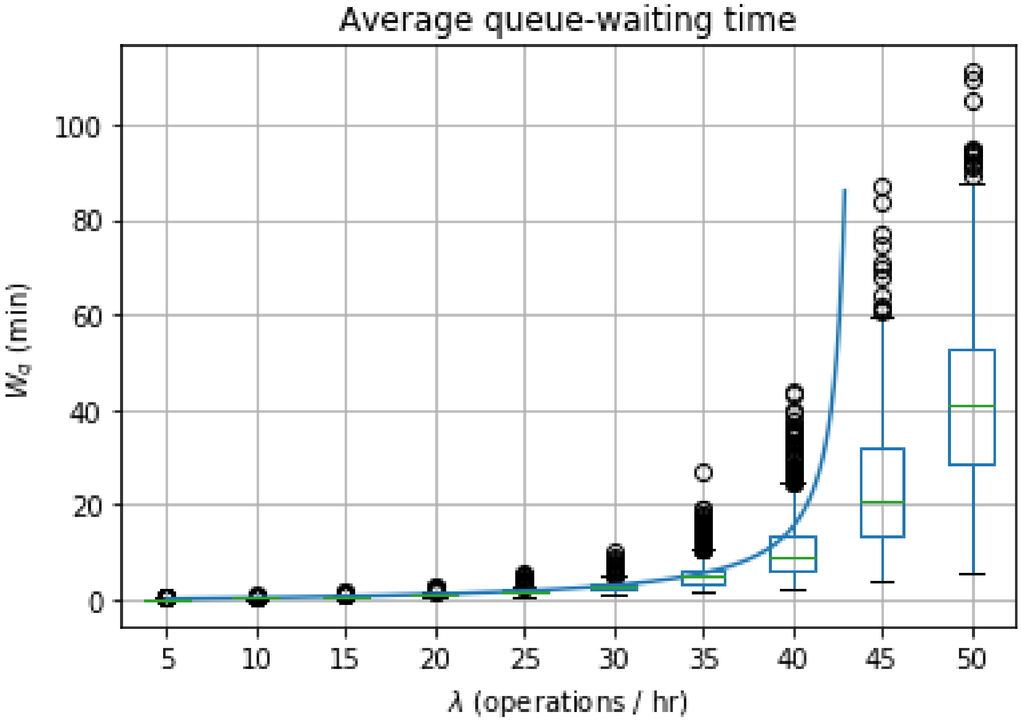
[93]:

```
df = pd.DataFrame(data)
df = df.transpose()
df = df.rename(columns=dict(zip([i for i in range(10)],rates)))
```

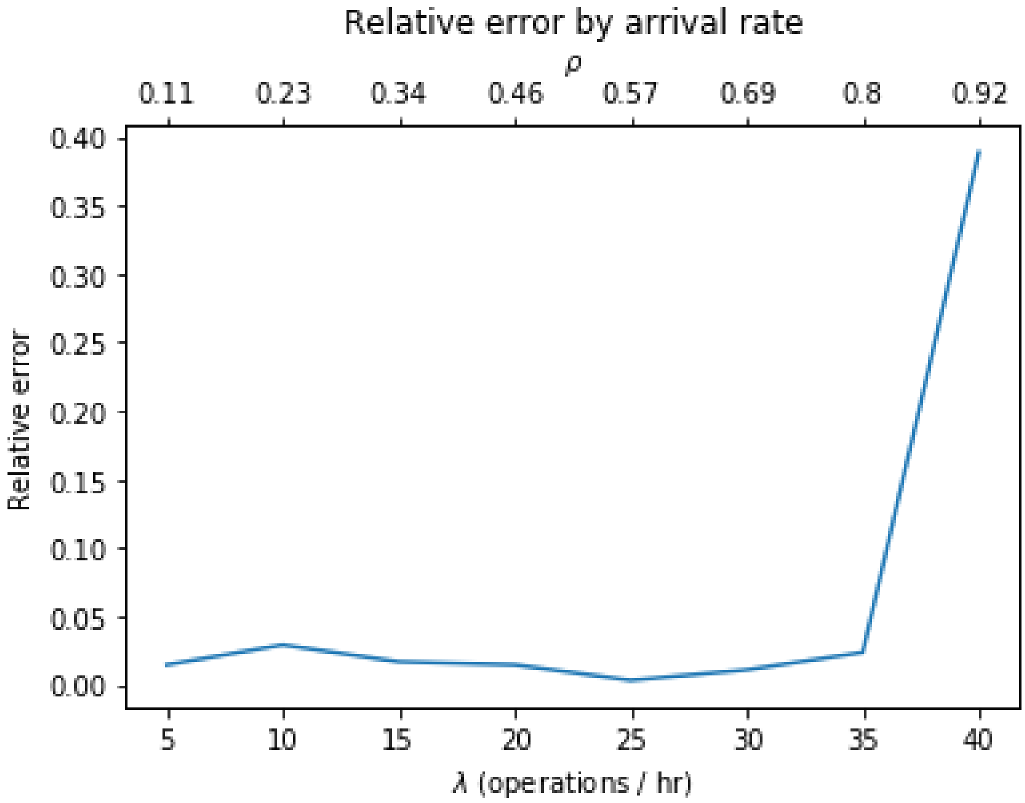
[94]:

```
df.boxplot()
x = np.arange(1, 8.6, 0.01)
```

```
y = [i*5*service_mean/60/(1/service_mean-i*5/60) for i in x]
plt.plot(x,y)
plt.title('Average queue-waiting time')
plt.xlabel(r'$\lambda$'+' (operations / hr)')
plt.ylabel(r'$W_q$'+' (min)')
plt.show()
```



[95]:
```
m = [rate*service_mean/60/(1/service_mean-rate/60) for rate in rates][:-2]
d = np.array(df.mean())[:-2]
rho = np.array(rates)*service_mean/60
err = abs((d-m)/d)
plt.title('Relative error by arrival rate')
plt.ylabel('Relative error')
plt.plot(rates[:-2],err)
plt.xlabel(r'$\lambda$'+' (operations / hr)')
plt.twiny()
plt.xlabel(r'$\rho$')
plt.plot(rates[:-2],err, alpha=0)
plt.gca().set_xticklabels([0]+[round(r,2) for r in rho])
plt.show()
```



[96]:

```
arr_frac = 0.5
dep_frac = 1 - arr_frac
arr_hold_mean = 1
dep_hold_mean = 1.5
rate_hr = 50
rate = rate_hr/60
service_mean = arr_frac * arr_hold_mean + dep_frac * dep_hold_mean
rho = rate*service_mean
print(rate, service_mean)
```

```
0.8333333333333334 1.25
```

[97]:

```
T = np.random.exponential(1/rate,num_ops)
np.mean(T)
```

[97]:

```
1.3176689143665428
```

[98]:

```
op_type = np.random.binomial(1, arr_frac, num_ops)
np.mean(op_type)
```

[98]:

```
0.5025
```

[99]:

```
S = [arr_hold_mean if b==1 else dep_hold_mean for b in op_type]
np.mean(S)
```

[99]:

```
1.24875
```

[100]:

```
#S = [np.random.exponential(arr_hold_mean)
#              if b==1
#              else np.random.exponential(dep_hold_mean) for b in op_type]
#np.mean(S)
```

[101]:

```
A = [0] + list(np.cumsum(T[:-1]))
```

[102]:

```
D = [0 for i in range(len(A))]
D[0] = S[0] + A[0]
for i in range(1,len(A)):
    D[i] = S[i] + max(D[i-1], A[i])
```

[103]:

```
D[-1]
```

[103]:

```
551.5958091442158
```

[104]:

```
W_q_n = [d-s-a for d,s,a in zip(D,S,A)]
```

[105]:

```
W_q = sum(W_q_n)/D[-1]
print(W_q)
```

```
4.952185617364083
```

[106]:

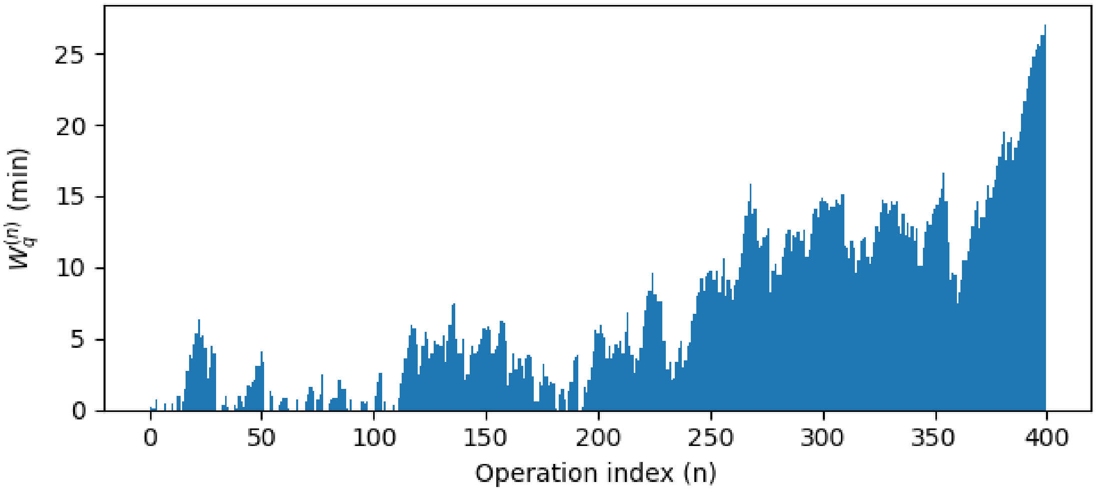
```
[i in range(len(W_q_n))]
```

[106]:

```
[True]
```

[107]:

```
plt.figure(figsize=(6.4,3),dpi=100)
plt.bar([i for i in range(len(W_q_n))],W_q_n,1)
#plt.title('Sample queue-waiting times')
plt.ylabel(r'$W_q^{(n)}$'+' (min)')
plt.xlabel('Operation index (n)')
plt.tight_layout()
#plt.savefig('path.pdf')
plt.show()
```



6 Problem 6

[109]:
```
c = 3

lam = 45/60
mu = 1/3
print(lam, mu)
```

```
0.75 0.3333333333333333
```

[110]:
```
r = lam / mu
print(r)
```

```
2.25
```

[111]:
```
rho = r / c
print(rho)
```

```
0.75
```

[112]:
```
n = r**c
d = factorial(c)*(1-rho)
s = sum([r**i/factorial(i) for i in range(c)])
p_0 = 1 / (n/d + s)
print(p_0)
```

```
0.07476635514018691
```

[113]:

```
L_q = n*rho/(d *(1-rho))*p_0
print(L_q)
```

```
1.703271028037383
```

[114]:

```
W = (1/mu) + L_q/lam
print(W)
```

```
5.271028037383177
```

7 Problem 7

[115]:

```
c = 5
lam = 75/60
mu = 1/2
print(lam, mu)
```

```
1.25 0.5
```

[116]:

```
r = lam / mu
print(r)
```

```
2.5
```

[117]:

```
rho = r / c
print(rho)
```

```
0.5
```

7.1 7 (b)

[118]:

```
n = r**c
d = factorial(c)*(1-rho)
s = sum([r**i/factorial(i) for i in range(c)])
p_0 = 1 / (n/d + s)
print(p_0)
```

```
0.08010012515644557
```

[119]:

```
p_n = sum([lam**i/factorial(i)/mu**i*p_0 for i in range(4)])
print(p_n)
```

```
0.7392574050896956
```

7.2 7 (c)

[120]:
```
L_q = n*rho/(d *(1-rho))*p_0
print(L_q)
```

```
0.1303712974551523
```

8 Problem 8

[121]:
```
lam = 18
mu = 20
print(lam, mu)
```

```
18 20
```

[122]:
```
r = lam / mu
print(r)
```

```
0.9
```

8.1 8 (a)

[123]:
```
c = 1
rho = r / c
print(rho)
```

```
0.9
```

[124]:
```
n = r**c
d = factorial(c)*(1-rho)
s = sum([r**i/factorial(i) for i in range(c)])
p_0 = 1 / (n/d + s)
print(p_0)
```

```
0.09999999999999998
```

[125]:
```
L_q = n*rho/(d *(1-rho))*p_0
print(L_q)
```

```
8.100000000000003
```

[126]:
```
L_q = rho**2/(1-rho)
print(L_q)
```

```
8.100000000000003
```

[127]:
```
r * 40 + (c - r) * 10 + L_q * 2
```

```
[127]: 53.2
```

8.2 8 (b)

```
[128]: c = 2
       rho = r / c
       print(rho)
```

```
0.45
```

```
[129]: n = r**c
       d = factorial(c)*(1-rho)
       s = sum([r**i/factorial(i) for i in range(c)])
       p_0 = 1 / (n/d + s)
       print(p_0)
```

```
0.37931034482758624
```

```
[130]: L_q = n*rho/(d *(1-rho))*p_0
       print(L_q)
```

```
0.22852664576802512
```

```
[131]: r * 40 + (c - r) * 10 + L_q * 2
```

```
[131]: 47.45705329153605
```

9 Problem 9 (3.32)

```
[132]: def approx(alpha, beta):
           return norm.pdf(beta)/(norm.pdf(beta) + beta * norm.cdf(beta)) - alpha
```

9.1 9 (a)

```
[133]: lam = 500 / 60
       mu = 1 / 2
       print(lam, mu)
```

```
8.333333333333334 0.5
```

```
[134]: r = lam / mu
       print(r)
```

```
16.666666666666668
```

[135]:

```
beta = newton(lambda beta: approx(0.10, beta), 1)
print(beta)
```

```
1.4201868881004411
```

[136]:

```
c_approx = r + beta * sqrt(r)
print(c_approx)
```

```
22.464555358728653
```

9.2 9 (b)

[137]:

```
lam = 500 / 60 * 1.6
mu = 1 / 2
print(lam, mu)
```

```
13.333333333333336 0.5
```

[138]:

```
r = lam / mu
print(r)
```

```
26.66666666666667
```

[139]:

```
c_approx = r + beta * sqrt(r)
print(c_approx)
```

```
34.00048022148687
```

9.3 9 (c)

[140]:

```
lam = 500 / 60
mu = 1 / 3
print(lam, mu)
```

```
8.333333333333334 0.3333333333333333
```

[141]:

```
r = lam / mu
print(r)
```

```
25.000000000000004
```

[142]:

```
c_approx = r + beta * sqrt(r)
print(c_approx)
```

```
32.10093444050221
```

9.4 9 (d)

[143]:
```
lam = 500 / 60
mu = 1 / 2
print(lam, mu)
```

```
8.333333333333334 0.5
```

[144]:
```
r = lam / mu
print(r)
```

```
16.666666666666668
```

[145]:
```
c = 24
rho = r / c
print(rho)
```

```
0.6944444444444445
```

[146]:
```
n = r**c
d = factorial(c)*(1-rho)
s = sum([r**i/factorial(i) for i in range(c)])
p_0 = 1 / (n/d + s)
print(p_0)
```

```
5.714528177635283e-08
```

[147]:
```
C_c_r = n*p_0/d
print(C_c_r)
```

```
0.06361426515341066
```

[148]:
```
L_q = n*rho/(d*(1-rho))*p_0
print(L_q)
```

```
0.14457787534866065
```

[149]:
```
W_q = L_q / lam
print(W_q)
```

```
0.017349345041839277
```

[150]:
```
lam = 500 / 60 * 1.6
mu = 1 / 2
print(lam, mu)
```

```
13.333333333333336 0.5
```

[151]:

```
r = lam / mu
print(r)
```

```
26.666666666666667
```

[152]:

```
c = 35
rho = r / c
print(rho)
```

```
0.7619047619047621
```

[153]:

```
n = r**c
d = factorial(c)*(1-rho)
s = sum([r**i/factorial(i) for i in range(c)])
p_0 = 1 / (n/d + s)
print(p_0)
```

```
2.5787312412854715e-12
```

[154]:

```
C_c_r = n*p_0/d
print(C_c_r)
```

```
0.08498227504737443
```

[155]:

```
L_q = n*rho/(d *(1-rho))*p_0
print(L_q)
```

```
0.2719432801515984
```

[156]:

```
W_q = L_q / lam
print(W_q)
```

```
0.020395746011369877
```

10 Problem 10 (3.37)

[157]:

```
beta = newton(lambda beta: approx(0.05, beta), 1)
print(beta)
```

```
1.7398362717905036
```

10.1 10 (a)

[158]:

```
lam = 300 / 60
mu = 1 / 2
print(lam, mu)
```

```
5.0 0.5
```

[159]:

```
r = lam / mu
print(r)
```

```
10.0
```

[160]:

```
c_approx = r + beta * sqrt(r)
print(c_approx)
```

```
15.50184537463375
```

10.2 10 (b)

[161]:

```
lam = 300 / 60 * 2
mu = 1 / 2
print(lam, mu)
```

```
10.0 0.5
```

[162]:

```
r = lam / mu
print(r)
```

```
20.0
```

[163]:

```
c_approx = r + beta * sqrt(r)
print(c_approx)
```

```
27.780784346886733
```