0 Introduction

The goal of this document is to fully characterize the Dunson day-specific probabilities model. In its current state it tries to provide full detail of the derivations described in *Bayesian Inferences on Predictors of Conception Probabilities*.

1 The model

1.1 Marginal probability of conception

The marginal probability of conception, obtained by integrating out the couple-specific frailty ξ_i, has form as follows.

$$
\begin{aligned}
&\mathbb{P}(Y_{ij} = 1 \,|\, \boldsymbol{X}_{ij}, \boldsymbol{U}_{ij}) \\
&\quad = \int_0^\infty \mathbb{P}\Big(Y_{ij}, \xi_i \,|\, \boldsymbol{X}_{ij}, \boldsymbol{U}_{ij}\Big)\, d\xi_i \\
&\quad = \int_0^\infty \mathbb{P}\Big(Y_{ij}, \xi_i \,|\, \boldsymbol{X}_{ij}, \boldsymbol{U}_{ij}\Big)\, \mathcal{G}(\xi_i;\, \phi, \phi)\, d\xi_i \\
&\quad = \int_0^\infty \left[1 - \prod_{k=1}^{K} (1 - \lambda_{ijk})^{X_{ijk}}\right] \mathcal{G}(\xi_i;\, \phi, \phi)\, d\xi_i \\
&\quad = 1 - \int_0^\infty \prod_{k=1}^{K} (1 - \lambda_{ijk})^{X_{ijk}}\, \mathcal{G}(\xi_i;\, \phi, \phi)\, d\xi_i \\
&\quad = 1 - \int_0^\infty \prod_{k=1}^{K} \left[\exp\left\{-\xi_i \exp\Big(\boldsymbol{u}'_{ijk}\boldsymbol{\beta}\Big)\right\}\right]^{X_{ijk}} \mathcal{G}(\xi_i;\, \phi, \phi)\, d\xi_i \\
&\quad = 1 - \int_0^\infty \prod_{k=1}^{K} \exp\left\{-\xi_i X_{ijk} \exp\Big(\boldsymbol{u}'_{ijk}\boldsymbol{\beta}\Big)\right\} \mathcal{G}(\xi_i;\, \phi, \phi)\, d\xi_i \\
&\quad = 1 - \int_0^\infty \exp\left\{-\xi_i \sum_{k=1}^{K} X_{ijk} \exp\Big(\boldsymbol{u}'_{ijk}\boldsymbol{\beta}\Big)\right\} \mathcal{G}(\xi_i;\, \phi, \phi)\, d\xi_i \\
&\quad = 1 - \left[\frac{\phi}{\phi + \sum_{k=1}^{K} X_{ijk} \exp\Big(\boldsymbol{u}'_{ijk}\boldsymbol{\beta}\Big)}\right]^{\phi}
\end{aligned}
$$

since

$$
\begin{aligned}
&\int_0^\infty \exp\left\{-\xi_i \sum_{k=1}^K X_{ijk} \exp\left(\boldsymbol{u}'_{ijk}\boldsymbol{\beta}\right)\right\} \mathcal{G}(\xi_i;\, \phi, \phi)\, d\xi_i \\
&\quad = \int_0^\infty \exp\left\{-\xi_i \sum_{k=1}^K X_{ijk} \exp\left(\boldsymbol{u}'_{ijk}\boldsymbol{\beta}\right)\right\} \frac{\phi^\phi}{\Gamma(\phi)}\, \xi_i^{\phi-1} d\xi_i \\
&\quad = \int_0^\infty \frac{\phi^\phi}{\Gamma(\phi)}\, \xi_i^{\phi-1} \exp\left\{-\xi_i \left[\phi + \sum_{k=1}^K X_{ijk} \exp\left(\boldsymbol{u}'_{ijk}\boldsymbol{\beta}\right)\right]\right\} d\xi_i \\
&\quad = \left[\frac{\phi}{\phi + \sum_{k=1}^K X_{ijk} \exp\left(\boldsymbol{u}'_{ijk}\boldsymbol{\beta}\right)}\right]^\phi \int_0^\infty \frac{\left[\phi + \sum_{k=1}^K X_{ijk} \exp\left(\boldsymbol{u}'_{ijk}\boldsymbol{\beta}\right)\right]^\phi}{\Gamma(\phi)} \\
&\qquad\qquad\qquad \times \xi_i^{\phi-1} \exp\left\{-\xi_i \left[\phi + \sum_{k=1}^K X_{ijk} \exp\left(\boldsymbol{u}'_{ijk}\boldsymbol{\beta}\right)\right]^\phi\right\} d\xi_i
\end{aligned}
$$

2 Posterior computation

Express the model as

$$
\begin{aligned}
Y_{ij} &= I\left(\sum_{k=1}^K W_{ijk} > 0\right), \\
W_{ijk} &\overset{ind}{\sim} \begin{cases} 0, & X_{ijk} = 0 \\ \text{Poisson}\left(\xi_i \exp\left(\boldsymbol{u}'_{ijk}\boldsymbol{\beta}\right)\right), & \text{else} \end{cases}
\end{aligned}
$$

for $k = 1, \ldots, K$,