0 Introduction

The goal of this document is to fully characterize the Dunson and Stanford day-specific probabilities model. In its current state it tries to provide full detail of the derivations described in *Bayesian Inferences on Predictors of Conception Probabilities*.

1 The day-specific probabilities model

1.1 Model specification

We wish to model the probability of a woman becoming pregnant for a given menstrual cycle as a function of her covariate status across the days of the cycle. Consider a study cohort and let us index

$$\begin{aligned} &\text{woman } i, && i = 1, \dots, n \\ &\text{cycle } j, && j = 1, \dots, n_i \\ &\text{day } k, && k = 1, \dots, K \end{aligned}$$

where day k refers to the k^{th} day out of a total of K days in the fertile window. Let us write day i,j,k as a shorthand for individual i, cycle j, and day k and similarly for cycle j,k. Then define

Y_{ij}	an indicator of conception for woman i, cycle j
V_{ijk}	an indicator of conception for woman i, cycle j, day k
X_{ijk}	an indicator of intercourse for woman i, cycle j, day k

Then writing $\boldsymbol{X}_{ij} = \left(X_{ij1}, \dots, X_{ijK}\right)$, we observe that

$$\begin{aligned}
&\mathbb{P}\Big(Y_{ij} = 1 \;\Big|\; \boldsymbol{X}_{ij},\ Y_{i1} = 0, \dots, Y_{i,j-1} = 0\Big) \\
&\quad = 1 - \mathbb{P}\Big(Y_{ij} = 0 \;\Big|\; \boldsymbol{X}_{ij},\ Y_{i1} = 0, \dots, Y_{i,j-1} = 0\Big) \\
&\quad = 1 - \mathbb{P}\Big(V_{ijk} = 0,\ k = 1, \dots, K \;\Big|\; \boldsymbol{X}_{ij},\ Y_{i1} = 0, \dots, Y_{i,j-1} = 0\Big) \\
&\quad = 1 - \prod_{k=1}^{K} \mathbb{P}\Big(V_{ijk} = 0 \;\Big|\; X_{ijk},\ Y_{i1} = 0, \dots, Y_{i,j-1} = 0,\ V_{ij1} = 0, \dots, V_{i,k-1} = 0\Big) \\
&\quad = 1 - \prod_{k=1}^{K} \Big\{1 - \mathbb{P}\Big(V_{ijk} = 1 \;\Big|\; X_{ijk},\ Y_{i1} = 0, \dots, Y_{i,j-1} = 0,\ V_{ij1} = 0, \dots, V_{i,k-1} = 0\Big)\Big\} \\
&\quad = 1 - \prod_{k=1}^{K} \Big\{1 - X_{ijk}\, \mathbb{P}\Big(V_{ijk} = 1 \;\Big|\; Y_{i1} = 0, \dots, Y_{i,j-1} = 0,\ V_{ij1} = 0, \dots, V_{i,k-1} = 0\Big)\Big\} \\
&\quad = 1 - \prod_{k=1}^{K} \Big\{1 - \mathbb{P}\Big(V_{ijk} = 1 \;\Big|\; Y_{i1} = 0, \dots, Y_{i,j-1} = 0,\ V_{ij1} = 0, \dots, V_{i,k-1} = 0\Big)\Big\}^{X_{ijk}}
\end{aligned}$$

With this result in mind, we now consider the Dunson and Stanford day-specific probabilities model. Using the same indexing scheme as above, define

$\boldsymbol{u}_{ijk}$	a covariate vector of length q for woman i, cycle j, day k
$\boldsymbol{\beta}$	a vector of length q of regression coefficients
ξ_i	woman-specific random effect

Then writing $\boldsymbol{U}_{ij} = \left(\boldsymbol{u}'_{ijk}, \ldots, \boldsymbol{u}'_{ijk}\right)'$, Dunson and Stanford propose the model:

$$\begin{aligned}
\mathbb{P}\left(Y_{ij} = 1 \mid \xi_i, \boldsymbol{X}_{ij}, \boldsymbol{U}_{ij}\right) = 1 - \prod_{k=1}^{K} (1 - \lambda_{ijk})^{X_{ijk}} \\
\lambda_{ijk} = 1 - \exp\left\{-\xi_i \exp\left(\boldsymbol{u}'_{ijk}\boldsymbol{\beta}\right)\right\} \\
\xi_i \sim \mathcal{G}(\phi, \phi)
\end{aligned} \tag{1}$$

From our previous derivation, we see that we may interpret λ_{ijk} as the day-specific probability of conception in cycle j from couple i given that conception has not already occured, or in the language of Dunson and Stanford, given intercourse only on day k.

Delving further, we see that λ_{ijk} is strictly increasing in $u_{ijkh}\beta_h$, where we are denoting u_{ijkh} to be the h^{th} term in $\boldsymbol{u}_{ijk}$ and similarly for β_h. When $\beta_h = 0$ then the h^{th} covariate has no effect on the day-specific probability of conception.

λ_{ijk} is also strictly increasing in ξ_i which as Dunson and Stanford suggest may be interpreted as a woman-specific random effect. The authors state that specifying the distribution of the ξ_i with a common parameters prevents nonidentifiability between $\mathbb{E}[\xi_i]$ and the day-specific parameters. Since $\text{Var}[\xi_i] = 1/\phi$ it follows that ϕ may be interpreted as a measure of variability across women.

1.1.1 Computation consideration

As an aside, we note that it may be more computationally convenient to calculate

$$\begin{aligned}
&\mathbb{P}\left(Y_{ij} = 1 \mid \xi_i, \boldsymbol{X}_{ij}, \boldsymbol{U}_{ij}\right) \\
&\quad = 1 - \prod_{k=1}^{K} (1 - \lambda_{ijk})^{X_{ijk}} \\
&\quad = 1 - \prod_{k=1}^{K} \left[\exp\left\{-\xi_i \exp\left(\boldsymbol{u}'_{ijk}\boldsymbol{\beta}\right)\right\} \right]^{X_{ijk}} \\
&\quad = 1 - \prod_{k=1}^{K} \exp\left\{-X_{ijk}\xi_i \exp\left(\boldsymbol{u}'_{ijk}\boldsymbol{\beta}\right)\right\}
\end{aligned}$$

1.2 Marginal probability of conception

The marginal probability of conception, obtained by integrating out the couple-specific frailty ξ_i, has form as follows.

$$\begin{aligned}
&\mathbb{P}(Y_{ij} = 1 \,|\, \boldsymbol{X}_{ij}, \boldsymbol{U}_{ij}) \\
&\quad = \int_0^\infty \mathbb{P}\left(Y_{ij}, \xi_i \,|\, \boldsymbol{X}_{ij}, \boldsymbol{U}_{ij}\right) d\xi_i \\
&\quad = \int_0^\infty \mathbb{P}\left(Y_{ij}, \xi_i \,|\, \boldsymbol{X}_{ij}, \boldsymbol{U}_{ij}\right) \mathcal{G}(\xi_i;\, \phi, \phi)\, d\xi_i \\
&\quad = \int_0^\infty \left[1 - \prod_{k=1}^{K} (1 - \lambda_{ijk})^{X_{ijk}}\right] \mathcal{G}(\xi_i;\, \phi, \phi)\, d\xi_i
\end{aligned}$$

$$
\begin{aligned}
&= 1 - \int_0^\infty \prod_{k=1}^K (1-\lambda_{ijk})^{X_{ijk}} \mathcal{G}(\xi_i;\, \phi, \phi)\, d\xi_i \\
&= 1 - \int_0^\infty \prod_{k=1}^K \Big[\exp\Big\{ -\xi_i \exp\Big(\boldsymbol{u}'_{ijk}\boldsymbol{\beta} \Big) \Big\} \Big]^{X_{ijk}} \mathcal{G}(\xi_i;\, \phi, \phi)\, d\xi_i \\
&= 1 - \int_0^\infty \prod_{k=1}^K \exp\Big\{ -\xi_i X_{ijk} \exp\Big(\boldsymbol{u}'_{ijk}\boldsymbol{\beta} \Big) \Big\} \mathcal{G}(\xi_i;\, \phi, \phi)\, d\xi_i \\
&= 1 - \int_0^\infty \exp\Big\{ -\xi_i \sum_{k=1}^K X_{ijk} \exp\Big(\boldsymbol{u}'_{ijk}\boldsymbol{\beta} \Big) \Big\} \mathcal{G}(\xi_i;\, \phi, \phi)\, d\xi_i \\
&= 1 - \left[\frac{\phi}{\phi + \sum_{k=1}^K X_{ijk} \exp\Big(\boldsymbol{u}'_{ijk}\boldsymbol{\beta} \Big)} \right]^\phi
\end{aligned}
$$

since

$$
\begin{aligned}
&\int_0^\infty \exp\Big\{ -\xi_i \sum_{k=1}^K X_{ijk} \exp\Big(\boldsymbol{u}'_{ijk}\boldsymbol{\beta} \Big) \Big\} \mathcal{G}(\xi_i;\, \phi, \phi)\, d\xi_i \\
&\quad = \int_0^\infty \exp\Big\{ -\xi_i \sum_{k=1}^K X_{ijk} \exp\Big(\boldsymbol{u}'_{ijk}\boldsymbol{\beta} \Big) \Big\} \frac{\phi^\phi}{\Gamma(\phi)}\, \xi_i^{\phi-1} d\xi_i \\
&\quad = \int_0^\infty \frac{\phi^\phi}{\Gamma(\phi)}\, \xi_i^{\phi-1} \exp\Big\{ -\xi_i \Big[\phi + \sum_{k=1}^K X_{ijk} \exp\Big(\boldsymbol{u}'_{ijk}\boldsymbol{\beta} \Big) \Big] \Big\} d\xi_i \\
&\quad = \left[\frac{\phi}{\phi + \sum_{k=1}^K X_{ijk} \exp\Big(\boldsymbol{u}'_{ijk}\boldsymbol{\beta} \Big)} \right]^\phi \int_0^\infty \frac{\Big[\phi + \sum_{k=1}^K X_{ijk} \exp\Big(\boldsymbol{u}'_{ijk}\boldsymbol{\beta} \Big) \Big]^\phi}{\Gamma(\phi)} \\
&\qquad\qquad \times \xi_i^{\phi-1} \exp\Big\{ -\xi_i \Big[\phi + \sum_{k=1}^K X_{ijk} \exp\Big(\boldsymbol{u}'_{ijk}\boldsymbol{\beta} \Big) \Big]^\phi \Big\} d\xi_i
\end{aligned}
$$

and the function inside the integral is a gamma density function.

1.2.1 Day-specific marginal probability of conception

Dunson and Stanford also point out the following remarkable result. The marginal day-specific probability of conception given that conception has not already occured is given by

$$
\mathbb{P}(Y = 1 \,|\, \boldsymbol{u}) = 1 - \left(\frac{\phi}{\phi + \exp(\boldsymbol{u}'\boldsymbol{\beta})} \right)^\phi
$$

which is in the form of the Aranda-Ordaz generalized linear model, and reduces to a logistic regression model for $\phi = 1$.

1.3 Prior specification

**** needs completed

2 Posterior computation

Express the data augmentation model as

$$
\begin{aligned}
&Y_{ij} = I\left(\sum_{k=1}^{K} X_{ijk} Z_{ijk} > 0\right), \\
&Z_{ijk} \sim \text{Poisson}\Big(\xi_i \exp\Big(\boldsymbol{u}'_{ijk}\boldsymbol{\beta}\Big)\Big), \qquad k = 1, \ldots, K
\end{aligned} \tag{2}
$$

Let us further define $W_{ijk} = X_{ijk} Z_{ijk}$ for all i, j, k.

2.1 Verifying the equivalence of the data augmentation model

Need to do

2.2 The full conditional distributions

Step 1. Writing $\boldsymbol{W}_{ij} = \big(W_{ij1}, \ldots, W_{ijK}\big)$ and letting $\boldsymbol{m} = \big(m_1, \ldots, m_K\big)$ be a vector of outcomes for $\boldsymbol{W}_{ij}$, we see first that for $Y_{ij} = 0$ we have

$$
\mathbb{P}\Big(\boldsymbol{W}_{ij} = \boldsymbol{m} \;\Big|\; Y_{ij} = 0, \boldsymbol{\beta}, \phi, \boldsymbol{\xi}, \text{data}\Big) \;=\; \begin{cases} 1, & \boldsymbol{m} = \boldsymbol{0} \\ 0, & \text{else} \end{cases}
$$

Next, for $Y_{ij} = 1$ we have

$$
\begin{aligned}
&\mathbb{P}\Big(\boldsymbol{W}_{ij} = \boldsymbol{m} \;\Big|\; Y_{ij} = 1, \boldsymbol{\beta}, \phi, \boldsymbol{\xi}, \text{data}\Big) \\
&\quad = \sum_{s=0}^{\infty} \mathbb{P}\Big(\boldsymbol{W}_{ij} = \boldsymbol{m},\ \textstyle\sum_k W_{ijk} = s \;\Big|\; Y_{ij} = 1, \boldsymbol{\beta}, \phi, \boldsymbol{\xi}, \text{data}\Big) \\
&\quad = \mathbb{P}\Big(\boldsymbol{W}_{ij} = \boldsymbol{m},\ \textstyle\sum_k W_{ijk} = \sum_k m_k \;\Big|\; Y_{ij} = 1, \boldsymbol{\beta}, \phi, \boldsymbol{\xi}, \text{data}\Big) \\
&\quad = \mathbb{P}\Big(\boldsymbol{W}_{ij} = \boldsymbol{m} \;\Big|\; \textstyle\sum_k W_{ijk} = \sum_k m_k,\ Y_{ij} = 1, \boldsymbol{\beta}, \phi, \boldsymbol{\xi}, \text{data}\Big) \\
&\qquad\qquad \times \mathbb{P}\Big(\textstyle\sum_k W_{ijk} = \sum_k m_k \;\Big|\; Y_{ij} = 1, \boldsymbol{\beta}, \phi, \boldsymbol{\xi}, \text{data}\Big)
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\pi&\left(\sum_{k=1}^{K} W_{ijk} \,\middle|\, Y_{ij}=1,\boldsymbol{\beta},\phi,\boldsymbol{\xi},\text{data}\right)\\
&= \pi\left(\sum_{k=1}^{K} W_{ijk} \,\middle|\, \sum_{k=1}^{K} W_{ijk}\geq 1,\boldsymbol{\beta},\phi,\boldsymbol{\xi},\text{data}\right)\\
&\sim \text{Poisson}\left(\xi_i \sum_{k:\, X_{ijk}=1} \exp\left(\boldsymbol{u}'_{ijk}\boldsymbol{\beta}\right)\right) \text{ truncated so that } \sum_{k=1}^{K} W_{ijk}\geq 1
\end{aligned}
$$

and

$$
\begin{aligned}
\pi&\left(\boldsymbol{W}_{ij} \,\middle|\, \sum_{k=1}^{K} W_{ijk},\ Y_{ij}=1,\boldsymbol{\beta},\phi,\boldsymbol{\xi},\text{data}\right)\\
&\sim \text{Multinomial}\left(\sum_{k=1}^{K} W_{ijk};\ \frac{X_{ij1}\xi_i\exp\left(\boldsymbol{u}'_{ij1}\boldsymbol{\beta}\right)}{\xi_i\sum_{k:\, X_{ijk}=1}\exp\left(\boldsymbol{u}'_{ijk}\boldsymbol{\beta}\right)},\ \ldots,\ \frac{X_{ijK}\xi_i\exp\left(\boldsymbol{u}'_{ijK}\boldsymbol{\beta}\right)}{\xi_i\sum_{k:\, X_{ijk}=1}\exp\left(\boldsymbol{u}'_{ijk}\boldsymbol{\beta}\right)}\right)
\end{aligned}
$$

Step 2. Define the following terms which will be of use in the following derivation. Denote

$$
\begin{array}{ll}
\widetilde{a}_h & a_h+\sum_{i,j,k} u_{ijkh}W_{ijk}\\
\widetilde{b}_h & b_h+\displaystyle\sum_{i,j,k:X_{ijk}=1}\xi_i\prod_{\ell\neq h}\gamma_\ell^{u_{ijk\ell}}\\
c_1 & p_h\exp\left\{-(\widetilde{b}_h-b_h)\right\}\\
c_2 & (1-p_h)\dfrac{C(a_h,b_h)\int_{\mathcal{A}_h}\mathcal{G}(\gamma;\,\widetilde{a}_h,\widetilde{b}_h)\,d\gamma}{C(\widetilde{a}_h,\widetilde{b}_h)\int_{\mathcal{A}_h}\mathcal{G}(\gamma;\,a_h,b_h)\,d\gamma}\\
\widetilde{p}_h & \dfrac{c_1}{c_1+c_2}
\end{array}
$$

Then

$$
\begin{aligned}
\pi&\left(\gamma_h\,|\,\boldsymbol{\gamma}_{(-h)},\phi,\boldsymbol{\xi},\boldsymbol{W},\text{data}\right)\\
&\propto \pi\left(\boldsymbol{W}\,|\,\boldsymbol{\xi},\gamma,\text{data}\right)\pi\left(\gamma_h\right)\\
&= \left(\prod_{i=1}^{n}\prod_{j=1}^{n_i}\prod_{k:X_{ijk}=1}\pi\left(W_{ijk}\,|\,\boldsymbol{\xi}_i,\gamma,\text{data}\right)\right)\pi\left(\gamma_h\right)\\
&\propto \left(\prod_{i=1}^{n}\prod_{j=1}^{n_i}\prod_{k:X_{ijk}=1}\left[\exp\left(u_{ijkh}\log\gamma_h\right)\right]^{W_{ijk}}\exp\left\{-\xi_i\exp\left(\sum_{\ell=1}^{q}u_{ijk\ell}\log\gamma_\ell\right)\right\}\right)\pi\left(\gamma_h\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\prod_{i=1}^{n} \prod_{j=1}^{n_i} \prod_{k: X_{ijk}=1} \gamma_h^{u_{ijkh} W_{ijk}} \exp\left\{ -\xi_i \prod_{\ell=1}^{q} \gamma_\ell^{u_{ijk\ell}} \right\} \right) \pi(\gamma_h) \\
&= \gamma_h^{\sum_{i,j,k} u_{ijkh} W_{ijk}} \exp\left\{ - \sum_{i,j,k: X_{ijk}=1} \xi_i \prod_{\ell=1}^{q} \gamma_\ell^{u_{ijk\ell}} \right\} \pi(\gamma_h) \\
&= \gamma_h^{\sum_{i,j,k} u_{ijkh} W_{ijk}} \exp\left\{ - \sum_{i,j,k: X_{ijk}=1} \xi_i \prod_{\ell=1}^{q} \gamma_\ell^{u_{ijk\ell}} \right\} \\
&\qquad\qquad \times \left[p_h I\big(\gamma_h = 1\big) \,+\, (1-p_h) I\big(\gamma_h \neq 1\big) \mathcal{G}_{\mathcal{A}_h}(\gamma_h; a_h, b_h) \right] \\
&= p_h\, I\big(\gamma_h = 1\big) \exp\left\{ - \sum_{i,j,k: X_{ijk}=1} \xi_i \prod_{\ell \neq h} \gamma_\ell^{u_{ijk\ell}} \right\} \\
&\qquad + (1-p_h) I\big(\gamma_h \neq 1\big) \gamma_h^{\sum_{i,j,k} u_{ijkh} W_{ijk}} \exp\left\{ - \sum_{i,j,k: X_{ijk}=1} \xi_i \prod_{\ell=1}^{q} \gamma_\ell^{u_{ijk\ell}} \right\} \mathcal{G}_{\mathcal{A}_h}(\gamma_h; a_h, b_h) \\
&= p_h\, I\big(\gamma_h = 1\big) \exp\left\{ - \sum_{i,j,k: X_{ijk}=1} \xi_i \prod_{\ell \neq h} \gamma_\ell^{u_{ijk\ell}} \right\} \\
&\qquad + (1-p_h) I\big(\gamma_h \neq 1\big) \frac{C(a_h, b_h)}{\int_{\mathcal{A}_h} \mathcal{G}(\gamma; a_h, b_h)\, d\gamma} \gamma_h^{a_h + \sum_{i,j,k} u_{ijkh} W_{ijk} - 1} \\
&\qquad \times \exp\left\{ -\gamma_h \left[b_h + \sum_{i,j,k: X_{ijk}=1} \xi_i \prod_{\ell \neq h} \gamma_\ell^{u_{ijk\ell}} \right] \right\} \\
&= p_h\, I\big(\gamma_h = 1\big) \exp\{ -(\widetilde{b}_h - b_h) \} \,+\, (1-p_h) I\big(\gamma_h \neq 1\big) \frac{C(a_h, b_h)}{\int_{\mathcal{A}_h} \mathcal{G}(\gamma; a_h, b_h)\, d\gamma} \gamma_h^{\widetilde{a}_h - 1} \exp\{ -\widetilde{b}_h \gamma_h \} \\
&= p_h\, I\big(\gamma_h = 1\big) \exp\{ -(\widetilde{b}_h - b_h) \} \,+\, (1-p_h) I\big(\gamma_h \neq 1\big) \\
&\qquad \times \frac{C(a_h, b_h) \int_{\mathcal{A}_h} \mathcal{G}(\gamma; \widetilde{a}_h, \widetilde{b}_h)\, d\gamma}{C(\widetilde{a}_h, \widetilde{b}_h) \int_{\mathcal{A}_h} \mathcal{G}(\gamma; a_h, b_h)\, d\gamma} \frac{C(\widetilde{a}_h, \widetilde{b}_h)}{\int_{\mathcal{A}_h} \mathcal{G}(\gamma; \widetilde{a}_h, \widetilde{b}_h)\, d\gamma} \gamma_h^{\widetilde{a}_h - 1} \exp\{ -\widetilde{b}_h \gamma_h \} \\
&= p_h\, I\big(\gamma_h = 1\big) \exp\{ -(\widetilde{b}_h - b_h) \} \,+\, (1-p_h) I\big(\gamma_h \neq 1\big) \frac{C(a_h, b_h) \int_{\mathcal{A}_h} \mathcal{G}(\gamma; \widetilde{a}_h, \widetilde{b}_h)\, d\gamma}{C(\widetilde{a}_h, \widetilde{b}_h) \int_{\mathcal{A}_h} \mathcal{G}(\gamma; a_h, b_h)\, d\gamma} \mathcal{G}_{\mathcal{A}_h}(\gamma; \widetilde{a}_h, \widetilde{b}_h) \\
&= c_1 I\big(\gamma_h = 1\big) \,+\, c_2 I\big(\gamma_h \neq 1\big) \mathcal{G}_{\mathcal{A}_h}(\gamma; \widetilde{a}_h, \widetilde{b}_h) \\
&\propto \frac{c_1}{c_1 + c_2} I\big(\gamma_h = 1\big) \,+\, \frac{c_2}{c_1 + c_2} I\big(\gamma_h \neq 1\big) \mathcal{G}_{\mathcal{A}_h}(\gamma; \widetilde{a}_h, \widetilde{b}_h) \\
&= \widetilde{p}_h I\big(\gamma_h = 1\big) \,+\, (1 - \widetilde{p}_h) I\big(\gamma_h \neq 1\big) \mathcal{G}_{\mathcal{A}_h}(\gamma; \widetilde{a}_h, \widetilde{b}_h)
\end{aligned}
$$