Exo mq si $\int_0^1 v \varphi \, dx = 0 \quad \forall \varphi \in C_c^1$

et $v \in L^1_{loc}(0,1)$

$\Rightarrow \quad v = 0$

Rmq $v \in L^p_{loc} \longrightarrow [v]$ classe d'equivalence

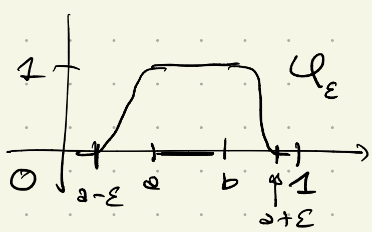
v n'est pas défine en tout points

$\rightsquigarrow \quad v = 0 \iff v(x) = 0$ pour p.t. $x \in [0,1]$

$\to$ En effet $v(x) = \begin{cases} 0 & \text{si } x \neq 1/2 \\ 1 & \text{si } x = 1/2 \end{cases}$

t.q. $\int_0^1 v \varphi \, dx = 0 \quad \forall \varphi \in C_c^1$

Solution



$0 < a < b < 1 \quad \forall \varepsilon > 0$

On a donc $\exists \varphi_\varepsilon \in C_c^\infty(0,1)$

t.q. $\varphi_\varepsilon(x) = 1$ si $x \in (a,b)$

$\varphi(x) = 0$ si $x \notin (a-\varepsilon, b+\varepsilon)$

$\varphi_\varepsilon(x) \in [0,1]$

$$0 = \int_0^1 v \varphi_\varepsilon \, dx = \int_a^b v \, dx + \int_{a-\varepsilon}^{a} v \varphi_\varepsilon \, dx + \int_b^{b+\varepsilon} v \varphi_\varepsilon \, dx$$

$\varepsilon \to 0$, par converg. dominée

$$\int_a^b v \, dx = 0 \quad \forall \, 0 < a < b < 1$$

mq $\quad \lim_{\varepsilon \downarrow 0} \int_{a-\varepsilon}^{a} v \cdot \varphi_\varepsilon \, dx = 0$

$$\int_{a-\varepsilon}^{a} v \, \varphi_\varepsilon \, dx = \int_{0}^{a} v(x) \underbrace{\chi_{(a-\varepsilon, a)}(x) \; \varphi_\varepsilon(x)} \, dx$$

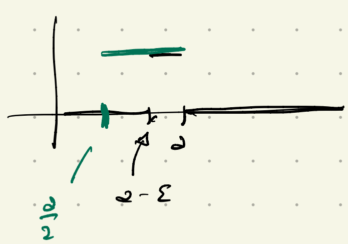
si $x \neq a$, $\quad \lim_{\varepsilon \downarrow 0} \; v(x) \underbrace{\chi_{(a-\varepsilon, a)}}(x) \, \varphi_\varepsilon(x) = 0$

en effet $\quad \chi_{(a-\varepsilon, a)}(x) = 0 \quad \forall \varepsilon < |a - x|$

Pour appliquer conv dominé

$$|v(x) \, \underbrace{\chi_{(a-\varepsilon, a)}}(x) \, \varphi_\varepsilon(x)| \leq |v(x)| \cdot \chi_{(a/2, a)}(x) \cdot \overbrace{|\varphi_\varepsilon(x)|}^{\leq 1}$$

$$\leq |v(x)| \, \chi_{(a/2, a)}(x) \quad \in L^1(0, a)$$



car $\quad v \in L^1_{loc}(0,1)$

et $\quad (a/2, a) \subset\subset (0,1)$

Conv. dominée $\Longrightarrow \quad \int_{a-\varepsilon}^{a} v(x) \, \varphi_\varepsilon(x) \, dx \xrightarrow[\varepsilon \downarrow 0]{} 0$

Thm de Lesbegue $\quad$ Pour p.t. $x \in (0,1)$

$$v(x) = \lim_{r \downarrow 0} \frac{1}{2r} \underbrace{\int_{x-r}^{x+r} v \, dy}_{=0} = 0$$

$\Longrightarrow \quad v(x) = 0 \quad$ p.t $\quad x \in (0,1)$.

Dérivé faible

•) Si $u \in L^1_{loc}(\Omega)$, $v \in L^1_{loc}(\Omega)$ est $\partial_{x_j} u$ ssi

$$\int_\Omega v \varphi = - \int_\Omega u \, \partial_{x_j} \varphi \quad \forall \varphi \in C^1_c$$

•) Si $u \in L^1_{loc}(\Omega)$, $w \in L^1_{loc}(\Omega, \mathbb{R}^d)$ est ∇u ssi

$$\int_\Omega w \cdot \varphi = - \int u \, \operatorname{div} \varphi \quad \forall \varphi \in C^1_c(\Omega, \mathbb{R}^d)$$

<u>Rmq</u> On a mq si la dérivé faible existe elle est <u>unique</u>

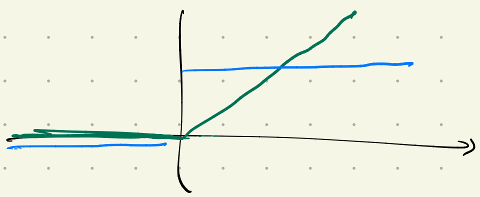
Si $u \in C^1(\Omega)$, l'IPP $\Rightarrow$ dérivé faible de u coincide avec la dérivé <u>forte</u>

<u>Exemples</u>

① $f(x) = \max(0, x)$ $\quad f : \mathbb{R} \longrightarrow \mathbb{R}$

f dérivable "fortement" si $x \neq 0$

$$f'(x) = \begin{cases} 0 & \text{si } x < 0 \\ 1 & \text{si } x > 0 \end{cases}$$



⇝ dérivé faible de f est

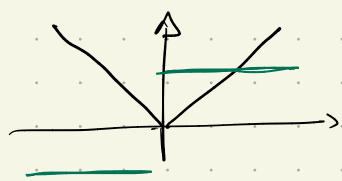
$$H(x) = \begin{cases} 0 & \text{si } x < 0 \\ 1 & \text{si } x \geq 0 \end{cases}, \quad H \in L^1_{loc}(\mathbb{R})$$

On le vérifie : $\varphi \in C^1_c(\mathbb{R})$

$$\int_{\mathbb{R}} H(x)\varphi(x)\,dx = \int_0^{\infty} 1 \cdot \varphi(x)\,dx = -\int_0^{\infty} x\,\varphi'(x)\,dx$$

$$= -\int_{\mathbb{R}} f(x)\,\varphi'(x)\,dx .$$

(2) $f(x) = |x| \longrightarrow$ dérivé faible est $\operatorname{sgn}(x)$



$$\operatorname{sgn}(x) = \chi_{(0,+\infty)}(x) - \chi_{(-\infty,0)}(x)$$

$\operatorname{sgn} \in L^1_{loc}$

(3) $H \in L^1_{loc}$ admet-elle une dérivé faible v ?

$$\int_{\mathbb{R}} v\,\varphi\,dx = -\int_{\mathbb{R}} H(x)\,\varphi'(x)\,dx$$

Or : $$-\int_{\mathbb{R}} H(x)\,\varphi'(x)\,dx = -\int_0^{\infty} 1 \cdot \varphi'(x)\,dx \overset{IPP}{=} \underbrace{\cancel{\int_0^{\infty} \frac{d}{dx}(1) \cdot \varphi(x)\,dx}}_{=0} - \varphi(x)\Big|_0^{\infty}$$

$$= \underbrace{-\varphi(\infty)}_{=0 \text{ car } \varphi \in C^1_c} + \varphi(0) \qquad = \varphi(0)$$

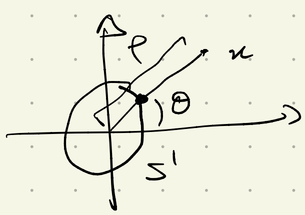
Si v dérivé faible : $\displaystyle\int_{\mathbb{R}} v \varphi \, dx = \varphi(0) \quad \forall \varphi \in C^1_c$

$$\Rightarrow \quad v = \delta_0 \notin L^1_{loc}$$

Donc H n'admet pas de dérivés faibles.

④ $\mathbb{R}^d$, $\quad u(x) = |x|^s \quad s \in \mathbb{R}$

•) Quand est-ce que $u \in L^1_{loc}(\mathbb{R}^d)$

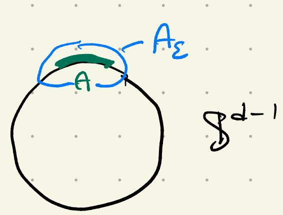


$x \in \mathbb{R}^d \rightsquigarrow \rho\omega$ où $\underset{\mathbb{R}_+}{\rho \in} = |x|$, $\omega = \dfrac{x}{|x|} \in \mathbb{S}^{d-1}$

$$|\omega| = \left|\frac{x}{|x|}\right| = 1$$

$$\mathbb{S}^{d-1} = \{ y \in \mathbb{R}^d \,/\, |y| = 1 \} \subset \mathbb{R}^d$$

On note σ_S la mesure sur $\mathbb{S}^{d-1}$

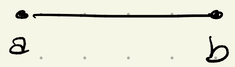


$A \subset \mathbb{S}^{d-1}$

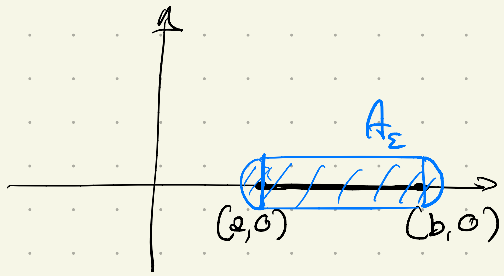
$A_\varepsilon = A + B_\varepsilon \subset \mathbb{R}^d$

$$\sigma_S(A) = \lim_{\varepsilon \downarrow 0} \frac{1}{(2\varepsilon)^{d-1}} \mathcal{L}^d(A_\varepsilon)$$

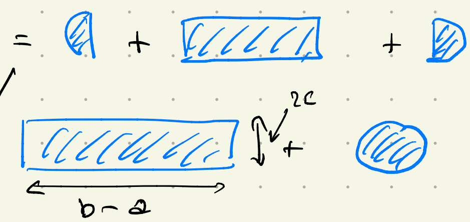
C'est le même chose que pour mesurer la longueur de :



$$A = [(a,0),(b,0)]$$



$$\mathcal{L}^2(A_\varepsilon) =$$



$$= 2\varepsilon\cdot(b-a) + \pi\varepsilon^2$$

$d=2$

$$\frac{1}{2\varepsilon}\mathcal{L}^2(A_\varepsilon) = b-a+\frac{\pi}{2}\cdot\varepsilon$$

$$\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\mathcal{L}^2(A_\varepsilon) = b-a = \text{longueur de } A$$

$\rightsquigarrow$

$$\int_{\mathbb{R}^d} f(x)\,dx = \int_0^\infty \int_{S^{d-1}} f(\rho\omega)\,d\sigma(\omega)\,\rho^{d-1}\,d\rho$$

$\boxed{\text{exo}}$

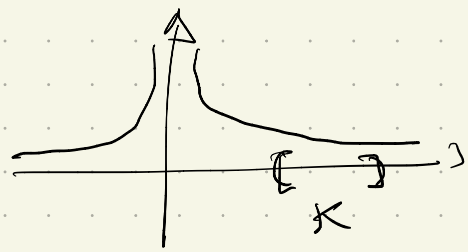
$$\int_{\mathbb{R}^d} f(g(y))\,dy = \int_{\mathbb{R}^d} f(x)\,|Jg|\,dx$$

$u \in L^1_{loc}(\mathbb{R}^d)$ $\qquad$ $K \subset\subset \mathbb{R}^d$

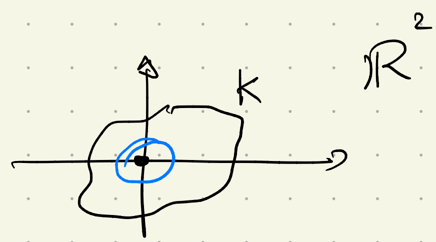
$$\int_K |u|\, dx \quad < +\infty \quad ? \qquad \forall K$$

(1) $0 \notin K$

$$\int_K |x|^s\, dx \le \underbrace{\max_{x\in K} |x|^s \cdot \mathcal{L}^d(K)}_{< +\infty}$$



(2) $0 \in K$



$$\int_K |x|^s\, dx = \underbrace{\int_{B_\varepsilon} |x|^s\, dx}_{} + \underbrace{\int_{K\setminus B_\varepsilon} |x|^s\, dx}_{< +\infty}$$

$$\int_{B_\varepsilon} |x|^s\, dx = \int_0^\varepsilon \int_{S^{d-1}} |\rho\omega|^s\, d\sigma_S(\omega)\, \rho^{d-1}\, d\rho$$

$$B_\varepsilon = \{\rho\omega \mid \rho < \varepsilon\}$$

$$|\rho\omega|^s = \rho^s |\omega|^s = \rho$$

$$= \int_0^\varepsilon \rho^s \int_{S^{d-1}} \cancel{\rho^s}\, d\sigma_S(\omega)\, \rho^{d-1}\, d\rho$$

$$= |S^{d-1}| \int_0^\varepsilon \rho^{s+d-1}\, d\rho$$

En général $$\int_0^{\varepsilon} t^{\alpha}\,dt = \begin{cases} \log t \Big|_0^{\varepsilon} = +\infty, & \alpha = -1 \\ \dfrac{t^{\alpha+1}}{\alpha+1}\Big|_0^{\varepsilon} & \alpha \neq -1 \end{cases}$$

$$\rightsquigarrow \int_0^{\varepsilon} t^{\alpha}\,dt < +\infty \quad \text{ssi} \quad \alpha > -1$$

$$\int_{B_\varepsilon} |x|^s\,dx < +\infty \iff s > -d$$

<u>Donc</u> $u \in L^1_{loc}(\mathbb{R}^d)$ ssi $s > -d$

•) $x \neq 0$, $$\nabla u(x) = \nabla(|x|^s) = s\,|x|^{s-1}\,\nabla(|x|) = s\,|x|^{s-2}\,x$$

$$\nabla(|x|) = \frac{x}{|x|}$$

"Si le grad faible existe il est $s|x|^{s-2}x$"

•) $v(x) = s|x|^{s-2}x$ $\quad v \in L^1_{loc}(\mathbb{R}^d)$?

$$|v| = s\,|x|^{s-1} \implies v \in L^1_{loc}(\mathbb{R}^d)$$

$$s-1 > -d$$
$$\Updownarrow$$
$$s > -d+1$$

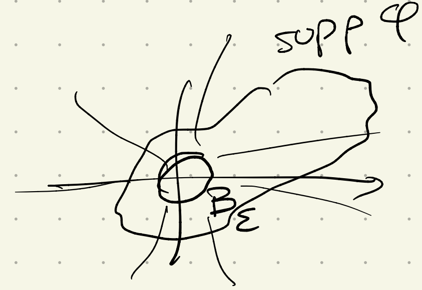
•) Soit $s > -d+1$ mq v est le grad faible de u.

$\underline{\text{mq}}$ $$\int_{\mathbb{R}^d} v(x)\cdot\varphi(x)\,dx = -\int_{\mathbb{R}^d} |x|^s \operatorname{div}\varphi(x)\,dx$$

$\forall \varphi \in C_c^1(\mathbb{R}^d, \mathbb{R}^d)$

Soit $\varepsilon > 0$,

$$\int_{\mathbb{R}^d \setminus B_\varepsilon} s|x|^{s-2} x \cdot \varphi(x)\,dx$$



$$= \int_{\mathbb{R}^d \setminus B_\varepsilon} \nabla u(x)\cdot\varphi(x)\,dx$$

thm divergence

$$= -\int_{\mathbb{R}^d \setminus B_\varepsilon} u(x)\operatorname{div}\varphi(x)\,dx + \underbrace{\int_{\partial B_\varepsilon} (u\varphi)\cdot n\, d\sigma_{\partial B_\varepsilon}}$$

$$\int_{\partial B_\varepsilon} (u\varphi)\cdot n\, d\sigma_{\partial B_\varepsilon} = \varepsilon^s \int_{\partial B_\varepsilon} \varphi(x)\cdot n\, d\sigma_{\partial B_\varepsilon}(x)$$

$u|_{\partial B_\varepsilon} = \varepsilon^s$ car $\partial B_\varepsilon = \{x \mid |x| = \varepsilon\}$

$\varepsilon y = x$ si $x \in \partial B_\varepsilon$
$\Rightarrow y \in \mathbb{S}^{d-1} = \partial B_1$

$$d\sigma_{\partial B_\varepsilon}(\varepsilon y) = \varepsilon^{d-1} d\sigma_S(y)$$

$$= \varepsilon^s \int_{\mathbb{S}^{d-1}} \varphi(\varepsilon y)\cdot n \ \varepsilon^{d-1} d\sigma_S(y)$$

$$= \varepsilon^{s+d-1} \int_{\mathbb{S}^{d-1}} \underbrace{\varphi(\varepsilon y)\cdot n}_{\leq |\varphi(\varepsilon y)|\cdot|n| = |\varphi(\varepsilon y)|} d\sigma_S(y)$$

$$\Rightarrow \left| \int_{\partial B_\varepsilon} (u\varphi)\cdot n \, d\sigma_{\partial B_\varepsilon} \right| \leq \underbrace{\varepsilon^{s+d-1}}\cdot \|\varphi\|_\infty |\mathbb{S}^{d-1}|$$

donc comme $s > -d+1$

$$\lim_{\varepsilon\downarrow 0} \int_{\partial B_\varepsilon} (u\varphi)\cdot n \, d\sigma_{\partial B_\varepsilon} = 0$$

On a donc

$$\lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^d\setminus B_\varepsilon} v(x)\cdot\varphi(x)\,dx = -\lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^d\setminus B_\varepsilon} u(x)\,\mathrm{div}\,\varphi(x)\,dx$$

Comme $|v\cdot\varphi| \leq \|\varphi\|_\infty |v| \Rightarrow |v\cdot\varphi| \in L^1(\mathbb{R}^d)$
et φ supp. cpt

$|u\,\mathrm{div}\,\varphi| \leq \|\mathrm{div}\,\varphi\|_\infty |u| \Rightarrow |u\,\mathrm{div}\,\varphi| \in L^1(\mathbb{R}^d)$

Donc par convergence dominée on a

$$\int_{\mathbb{R}^d} v(x) \cdot \varphi(x)\, dx = -\int_{\mathbb{R}^d} u(x) \operatorname{div} \varphi(x)\, dx$$

$\Rightarrow$ v gradient faible de u ! $\quad (s > -d+1)$

<u>Proposition</u> $\quad u \in L^1_{loc}$ faiblement dérivable
$\varphi \in C^1_c(\mathbb{R}^d)$

$\Rightarrow$ φu est faib. dér et on a

$$\nabla(\varphi u) = u \nabla\varphi + \varphi \nabla u$$

Formule de Leibniz

<u>Rmq</u> $\quad u, v \in L^1_{loc}$ faib. dérivables

$u \cdot v \notin L^1_{loc}$ en général

sur $\mathbb{R}$ $\quad u(x) = |x|^{-1/2} \in L^1_{loc}$ $\quad -\frac{1}{2} > -1$

$v(x) = |x|^{-1/2} \in L^1_{loc}$

mais $\quad u(x) v(x) = |x|^{-1} \notin L^1_{loc}(\mathbb{R})$

<u>Preuve</u> $\quad \varphi u \in L^1(\mathbb{R}^d)$ car φu a supp. cpt
et $|\varphi u| \leq \|\varphi\|_\infty \cdot |u|$.

$$\int_{\mathbb{R}^d} |\varphi u|\, dx = \int_{\operatorname{supp}\varphi} |\varphi u|\, dx \leq \|\varphi\|_\infty \int_{\operatorname{supp}\varphi} |u|\, dx \underset{u \in L^1_{loc}}{<} +\infty$$

$\nabla\varphi \in C_c^0(\mathbb{R}^d, \mathbb{R}^d)$ donc

$\nabla(\varphi u),\ \nabla\varphi \cdot u,\ \varphi\nabla u \in L^1_{loc}$

mq $\nabla(\varphi u) = u\nabla\varphi + \varphi\nabla u$ faiblement

$\psi \in C_c^1(\mathbb{R}^d, \mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \psi \cdot (\underset{\uparrow \in C_c^1(\mathbb{R}^d)}{\varphi} \nabla u) = \int_{\mathbb{R}^d} (\overbrace{\varphi\psi}^{\in C_c^1(\mathbb{R}^d, \mathbb{R}^d)}) \cdot \nabla u$$

déf der. faible

$$= -\int_{\mathbb{R}^d} u \operatorname{div}(\varphi\psi)$$

$$\operatorname{div}(\varphi\psi) = \nabla\varphi \cdot \psi + \varphi \operatorname{div}(\psi)$$

$$= -\int_{\mathbb{R}^d} (u\nabla\varphi \cdot \psi + u\varphi \operatorname{div}\psi)$$

$$\Rightarrow \int_{\mathbb{R}^d} \psi \cdot (\varphi\nabla u + u\nabla\varphi) = -\int u\varphi \operatorname{div}\psi = \int \nabla(u\varphi)\cdot\psi$$

$\forall \psi \in C_c^1(\mathbb{R}^d, \mathbb{R}^d)$

$$\Rightarrow \nabla(u\varphi) = \varphi\nabla u + u\nabla\varphi$$

$\square$

ESPACES DE SOBOLEV $\quad \mathbb{R}^d$

Notation

multi-indice $\quad \alpha \in \mathbb{N}^d \qquad \alpha = (\alpha_1, \dots, \alpha_d)$

$$|\alpha| = \alpha_1 + \dots + \alpha_d \qquad \alpha! = (\alpha_1!) \dots (\alpha_d!)$$

$\alpha, \beta \in \mathbb{N}^d \qquad \alpha \leq \beta$ si $\alpha_j \leq \beta_j \quad \forall j \in \{1, \dots, d\}$

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!\,\gamma!} \quad \text{où} \quad \alpha = \beta + \gamma$$

$$x \in \mathbb{R}^d \qquad x^\alpha = (x_1^{\alpha_1}) \dots (x_d^{\alpha_d})$$

$$\partial^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d}$$

Lemme $\quad f, g : \Omega \subset \mathbb{R}^d \longrightarrow \mathbb{R}$

$$\partial^\alpha(fg) = \sum_{\beta,\, \beta \leq \alpha} \binom{\alpha}{\beta} (\partial^\beta f)\, \partial^{\alpha-\beta} g$$

Formule de Leibniz

$$f(x+h) = \sum_{\alpha,\, |\alpha| < m} \frac{1}{\alpha!} h^\alpha \partial^\alpha f(x) + \sum_{\alpha,\, |\alpha| = m} \frac{h^\alpha}{\alpha!} \int_0^1 \partial^\alpha f(x+th) \cdot m(1-t)^{m-1}\, dt$$

Formule de Taylor

Ex $\quad f : \mathbb{R}^2 \longrightarrow \mathbb{R} \qquad m = 2$

$|\alpha| = 0 \longrightarrow \alpha = (0,0)$

$|\alpha| = 1 \longrightarrow \alpha = (1,0)$, $\alpha = (0,1)$

$$f(x+h_1, y+h_2) = \overbrace{f(x,y)}^{|\alpha|=0} + \overbrace{h_1 \partial_x f(x,y) + h_2 \partial_y f(x,y)}^{|\alpha|=1} + \underbrace{\frac{1}{2}\left[\underbrace{h_1^2}_{\alpha=(2,0)} \partial_x^2 f(x,y) + \underbrace{h_1 h_2}_{\alpha=(1,1)} \partial_x \partial_y f(x,y) + \underbrace{h_2^2}_{\alpha=(0,2)} \partial_y^2 f(x,y)\right]}_{|\alpha|=2}$$

$|\alpha| = 2$
$\alpha = (1,1)$
$\alpha = (2,0)$
$\alpha = (0,2)$

$+ \dots$ (reste intégral)

Déf L'espace de Sobolev $W^{m,p}(\Omega)$ où $m \in \mathbb{N}$, $p \in [1,\infty]$ est l'espace des fcts $u \in L^p(\Omega)$ tel que

$$\underbrace{\partial^\alpha u}_{\text{dérivés faibles}} \in L^p(\Omega) \quad \forall \alpha \in \mathbb{N}^d, \ |\alpha| \le m.$$

équipé avec la norme :

$$\|u\|_{W^{m,p}(\Omega)} := \Big(\sum_{\alpha, |\alpha| \le m} \|\partial^\alpha u\|_p^p \Big)^{1/p}$$

Rmq •) On pourrait aussi choisir

$$\sum_{|\alpha| \le m} \|\partial^\alpha u\|_p \quad \text{ou} \quad \max_{|\alpha| \le m} \|\partial^\alpha u\|_p$$

sont equivalents à $\|\cdot\|_{W^{m,p}(\Omega)}$

Exo : mq $L^p(\Omega) \subset L^1_{loc}(\Omega)$

Exo : mq $\|\cdot\|_{W^{m,p}(\Omega)}$ est une norme

•) Si $m \ge m'$ alors $W^{m,p}(\Omega) \subset W^{m',p}(\Omega)$

•) $u \in W^{m+1,p}(\Omega) \iff u, \partial_1 u, \dots, \partial_d u \in W^{m,p}(\Omega)$

Proposition $\left(W^{m,p}(\Omega), \|\cdot\|_{W^{m,p}(\Omega)} \right)$ est un espace de Banach

Preuve $\boxed{m=1}$ $W^{1,p}(\Omega)$ est complet

$$W^{1,p}(\Omega) = \{ u \in L^p(\Omega) \, / \, \partial_j u \in L^p(\Omega) \ \forall j \in \{1, \dots, d\} \}$$

$$\Phi : W^{1,p}(\Omega) \longrightarrow \underbrace{L^p(\Omega; \mathbb{R}^{d+1})}_{\text{Banach}}$$

Si $\exists \Phi$ linéaire, continue, injective t.q. $\Phi(W^{1,p}(\Omega))$ fermé

$\rightsquigarrow$ Φ est un isomorphisme sur son image

Son image $\underbrace{\Phi(W^{1,p}(\Omega))}_{\substack{\text{fermé} \\ \Rightarrow \text{ Banach}}} \subset \underbrace{L^p(\Omega; \mathbb{R}^{d+1})}_{\text{Banach}}$

$$\boxed{\begin{array}{l} \phi : X \to Y \\ \text{isomorphisme} \\ \text{et } Y \text{ Banach} \\ \Rightarrow X \text{ Banach} \end{array}}$$

$\Rightarrow$ $W^{1,p}(\Omega)$ Banach

Posons : $\Phi : u \in W^{1,p}(\Omega) \longmapsto (u, \partial_1 u, \partial_2 u, \dots, \partial_d u) \in L^p(\Omega, \mathbb{R}^{d+1})$

•) Φ linéaire (car $\partial_j(u+v) = \partial_j u + \partial_j v$)

•) Φ injective $\quad \ker \Phi = \{0\}$ triviale

•) Φ continue

$$\|\Phi(u)\|_{L^p(\Omega, \mathbb{R}^{d+1})} = \left(\sum_{j=1}^{d+1} \|\Phi(u)_j\|_p^p \right)^{1/p} = \left(\sum_{j=1}^{d} \|\partial_j u\|_p^p + \|u\|_p^p \right)^{1/p} = \|u\|_{W^{1,p}(\Omega)}$$

•) mq $\Phi(W^{1,p}(\Omega))$ fermé dans $L^p(\Omega, \mathbb{R}^{d+1})$

Soit $(v_k)_k \subset \Phi(W^{1,p}(\Omega))$

t.q. $v_k \longrightarrow v \in L^p(\Omega, \mathbb{R}^{d+1})$

Il faut mq $v \in \Phi(W^{1,p}(\Omega))$

$$v_k = (u_k, \partial_1 u_k, \dots, \partial_d u_k), \qquad u_k \in W^{1,p}(\Omega)$$

$$v = (v^0, v^1, \dots, v^d) \in L^p(\Omega, \mathbb{R}^{d+1})$$

$$\Rightarrow \quad u_k \xrightarrow{L^p(\Omega)} v^0$$

$$\partial_j u_k \xrightarrow{L^p(\Omega)} v^j \qquad \forall j \in \{1, \dots, d\}$$

Donc, il faut mq $\quad v^j = \partial_j v^0 \quad \forall j$

$\forall \varphi \in C^1_c(\Omega)$ il faut que

$$\int_\Omega v^j \varphi \, dx = -\int_\Omega v^0 \, \partial_j \varphi \, dx$$

On a :

$$\int_\Omega v^0 \overbrace{\partial_j \varphi}^{\in C_c^\infty(\Omega) \subset L^{p'}(\Omega)} dx = \lim_{k \to \infty} \int u_k \, \partial_j \varphi \, dx$$

conv. forte $\Rightarrow$ conv. faible

$$= \lim_{k \to +\infty} \left[- \int_\Omega \partial_j u_k \cdot \varphi \, dx \right]$$

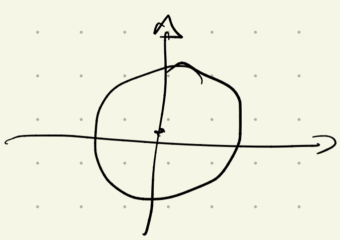
$$\overset{!}{=} - \int_\Omega v^j \varphi \, dx$$

$$\left| \int_\Omega (\partial_j u_k - v^j) \varphi \, dx \right| \le \underline{\| \partial_j u_k - v^j \|_p} \; \underbrace{\| \varphi \|_{p'}}_{< +\infty} \xrightarrow[k \to +\infty]{} 0.$$

$\varphi \in C^1_c(\Omega) \subset L^{p'}$

$\Rightarrow \quad v \in \Phi(W^{1,p}(\Omega))$ et on conclut $\square$

Exemple $\qquad B_1 \subset \mathbb{R}^d$



$u(x) = |x|^s \quad \in L^1_{loc}(\Omega)$ si $s > -d$

$\nabla u(x) = s |x|^{s-2} x \in L^1_{loc}(\Omega), \quad s > -d+1$

$\Rightarrow \quad u \in W^{1,1}(B) \quad$ si $\quad s > -d+1$

Exo mq $u \in W^{1,p}(B)$ si $s > -\frac{d}{p} + 1$

Proposition Soit $p \in [1,+\infty]$, $u \in L^p$

1. $u \in W^{1,p}$
2. $\exists c_u > 0$ t.q. $\int u \, \partial_j \varphi \leq c_u \|\varphi\|_{p'}$, $\forall \varphi \in C_c^1$, $j \in \{1,\dots,d\}$
3. $\exists c_u > 0$ $\|\tau_h u - u\|_p \leq c_u |h|$
4. $\exists (u_n)_n \subset W^{1,p}$ bornée en $W^{1,p}(\Omega)$ t.q. $u_n \xrightarrow{L^p} u$

Donc : $1 \Rightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4$ $\forall p \in [1,+\infty]$

$\Leftarrow$ si $p > 1$

De plus $c_u = c\|\nabla u\|_p$ en 2 et 3 où $c > 0$.

Preuve $1 \Rightarrow 2$:

$$\int u \, \partial_j \varphi \, dx = -\int (\partial_j u)\varphi \leq \underbrace{\|\partial_j u\|_p}_{c_u} \cdot \|\varphi\|_{p'}$$

Hölder

$$c_u \leq \|\nabla u\|_p := \left(\sum_{j=1}^d \|\partial_j u\|_p^p\right)^{1/p}$$

$2 \Rightarrow 1$ si $p > 1$:

$$\Phi_u : \underset{C_c^1}{\varphi} \longmapsto \underbrace{\int u \, \partial_j \varphi}_{\in \mathbb{R}} \quad \text{linéaire}$$

$2 \Leftrightarrow |\Phi_u(\varphi)| \leq c_u \|\varphi\|_{p'}$ $\forall \varphi \in C_c^1$ $\Leftrightarrow$ Φ_u continue sur $(C_c^1, \|\cdot\|_{p'})$

$C_c^1 \subset L^{p'}$ dense $\Rightarrow$ $\Phi_u : L^{p'} \longrightarrow \mathbb{R}$

linéaire et continue

$\Rightarrow$ $\Phi_u \in (L^{p'})' \cong L^p$ $\boxed{p' < \infty}$

car $p > 1$

Thm représentation de Riesz

on rappel que

$(L^\infty)' \underset{\neq}{\supset} L^1$

c-à-d : $\exists!\, v \in L^p$ t.q.

$$\int u\, \partial \varphi = \Phi_u(\varphi) = \int v \varphi \qquad \forall \varphi \in L^{p'} \supset C_c^1$$

$\Rightarrow$ $v = -\partial u \in L^p$ $\Rightarrow$ $u \in W^{1,p}$.

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