$$\mathcal{E}' \subset \mathcal{S}' \subset \mathcal{D}'$$

$\mathcal{E}'$: distr à support compact

$\mathcal{S}'$: distr. tempérés

$\mathcal{D}'$: distributions

Opérations sur les distributions

$$u \in L^1_{loc} \rightsquigarrow T_u \in \mathcal{D}', \quad \langle T_u, \varphi \rangle = \int_{\mathbb{R}^d} u \varphi \, dx \quad \forall \varphi \in \mathcal{D}$$

① Translations $\quad h \in \mathbb{R}^d, \quad \tau_h u(x) = u(x-h) \quad \forall x \in \mathbb{R}^d$

Pour étendre τ_h aux distr. on veut que

$$\tau_h u \longrightarrow T_{\tau_h u} = \tau_h T_u$$

$$i : L^1_{loc} \hookrightarrow \mathcal{D}'$$

$$\begin{tikzcd}
u \arrow[r, mapsto, "\tau_h"] \arrow[d, "i"'] & \tau_h u \arrow[d, "i"] \\
T_u \arrow[r, mapsto, "\tau_h"'] & T_{\tau_h u} = \tau_h T_u
\end{tikzcd}$$

extension

$\tau_h : \mathcal{D}' \to \mathcal{D}'$

$$\langle T_{\tau_h u}, \varphi \rangle = \int_{\mathbb{R}^d} \tau_h u(x) \varphi(x) \, dx = \int_{\mathbb{R}^d} \underbrace{u(x-h)}_{y} \varphi(x) \, dx$$

$$= \int_{\mathbb{R}^d} u(y) \varphi(y+h) \, dy = \langle T_u, \tau_{-h} \varphi \rangle$$

Déf $\quad T \in \mathcal{D}', \quad h \in \mathbb{R}^d \Rightarrow \tau_h T \in \mathcal{D}'$ définie par

$$\langle \tau_h T, \varphi \rangle = \langle T, \tau_{-h} \varphi \rangle \quad \forall \varphi \in D$$

Ex : δ_0 masse de Dirac

$\varphi \in D$, $\quad \langle \tau_h \delta_0 , \varphi \rangle := \langle \delta_0 , \tau_{-h} \varphi \rangle = \tau_{-h} \varphi(0) = \varphi(h)$

$$\Rightarrow \quad \tau_h \delta_0 = \delta_h$$

② Dérivation

$u \in L^1_{loc}$, $\quad \partial_x u \in L^1_{loc}$ dérivée faible

$$\underbrace{\langle T_{\partial_x u}, \varphi \rangle}_{D} = \int_{\mathbb{R}^d} \partial_x u \cdot \varphi \, dx$$

$$= - \int u \, \partial_x \varphi \, dx = - \langle T_u, \partial_x \varphi \rangle$$

Déf Soit $T \in D'$, $\alpha \in \mathbb{N}^d \Rightarrow \partial^\alpha T$ est donnée par

$$\langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle \quad \forall \varphi \in D$$

Exo mq $\partial^\alpha T$ est continue et linéaire sur D

si T a ordre $m \Rightarrow \partial^\alpha T$ a ordre $m + |\alpha|$

Ex $\quad H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \qquad H \in L^1_{loc}$

$$H' := T_H' = \delta_0$$

$$\langle H', \varphi \rangle := - \langle H, \varphi' \rangle = - \int_0^\infty \varphi'(x) \, dx = - \big(\varphi(\infty) - \varphi(0) \big)$$

$$= \varphi(0) = \langle \delta_0, \varphi \rangle$$

$$\delta_0' = ?$$

$\varphi \in \mathcal{D}$, $\quad \langle \delta_0', \varphi \rangle := -\langle \delta_0, \varphi' \rangle = -\varphi'(0)$

$$\langle \delta_0^{(m)}, \varphi \rangle = (-1)^m \varphi^{(m)}(0)$$

③ Multiplication

$a : \mathbb{R}^d \longrightarrow \mathbb{R}$, $\quad u \in L^1_{loc}$

$(au)(x) = a(x)\,u(x) \qquad au \in L^1_{loc}$ si $a \in L^\infty_{loc}$

Ex $\quad u(x) = \dfrac{1}{\sqrt{|x|}}$ $\quad x \in \mathbb{R}$, $\quad u \in L^1_{loc}(\mathbb{R})$

-) $[a,b] \subset \mathbb{R}$ $\quad$ si $a > 0 \Rightarrow \displaystyle\int_a^b |u| \leq \frac{1}{\sqrt{a}}(b-a) < +\infty$

$b < 0 \Rightarrow \displaystyle\int_a^b |u| \leq \frac{1}{\sqrt{|b|}}\,|b-a| < +\infty$

si $a < 0 < b \Rightarrow \displaystyle\int_a^b |u|\,dx = \int_0^b \frac{dx}{\sqrt{x}} + \int_0^a \frac{dx}{\sqrt{x}}$

$$= 2\sqrt{x}\Big|_0^b + 2\sqrt{x}\Big|_0^a$$

$$= 2\sqrt{b} + 2\sqrt{a} \quad < +\infty.$$

Par contre: $\quad u(x)^2 = u(x)\cdot u(x) = \dfrac{1}{|x|} \notin L^1_{loc}$

$$\int_{-\varepsilon}^{\varepsilon} \frac{1}{|x|}\,dx = +\infty.$$

(En général $u \in L^p_{loc}$, $a \in L^{p'}_{loc} \Rightarrow au \in L^1_{loc}$)
Hölder

Si $au \in L^1_{loc}$ $\qquad aT_u = T_{au}$

$$\varphi \in D, \quad \langle T_{au}, \varphi \rangle = \int_{\mathbb{R}^d} a(x)u(x)\varphi(x)\,dx = \langle T_u, a\varphi \rangle$$

<u>Def</u> $T \in D'$ $\quad a : \mathbb{R}^d \longrightarrow \mathbb{R}$, aT est définie par

$$\langle aT, \varphi \rangle = \langle T, a\varphi \rangle \qquad \varphi \in D$$

- $\to$ $T \in D'$, $a \in D \Rightarrow aT \in D'$
- $\to$ $T \in \mathcal{E}'$, $a \in \mathcal{E} \Rightarrow aT \in \mathcal{E}'$
- $\to$ $T \in \mathcal{S}'$, $a \in \mathcal{O}_M \Rightarrow aT \in \mathcal{S}'$

<u>Rmq</u> <u>En général</u> on ne peut pas multiplier deux distributions !

④ Convolution

$u \in L^1_{loc}$, $\quad \varphi \in C^\infty_c$

$$(u * \varphi)(x) = \int_{\mathbb{R}^d} u(y)\varphi(x-y)\,dy = \langle T_u, \varphi(x - \cdot) \rangle$$

<u>Def</u> Si T distribution, φ fct test

$$(T * \varphi)(x) = \langle T, \underbrace{\varphi(x - \cdot)}_{\text{fct test}} \rangle \qquad \forall x \in \mathbb{R}^d$$

Exo $C_\varphi : T \longmapsto T * \varphi$ linéaire

$C_\varphi : \mathcal{D}' \longrightarrow \mathcal{E}$ continu, si $\varphi \in \mathcal{D}$

$C_\varphi : \mathcal{E}' \longrightarrow \mathcal{D}$ continu, si $\varphi \in \mathcal{D}$

$C_\varphi : \mathcal{E}' \longrightarrow \mathcal{E}$ continu, si $\varphi \in \mathcal{E}$

ϕ test

$$\langle \underbrace{T * \varphi}, \phi \rangle = \int_{\mathbb{R}^d} (T * \varphi)(x)\, \phi(x)$$

$$= \int_{\mathbb{R}^d} \langle T, \underbrace{\varphi(x - \cdot)\rangle \phi(x)}_{\text{test}} dx$$

justifier $\longrightarrow$

$$= \left\langle T, \underbrace{y \longmapsto \int \varphi(\overset{z}{\widetilde{x-y}})\, \phi(\overset{z+y}{x})\, dx}_{\text{test}} \right\rangle$$

$$= \left\langle T, \int_{\mathbb{R}^d} \varphi\, \tau_{-y} \phi \right\rangle$$

On observe que φ test, $\displaystyle\int_{\mathbb{R}^d} \varphi\, \tau_{-y} \phi\, dx = \langle T_\varphi, \tau_{-y}\phi \rangle$
$= \langle \tau_y T_\varphi, \phi \rangle$

Def T, U distributions, $T * U$ est la distr. t.q.

$$\langle T * U, \phi \rangle = \langle T, y \longmapsto \langle \tau_y U, \phi \rangle \rangle \quad \forall \phi \text{ test}$$

5 Transformé de Fourier

$x, \xi \in \mathbb{R}^d$ on pose $e_\xi(x) = e^{i\langle x, \xi \rangle}$

$|e_\xi(x)| = 1,$

$$\boxed{\begin{aligned}\partial^\alpha e_\xi(x) &= \partial^\alpha[i\langle x,\xi\rangle]\, e_\xi(x)\\ &= i^\alpha\Big(\cancel{\sum_{i=1}^d c_i \xi_i}\Big)\, e_\xi(x)\end{aligned}}$$

FAUX $\quad \xi^\alpha$

~~où $c_i = 1$ si $\alpha_i = 1$~~
~~$c_i = 0$ sinon~~

$|\partial^\alpha e_\xi(x)| \leq c|\xi|$

$\Rightarrow \quad \|\partial^\alpha e_\xi\|_\infty \leq c|\xi|^\alpha$

$\Rightarrow \quad e_\xi \in \mathcal{O}_M$

$$\boxed{\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}}$$

__Prop__ $\quad \xi \in \mathbb{R}^d \xrightarrow{i} e_\xi \in \mathcal{O}_M$ cette application est une injection continue

__Preuve__ $\quad (\xi_n)_n \subset \mathbb{R}^d, \quad \xi_n \longrightarrow \xi_0 \in \mathbb{R}^d$

__mq__ $\quad e_{\xi_n} \xrightarrow{\mathcal{O}_M} e_{\xi_0}$

$\forall \alpha \in \mathbb{N}^d, \ \exists p \in \mathbb{N} \ / \quad \|\Lambda^{-p}(\partial^\alpha e_{\xi_n} - \partial^\alpha e_\xi)\|_\infty \longrightarrow 0$

$$\begin{aligned}
&\left|\left(\partial^\alpha e_{\xi_n} - \partial^\alpha e_\xi\right)(x)\right| \\
&= \left|\sum_i c_i\left[(\xi_n)_i\, e_{\xi_n}(x) - (\xi_0)_i\, e_{\xi_0}(x)\right]\right| \\
&\leq c\left|\xi_n e_{\xi_n}(x) - \xi_0 e_{\xi_0}(x)\right| \\
&\leq c\left(\left|\xi_n e_{\xi_n}(x) \pm \xi_0 e_{\xi_n}(x) - \xi_0 e_{\xi_0}(x)\right|\right) \\
&\leq c\left(|\xi_n - \xi_0| + |\xi_0| \cdot |e_{\xi_n}(x) - e_{\xi_0}(x)|\right)
\end{aligned}$$

$$\Rightarrow \|\Lambda^{-p}(\partial^\alpha e_{\xi_n} - \partial^\alpha e_\xi)\|_\infty \leq c\Big(|\xi_n - \xi_0| \cdot \|\Lambda^{-p}\|_\infty + |\xi_0|\, \|\Lambda^{-p}(e_{\xi_n} - e_{\xi_0})\|_\infty\Big)$$

$\boxed{\underline{\text{Exo}} \quad \text{mq} \quad \exists p \in \mathbb{N} \ / \quad \| \Delta^{-p} (e_{\xi_n} - e_\xi) \|_\infty \xrightarrow[n]{} 0}$

$\square$

Def Soit $\varphi \in \mathcal{S}$, alors la TF de φ est

$$\hat{\varphi}(\xi) = \int_{\mathbb{R}^d} \varphi(x) e^{-i\langle x, \xi \rangle} dx = \int_{\mathbb{R}^d} \varphi \, \overline{e_\xi} \, dx$$

Théorème Si $\varphi \in \mathcal{S} \Rightarrow \hat{\varphi} \in \mathcal{S}$

Preuve ① $\hat{\varphi} \in C^\infty$, $\alpha \in \mathbb{N}^d$, $\xi \in \mathbb{R}^d$

$$\partial^\alpha \hat{\varphi}(\xi) = \partial^\alpha \int_{\mathbb{R}^d} \varphi(x) e^{-i\langle x, \xi \rangle} dx$$

$$= \int_{\mathbb{R}^d} \varphi(x) \underbrace{\partial^\alpha \left(e^{-i\langle x, \xi \rangle} \right)} dx$$

$$= \int_{\mathbb{R}^d} \varphi(x) (-i x)^\alpha e^{-i\langle x, \xi \rangle} dx$$

$$= \underbrace{\widehat{(-ix)^\alpha \varphi}}_{\in \mathcal{S}} (\xi)$$

$$\partial^\alpha \hat{\varphi}(\xi) = \widehat{(-ix)^\alpha \varphi}(\xi)$$

Il suffit donc de prouver $\hat{\varphi} \in C^0$, $\varphi \in \mathcal{S}$,

$$\lim_{\xi \to \xi_0} \hat{\varphi}(\xi) = \int_{\mathbb{R}^d} \lim_{\xi \to \xi_0} \varphi(x) e^{-i\langle \xi, x \rangle} dx = \hat{\varphi}(\xi_0)$$

(2) $\widehat{\partial^\alpha \varphi}(\xi) = (i\xi)^\alpha \hat{\varphi}(\xi)$

$$\widehat{\partial^\alpha \varphi}(\xi) = \int_{\mathbb{R}^d} \partial^\alpha \varphi(x)\, e^{-i\langle x, \xi\rangle}\, dx$$

IPP

$$= (-1)^{|\alpha|} \int_{\mathbb{R}^d} \varphi(x)\, \underbrace{\partial^\alpha\left(e^{-i\langle x,\xi\rangle}\right)}_{(-i\xi)^\alpha e^{-i\langle x,\xi\rangle}} dx$$

$$= \underbrace{(-1)^{|\alpha|}(-i\xi)^\alpha}_{(i\xi)^\alpha}\; \hat{\varphi}(\xi)$$

(3) $\hat{\varphi} \in \mathcal{S}$:

$\Updownarrow$

$\forall \alpha, \beta \in \mathbb{N}^d$, $\quad \xi^\alpha \partial^\beta \hat{\varphi}(\xi)$ bornée

$$\xi^\alpha \partial^\beta \hat{\varphi}(\xi) = \xi^\alpha\, \widehat{(-ix)^\beta \varphi}(\xi)$$

$$= (-i)^{|\alpha|} (i\xi)^\alpha\, \widehat{(-ix)^\beta \varphi}(\xi)$$

(2)

$$= (-i)^{|\alpha|}\, \widehat{\partial^\alpha\left[x \mapsto (-ix)^\beta \varphi(x)\right]}(\xi)$$

$\varphi \in \mathcal{S} \Rightarrow x \mapsto \underbrace{\partial^\alpha (-ix)^\beta \varphi(x)}_{\psi \in \mathcal{S}}$ est dans L^1

en effet $\Lambda^p \psi$ borné

$$\|\psi\|_1 = \|\Lambda^p \psi \cdot \Lambda^{-p}\|_1$$

$$\leq \underbrace{\|\Lambda^{-p}\|_1}_{<+\infty} \cdot \|\Lambda^p \psi\|_\infty \qquad p \geq 2$$

Si $\varphi \in L^1 \Rightarrow \hat{\varphi} \in L^\infty$

$$|\hat{\varphi}(\xi)| = \left| \int_{\mathbb{R}^d} \varphi(x) \underbrace{e^{-i\langle x, \xi \rangle}}_{|\cdot| = 1} dx \right| \leq \|\varphi\|_1$$

$\square$

$$\mathcal{F} : \mathcal{S} \longrightarrow \mathcal{S} \quad , \quad \mathcal{F}^{-1} : \mathcal{S} \longrightarrow \mathcal{S}$$

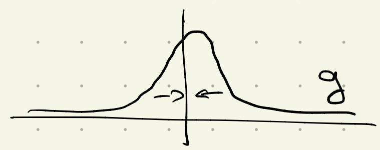
Prop (Formule inversion) $\hat{\varphi} \in \mathcal{S} \Rightarrow$

$$\varphi(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{\varphi}(\xi)\, e^{i\langle x, \xi \rangle}\, d\xi$$

et $\varphi \in \mathcal{S}$

Preuve •) $\varphi \in \mathcal{S}$ se démontre comme avant

•) ① $g(x) = e^{-|x|^2/2}$



$$\sigma_r g(x) := g(x/r)$$

$$\int_{\mathbb{R}^d} \hat{\varphi}(\xi)\, \sigma_r g(\xi)\, e^{+i\langle x, \xi \rangle}\, d\xi$$

$$= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \varphi(y)\, e^{-i\langle y, \xi \rangle}\, dy \right) \sigma_r g(\xi)\, e^{i\langle x, \xi \rangle}\, d\xi$$

$$= \int_{\mathbb{R}^d} g(\xi/r)\, e^{i\langle x, \xi \rangle} \left(\int_{\mathbb{R}^d} \varphi(y)\, e^{-i\langle y, \xi \rangle}\, dy \right) d\xi$$

$$\lim_{r \to +\infty} \int \hat{\varphi}(\xi)\, \sigma_r g(\xi)\, e^{i\langle x, \xi \rangle}\, d\xi = \int \hat{\varphi}(\xi)\, e^{i\langle x, \xi \rangle}\, d\xi$$

parce que $\tau_\pi g(\xi) = g(\xi/\pi) \xrightarrow[\pi\to\infty]{} g(0) = 1$

et on applique conv. dominée

$$\lim_{\pi\to+\infty} \int_{\mathbb{R}^d} g(\xi/\pi)\, e^{i\langle x,\xi\rangle} \left(\int_{\mathbb{R}^d} \varphi(y)\, e^{-i\langle y,\xi\rangle} dy \right) d\xi$$

$$= \lim_{\pi\to\infty} \int_{\mathbb{R}^d} \varphi(y) \underbrace{\left(\int_{\mathbb{R}^d} g(\xi/\pi) \underbrace{e^{i\langle x,\xi\rangle} e^{-i\langle y,\xi\rangle}}_{\substack{\shortparallel \\ e^{i\langle x-y,\xi\rangle}}} d\xi \right)}_{\substack{\shortparallel \\ \widehat{\tau_\pi g}(y-x) = \pi^d \hat{g}(\pi(y-x))}} dy$$

$$= \lim_{\pi\to+\infty} \int_{\mathbb{R}^d} \varphi(y)\, \hat{g}(\underbrace{\pi(y-x)}_{z}) \cdot \underbrace{\pi^d dy}$$

$y = \frac{z}{\pi} + x$

$dz = \pi^d dy$

$$= \lim_{\pi\to+\infty} \int_{\mathbb{R}^d} \varphi(x + z/\pi)\, \hat{g}(z)\, dz$$

$\hat{g} \in \mathcal{S}$, car $g \in \mathcal{S}$

conv. dominé

$$= \varphi(x) \int_{\mathbb{R}^d} \hat{g}(z)\, dz$$

On a mq $\varphi(x) \int_{\mathbb{R}^d} \hat{g}(z) dz = \int_{\mathbb{R}^d} \varphi(\xi)\, e^{i\langle x,\xi\rangle} d\xi$

② Calculons $\int_{\mathbb{R}^d} \hat{g}(z)\, dz$:

$d = 1$: $g(x) = e^{-x^2/2}$

$g'(x) = -x\, g(x) = -i(-ix\, g(x))$

FT

$$(-i)\quad i\xi\,\hat{g}(\xi) = -i\,\hat{g}'(\xi)$$

$$\xi\,\hat{g}(\xi) = -\hat{g}'(\xi) \qquad \text{EDO}$$

Sol. générale : $\hat{g}(0)e^{-\xi^2/2} = \hat{g}(\xi)$

$$\hat{g}(0) = \int_{\mathbb{R}} g(x)\,dx = \int_{\mathbb{R}} e^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}}$$

$$\rightsquigarrow \quad \hat{g}(\xi) = \frac{1}{\sqrt{2\pi}}\,e^{-\xi^2/2}$$

$$\int_{\mathbb{R}} \hat{g}(\xi)\,d\xi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\xi^2/2}\,d\xi = \frac{1}{2\pi} = \frac{1}{(2\pi)^d}$$

En général : $d \geq 1$ $\qquad \hat{g}(\xi) = \frac{1}{(2\pi)^{d/2}}\,e^{-|\xi|^2/2}$

$$\int_{\mathbb{R}^d} \hat{g}(\xi)\,d\xi = \frac{1}{(2\pi)^d} \qquad \square$$

Rmq $\quad \varphi, \psi \in \mathcal{S}$

$$\langle T_\varphi, \hat{\psi} \rangle_{\hat{\mathcal{S}}} = \int \varphi(\xi)\,\hat{\psi}(\xi)\,d\xi$$

$$= \int_{\mathbb{R}^d} \varphi(\xi) \int_{\mathbb{R}^d} \psi(x)\,e^{-i\langle x,\xi\rangle}\,dx\,d\xi$$

Fubini

$$= \int_{\mathbb{R}^d} \psi(x) \int_{\mathbb{R}^d} \varphi(\xi)\,e^{-i\langle x,\xi\rangle}\,d\xi\,dx$$

$$= \int_{\mathbb{R}^d} \varphi(x)\, \hat{\varphi}(x)\, dx$$

$$= \langle T_{\hat{\varphi}}, \varphi \rangle$$

Exo Avec la même idée mq $\forall \varphi, \psi \in \mathcal{S}$ on a
+ form. inversion

Parseval-Plancherel : $\int_{\mathbb{R}^d} \hat{\varphi}(\xi)\overline{\hat{\psi}(\xi)}\, d\xi = \int_{\mathbb{R}^d} \varphi(x)\overline{\psi(x)}\, dx$

$\Rightarrow \mathcal{F} : L^2 \longrightarrow L^2$ (car $\mathcal{S}$ dense dans L^2)
est une isométrie

Def Si $T \in \mathcal{S}'$, sa TF est $\hat{T} \in \mathcal{S}'$ donnée par

$$\langle \hat{T}, \varphi \rangle = \langle T, \hat{\varphi} \rangle \quad \forall \varphi \in \mathcal{S}$$

Rmq toute propriétés qu'on a vu restent vrai en $\mathcal{S}'$

- La formule d'inversion :

$$\langle \underbrace{\mathcal{F}^{-1} T}_{\in \mathcal{S}'}, \varphi \rangle = \langle T, \mathcal{F}^{-1}\varphi \rangle \quad \forall \varphi \in \mathcal{S}'$$

$\rightsquigarrow \mathcal{F} : \mathcal{S}' \longrightarrow \mathcal{S}'$ isomorphisme

- $\partial^\alpha \hat{\varphi} = \widehat{(-ix)^\alpha \varphi} \longrightarrow \partial^\alpha \hat{T} = \widehat{\underbrace{(-ix)^\alpha T}_{\in \mathcal{O}_M}}$

$\Downarrow$
$(-ix)^\alpha T \in \mathcal{S}'$

<u>Exo</u> δ_0 masse de Dirac $\delta_0 \in \mathcal{S}'$

Calculons $\hat{\delta}_0$:

$$\langle \hat{\delta}_0, \varphi \rangle := \langle \delta_0, \hat{\varphi} \rangle = \hat{\varphi}(0) = \int_{\mathbb{R}^d} \varphi(x)\,dx$$

$$\hat{\delta}_0(x) = 1 \in L^1_{loc} \qquad \hat{\delta}_0 = T_1$$

$h \in \mathbb{R}^d$: $\hat{\delta}_h(x) = e^{-i\langle h, x\rangle} \longrightarrow \hat{\delta}_h = T_{e_{-h}}$

<u>Lemme</u> : $\varphi, \psi \in L^2 \Rightarrow \widehat{\varphi * \psi}(\xi) = \hat{\varphi}(\xi)\,\hat{\psi}(\xi) \quad \forall \xi \in \mathbb{R}^d$

L

<u>Preuve</u> $\varphi, \psi \in \mathcal{S}$

$$\widehat{\varphi * \psi}(\xi) = \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \varphi(y)\,\psi(x-y)\,dy \right] e^{-i\langle x, \xi\rangle}\,dx$$

$$= \int_{\mathbb{R}^d} \varphi(y) \int_{\mathbb{R}^d} \psi(x-y)\, \underbrace{e^{-i\langle x, \xi\rangle}}_{= e^{-i\langle x-y, \xi\rangle} e^{-i\langle y, \xi\rangle}}\,dx\,dy.$$

$$= \int_{\mathbb{R}^d} \varphi(y)\, e^{-i\langle y, \xi\rangle} \underbrace{\int_{\mathbb{R}^d} \psi(\underbrace{x-y}_{z})\, e^{-i\langle \underbrace{x-y}_{z}, \xi\rangle}\,\underbrace{dy}_{dz}}_{\hat{\psi}(\xi)}\,dx$$

$$= \hat{\varphi}(\xi)\,\hat{\psi}(\xi)$$

$\square$

EQUATIONS AUX DÉRIVÉS PARTIELLES

$\mathbb{R}^d$ Pb étant donné operateur linéaire de dérivation partielle $P(\partial)$ chercher u t.q.

$$P(\partial)u = g \quad \text{où } g \text{ fct donnée}$$

$$P(\partial) = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha, \quad m = \text{ordre de } P(\partial)$$

Ex :

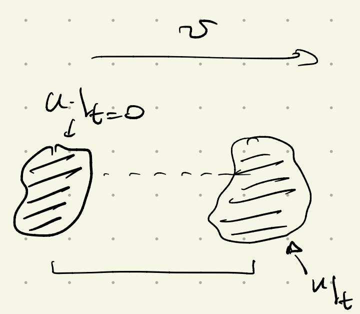
- Probléme de Dirichlet $(f=0)$ ou de Poisson

$$\Delta u = f \qquad \Delta u := \sum_{i=1}^{n} \partial_{x_i}^2 u$$

- Equation du transport : $\partial_t u + v \cdot \nabla u = 0$

où $u : [0,+\infty) \times \mathbb{R}^d \longrightarrow \mathbb{R}$

$v : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ champ de vitesses.

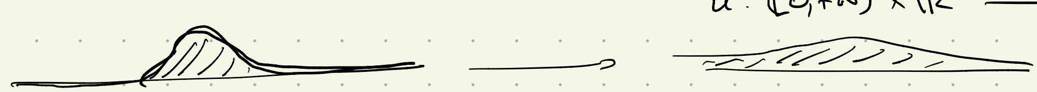


$$\partial_t u + v\,\partial_x u = 0 \qquad x \in \mathbb{R}$$

$$\hookrightarrow u(t,x) = u(0, x - tv)$$

- Equation de la chaleur : $\partial_t u = \Delta u$ u = temperat.

$u : [0,+\infty) \times \mathbb{R}^d \longrightarrow \mathbb{R}$



$$\partial_t u = \partial_x^2 u \quad \text{en dim} = 1$$

- Equation des ondes : $\partial_t^2 u = \Delta u$

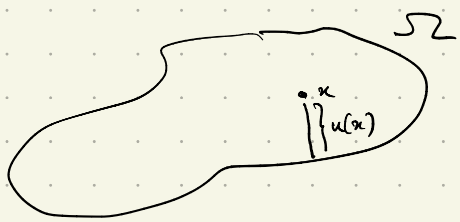
•) Equation de Schrödinger $\quad i\,\partial_t u = \Delta u + Vu$

$u : [0,+\infty) \times \mathbb{R}^d \longrightarrow \mathbb{C}$

$V : \mathbb{R}^d \longrightarrow \mathbb{R}$ potentiel

•) Equation non-linéaire : Eq. Eikonale

$u : \Omega \longrightarrow \mathbb{R}$



$$\begin{cases} \|\nabla u\| = 1 \\ u|_{\partial\Omega} = 0 \end{cases}$$

$\rightsquigarrow$ la solution est la distance au bord

Solution fondamentale

$$P(\partial)u = g$$

Ansatz $\quad u = F * g \quad$ est solution

$$g = P(\partial)(F * g) = [P(\partial)F] * g \iff P(\partial)F = \delta_0$$

But déterminer F (solution fondamentale)

$\Rightarrow$ on pourra resoudre le pb $\forall g$

Rmq $\quad F$ n'est pas unique en général !

Par ex. $\quad F$ est sol fond. de Δ

$F+c$ est sol fond de Δ $\forall c \in \mathbb{R}$