Sol fondamentale du Laplacien

$\mathbb{R}^d$, $\quad \Delta = \sum_{i=1}^{d} \partial_{x_i}^2$

Probleme de Poisson
$$\Delta u = f$$

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-i\langle x,\xi\rangle} dx$$

On cherche K t.q. $\Delta K = \delta_0$

$$K \in \mathcal{S}' \Rightarrow \Delta K \in \mathcal{S}' \Rightarrow \widehat{\Delta K} \in \mathcal{S}'$$

$$\widehat{\Delta K}(\xi) = \mathcal{F}\left(\sum_{i=1}^{d} \partial_{x_i}^2 K\right) = \sum_{i=1}^{d} \widehat{\partial_{x_i}^2 K}(\xi)$$

$$= \sum_{i=1}^{d} \underbrace{(+i\xi_i)^2}_{-|\xi_i|^2} \hat{K}(\xi) = -|\xi|^2 \hat{K}(\xi)$$

$$\Delta K = \delta_0 \underset{\mathcal{F}_1 : \mathcal{S}' \to \mathcal{S}'}{\Longleftrightarrow} \widehat{\Delta K} = \hat{\delta}_0 \Longleftrightarrow -|\xi|^2 \hat{K}(\xi) = 1$$

$$\rightsquigarrow \hat{K}(\xi) = -\frac{1}{|\xi|^2} \quad \forall \xi \neq 0$$

On observe que $-|\xi|^{-2} \in L^1_{loc} \Leftrightarrow -2 > -d \Leftrightarrow \underline{d > 2}$

Pour $d \geq 3 \Rightarrow \hat{K}(\xi) = -|\xi|^{-2} \in \mathcal{S}' \Rightarrow K(x) = \mathcal{F}_1^{-1}(-|\xi|^{-2})$

Proposition $\quad d > 2$, $\quad 0 < k < d$, $\quad F_k$ = distr. associée à $|\xi|^{-k}$

$$\Rightarrow \mathcal{F}_1^{-1}(F_k) = \frac{\Gamma\left(\frac{d-k}{2}\right)}{\Gamma(k/2)} c(k,d) |x|^{k-d}$$

Ici $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx \quad (Re(z) > 0)$

Preuve $\quad k>0$, $r>0$

$$\int_0^\infty e^{-\overbrace{rt}^{x}} t^{k-1}\,dt = \int_0^\infty e^{-x} \frac{x^{k-1}}{r^{k-1}} \cdot \frac{dx}{r} = r^{-k}\Gamma(k)$$

$$\rightsquigarrow \quad r^{-k} = \frac{1}{\Gamma(k)} \int_0^\infty e^{-rt} t^{k-1}\,dt$$

(o) $$F_k(\xi) = |\xi|^{-k} = (|\xi|^2)^{-k/2} = \frac{1}{\Gamma(k/2)} \int_0^\infty e^{-|\xi|^2 t}\, t^{\frac{k}{2}-1}\,dt$$

On observe que $\underbrace{\mathcal{F}\left(e^{-|\xi|^2 t}\right)(x) = \frac{1}{(2t)^{d/2}} e^{-\frac{|x|^2}{4t}}}_{\boxed{Exo}}$

$$\langle e^{-|\xi|^2 t}, \hat{\varphi} \rangle = \langle \widehat{e^{-|\xi|^2 t}}, \varphi \rangle$$

$$\int_{\mathbb{R}^d} e^{-|\xi|^2 t} \hat{\varphi}(\xi)\,d\xi \qquad\qquad \frac{1}{2^{d/2}} \int_{\mathbb{R}^d} t^{-d/2} e^{-|x|^2/4t} \varphi(x)\,dx$$

$$\Downarrow \int_0^\infty t^{k/2-1} \cdot\,dt$$

$$\underbrace{\int_0^\infty t^{k/2-1} \int_{\mathbb{R}^d} e^{-|\xi|^2 t} \hat{\varphi}(\xi)\,d\xi\,dt}_{LHS} = \underbrace{\frac{1}{2^{d/2}} \int_0^\infty t^{k/2-1} \int_{\mathbb{R}^d} t^{-d/2} e^{-|x|^2/4t} \varphi(x)\,dx\,dt}_{RHS}$$

LHS: Ici on justifie facilement les hypothèses de Fubini

$$LHS = \int_{\mathbb{R}^d} \hat{\varphi}(\xi) \int_0^\infty t^{k/2-1} e^{-|\xi|^2 t}\,dt\,d\xi \overset{(o)}{=} \int_{\mathbb{R}^d} \hat{\varphi}(\xi)\Gamma(\tfrac{k}{2}) F_k(\xi)\,d\xi$$

$$= \langle \Gamma(\tfrac{k}{2}) F_k, \hat{\varphi} \rangle$$

RHS : Par Fubini

$$\text{RHS} = \frac{1}{2^{d/2}} \int_{\mathbb{R}^d} \varphi(x) \left(\int_0^\infty t^{\frac{k-d}{2}-1} e^{-|x|^2/4t} \, dt \right) dx$$

(o)

$$\boxed{F_k(\xi) = \frac{1}{\Gamma(k/2)} \int_0^\infty e^{-|\xi|^2 t} \, t^{\frac{k}{2}-1} \, dt}$$

Posons $s = \frac{1}{4t}$

$t = (4s)^{-1}$

$$ds = -\frac{1}{4t^2} dt = -4s^2 \, dt$$

$$= \frac{1}{2^{d/2}} \int_{\mathbb{R}^d} \varphi(x) \left(\int_0^\infty (4s)^{1-\frac{k-d}{2}} e^{-|x|^2 s} \cdot \frac{1}{4s^2} \, ds \right) dx$$

$$= \frac{1}{2^{d/2}} \int_{\mathbb{R}^d} \varphi(x) \int_0^\infty (4s)^{\frac{d-k}{2}-1} e^{-|x|^2 s} \, ds \, dx$$

$k = d-k$

$$= \frac{2^{(d-k)-2}}{2^{d/2}} \int_{\mathbb{R}^d} \varphi(x) \Gamma\left(\frac{d-k}{2}\right) |x|^{-(d-k)} \, dx$$

$$= \left\langle 2^{\frac{d}{2}-k-2} \cdot \Gamma\left(\frac{d-k}{2}\right) |x|^{-(d-k)}, \varphi \right\rangle$$

Donc, comme LHS = RHS, $\forall \varphi \in \mathcal{S}$

$$\left\langle \Gamma\left(\tfrac{k}{2}\right) F_k, \hat{\varphi} \right\rangle = \left\langle 2^{\frac{d}{2}-k-2} \Gamma\left(\frac{d-k}{2}\right) |x|^{-(d-k)}, \varphi \right\rangle$$

$$\parallel$$

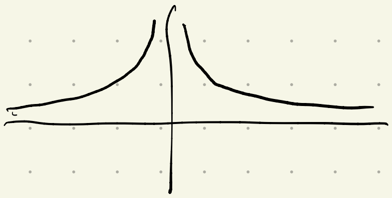
$$\left\langle \Gamma\left(\tfrac{k}{2}\right) \hat{F}_k, \varphi \right\rangle$$

$$\Rightarrow \quad \hat{F}_k(x) = 2^{\frac{d}{2}-k-2} \frac{\Gamma\left(\frac{d-k}{2}\right)}{\Gamma(k/2)} |x|^{-(d-k)}$$

Il suffit d'observer que $\hat{F}_k(x) = \dfrac{\mathcal{F}^{-1}(F_k)(x)}{(2\pi)^{d/2}}$ $\boxed{\text{Exo}}$

$$\Rightarrow \quad \mathcal{F}^{-1}(F_k)(x) = \frac{2^{d/2-k-2}}{(2\pi)^{d/2}} \, \frac{\Gamma\left(\frac{d-k}{2}\right)}{\Gamma(k/2)} \, |x|^{k-d} . \qquad \square$$

Comme $d \geq 3$, $\hat{K}(\xi) = -|\xi|^2$



$$K(x) = c(k,d) \, \frac{\Gamma\left(\frac{d-2}{2}\right)}{\underbrace{\Gamma(1)}_{=1}} \, |x|^{2-d}$$

Donc $\Delta u = g \Leftarrow u(x) = \int_{\mathbb{R}^d} c\,|y|^{2-d} g(x-y)\, dy$.

Le cas $d=2$ On choisit $k<2$, on remplace Δ par $\Delta^{k/2}$

où $\Delta^{k/2} f = \mathcal{F}^{-1}\left(|\xi|^{k/2} \hat{f}(\xi) \right) \quad \forall f \in \mathcal{S}'$

K_k sol fond de $\Delta^{k/2}$

$\Updownarrow$

$$\Delta^{k/2} K_k = \delta_0 \iff |\xi|^{k/2} \hat{K}_k(\xi) = 1 \iff \hat{K}_k(\xi) = |\xi|^{-k/2}$$

comme $k<2 \Rightarrow \hat{K}_k \in L^1_{loc}$

$\Rightarrow$ Par la Prop $\quad K_k(x) = \underbrace{c(k,2)\, \frac{\Gamma\left(\frac{2-k}{2}\right)}{\Gamma(k/2)}}_{C_k} \, |x|^{k-2}$

On remplace K_k par $K_k' = K_k - C_k$

$$\Delta^{k/2} K_k' = \Delta^{k/2} K_k - \overset{0}{\Delta^{k/2} C_k} = \Delta^{k/2} K_k = \delta_0$$

$$K'_k = c_k\left(|x|^{k-2} - 1\right) = c(k,2)\,\frac{\Gamma\left(\frac{2-k}{2}\right)}{\Gamma(k/2)}\left(|x|^{k-2}-1\right)$$

$$= 2\,\frac{c(k,2)}{\Gamma(k/2)}\cdot\Gamma\left(\frac{2-k}{2}+1\right)\ \frac{|x|^{k-2}-1}{2-k}$$

en effet $\quad \Gamma\left(\frac{2-k}{2}+1\right) = \frac{2-k}{2}\cdot\Gamma\left(\frac{2-k}{2}\right)$

$$\lim_{k\to 2} K'_k(x) = 2\,c(2,2)\lim_{k\to 2}\frac{|x|^{k-2}-1}{2-k}$$

$$\left(\frac{d}{dk}e^{(k-2)\lg x} = e^{(k-2)\lg x}\cdot\lg x\right.$$

$$|x|^{k-2} = 1 + \partial_k\left(|x|^{k-2}\right)\Big|_{k=2}\cdot(k-2) + \dots$$

$$= 1 + (k-2)\cdot\lg|x| + \dots$$

$$= \tilde{c}\lim_{k\to 2}\left(\lg|x| + \dots\right) = \tilde{c}\lg|x|.$$

$$\forall x \quad \lim_{k\to 2} K'_k(x) = \tilde{c}\lg x =: K_2(x)$$

Il faut mq $\quad K_2 \in \mathcal{S}'$

est-ce que $\quad \lg|x| \in L^1_{loc}(\mathbb{R}^2)$?

$B_\varepsilon \subset \mathbb{R}^2$

$$\int_{B_\varepsilon}|\lg|x||\,dx \underset{\text{polaires}}{=} 2\pi\varepsilon\int_0^\varepsilon \lg\rho\cdot\rho\,d\rho < +\infty$$

$\Rightarrow K_2 \in \mathcal{S}'$

Mq $\quad \Delta K_2 = \delta_0 \iff \widehat{\Delta K_2} = 1 \iff \hat{K}_2 = \frac{1}{|\xi|^2} \quad \xi \neq 0$

$$\langle \hat{K}_2, \varphi\rangle = \int_{|\xi|\le 1}\frac{\phi(\xi)-\phi(0)}{|\xi|^2}\,d\xi + \int_{|\xi|>1}\frac{\phi(\xi)}{|\xi|^2}\,d\xi$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{vp\left(\frac{1}{|\xi|^2}\right)}$$

cette déf donne un élement de $\mathcal{S}'$

qui satisfait $\Delta K_2 = 0$ par déf

Comme $\hat{K}_k \longrightarrow \hat{K}_2$ on obtient que

$\hat{K}_2$ est la FT de $K_2(x) = c \log|x|$

SYMBOLE D'UN OPÉRATEUR

$$P(\partial) = \sum_{|\alpha| \le m} c_\alpha \partial^\alpha \qquad\qquad \widehat{\partial^\alpha u}(\xi) = (-i\xi)^\alpha \hat{u}(\xi)$$

$$\updownarrow$$

$$D^\alpha = (-i)^{|\alpha|} \partial^\alpha = (-i\partial)^\alpha, \qquad P(D) = \sum_{|\alpha| \le m} b_\alpha D^\alpha$$

$$\widehat{P(D)u}(\xi) = \underline{\sum_{|\alpha| \le m} b_\alpha \xi^\alpha} \hat{u}(\xi) = p(\xi)\hat{u}(\xi)$$

Def $p(\xi) = \sum_{|\alpha| \le m} b_\alpha \xi^\alpha$ est le <u>symbole</u> de $P(D)$

$p_m(\xi) = \sum_{|\alpha| = m} b_\alpha \xi^\alpha$ est <u>symbole principale</u> de $P(D)$

Déf Un opérateur $P(D)$ est <u>elliptique</u> ssi

$p_m(\xi) \neq 0 \quad \forall \xi \in \mathbb{R}^d \setminus \{0\}$

Ex : •) Le laplacien est elliptique

$$P(\partial)=\sum_{i=1}^{d}\partial_{x_i}^2 \longrightarrow P(D)=\sum_{i=1}^{d}-D_i^2$$

$$\downarrow$$

$$p(\xi)=p_2(\xi)=\sum_{i=1}^{d}-\xi_i^2=-|\xi|^2$$

•) L'operateur de la chaleur ne l'est pas.

$$\partial_t u=\Delta u \quad \mathbb{R}^{d+1} \Longleftrightarrow \underbrace{(\partial_t-\Delta)}_{\text{op. de la chaleur}}u=0 \quad \text{in } \mathbb{R}^{d+1}$$

$$\mathbb{R}^{d+1}\ni(\underset{\overset{\updownarrow}{t}}{x_0},x_1,\dots,x_d)$$

$$P(\partial)=\partial_{x_0}-\sum_{i=1}^{d}\partial_{x_i}^2 \longrightarrow P(D)=-iD_0-\sum_{i=1}^{d}D_i^2$$

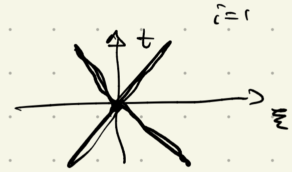
$$p(\xi)=-i\xi_0-\sum_{i=1}^{d}\xi_i^2, \quad p_2(\xi)=-\sum_{i=1}^{d}\xi_i^2$$

$$p_2(\xi_0,0,\dots,0)=0 \quad \forall \xi_0\in\mathbb{R}$$

•) L'opérateur des ondes n'est pas elliptique

$$\partial_t^2-\Delta \longrightarrow p_2(\xi)=\xi_0^2-\sum_{i=1}^{d}\xi_i^2$$

$$p_2\Big|_{\xi_0=\sum_{i=1}^{d}\xi_i^2}\equiv 0$$



But $\quad P(D)u=g$ et $P(D)$ est elliptique

Si $g\in W^{2,k} \Rightarrow u\in W^{2,k+m}$

On introduit les espace de Sobolev fractionnaires

On admet que $\mathcal{S} \subset L^2 \subset \mathcal{S}'$, $\mathcal{S}$ dense en L^2
($\mathcal{S} \supset C_c^\infty(\mathbb{R}^d)$)

<u>Lemme</u> $k \in \mathbb{N}$, $f \in \{ T \in \mathcal{S}' \,/\, \partial^\alpha T \in L^2 \text{ pour } |\alpha| \leq k \} = W^{2,k}$

$$\Updownarrow$$

$$\underset{\substack{\parallel \\ \xi \longmapsto (1+|\xi|^2)^{k/2} \hat{f}(\xi)}}{\Lambda^k \hat{f}} \in L^2$$

<u>Preuve</u> $\Rightarrow$ $f \in W^{2,k}$ mq $\Lambda^k \hat{f} \in L^2$

$$(1+|\xi|^2)^{k/2} \leq C_0 (1+|\xi|^k) \leq C_0 \Big(1 + C \sum_{j=1}^d |\xi_j|^k\Big)$$

$$\|\Lambda^k \hat{f}\|_2 \leq C_0 \Big\| |\hat{f}| + C |\hat{f}| \sum_{j=1}^d |\xi_j|^k \Big\|_2$$

Triang.

$$\leq C_0 \|\hat{f}\|_2 + C \sum_{j=1}^d \| \underbrace{|\xi_j|^k |\hat{f}|} \|_2 \qquad |\widehat{\partial_j^k f}| = |\xi_j|^k |\hat{f}|$$

$$= C_0 \|\hat{f}\|_2 + C \sum_{j=1}^d \| \widehat{\partial_j^k f} \|_2$$

$$= C_0 \|f\|_2 + C \sum_{j=1}^d \|\partial_j^k f\|_2 < +\infty \qquad f \in W^{2,k}$$

$\Leftarrow$ $\Lambda^k \hat{f} \in L^2$ mq $\forall |\alpha| \leq k$ $\partial^\alpha f \in L^2$

Il suffit de mq $\widehat{\partial^\alpha f} \in L^2$

$$\underset{\substack{\parallel \\ (-i\xi)^\alpha \hat{f}(\xi)}}{\widehat{\partial^\alpha f}}$$

Donc $|\widehat{\partial^\alpha f}| \leq |\xi|^{|\alpha|} |\hat{f}(\xi)|$

Si $|\xi| \geq 1 \Rightarrow |\xi|^{|\alpha|} \leq |\xi|^k \leq (1+|\xi|^2)^{k/2} = \Lambda^k(\xi)$

$|\xi| < 1 \Rightarrow |\xi|^{|\alpha|} \leq 1 \leq (1+|\xi|^2)^{k/2} = \Lambda^k(\xi)$

$$\|\widehat{\partial^\alpha f}\|_2 \leq \|\Lambda^k \hat{f}\|_2 < +\infty \Rightarrow f \in W^{k,2}$$

Le fait que $f \in \mathcal{S}'$: on vient de mq $\hat{f} \in L^2$

$\Rightarrow (\hat{f} \in \mathcal{S}' \Leftrightarrow f \in \mathcal{S}')$

□

$$W^{k,2} = \{ T \in \mathcal{S}' / \underbrace{\Lambda^k}_{\underset{\mathcal{O}_M}{\cap}} T \in \mathcal{S}' \} \; (= H^k)$$

<u>Déf</u> Soit $s \in \mathbb{R}$

$$H^s = \{ T \in \mathcal{S}' / \Lambda^s T \in \mathcal{S}' \}$$

On dit que $u \in H^s_{loc}$ si $\forall K \subset\subset \mathbb{R}^d \; \exists v \in H^s$ t.q. $u = v$ sur K

$\Updownarrow$

$T \in \mathcal{S}'$ est t.q. $T \in H^s_{loc} \Leftrightarrow \varphi T \in H^s \; \forall \varphi \in C_c^\infty$

<u>Lemme</u> Si $P(D)$ est elliptique d'ordre m

Si $u \in H^s$, $\underbrace{P(D)u}_{=g} \in H^s \Rightarrow u \in H^{s+m}$

<u>Preuve</u> $P(D)$ elliptique d'ordre m

$\Leftrightarrow p_m(\xi) \neq 0 \quad \forall \xi \in \mathbb{R}^d \setminus \{0\}$

$\underline{\text{mq}}$ $\exists C, R > 0$ / $|p(\xi)| \geq C|\xi|^m \quad \forall |\xi| \geq R$

$\boxed{\begin{array}{l}\text{Exo}\\ \underline{\text{mq}}\ \cdot \\ \Rightarrow P(D) \\ \text{elliptique}\end{array}}$

$$p_m(\xi) = \sum_{|\alpha|=m} c_\alpha \xi^\alpha \Rightarrow p_m(\lambda\xi) = \sum_{|\alpha|=m} c_\alpha (\lambda\xi)^\alpha = \lambda^m p_m(\xi) \quad \forall \lambda > 0$$

$\rightsquigarrow$ p_m homogène de dégré m.

$\Rightarrow$ $\xi \in \mathbb{R}^d$, $\xi = |\xi| \cdot \omega$ où $|\omega| = 1$

$$|p_m(\xi)| = |p_m(|\xi|\omega)| = |\xi|^m |p_m(\omega)| \geq C_1 |\xi|^m \quad \text{où} \quad C_1 = \min_{|\omega|=1} p_m(\omega)$$

$$|(p - p_m)(\xi)| = \Big|\sum_{|\alpha| \leq m-1} c_\alpha \xi^\alpha\Big| \leq \sum_{|\alpha| \leq m-1} |c_\alpha| \, |\xi|^{|\alpha|} \leq C_2 |\xi|^{m-1} \quad \text{si} \quad |\xi| \geq 1 \quad \left(\Rightarrow |\xi|^{|\alpha|} \leq |\xi|^{m-1}\right)$$

$$|p(\xi)| \geq |p_m(\xi)| - |(p-p_m)(\xi)| \geq C_1 |\xi|^m - C_2 |\xi|^{m-1} \quad \text{si} \quad |\xi| \geq 1$$

Si $|\xi| \geq \dfrac{2C_2}{C_1} \Rightarrow |\xi|^m \geq \dfrac{2C_2}{C_1} |\xi|^{m-1}$

$$\geq \frac{C_1}{2} |\xi|^m + \underbrace{\frac{C_1}{2}\left(|\xi|^m - C_2 \cdot \frac{2}{C_1} |\xi|^{m-1}\right)}_{\geq 0 \quad \text{si} \quad |\xi| \geq \frac{2C_2}{C_1}}$$

$\underline{\text{mq}}$ $u \in H^{m+s}$ si $u \in H^s$ et $P(D)u \in H^s$

$$u \in H^s \implies (1+|\xi|^2)^{s/2}\,\hat{u}(\xi) \in L^2$$

$$P(D)u \in H^s \implies (1+|\xi|^2)^{s/2}\,p(\xi)\,\hat{u}(\xi) \in L^2$$

mq $$(1+|\xi|^2)^{\frac{s+m}{2}}\,\hat{u}(\xi) \in L^2 \iff u \in H^{s+m}$$

$$|p(\xi)| \geq C|\xi|^m \quad \text{si} \quad |\xi| \geq R, \quad R \geq 1$$

Si $|\xi| > R \geq 1$: $(1+|\xi|^2)^{\frac{m}{2}} \leq C_0|\xi|^m \leq \frac{C_0}{C}|p(\xi)|$

$$\int_{\mathbb{R}^d} \left|(1+|\xi|^2)^{\frac{s+m}{2}}\,\hat{u}(\xi)\right|^2 d\xi = \underbrace{\int_{|\xi|\leq R} \cdots}_{A} + \underbrace{\int_{|\xi|>R} (1+|\xi|^2)^{s+m}\,|\hat{u}(\xi)|^2\,d\xi}_{B}$$

$$B \leq \left(\frac{C_0}{C}\right)^2 \int_{|\xi|>R} (1+|\xi|^2)^s\,|p(\xi)|^2\,|\hat{u}(\xi)|^2\,d\xi \leq \|\Lambda^s p\,\hat{u}\|_2^2$$

$$A = \int_{|\xi|\leq R} (1+|\xi|^2)^{s+m}\,|\hat{u}(\xi)|^2\,d\xi \leq (1+R^2)^m \int_{|\xi|\leq R} (1+|\xi|^2)^s\,|\hat{u}(\xi)|\,d\xi$$

$$\|\Lambda^{s+m}\hat{u}\|_2^2 \underbrace{\leq}_{<+\infty} \left(\frac{C_0}{C}\right)\|\Lambda^s p\hat{u}\|_2^2 + (1+R^2)^m\,\|\Lambda^s\hat{u}\|_2^2 .$$

$\square$

Théorème P operateur différentiel elliptique à coeffs. constants d'ordre m

hypo ellipticity

$\Omega \subset \mathbb{R}^d$, $u \in \mathcal{D}'(\Omega)$

Si $Pu = g$ où $g \in H^s_{loc}(\Omega) \Rightarrow u \in H^{s+m}_{loc}(\Omega)$

de plus Si $\underbrace{g \in C^\infty \Rightarrow u \in C^\infty}_{\text{ipoellipticité}}$

Il faut prouver que il existe une inverse à gauche $\underset{P^+}{\wedge}$ de P t.q.

$$P^+ : H^s \longrightarrow H^s \quad \not\Rightarrow \left(g \in H^s \Rightarrow u \in H^s\right)$$

$Pu = g \rightsquigarrow u = P^+ g = F * g$ où F sol. fond

mq $P^+(H^s) \subset H^s \rightsquigarrow$ trouver de bornes sur F

Ex Δ en $\mathbb{R}^d$, $d \geq 3$: $F(x) = c\,|x|^{2-d}$

$$\hat{F}(\xi) = -\frac{1}{|\xi|^2}$$

$$g \in H^s \longrightarrow (1+|\xi|^2)^{s/2}\,\hat{g} \in L^2$$

mq $F * g \in H^s \longrightarrow (1+|\xi|^2)^{s/2}\,\hat{F}(\xi)\,\hat{g}(\xi) \in L^2$

$$\int_{\mathbb{R}^d} (1+|\xi|^2)^s \left|\hat{g}(\xi)\right|^2 \cdot \frac{d\xi}{|\xi|^4} < +\infty$$

$\rightsquigarrow \quad \Delta u \in L^2 \Rightarrow u \in H^2 = W^{2,2}$

$$\int \Big| \sum_{j=1}^{d} \partial_j^2 u \Big|^2 dx < +\infty \qquad\qquad \int |\partial_j u|^2 < +\infty$$

$$\int |\partial_j^2 u|^2 < +\infty$$

$\to$ Evans, <u>Partial differential equations</u> (Chp 2)

$$\to \quad \begin{cases} \partial_t u = \Delta u + \chi_\omega f \\ u|_{t=0} = u_0, \quad u_0 \geq 0 \\ \nabla u \cdot n_\Omega = 0 \end{cases}$$

$$\forall t \quad \int_\Omega u(t,x)\,dx = \int u_0(x)\,dx, \qquad u(t,x) \xrightarrow[t\to+\infty]{} \frac{1}{|\Omega|}\int_\Omega u_0(x)$$

$$u(t,x) \geq 0 \quad \forall t, x \in \Omega$$

<u>Question controle</u> $\exists ?\ f(t,x) \ /\ u_f(t,x) \xrightarrow[t\to T]{L^2} 0$

$T > 0$

$\rightsquigarrow$ Coron, <u>Control and non-linearity</u>