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OUTILS ANALYSE EDP

1^{er}
- CONVOLUTION
- DERIVES FAIBLES
- ESPACES DE SOBOLEV
- DISTRIBUTIONS ET TRANSF. FOURIER

2^{eme}
- EDP $\to$ TRANSPORT, $\to$ CHALEUR
- CONTROLE D'EDP

NOTATIONS :

$\to$ $L^p(\mathbb{R}^n) = \{ u : \mathbb{R}^n \to \mathbb{R}$ mesurables t.q. $\|u\|_p < +\infty \}$

$$\|u\|_p = \left(\int_{\mathbb{R}^n} |u|^p \right)^{1/p} \qquad p \in [1, +\infty)$$

$$\|u\|_\infty = \sup_{x \in \mathbb{R}^n} |u(x)|$$

$\to$ Exposant conjugué à $p \in [1, \infty]$, $p' \in [1, +\infty]$ t.q. $\frac{1}{p} + \frac{1}{p'} = 1$

$p = 2 \longrightarrow p' = 2$

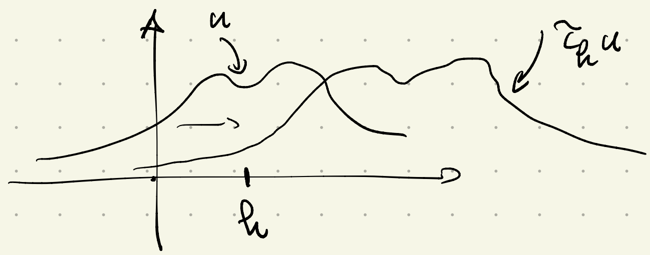
$p = 1 \longrightarrow p' = +\infty$

$p = +\infty \longrightarrow p' = 1$

$$\|u\|_p = \sup \left\{ \int_{\mathbb{R}^d} u \cdot v \;\Big/\; v \in L^{p'}(\mathbb{R}^d) \text{ t.q. } \|v\|_{p'} \leq 1 \right\}$$

→ <u>Translation</u> $h \in \mathbb{R}^n$, $u : \mathbb{R}^n \longrightarrow \mathbb{R}$

$$\tau_h u(x) = u(x - h)$$



→ <u>Espaces fonctionnelles</u> : $\Omega \subset \mathbb{R}^n$ ouvert

•) $C^0(\Omega) = \{ u : \Omega \to \mathbb{R} \,/\, u \text{ continue} \}$

•) $C^0_c(\Omega) = \{ u \in C^0(\Omega) \,/\, \text{supp } u \subset\subset \Omega \}$

compactem. contenu

•) $C^0_0(\Omega) = \{ u \in C^0(\Omega) \,/\, u(x) \underset{x \to \partial\Omega}{\longrightarrow} 0 \}$

•) $C^0_b(\Omega) = \{ u \in C^0(\Omega) \,/\, \exists M > 0 \text{ t.q. } |u(x)| \leq M \;\; \forall x \in \Omega \}$

•) $C^0_u(\Omega) = \{ u : \Omega \to \mathbb{R} \,/\, u \text{ uniform. continue} \}$

$$C^0_c(\Omega) \subset C^0_0(\Omega) \subset C^0(\Omega)$$

$$C^0_c(\Omega) \subset C^0_b(\Omega)$$

$$C^0_c(\Omega) \subset C^0_u(\Omega) \quad \to \quad \text{Thm de Heine}$$

•) Sur $C^0(\Omega)$ on considère la topologie de la convergence uniforme sur les compactes

$u : \Omega \longrightarrow \mathbb{R}$ $\qquad K \subset\subset \Omega$

$$\to \quad \|u\|_{L^\infty(K)} = \sup_{x \in K} |u(x)| \quad < +\infty$$

$(u_n)_n \subset C^0(\Omega)$, $\quad u_n \longrightarrow u \in C^0(\Omega)$ ssi

$$\forall K \subset\subset \Omega \qquad \|u_n - u\|_{L^\infty(K)} \xrightarrow[n]{} 0$$

Rmq en général on ne peut pas considérer

$$\|u\|_\infty = \sup_{x \in \Omega} |u(x)|$$

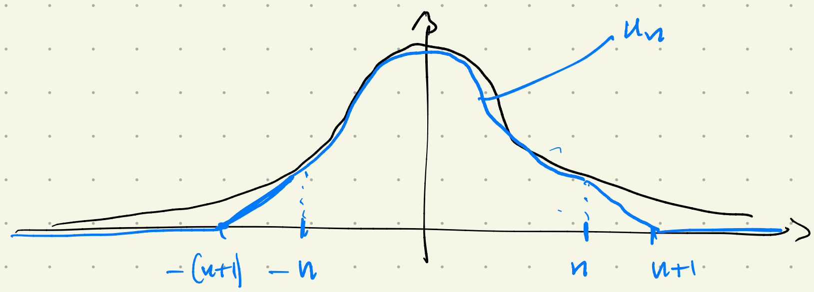
car on pourrait avoir $\|u\|_\infty = +\infty$

$$\left(\Omega = \mathbb{R}, \qquad u(x) = x \right)$$

•) $C^0_c(\Omega)$ $\quad$ ici $\|u\|_\infty < +\infty \quad \forall u \in C^0_c(\Omega)$

$\rightsquigarrow \left(C^0_c(\Omega), \|\cdot\|_\infty \right)$ est un evn mais $C_c^0(\Omega)$ n'est pas complet

($\Omega = \mathbb{R}$



$u(x) = e^{-x^2}$ $\quad u \in C^0(\mathbb{R})$, $u \notin C_c^0(\Omega)$

$(u_n)_n$ de Cauchy dans $C_c^0(\mathbb{R})$:

$n, k \in \mathbb{N}$, $\quad \|u_n - u_{n+k}\|_\infty = \sup_{x \in \mathbb{R}} |u_n(x) - u_{n+k}(x)|$

$u_n(x) = u_{n+k}(x)$ si $|x| < n$ ou $|x| > n+k+1$

$u_n(x) = u_{n+k}(x) = e^{-x^2}$ $\qquad u_n(x) = u_{n+k}(x) = 0$

$$\leq \sup_{n < |x| < n+k+1} |u_n(x) - u_{n+k}(x)|$$

$$\leq \sup_{n < |x| < n+k+1} \left(|u_n(x)| + |u_{n+k}(x)| \right)$$

$$\leq \sup |u_n(x)| + \sup |u_{n+k}(x)|$$

$$= u_n(n) + u_{n+k}(n) = 2e^{-n^2} \xrightarrow[n \to +\infty]{} 0 \,.\ )$$

$\rightsquigarrow$ mais $\|u_n - u\|_\infty \xrightarrow[n]{} 0$

•) Si on fait la complétion de $(C_c^0(\Omega), \|\cdot\|_\infty)$

on obtient $(C_0^0(\Omega), \|\cdot\|_\infty)$

•) $\left(C^0_b(\Omega), \|\cdot\|_\infty\right)$ est un espace de Banach

et $\underbrace{C^0_0(\Omega) \subset C^0_b(\Omega) \text{ sev fermé.}}_{\text{exo}}$

•) $\underbrace{C^0_u(\Omega) \text{ n'est pas un evn}}$ (comme $C^0(\Omega)$)

$\left(\Omega = \mathbb{R}, \quad u(x) = x, \quad u \in C^0_u(\mathbb{R})\right)$

On peut définir $\underbrace{C^0_{b,u}(\Omega)}_{\text{evn avec norme } \|\cdot\|_\infty} = C^0_b(\Omega) \cap C^0_u(\Omega)$

Exercice: $u \in C^0_u(\mathbb{R}^n)$ ssi $\|\tau_h u - u\|_\infty \xrightarrow[h\to 0]{} 0$

Obs: $\forall x \in \mathbb{R}^d$, $u \in C^0(\mathbb{R}^d) \iff |\tau_h u(x) - u(x)| \xrightarrow[h\to 0]{} 0$

$$|\tau_h u(x) - u(x)| = |u(x-h) - u(x)|$$

$\iff \forall x_0 \in \mathbb{R}^d \quad \lim_{x\to x_0} u(x) = u(x_0)$

Exercice: Est-ce que $\left(C^0_{b,u}(\mathbb{R}^d), \|\cdot\|_\infty\right)$ est complet ?

Lemme $u \in L^p(\mathbb{R}^d)$, $p < +\infty$ ou $u \in C^0_{b,u}(\mathbb{R}^d)$, $p = \infty$

on a $\lim\limits_{r \downarrow 0} \sup\limits_{|h|<r} \| \tau_h u - u \|_p = 0$

Preuve (1) On commence par considérer $u \in C^0_c$

$$\| \tau_h u - u \|_p^p = \int_{\mathbb{R}^d} |u(x-h) - u(x)|^p \, dx$$

On sait que $\underbrace{\sup\limits_{|h|<r} |\underset{\in C^0_c}{u}(x-h) - u(x)|^p \xrightarrow[r \downarrow 0]{} 0}_{\text{(exercice)}}$

Admettons

$$\lim_{r \downarrow 0} \sup_{|h|<r} \int_{\mathbb{R}^d} |u(x-h) - u(x)|^p \, dx = \int_{\mathbb{R}^d} \underbrace{\lim_{r \downarrow 0} \sup_{|h|<r} |u(x-h) - u(x)|^p}_{=0} \, dx$$

Pour montrer on utilise thm conv. dominé

mq $\exists g \in L^1(\mathbb{R}^d)$ t.q.

$$\sup_{|h|<r} |u(x-h) - u(x)|^p \leq g(x) \quad \text{p.p.}$$

On calcule: $\sup\limits_{|h|<r} |u(x-h) - u(x)|^p \leq$

$(a+b)^p \leq c(a^p + b^p)$

$$\leq \sup_{|h|<r} \left(|u(x-h)| + |u(x)| \right)^p$$

$$\leq c \left(\sup_{|h|<r} \overbrace{|u(x-h)|^p}^{=0 \text{ si } x \notin (\operatorname{supp} u)_r} + \sup_{|h|<r} \overbrace{|u(x)|^p}^{=0 \text{ si } x \notin \operatorname{supp} u} \right)$$

K cpt

$K_r = K + B_r$
$\stackrel{!}{=} \{x \in \mathbb{R}^d / \inf_{y \in K} |x-y| < r\}$

$$\stackrel{!}{\leq} \underbrace{\underbrace{c\|u\|_\infty^p}_{\notin L^1(\mathbb{R}^d)} \chi_{(\text{supp}\, u)_1}(x)}_{= g(x)} \Rightarrow \text{supp}\, g \subset\subset \mathbb{R}^d \Rightarrow g \in L^1(\mathbb{R}^d)$$

$r < 1$

Donc on a montré le résultat pour $u \in C_c^0 \subset L^p$

(2) $u \in L^p$ et on montre le résultat par densité.

$$\exists (u_n)_n \subset C_c^0 \;/\; u_n \xrightarrow{L^p} u$$

$$\|\tau_h u - u\|_p = \|\tau_h u \overbrace{- \tau_h u_n + \tau_h u_n}^{=0} \overbrace{- u_n + u_n}^{=0} - u\|_p$$

$$\leq \|\tau_h u - \tau_h u_n\|_p + \underline{\|\tau_h u_n - u_n\|_p} + \underline{\|u_n - u\|_p}$$

On observe que $(\tau_h u - \tau_h u_n)(x) = u(x-h) - u_n(x-h) \stackrel{!}{=} \tau_h[u - u_n]$

$$\|\tau_h u - \tau_h u_n\|_p^p = \|\tau_h(u_n - u)\|_p^p$$
$$\stackrel{!}{=} \int_{\mathbb{R}^d} |u_n(\underbrace{x-h}_{=y}) - u(x-h)|^p \, dx \qquad dy = d(x-h) = dx$$
$$\stackrel{!}{=} \int_{\mathbb{R}^d} |u_n(y) - u(y)|^p \, dy = \|u_n - u\|_p^p$$

$$\leq 2\|u_n - u\|_p + \|\tau_h u_n - u_n\|_p$$

$$\Rightarrow \lim_{r \downarrow 0} \sup_{|h|<r} \|\tau_h u - u\|_p \leq 2\|u_n - u\|_p + \underbrace{\cancel{\lim_{r \downarrow 0} \sup_{|h|<r} \|\tau_h u_n - u_n\|_p}}_{(1)\; = 0}$$

Donc on a mq

$$0 \leq \lim_{r \downarrow 0} \sup_{|h| < r} \|\tau_h u - u\|_p \leq \lim_{n \to +\infty} 2 \|u_n - u\|_p = 0$$

Le cas $p = +\infty$ correspond à l'exercice de ce matin. $\square$

<u>NOTATION</u> $\mathcal{M}^1(\Omega) = \{$ mesures de Radon bornées $\}$

norme de la variation totale

$$\|\mu\| = \int_\Omega d|\mu| = \sup \left\{ \int \varphi \, d\mu \,/\, \varphi \in C_c^0, \ \|\varphi\|_\infty \leq 1 \right\}$$

<u>Ex</u> : •) Si $\Omega = [0,1] \subset \mathbb{R}$, $\mu \in \mathcal{M}^1(\Omega)$ $\mu([a,b]) \geq 0$ $\forall 0 \leq a < b \leq 1$

$$\Rightarrow \|\mu\| = \mu([0,1]) = \int_0^1 d\mu$$

•) Si $\Omega \subset \mathbb{R}^d$ borné et $\mu = f(x)\,dx$ où $f \in C_c^0$

$$\Rightarrow \|\mu\| = \int_\Omega |f(x)|\,dx$$

En effet $\mathcal{M}^1(\Omega) = \left(C_c^0(\Omega)\right)'$

<u>Rmq</u> $u \in L^1(\Omega) \rightsquigarrow \mu_u \in \mathcal{M}^1(\Omega)$ $\left(\Rightarrow L^1(\Omega) \subset \mathcal{M}^1(\Omega)\right)$

$$\mu_u(A) = \int_A u(x)\,dx \qquad A \subset \Omega \text{ mesurable}$$

<u>Prop</u> $\|\mu_u\| = \|u\|_1 = \int_\Omega |u(x)|\,dx$

<u>Preuve</u> $\varphi \in C_c^0$, $\|\varphi\|_\infty \leq 1$

$$\int_\Omega \varphi \, d\mu_u = \int_\Omega \varphi(x) u(x) \, dx \leq \int_\Omega |\varphi(x)| \cdot |u(x)| \, dx$$

$$\leq \underbrace{\|\varphi\|_\infty}_{\leq 1} \cdot \underline{\int_\Omega |u(x)| \, dx}$$

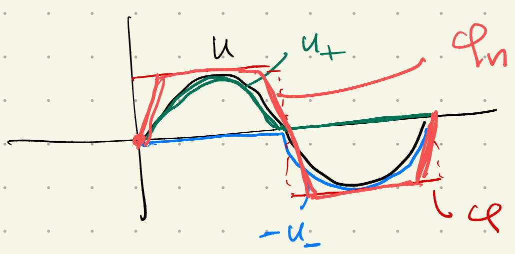
$$\|\mu_u\| = \sup \left\{ \left| \int_\Omega \varphi \, d\mu_u \right| \; \varphi \in C_c^0, \|\varphi\|_\infty \leq 1 \right\} \leq \int_\Omega |u(x)| dx$$

Il faut mq $\|\mu_u\| \geq \|u\|_1$

on va trouver $(\varphi_n)_n \subset C_c^0$, $\|\varphi_n\|_\infty \leq 1$

t.q. $\lim\limits_{n \to +\infty} \int_\Omega \varphi_n \, d\mu_u = \int_\Omega |u(x)| \, dx$

<u>observons</u> : $u \in L^1 \Rightarrow u = u_+ - u_-$ où $u_+, u_- \geq 0$



$$\int_\Omega |u(x)| \, dx = \int_\Omega u_+(x) dx + \int_\Omega u_-(x) \, dx$$

Si $\varphi(x) = \begin{cases} 1 & \text{si } x \in \{u(x) > 0\} \Leftrightarrow u_+(x) > 0 \\ -1 & \text{si } x \in \{u(x) < 0\} \Leftrightarrow u_-(x) > 0 \end{cases}$

$$\int_\Omega \varphi d\mu_u = \int_\Omega \varphi(x) u(x) \, dx = \int_{u_+(x)>0} u(x) \, dx - \int_{u_-(x)>0} u(x) \, dx$$

$$= \int u_+ \, dx + \int u_- \, dx = \int |u(x)| \, dx$$

Pb φ n'est pas continue, ni a support cpt !

Il faut donc trouver $(\varphi_n)_n \subset C_c^0$, $\|\varphi_n\|_\infty \leq 1$

t.q. $\int_\Omega \varphi_n \, d\mu_u \longrightarrow \int_\Omega \varphi \, d\mu_u = \int |u(x)| \, dx$

On verra plus tard comment ça est possible. $\square$

Def $a \in \mathbb{R}^d$ la <u>masse de Dirac</u> en a est

$$\delta_a \in \mathcal{M}^1(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} \varphi \, d\delta_a = \varphi(a)$$

$$\Rightarrow \quad \|\delta_a\| = 1$$

Convolution $\Omega = \mathbb{R}^d$

Déf $u \in L^p$, $v \in L^q$, $p, q \in [1, +\infty]$

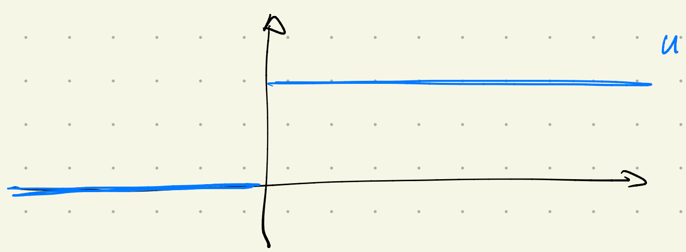
$$u * v(x) = \int_{\mathbb{R}^d} u(x-y)\, v(y)\, dy \quad \text{p.t. } x \in \mathbb{R}^d$$

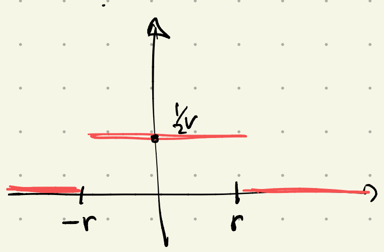
Rmq •) $u * v(x) = \int_{\mathbb{R}^d} u(z)\, v(x-z)\, dz$ en posant $z = x - y$ $\Rightarrow y = x - z$

$= v * u(x)$

Exemple $u(x) = \begin{cases} 0 & \text{si } x < 0 \\ 1 & \text{si } x \geq 0 \end{cases}$ Heviside

$v(x) = \frac{1}{2r} \chi_{[-r,r]}(x)$ $\int_{\mathbb{R}} v(x)\, dx = 1$

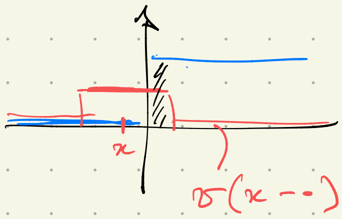




$$u * v(x) = \int_{\mathbb{R}} u(y)\, \underbrace{v(x-y)}_{\frac{1}{2r}\chi_{[-r,r]}(x-y)}\, dy$$

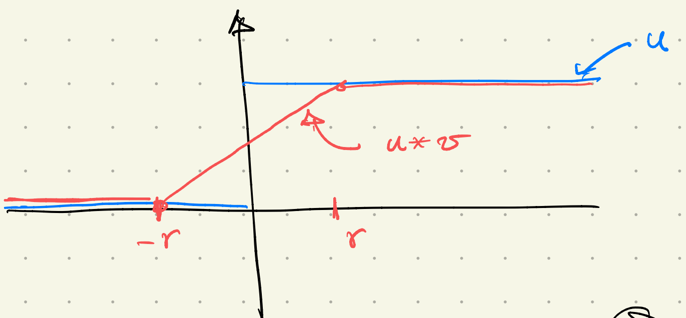
$\frac{1}{2r}\chi_{[-r,r]}(x-y) = \begin{cases} \frac{1}{2r} & \text{si } |x-y| < r \\ 0 & \text{sinon} \end{cases}$

$= \frac{1}{2r} \int_{x-r}^{x+r} u(y)\, dy$



$$u*v(x) = \begin{cases} 0 & x<-r \qquad \Rightarrow x+r<0 \\ \frac{1}{2r}\int_0^{x+r} 1 \, dy = \frac{x+r}{2r} & -r<x<r \\ \frac{1}{2r}\int_{x-r}^{x+r} 1 \, dy = 1 & x>r \qquad \Rightarrow 0<x-r<x+r \end{cases}$$

$$= \begin{cases} 0 & x<-r \\ \frac{x+r}{2r} & x\in[-r,r] \\ 1 & x>r \end{cases}$$



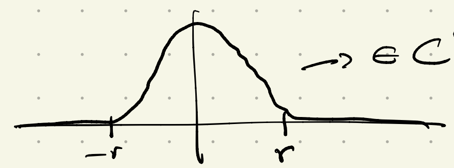
u discontinue

$\rightsquigarrow$

$u*v$ continue

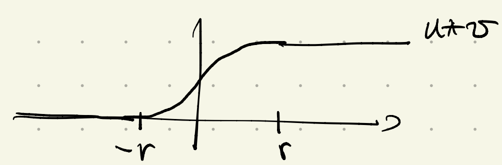
On verra $u*v_r \xrightarrow[r\downarrow 0]{L^1} u$

Si on remplace $v = \frac{1}{2r}\chi_{[-r,r]}$ par



$\to \in C^\infty$

$\Rightarrow \quad u*v \in C^\infty$



Lemme

$p, q, r \in [1,+\infty]$, $\quad \frac{1}{p}+\frac{1}{q} = 1+\frac{1}{r}$

$u\in L^p(\mathbb{R}^d)$, $\quad v\in L^q(\mathbb{R}^d)$, $\quad w\in L^{r}(\mathbb{R}^d)$

$$\left|\iint u(x)\, v(y)\, w(x+y)\, dx\, dy\right| \le \|u\|_p \|v\|_q \|w\|_{r'}$$

En particulier :

$$\|u * v\|_r \leq \|u\|_p \cdot \|v\|_q$$

et $\quad \operatorname{supp}(u*v) \subset \operatorname{adh}(\operatorname{supp} u + \operatorname{supp} v)$

<u>Preuve</u> (1) $u, v, w \in C_c^0(\mathbb{R}^d)$

Il suffit de montrer le résultat dans le cas $u, v, w \geq 0$

$$u(x)v(y)w(x+y) = \left(u(x)^p w(x+y)^{r'}\right)^a \left(v(y)^q w(x+y)^{r'}\right)^b \times \left(u(x)^p v(y)^q\right)^c$$

$$\begin{cases} pa + pc = 1 \\ ar' + br' = 1 \\ qb + qc = 1 \end{cases} \longrightarrow \quad a = \frac{1}{p} - \frac{1}{r}, \quad b = \frac{1}{q} - \frac{1}{r}, \quad c = \frac{1}{r}$$

$$a + b + c = 1 \qquad \left(\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}\right)$$

<u>Inégalité d'Hölder</u> (on le verra demain).

⋮