CORTICAL-INSPIRED FUNCTIONAL LIFTING FOR IMAGE INPAINTING

DARIO PRANDI

CNRS - L2S, CENTRALESUPÉLEC

SIAM conference on Imaging Science 2018

Mini-symposium "Geometry-driven anisotropic approaches for imaging problems"

Understand and exploit the advantages of the human visual system for image processing

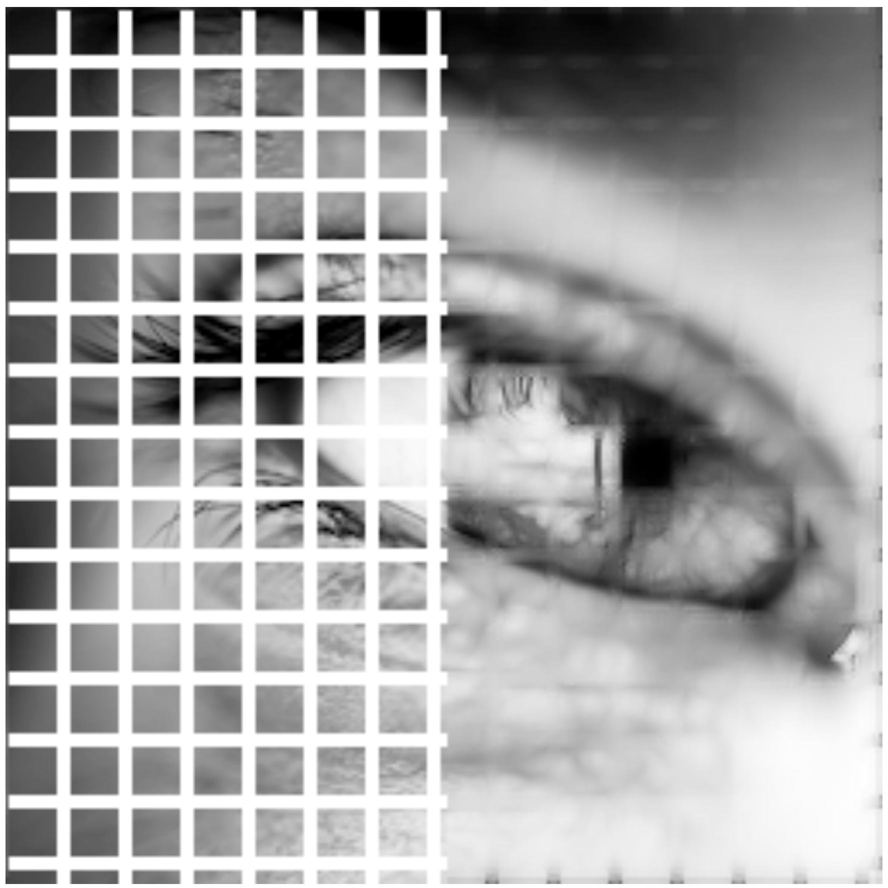
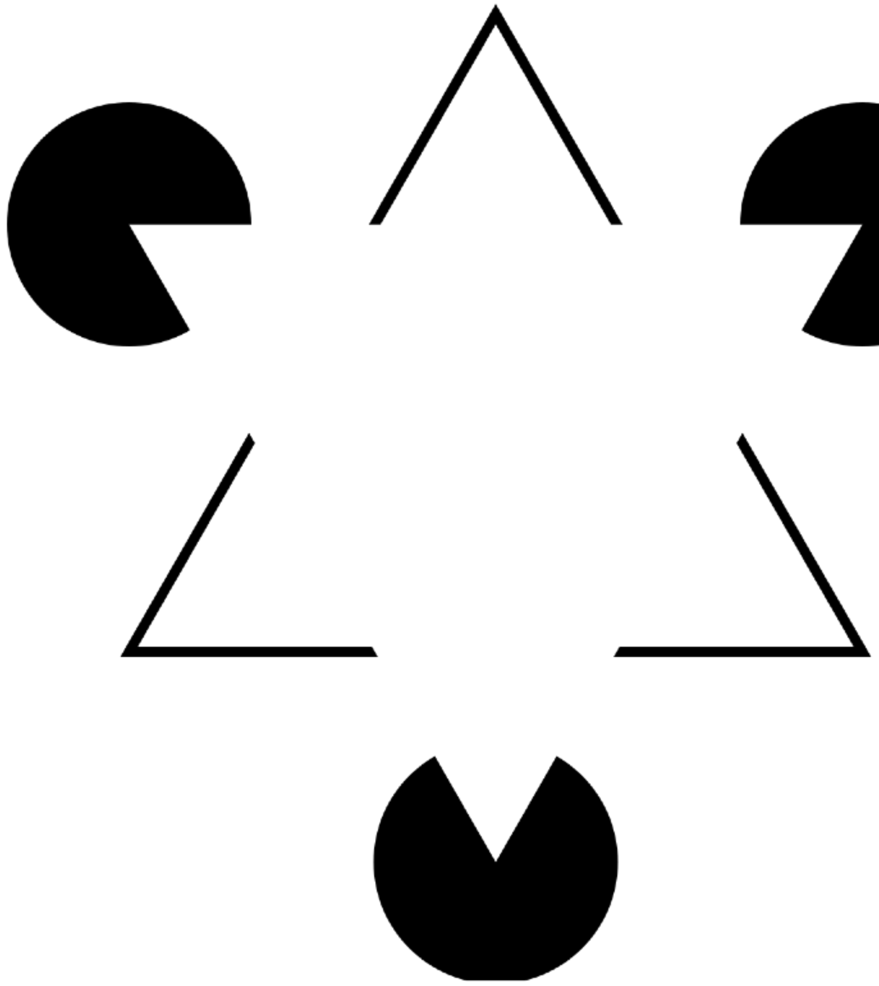


Image reconstruction

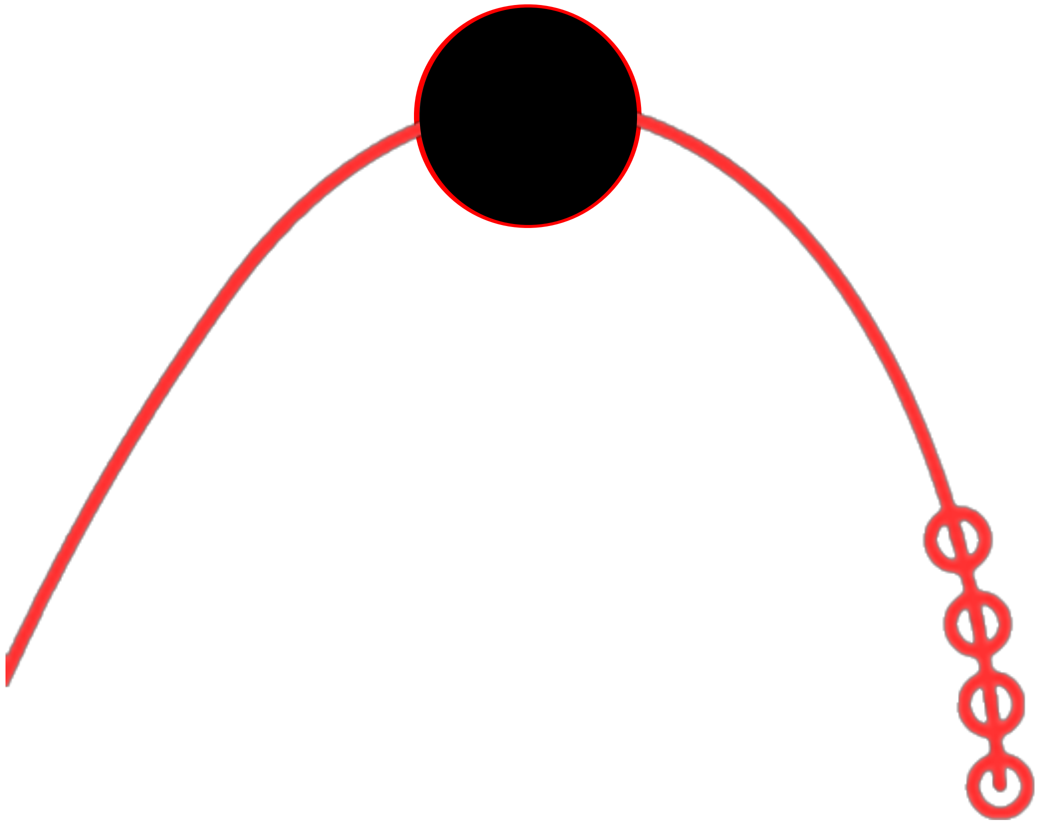
In collaboration with:
U. Boscain, J.-P. Gauthier,
R. Chertovskii, A. Remizov

Image recognition

In collaboration with:
J.-P. Gauthier, A. Bohi,
F. Bouchara, V. Guis



Kanisza triangle



CURVE RECONSTRUCTION

THE PROBLEM

Data :

$$\gamma : [0, a] \cup [b, 1] \to \mathbb{R}^2$$

$$\gamma(t) = (x(t), y(t))$$

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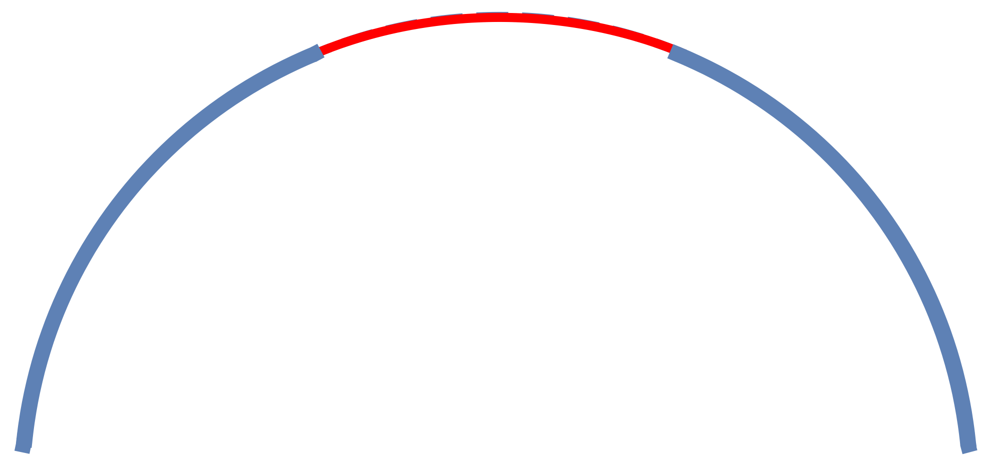

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$$\gamma(t) = (x(t), y(t))$$

Obtain a reasonable reconstruction

$$\gamma : [0, 1] \to \mathbb{R}^2$$

NAIVE IDEA : MINIMISE THE LENGTH

Complete the curve by finding $\tilde{\gamma} : [a, b] \to \mathbb{R}^2$ such that:

$$\tilde{\gamma}(a) = \gamma(a) \qquad \tilde{\gamma}(b) = \gamma(b)$$

$$\ell(\gamma) = \int_a^b \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} \to \min$$

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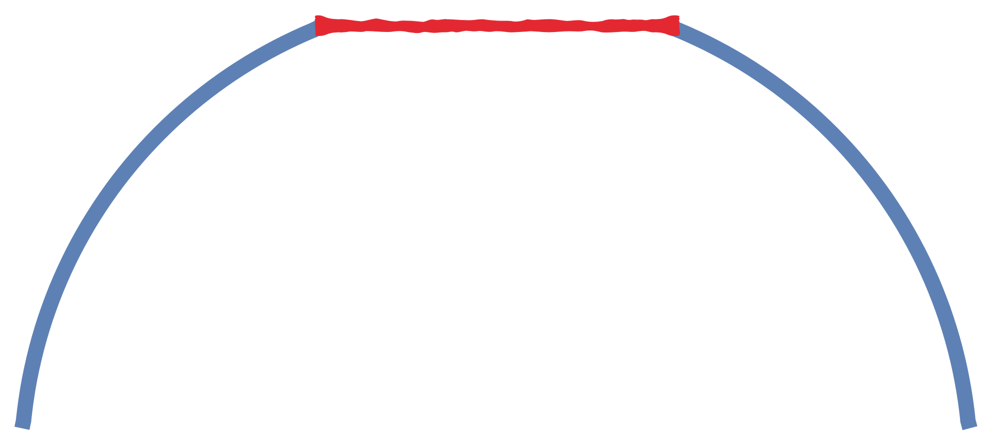

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NOT "REASONABLE" → The length is always minimised by straight lines

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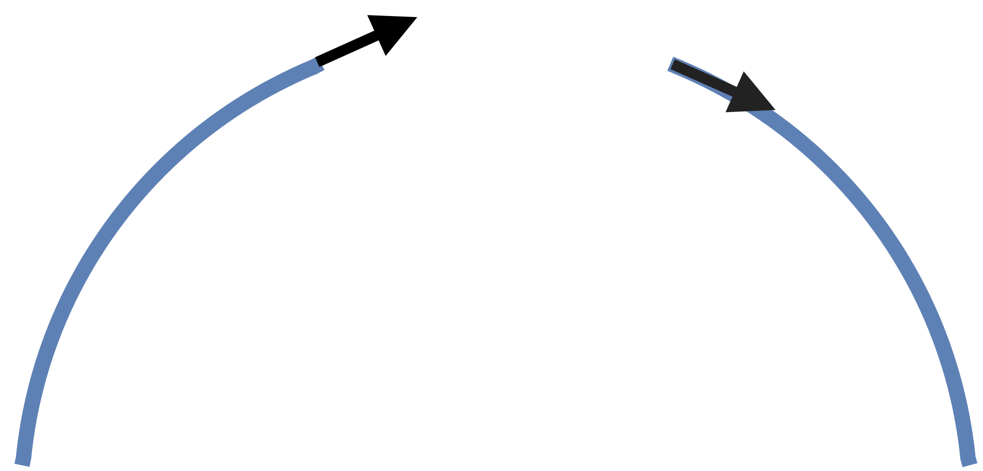

We ignore the direction of γ

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MUMFORD METHOD: EULER'S ELASTICA

D. Mumford (1994) proposed to look for a solution $\tilde{\gamma}$ of the system

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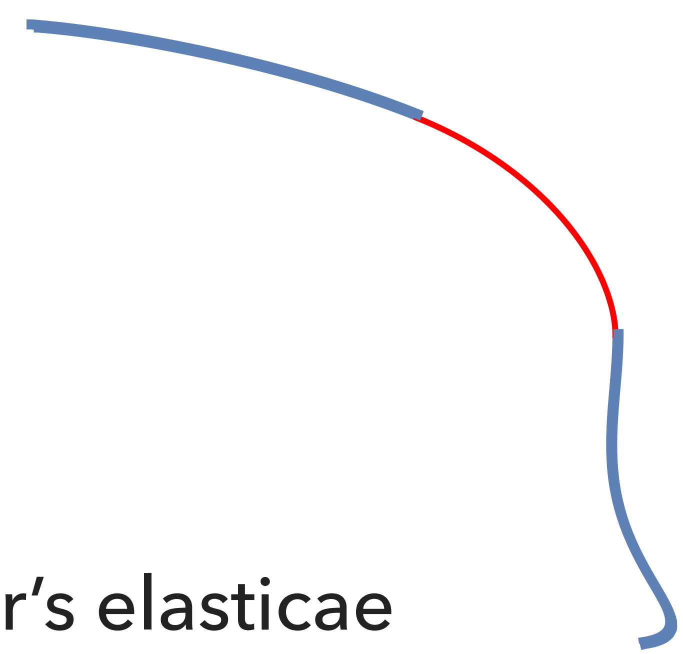
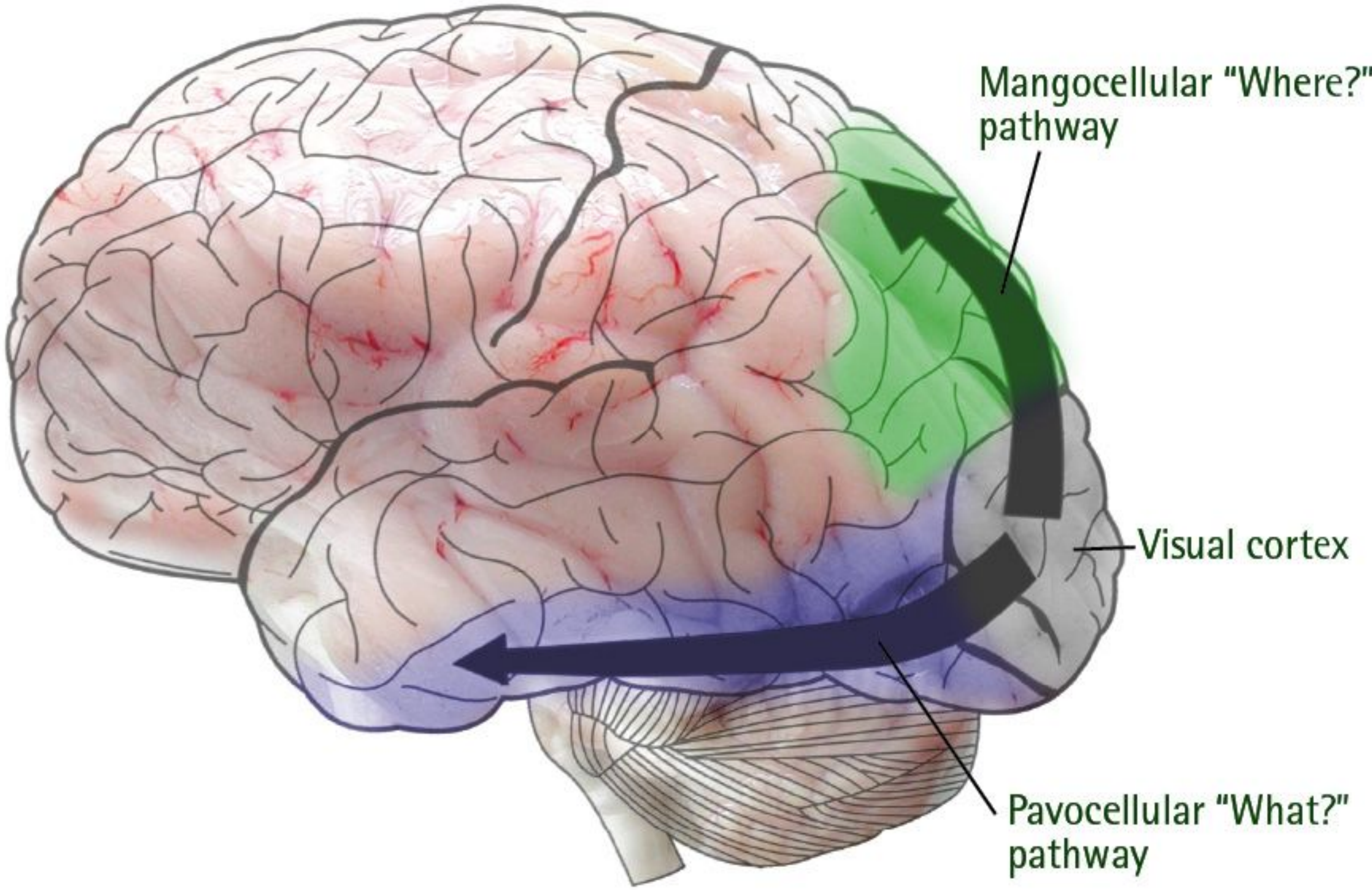


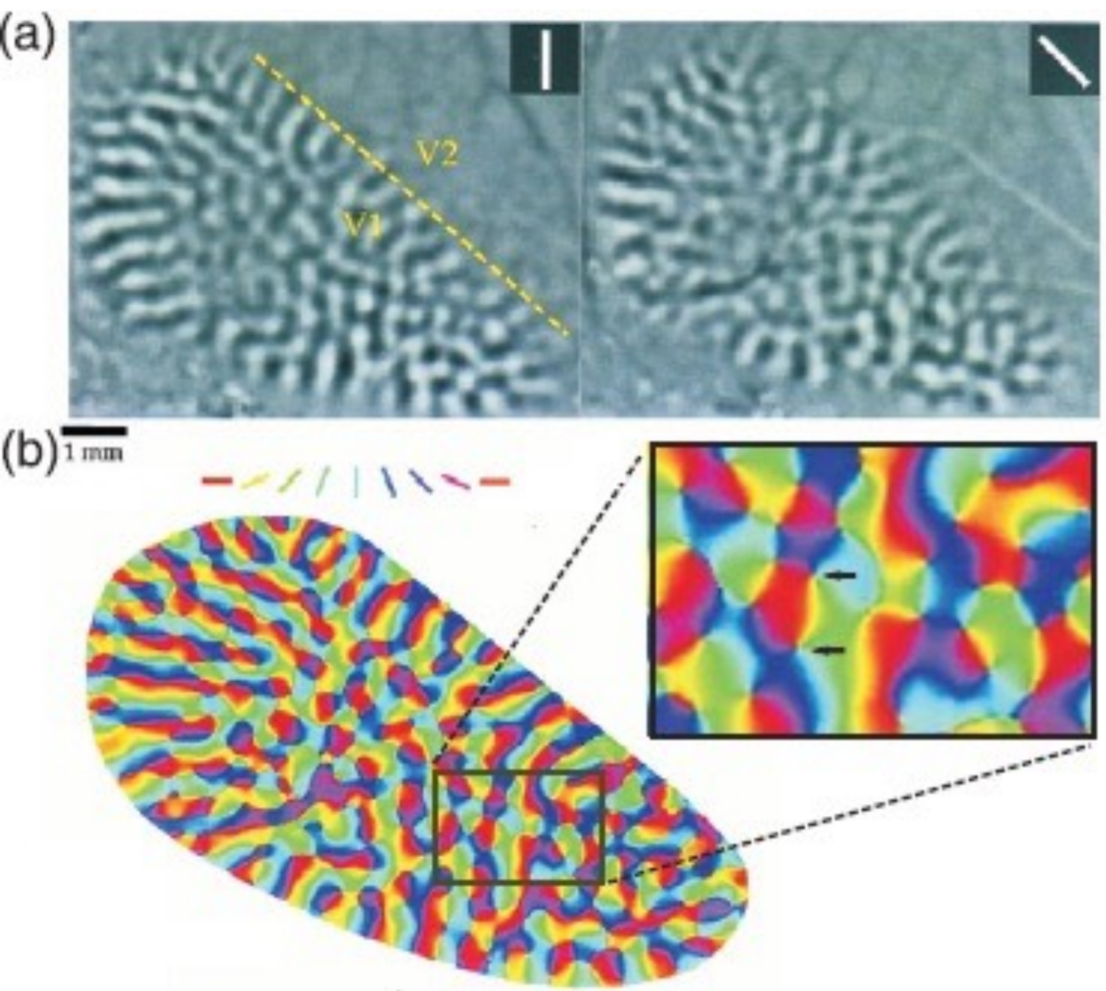
The solutions are called Euler's elasticae

WHY IT SEEMS MORE "REASONABLE" TO RECONSTRUCT CURVES À LA MUMFORD?

The answer lies probably in how our brain processes images



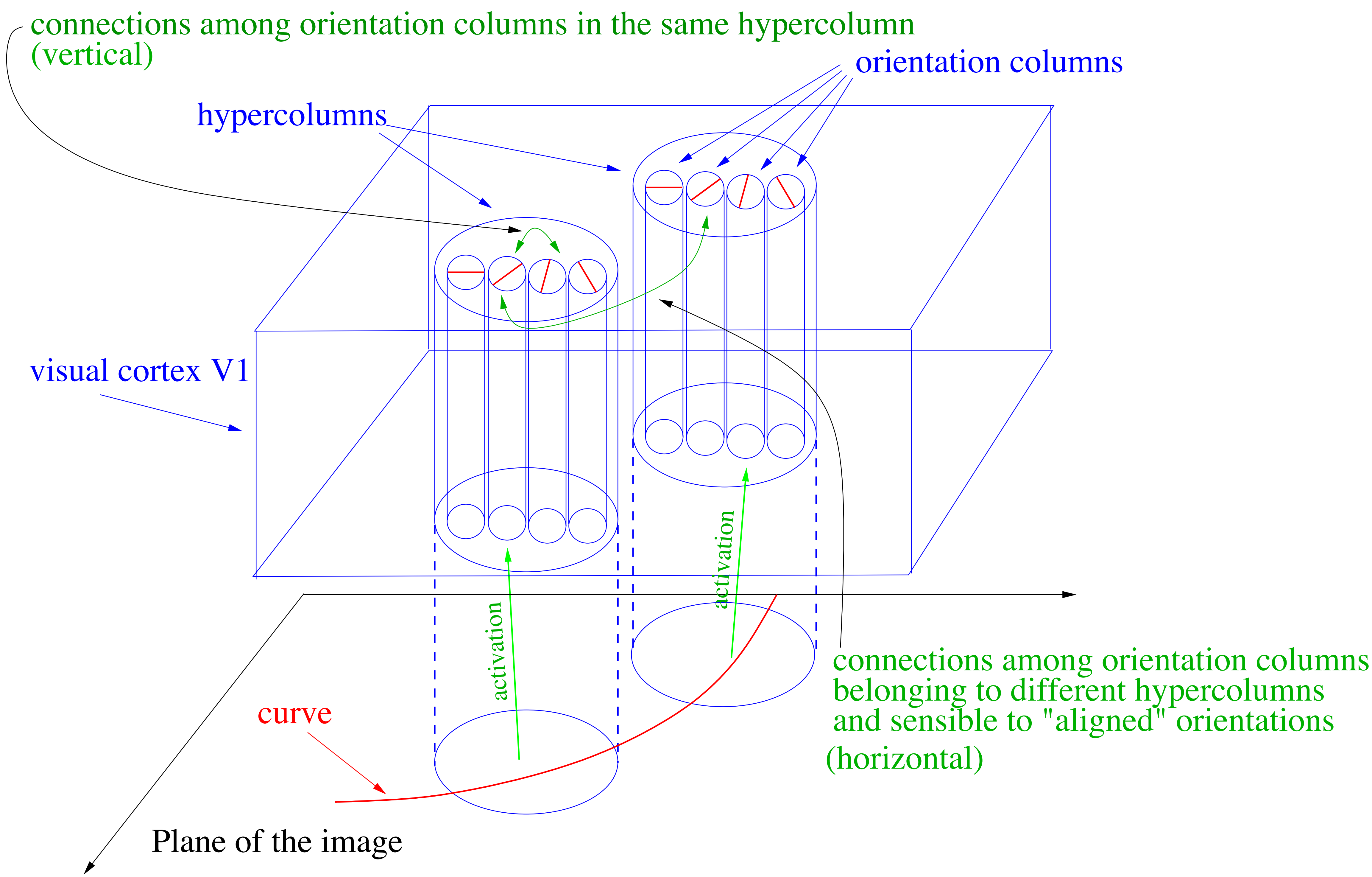
MATHEMATICAL MODEL OF THE PRIMARY VISUAL CORTEX V1



Sensibility to local orientations
in a tree shrew
Kaschube *et al.* (2008)

Neurons are sensible
to position
and
local orientations

Hubel et Wiesel
(Nobel prize '81)

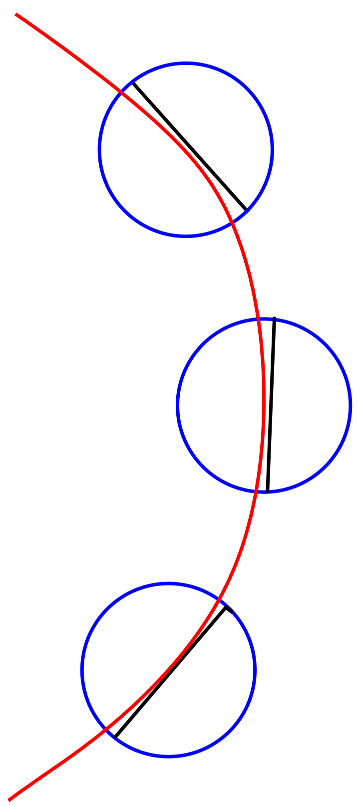

connections among orientation columns in the same hypercolumn
(vertical)
orientation columns
hypercolumns
visual cortex V1
activation
activation
curve
connections among orientation columns
belonging to different hypercolumns
and sensible to "aligned" orientations
(horizontal)
Plane of the image

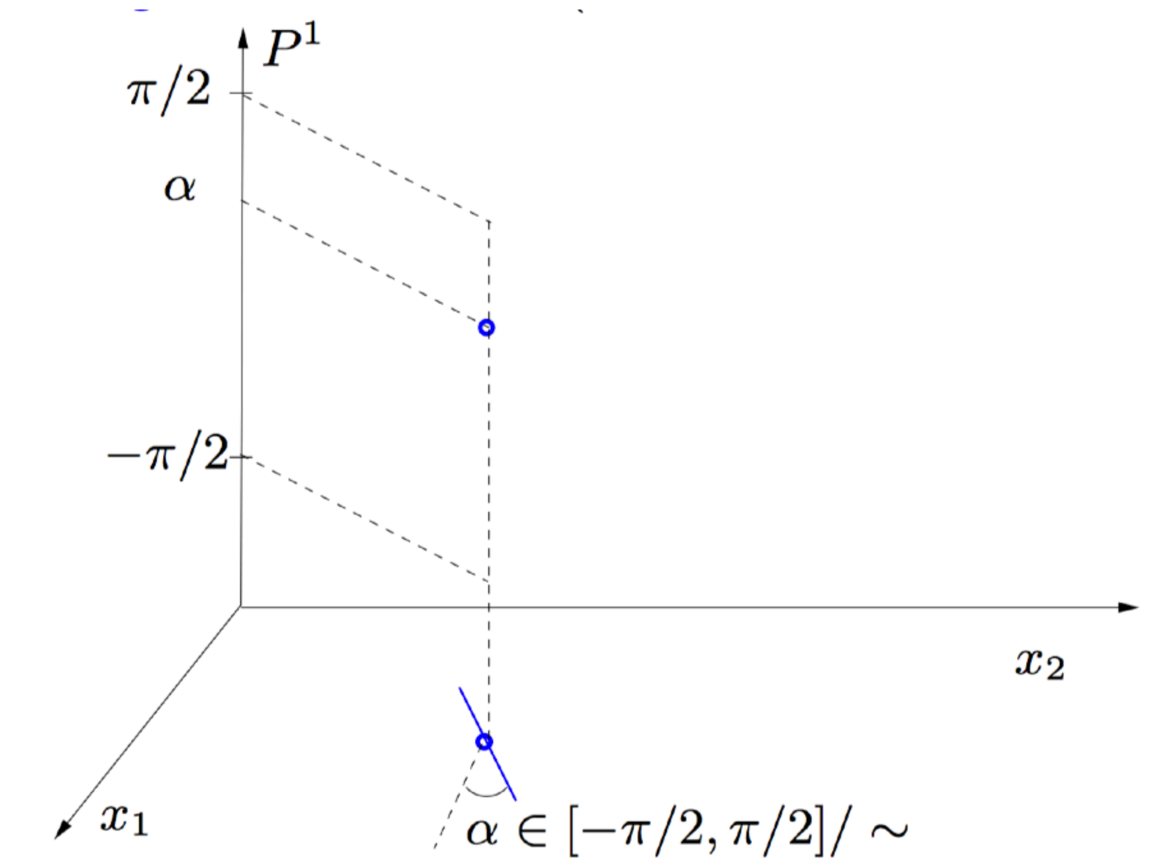
MODELLING V1

[Hoffman '89, Petitot-Tondut '99, Citti-Sarti '06]

- Neurons of V1 ⟷ Points of the Lie group:

$$SE(2) = \mathbb{R}^2 \times \mathbb{S}^1$$



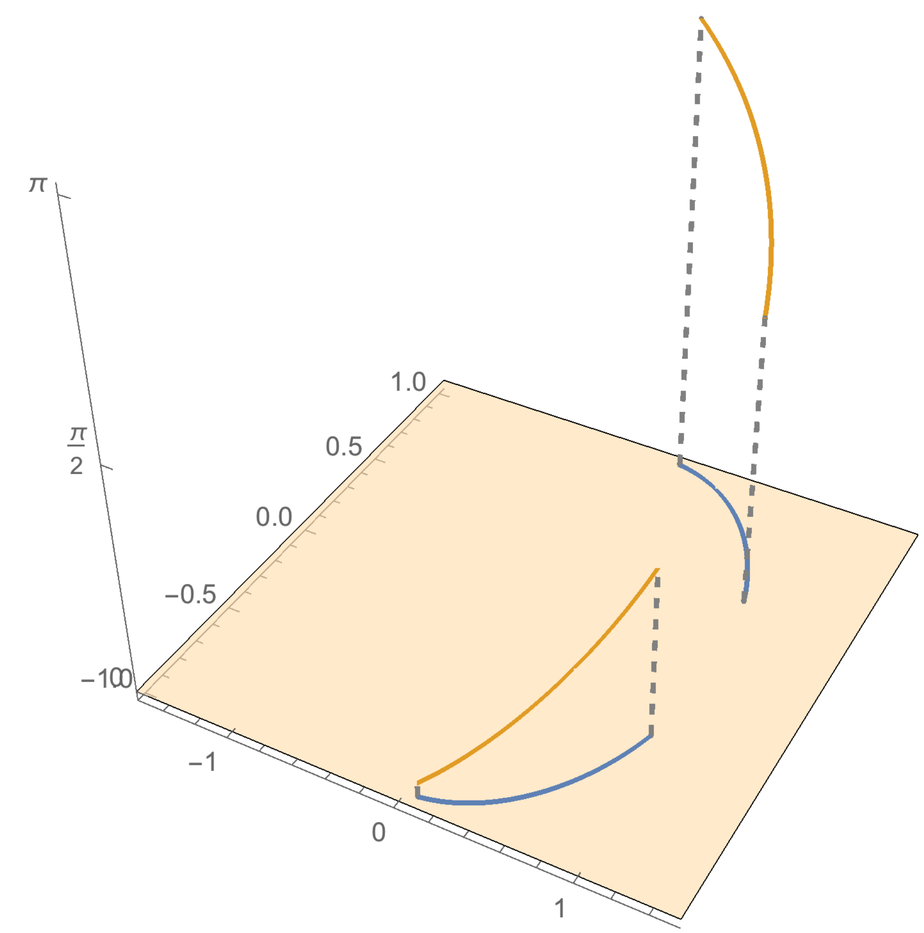


MODELLING V1

- Neurons of V1 $\longleftrightarrow$ Points of the Lie group:

$$SE(2) = \mathbb{R}^2 \times \mathbb{S}^1$$

- All planar curves can be lifted to $SE(2)$



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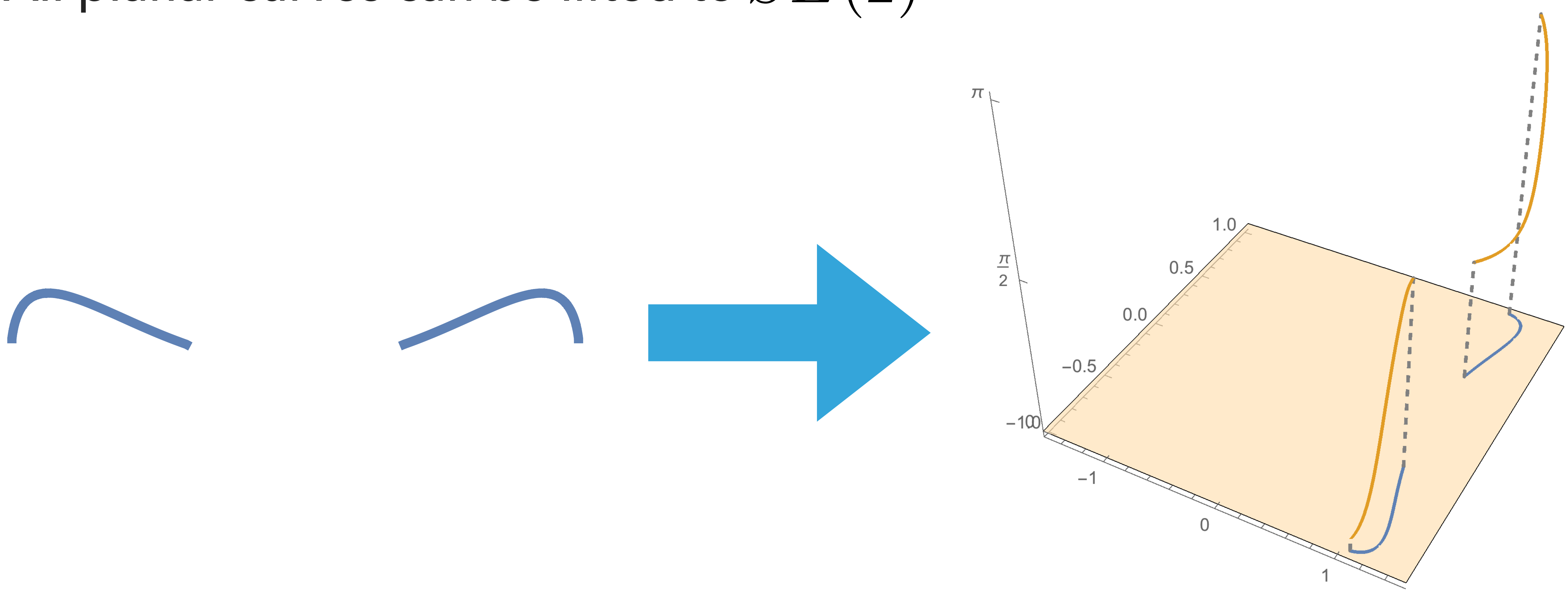

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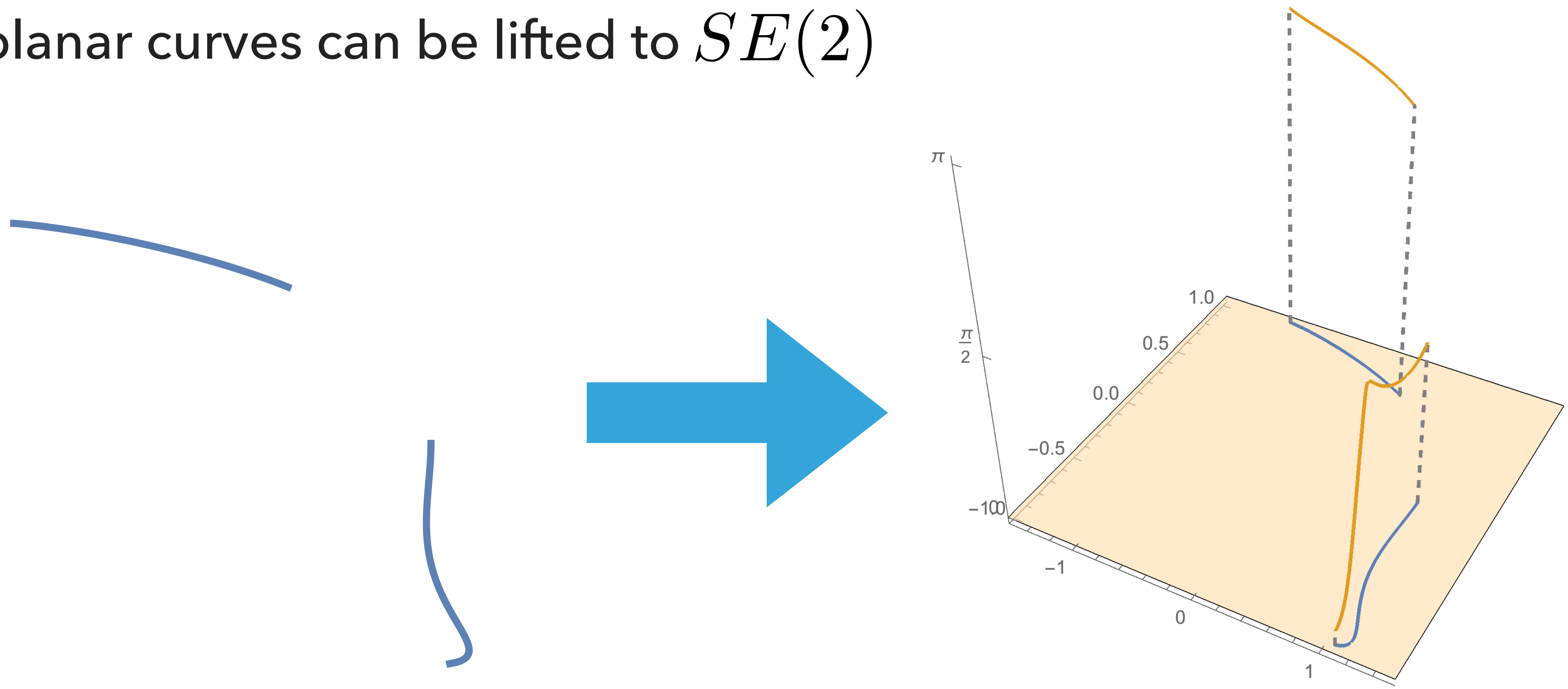

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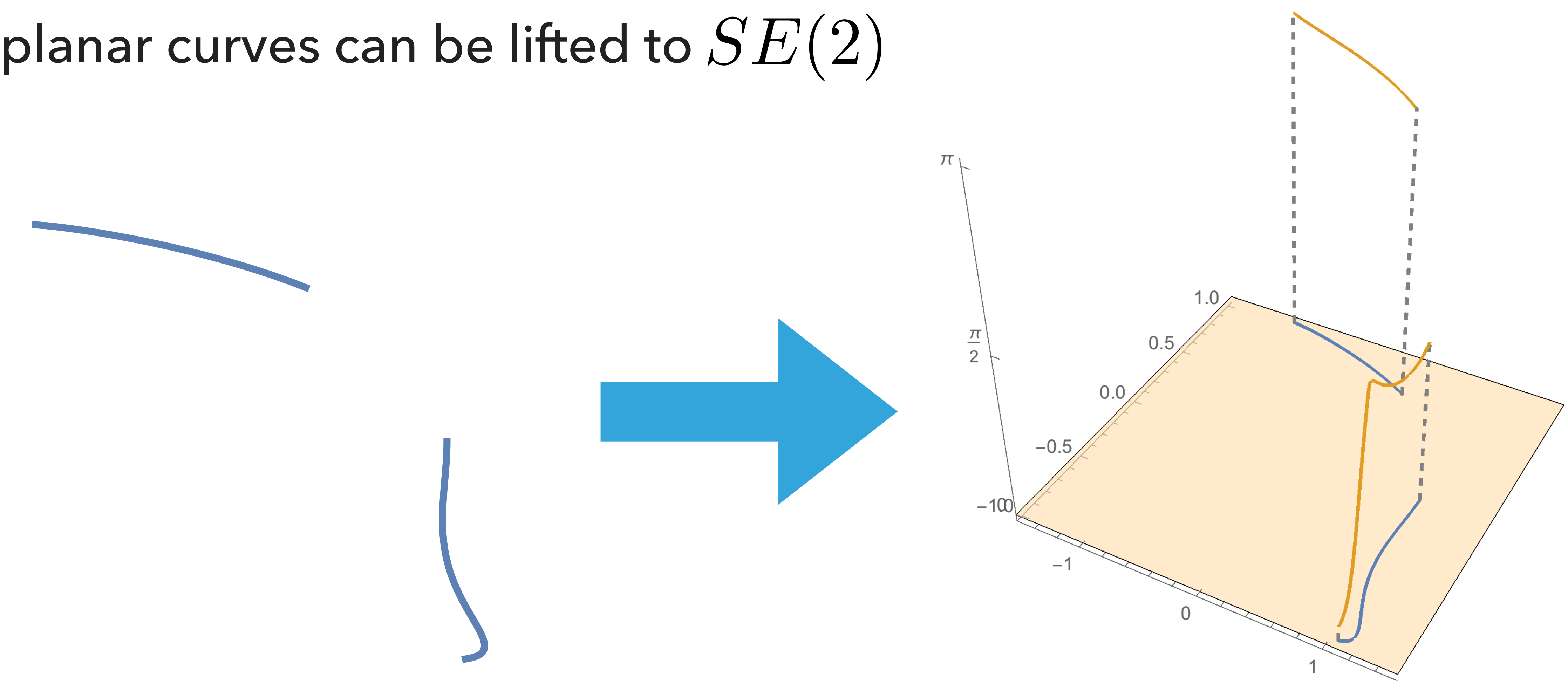


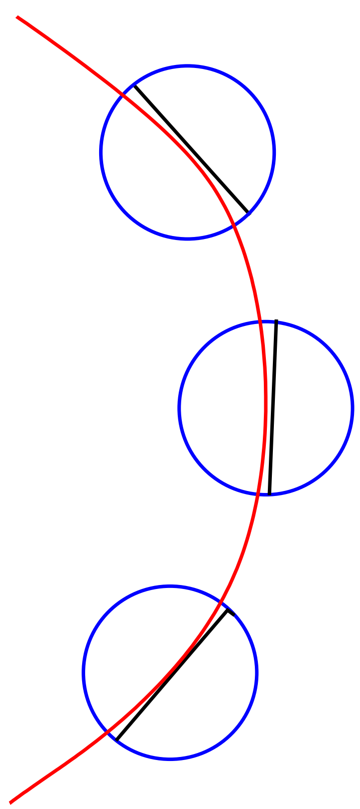
- There exist curves in $SE(2)$ that are not lifts of planar curves

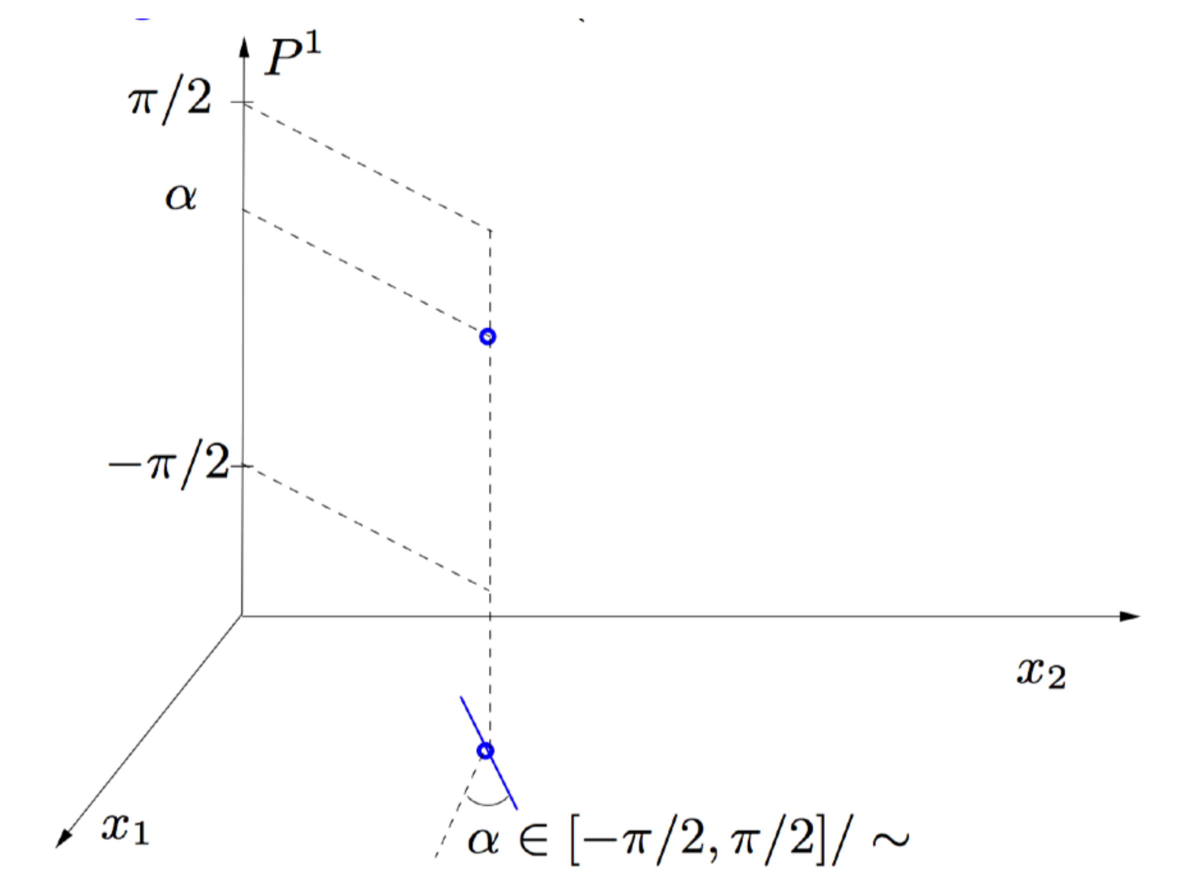
LIFT OF CURVES TO V1

[Boscain *et al* '12, Sachkov *et al* '12]

- $\Gamma : [0,1] \to SE(2)$ is the lift of a planar curve only if:

$$\Gamma(t) = (x(t), y(t), \theta(t)) \quad \text{with} \quad \theta(t) = \arctan\left(\frac{\dot{y}(t)}{\dot{x}(t)}\right)$$





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- That means that the curve is a solution to the Reed-Shepp car control system:

$$\exists u, v : [0,1] \to \mathbb{R}, \qquad \begin{cases} \dot{x}(t) = u(t)\cos\theta(t) \\ \dot{y}(t) = u(t)\sin\theta(t) \\ \dot{\theta}(t) = v(t) \end{cases}$$

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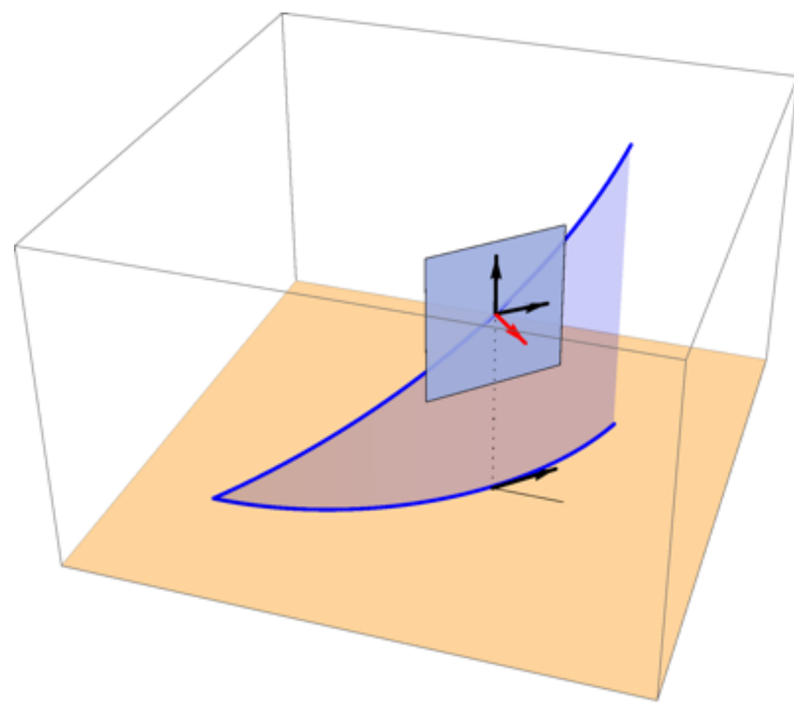

(Image by R. Duits)

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- Lifted curves can go straight and/or change direction, but cannot move in the transverse direction!

CURVE RECONSTRUCTION

- Neurophysiological principle

V1 RECONSTRUCT CURVES BY MINIMISING THE ENERGY REQUIRED TO ACTIVATE UNEXCITED NEURONS

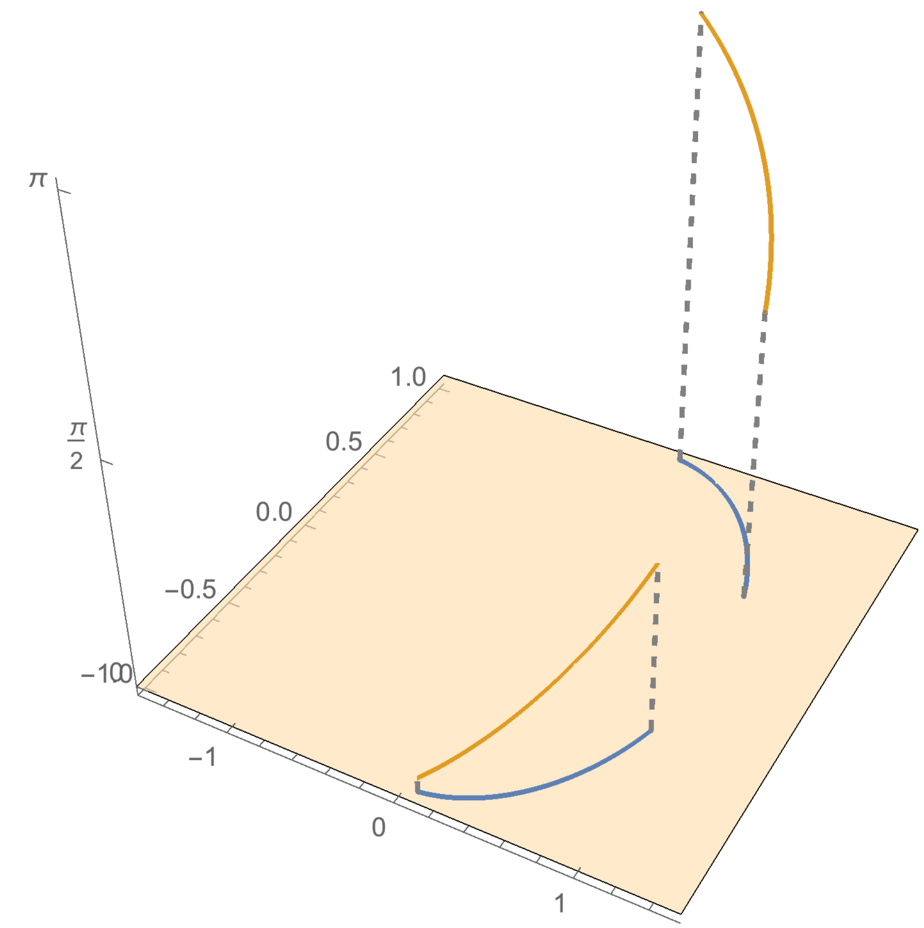
- This means that we look to minimise the cost:

$$\int_a^b u(t)^2 + v(t)^2 \, dt \sim \int_a^b (1 + \kappa(t)^2) \, dt$$

- If we consider curves that can only go forward (Dubins' car model) we recover the Mumford method

APPLICATIONS

It is possible to find the minimising curves by applying the Pontryiagin Maximum Principle



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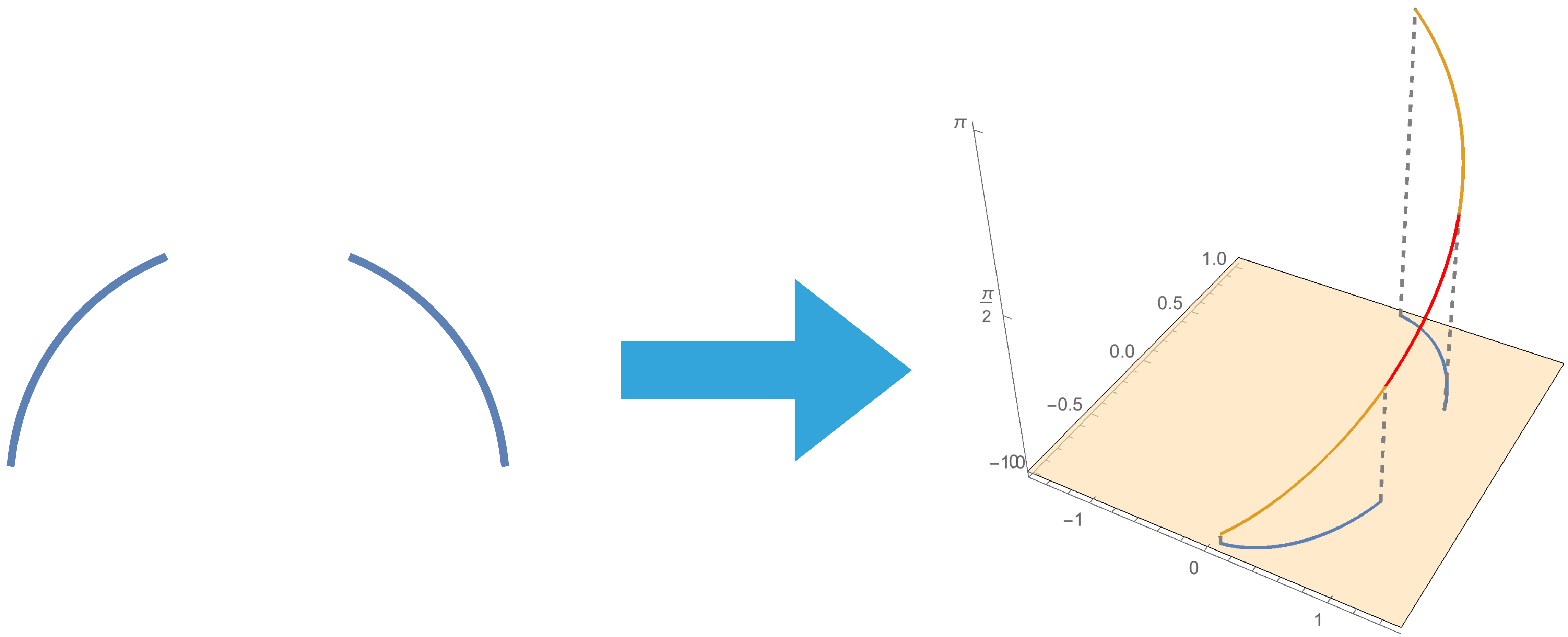

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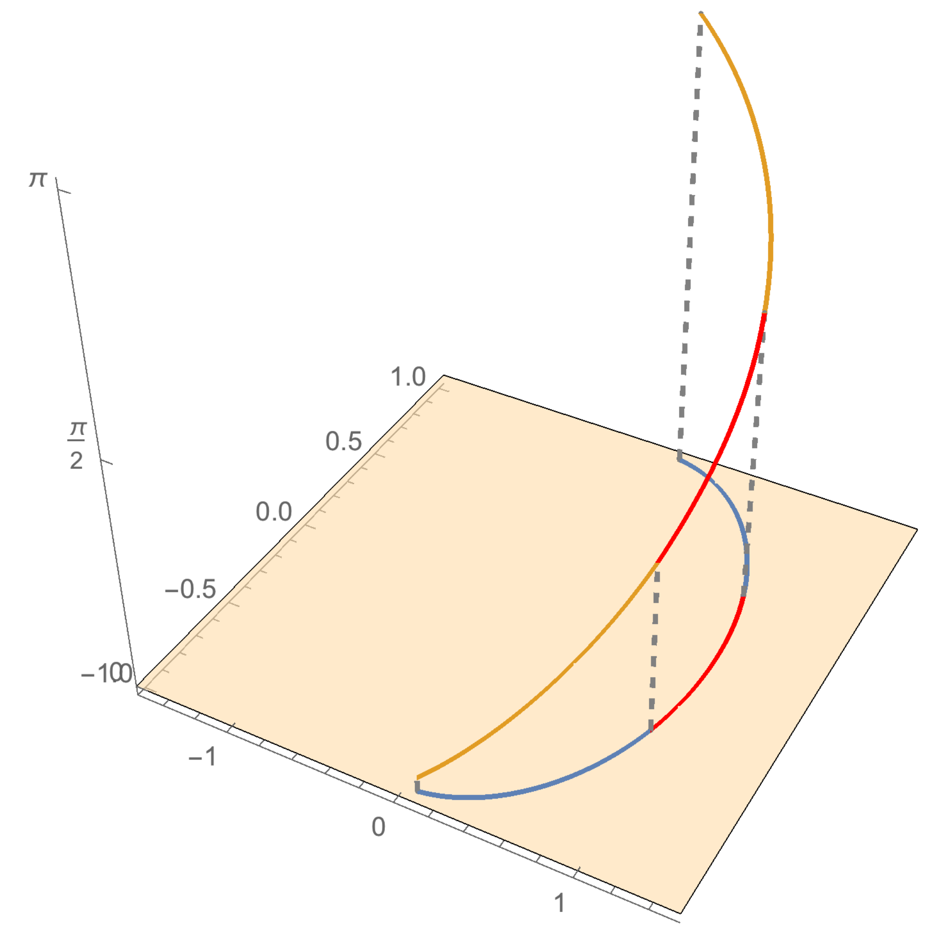

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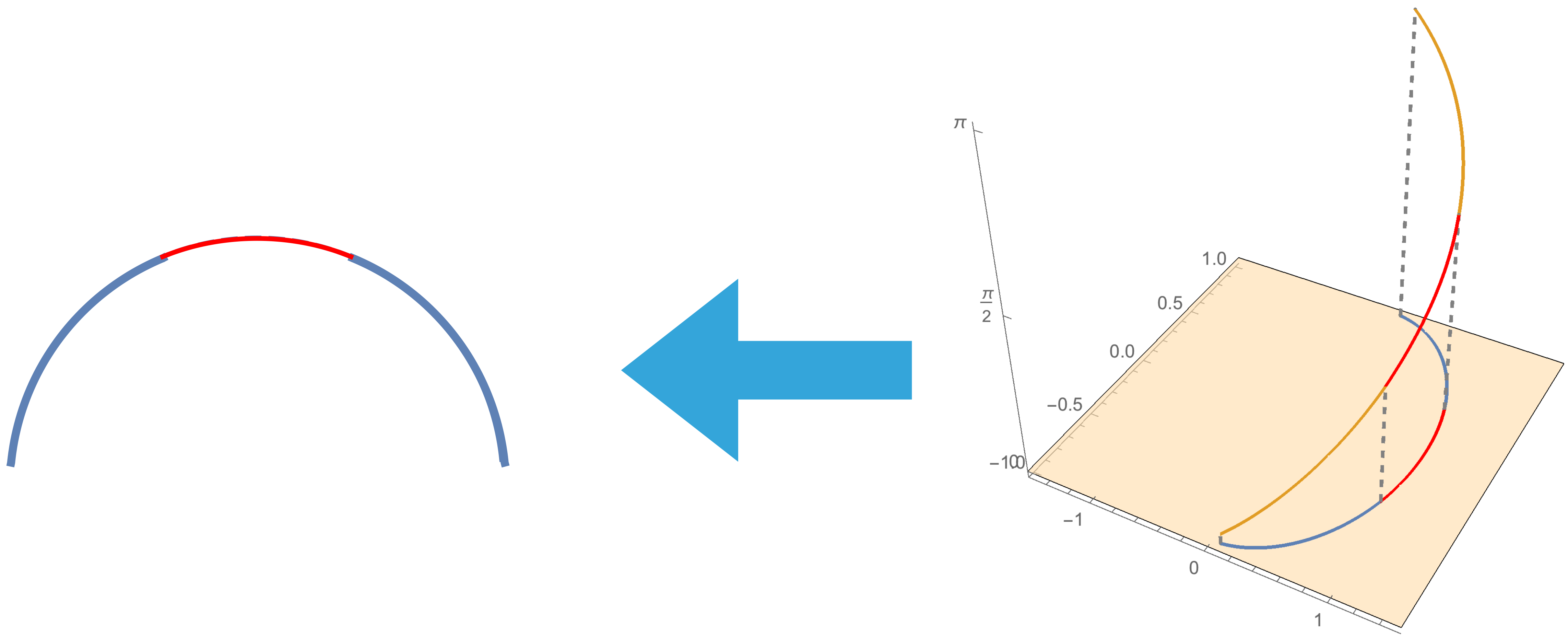

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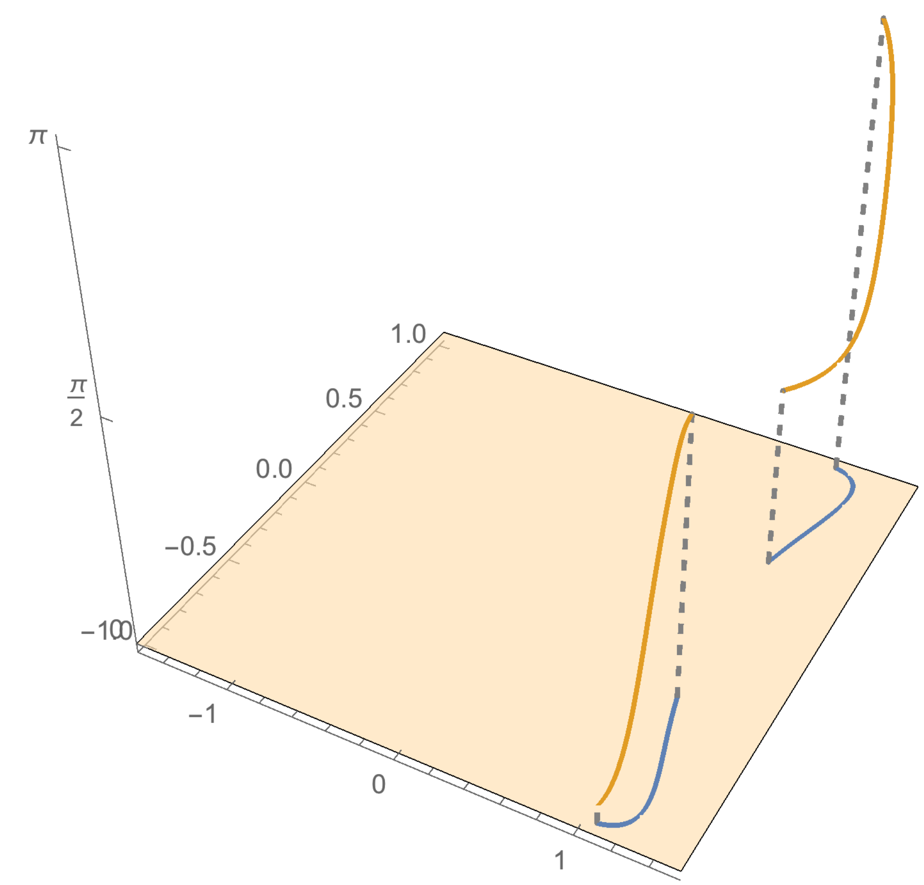

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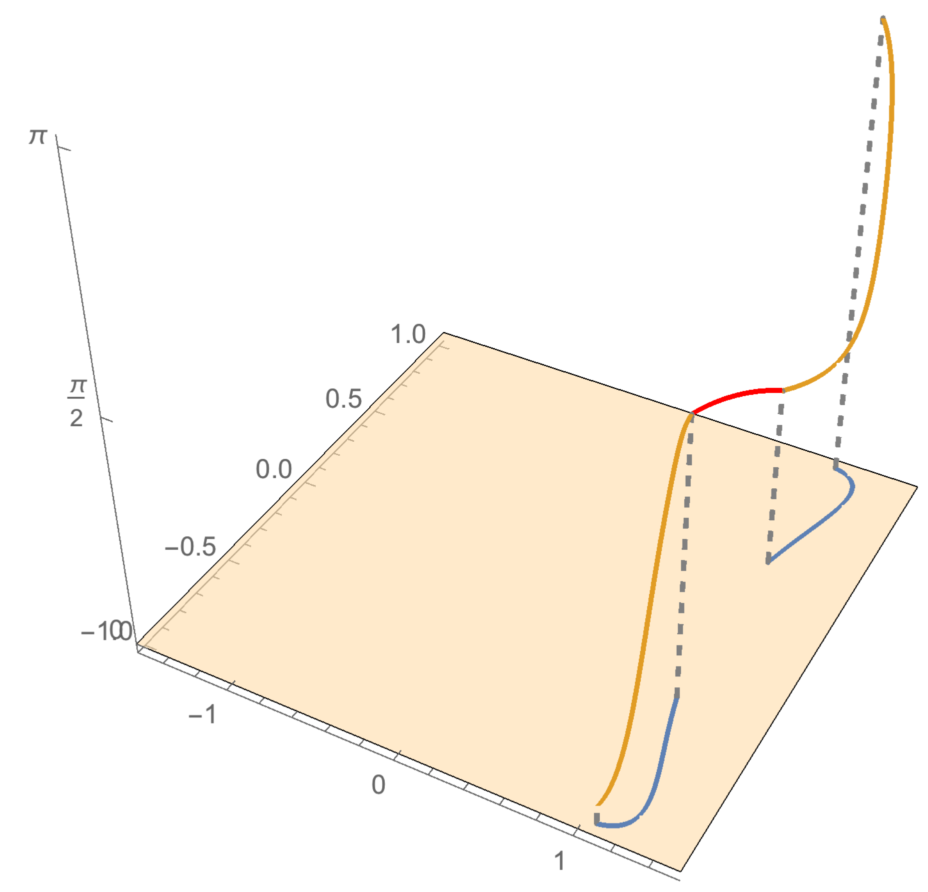

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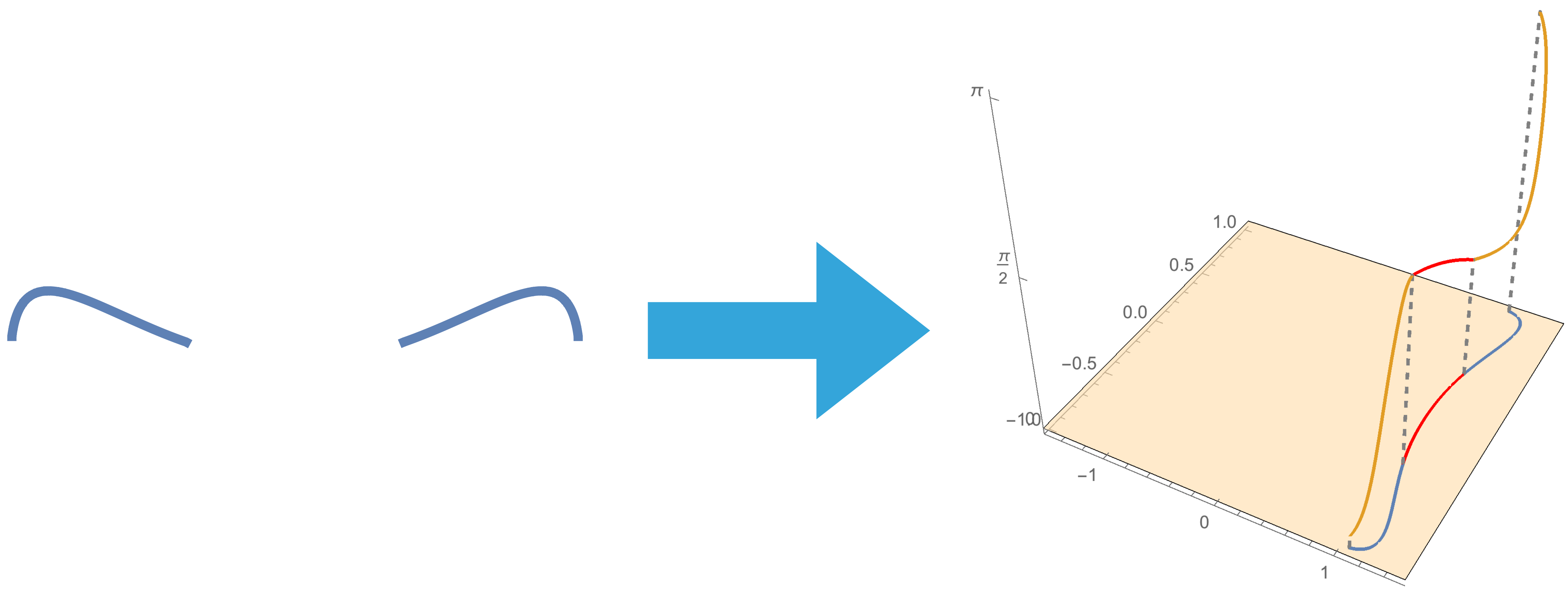

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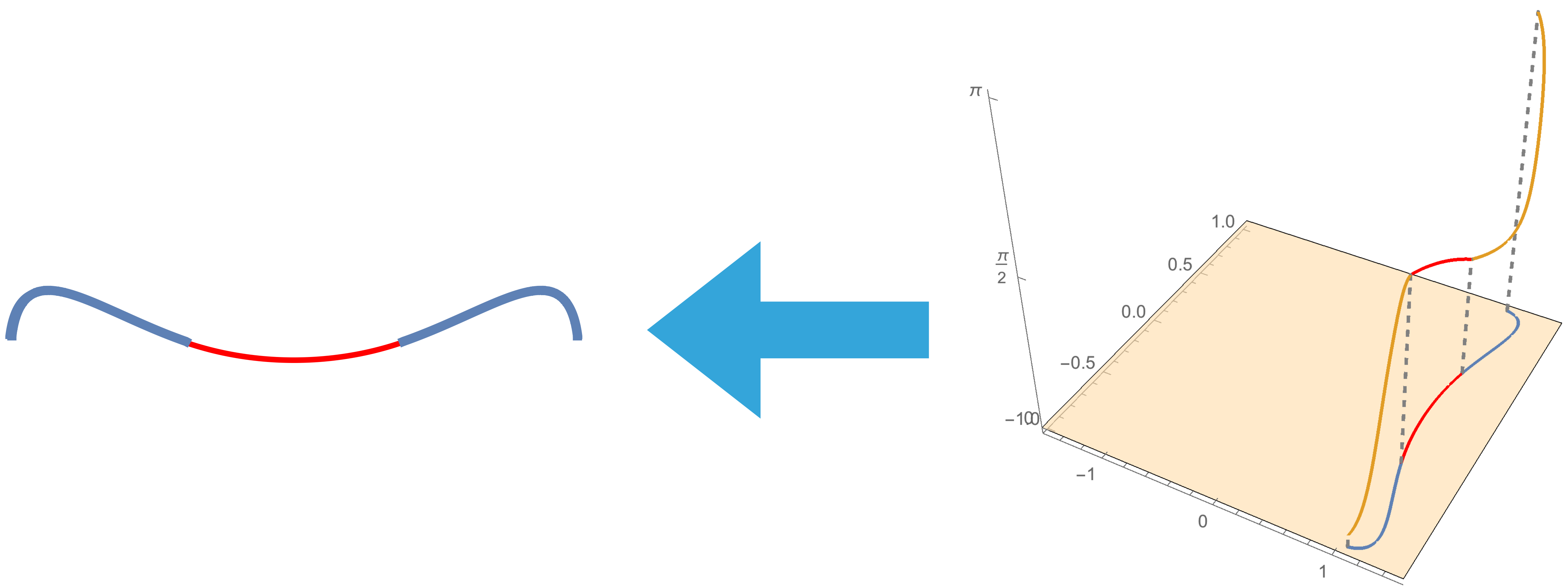

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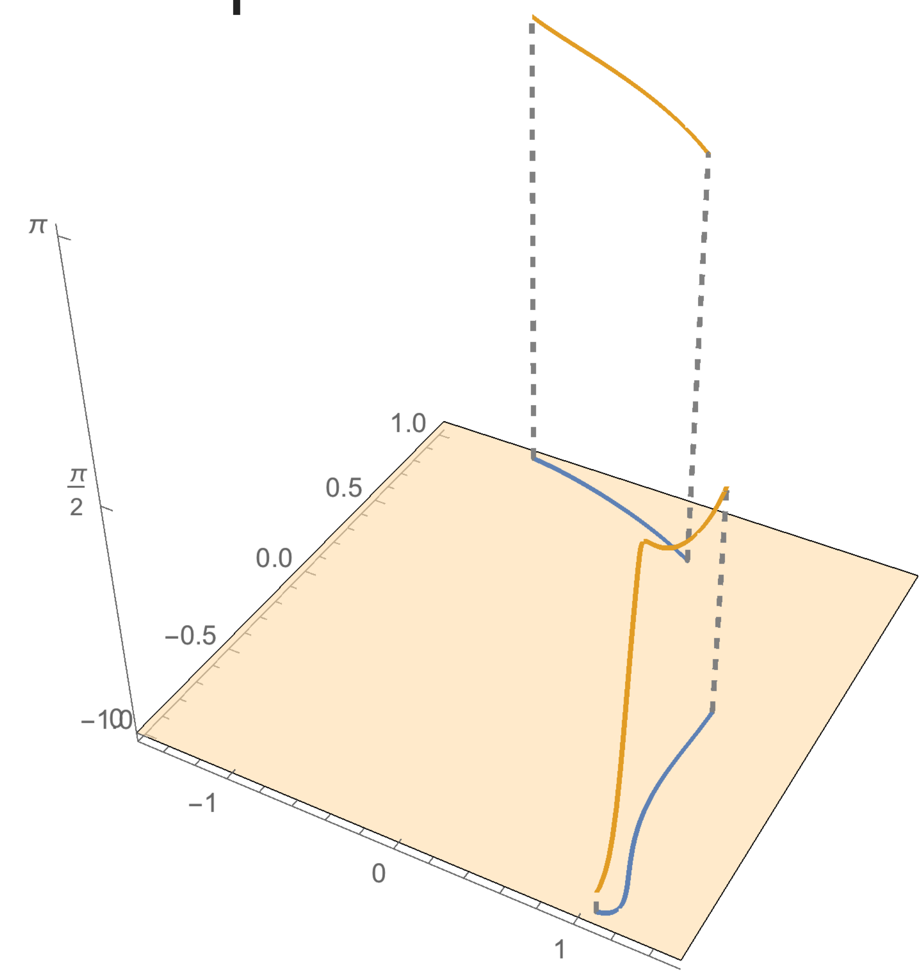

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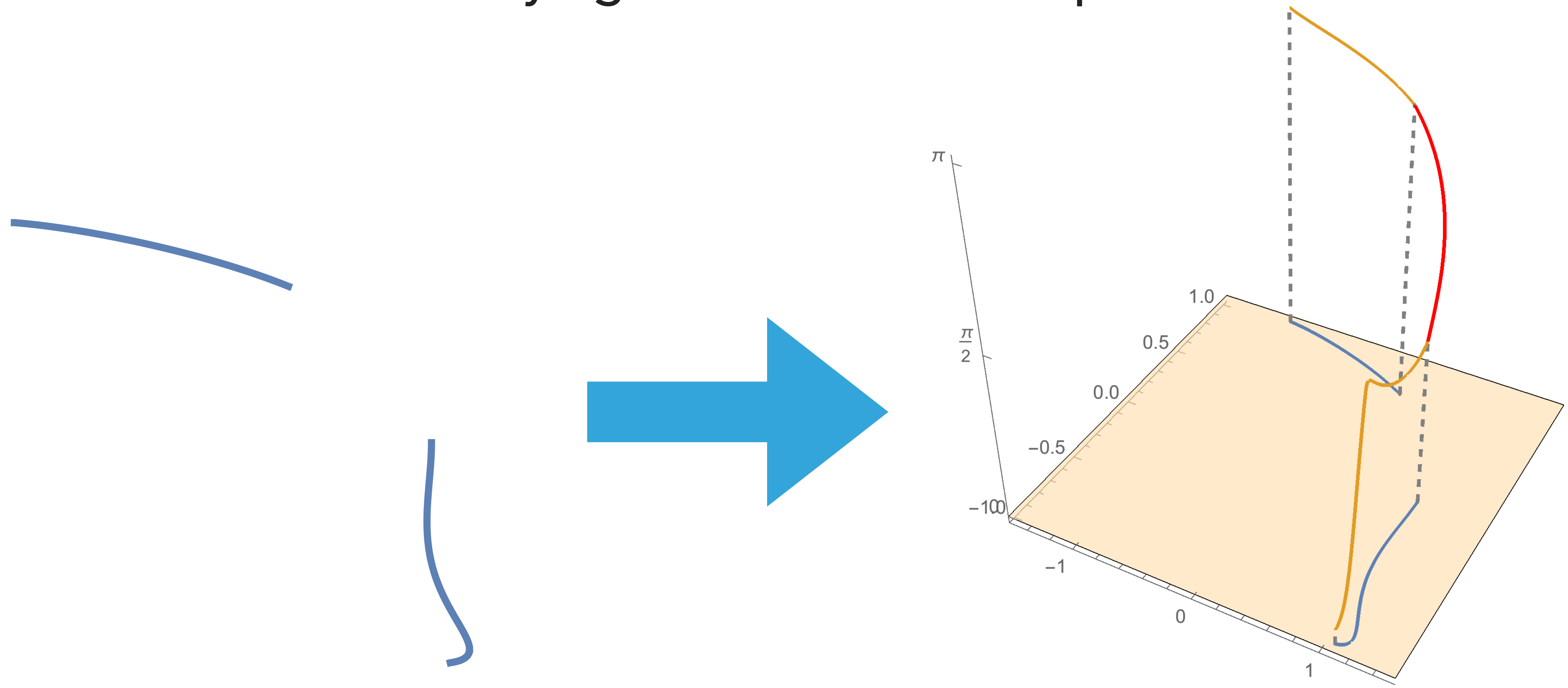

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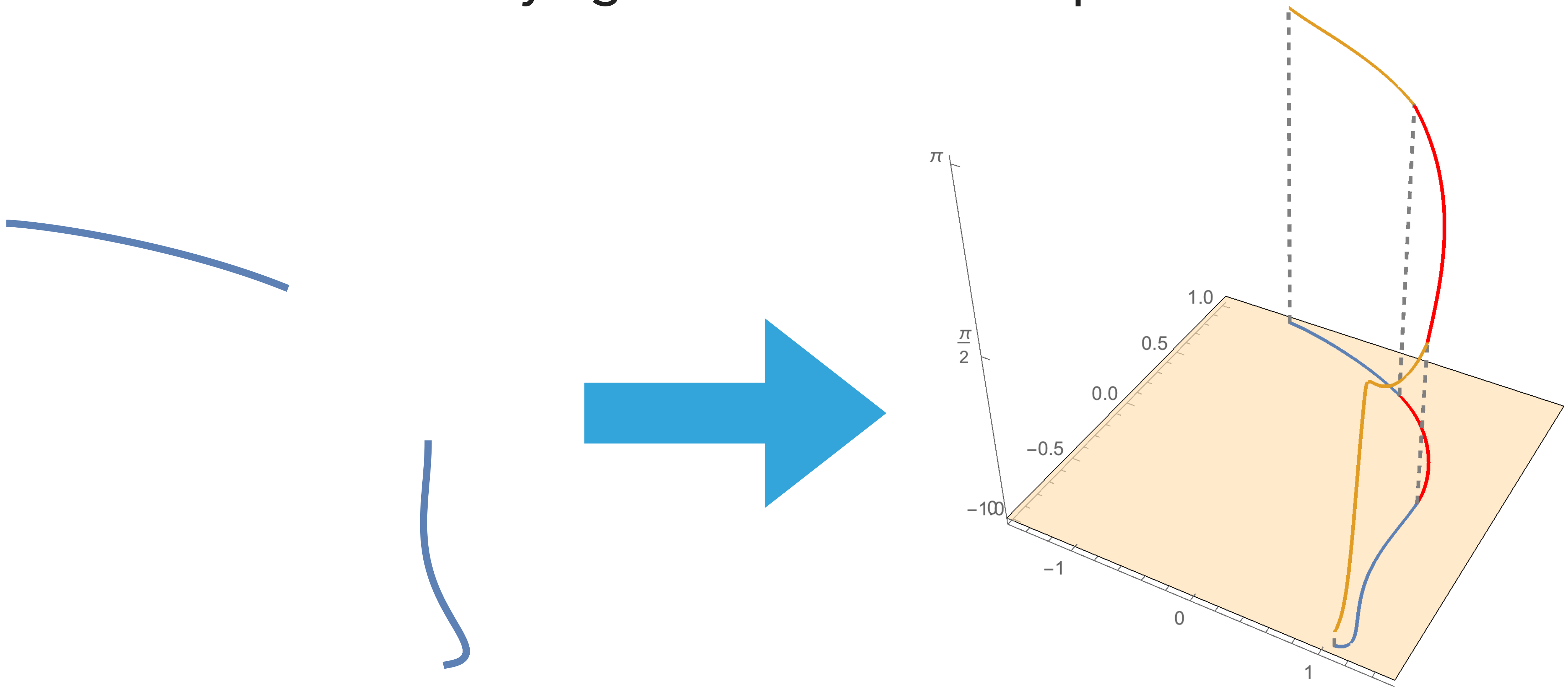

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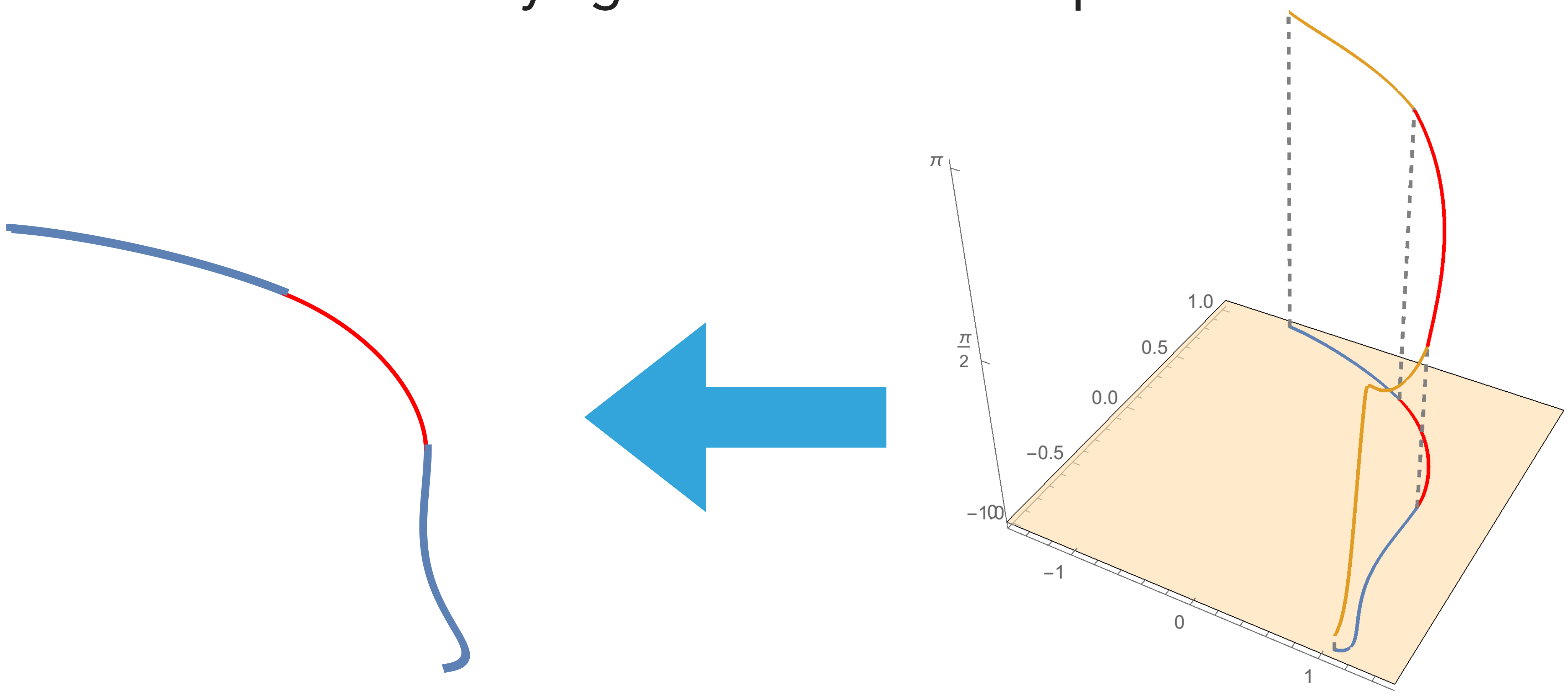


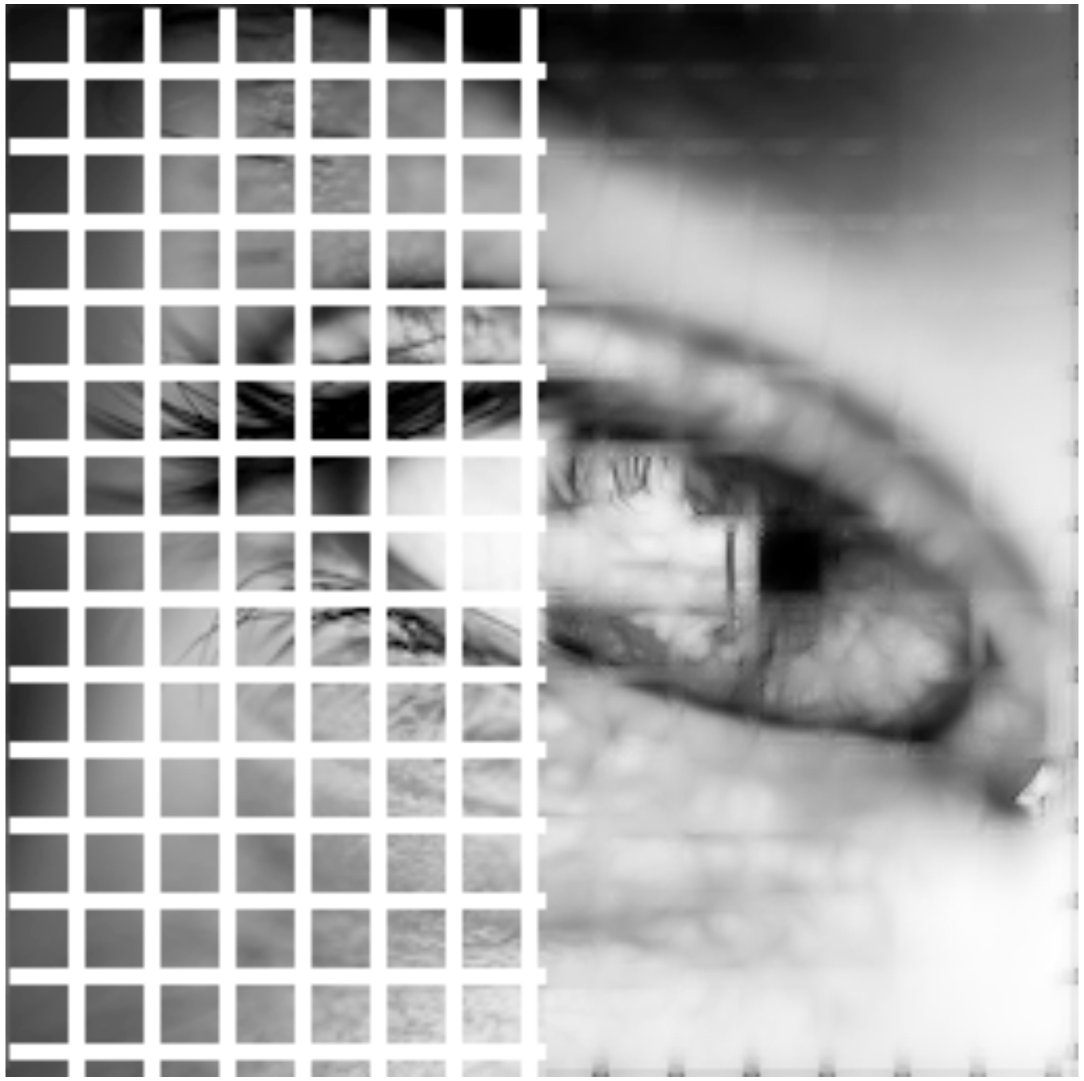
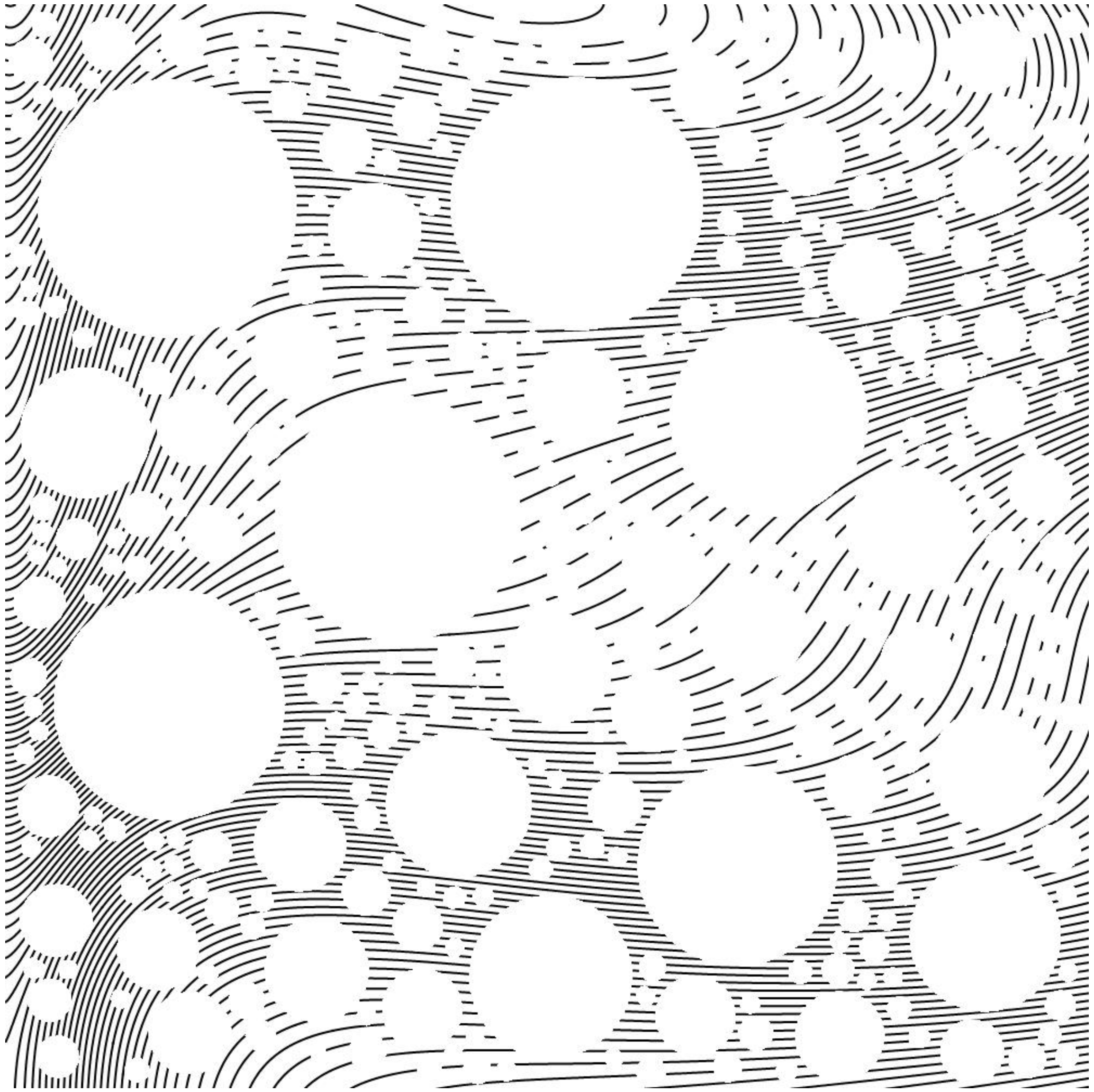


IMAGE RECONSTRUCTION

APPLICATION TO IMAGES

- We can reconstruct images by completing their level lines



(Image by Sachkov et al.)

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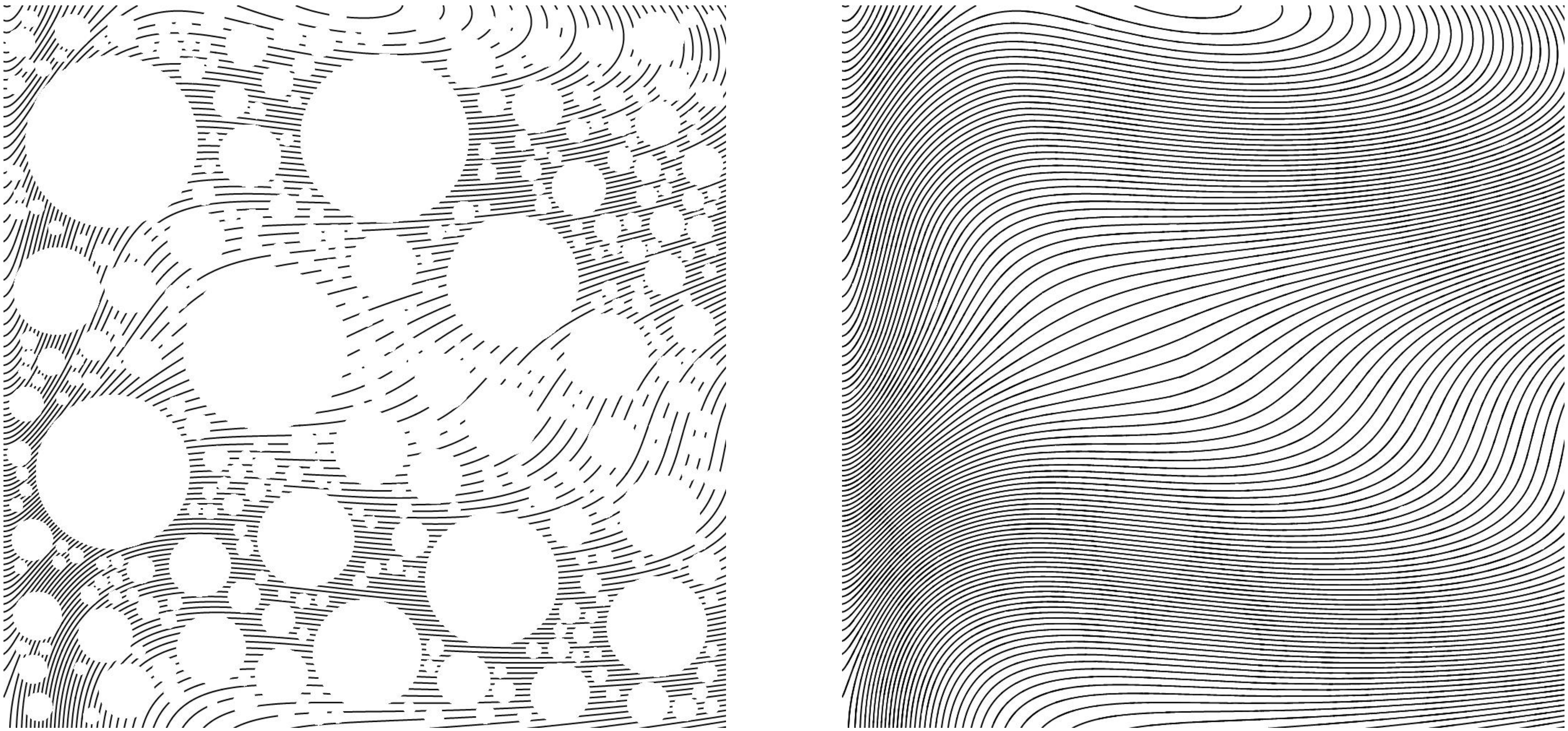

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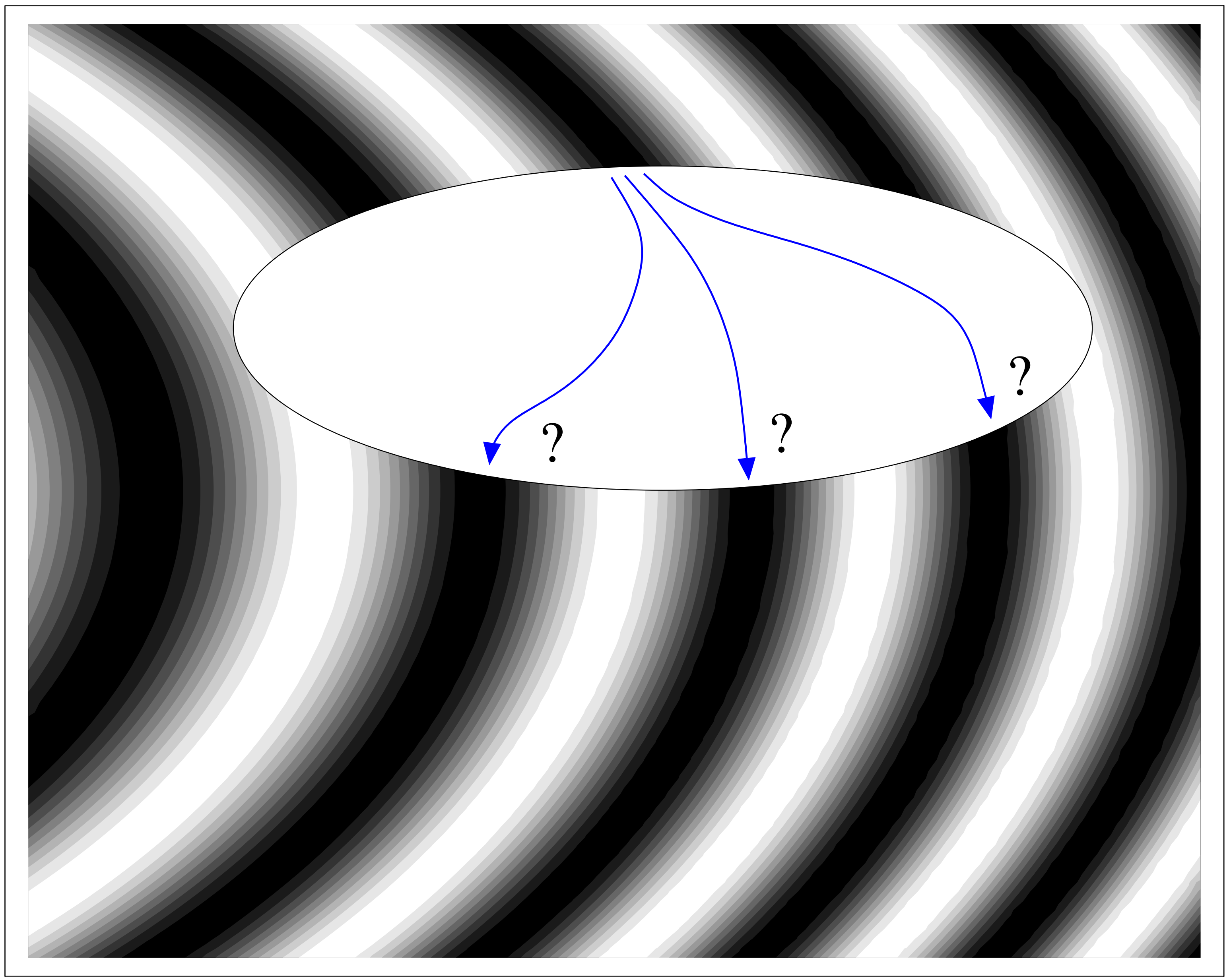
PROBLEM

- Natural images are more complex than simple contours



One has to manually input how level lines are connected

CITTI-PETITOT-SARTI MODEL OF V1

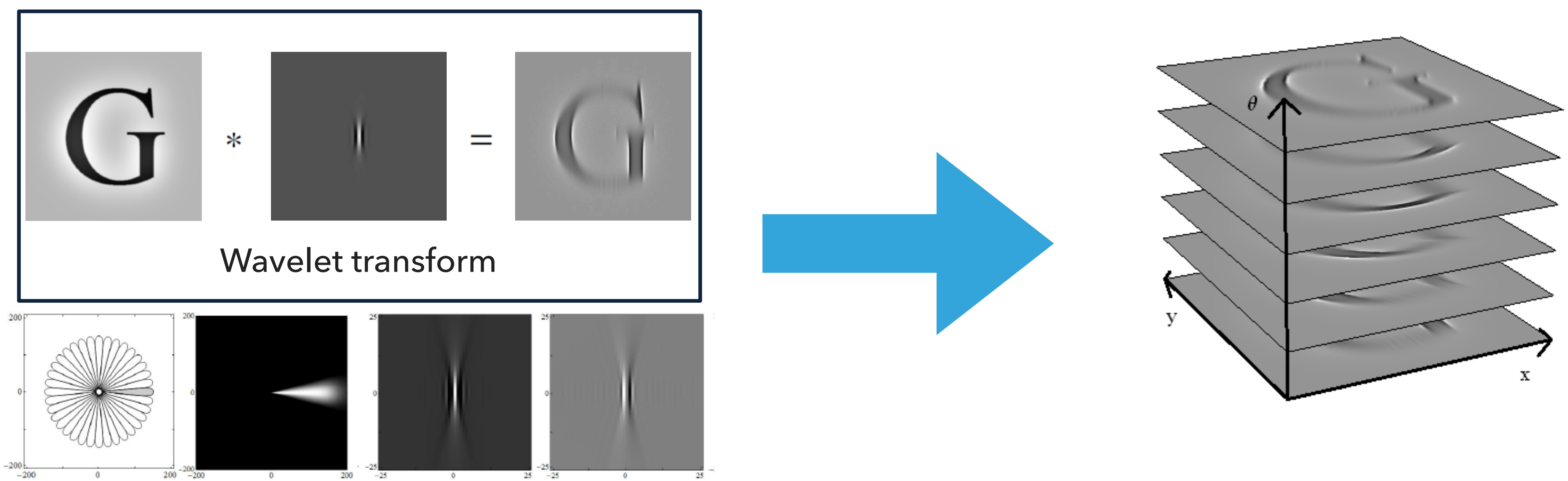
[Petitot-Tondut '99, Duits '05, Citti-Sarti '06]

- Neurons of V1 $\longleftrightarrow$ Points of the Lie group:

$$SE(2) = \mathbb{R}^2 \times \mathbb{S}^1$$

- Images : functions $f \in L^2(\mathbb{R}^2)$
- Linear lift operator (which repects the roto-translation symmetries)

$$\mathcal{L} : L^2(\mathbb{R}^2) \to L^2(SE(2))$$



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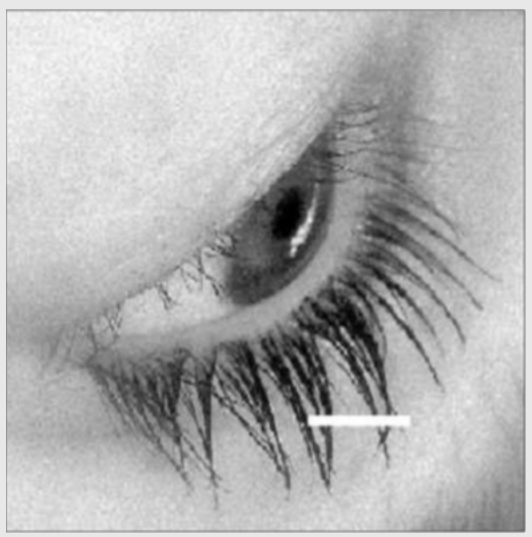
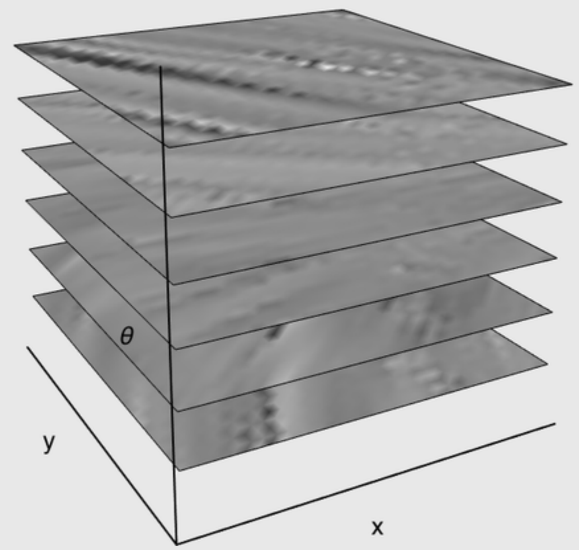
PIPELINE

Image



$$f : \mathbb{R}^2 \to \mathbb{R}$$

"Cortical activation"



$$\mathcal{L}f : SE(2) \to \mathbb{R}$$

Processed image

$$\tilde{f} : \mathbb{R}^2 \to \mathbb{R}$$

Processing

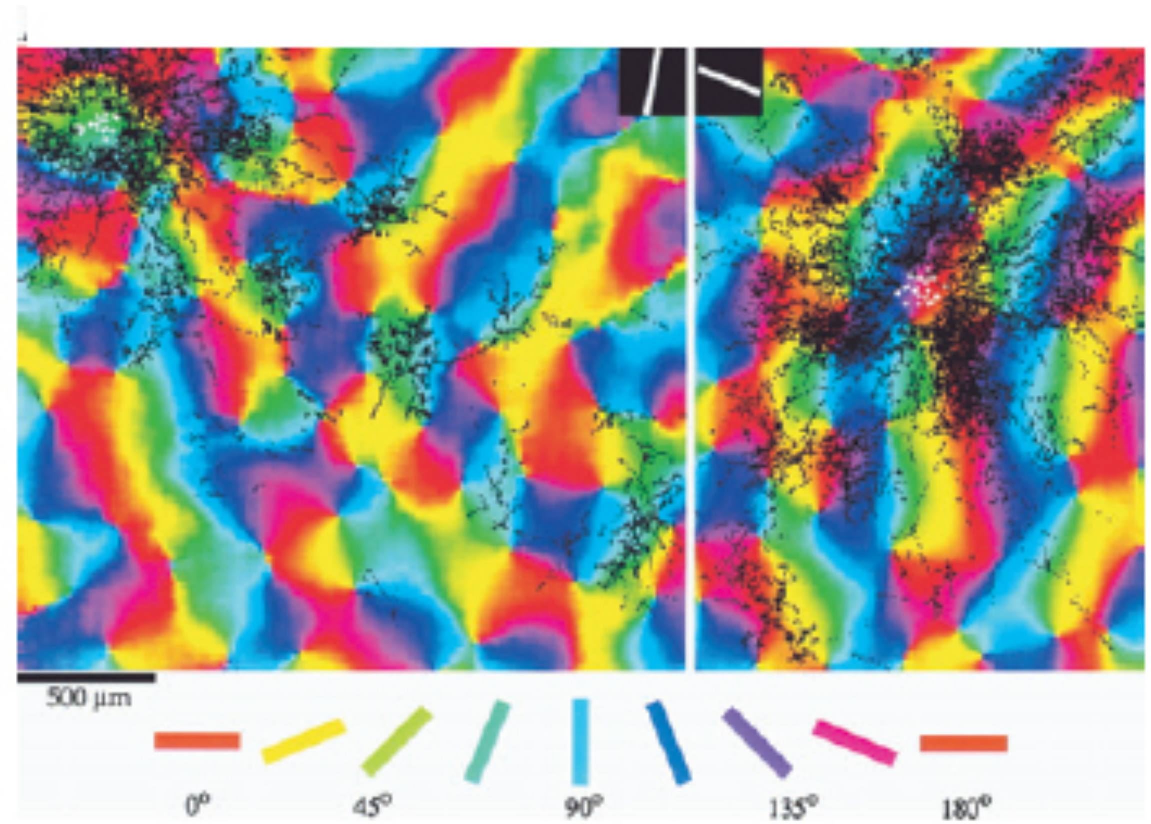
HORIZONTAL CONNECTIVITY

Cortical connections have the structure of the control system:

$$\frac{d}{dt}(x,y,\theta)^T = X(x,y,\theta)u + \Theta(x,y,\theta)v$$

$$X(x,y,\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \qquad \Theta(x,y,\theta) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

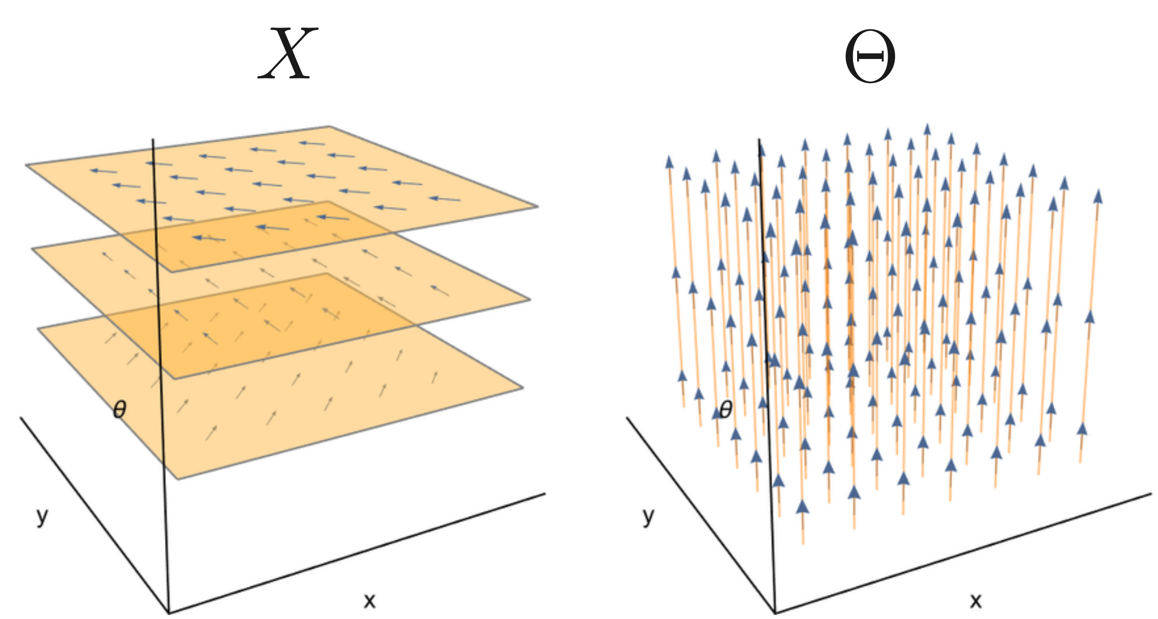
(Reeds-Shepp car)



Modelled by the stochastic process:

$$dZ_t = X\,\underline{dW_1} + \Theta\,\underline{dW_2}$$

Brownian motions



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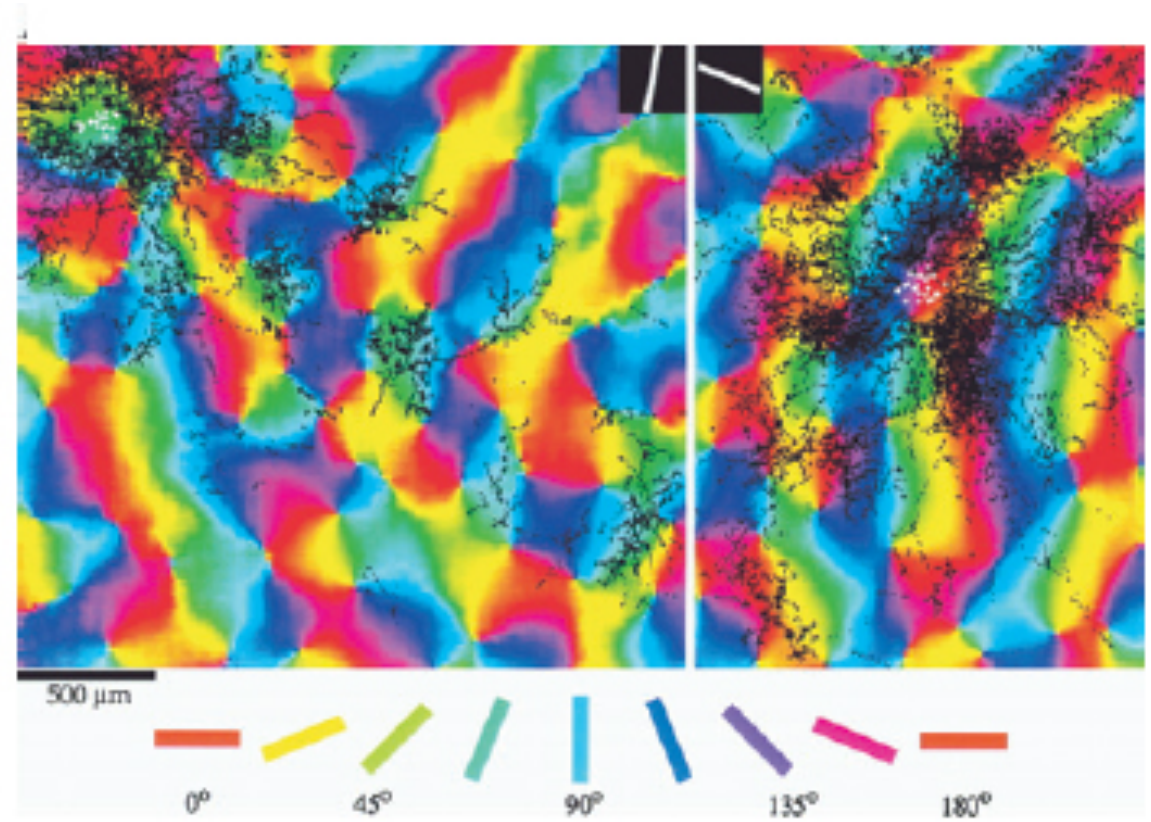

Modelled by the stochastic process:

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Brownian motions

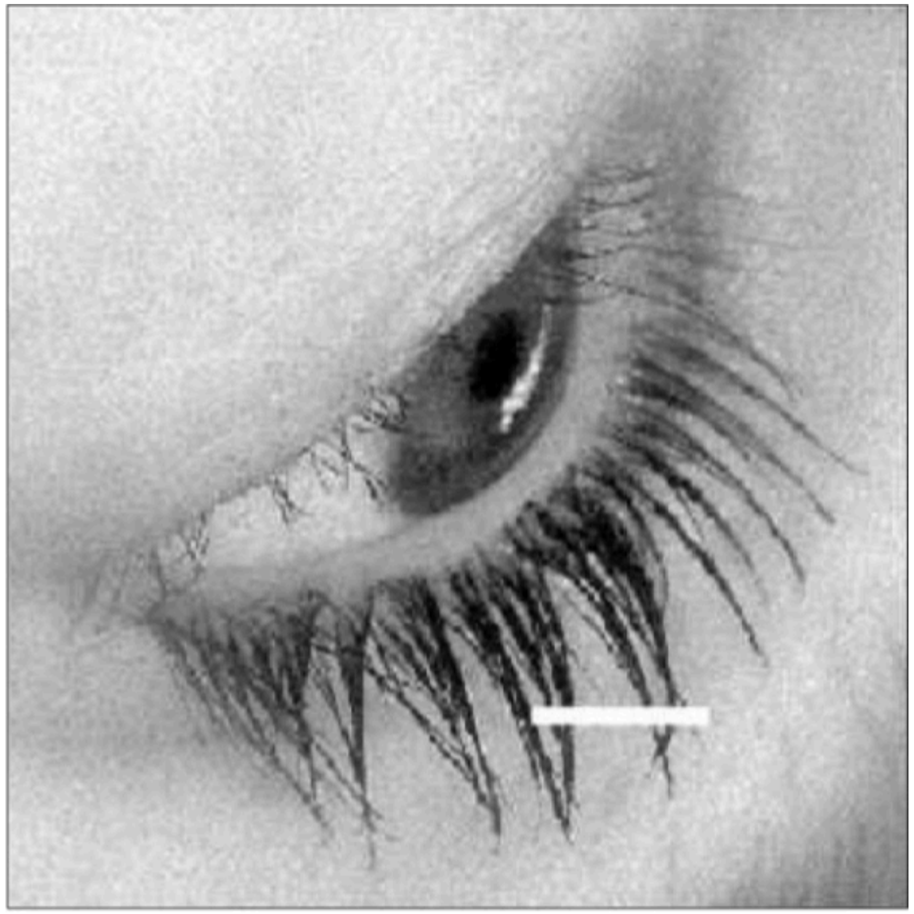
Highly anisotropic evolution:

$$\partial_t \psi = \Delta\psi \qquad \psi \in L^2(SE(2))$$

$$\Delta = \underbrace{(\cos\theta\,\partial_x + \sin\theta\,\partial_y)^2}_{X^2} + \underbrace{\partial_\theta^2}_{\Theta^2}$$

IMAGE RECONSTRUCTION

[Boscain *et al* '12-'14-'18, P-Gauthier '18]



Spontaneous evolution of V1

$$\begin{cases} \partial_t \Psi = \big(\underbrace{X^2 + Y^2}_{\Delta}\big)\Psi \\ \Psi|_{t=0} = \mathcal{L} f \end{cases}$$

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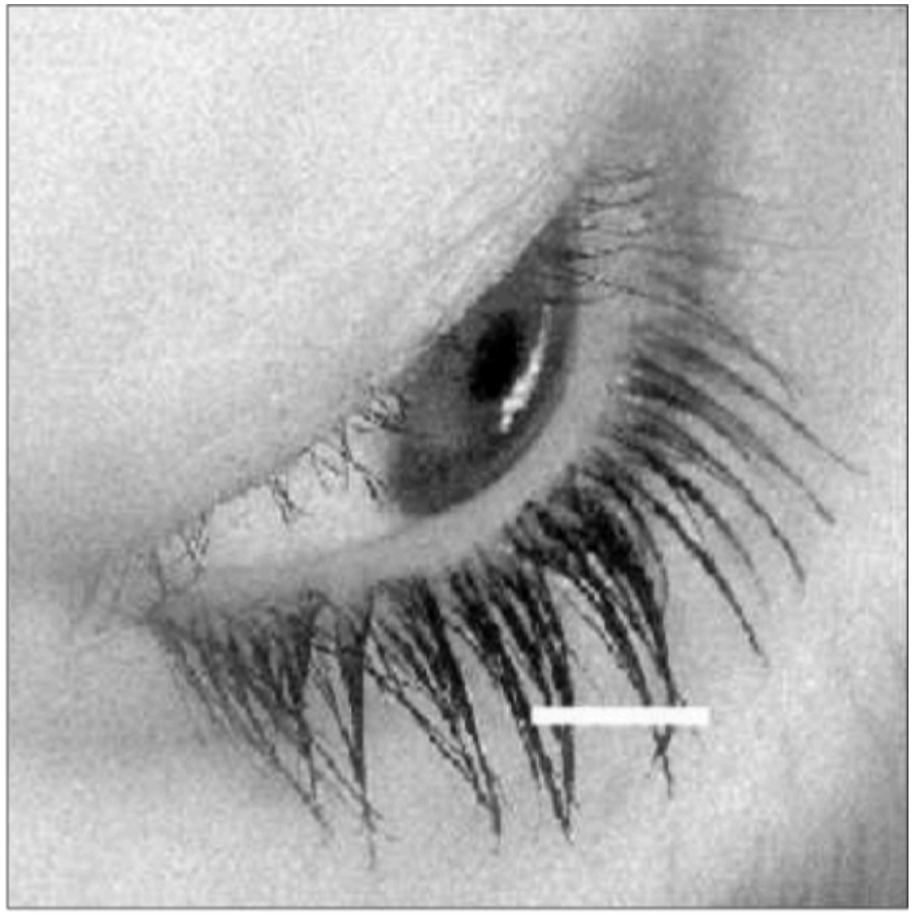

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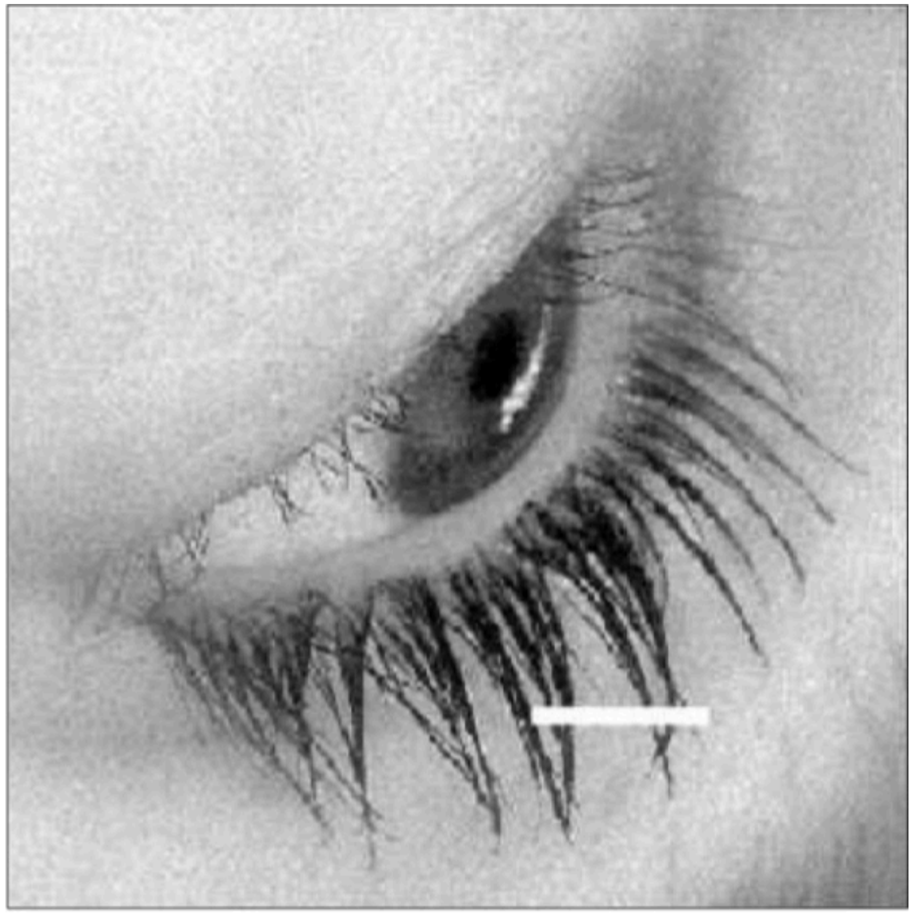
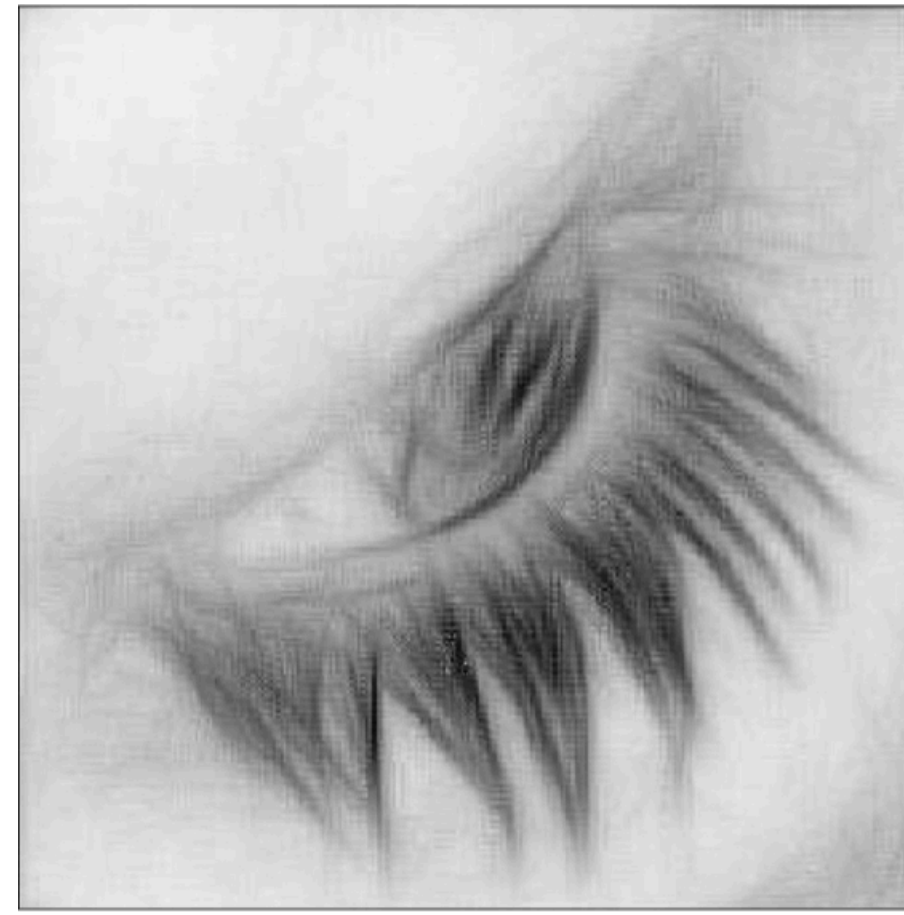
HIGHLY ANISOTROPIC EQUATION → VERY DIFFICULT NUMERICAL INTEGRATION

IMAGE RECONSTRUCTION

[Boscain *et al* '12-'14-'18, P-Gauthier '18]



Non-commutative Fourier analysis



Spontaneous evolution of V1

$$\begin{cases} \partial_t \Psi = \big(\underbrace{X^2 + Y^2}_{\Delta}\big)\Psi \\ \Psi|_{t=0} = \mathcal{L} f \end{cases}$$

HIGHLY ANISOTROPIC EQUATION → VERY DIFFICULT NUMERICAL INTEGRATION

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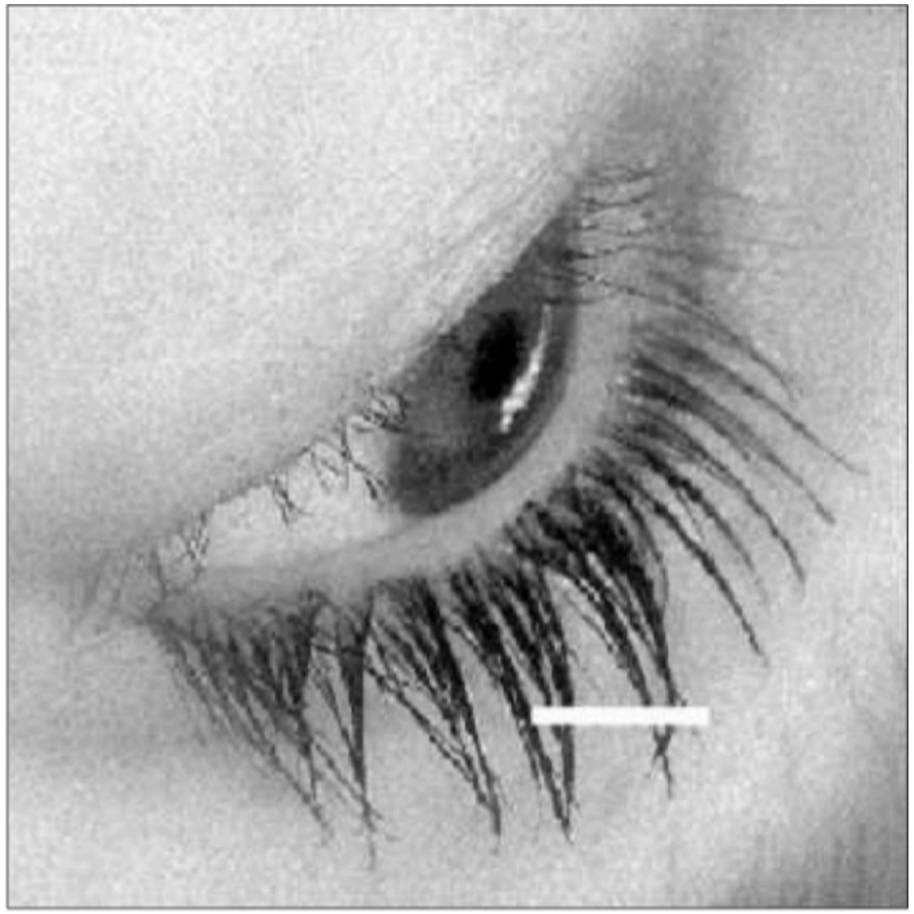
[Boscain *et al* '12-'14-'18, P-Gauthier '18]



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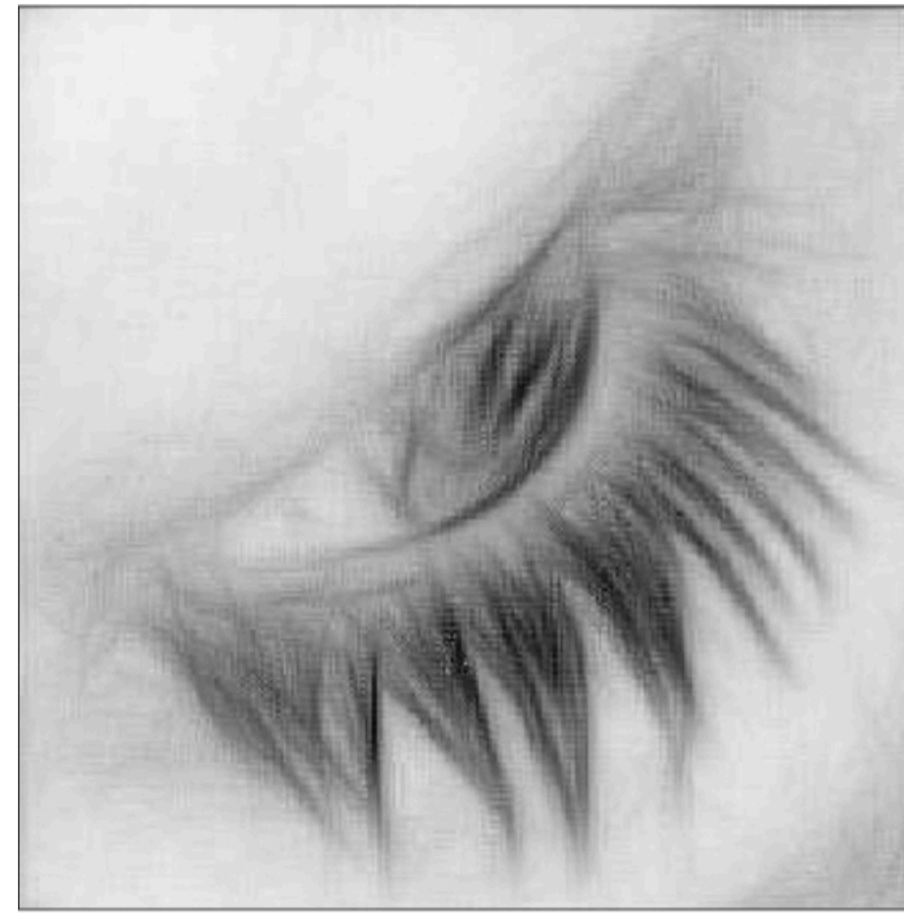

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HIGHLY ANISOTROPIC EQUATION → **VERY DIFFICULT NUMERICAL INTEGRATION**

Remarks:

- No informations on the location of the corruption
- Fast computational time: O (Highly parallelisable, 5s pour 512x512px sur GPU)

NUMERICAL INTEGRATION

$$\begin{cases} \partial_t \Psi = (\underbrace{X^2 + Y^2}_{\Delta})\Psi \\ \Psi|_{t=0} = \mathcal{L}f \end{cases}$$

- Finite difference schemes [Citti, Sarti *et al*]
- numerical implementation of convolution kernels [Duits *et al*]
- Sparse Non-Negative Stencils [Mirebeau *et al*]

We chose:

- numerical integration of the (non-commutative) Fourier Transform of the hypoelliptic diffusion equation on $SE(2)$:

Solve the decoupled Mathieu equations, for $(\lambda, \mu) \in \mathbb{R}^2$,

$$\begin{cases} \partial_t \hat{\Psi}_{\lambda,\mu}(\theta) = \partial_\theta^2 \hat{\Psi}_{\lambda,\mu}(\theta) - 2\pi^2(\lambda\cos\theta + \mu\sin\theta)^2 \Psi_{\lambda,\mu}(\theta) \\ \Psi_{\lambda,\mu}|_{t=0} = \widehat{\mathcal{L}f}_{\lambda,\mu} \end{cases}$$

NUMERICAL INTEGRATION

HIGHLY ANISOTROPIC EQUATION → VERY DIFFICULT NUMERICAL INTEGRATION

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DYNAMIC RECONSTRUCTION

- Can we obtain better results by exploiting information on the location of the corruption?

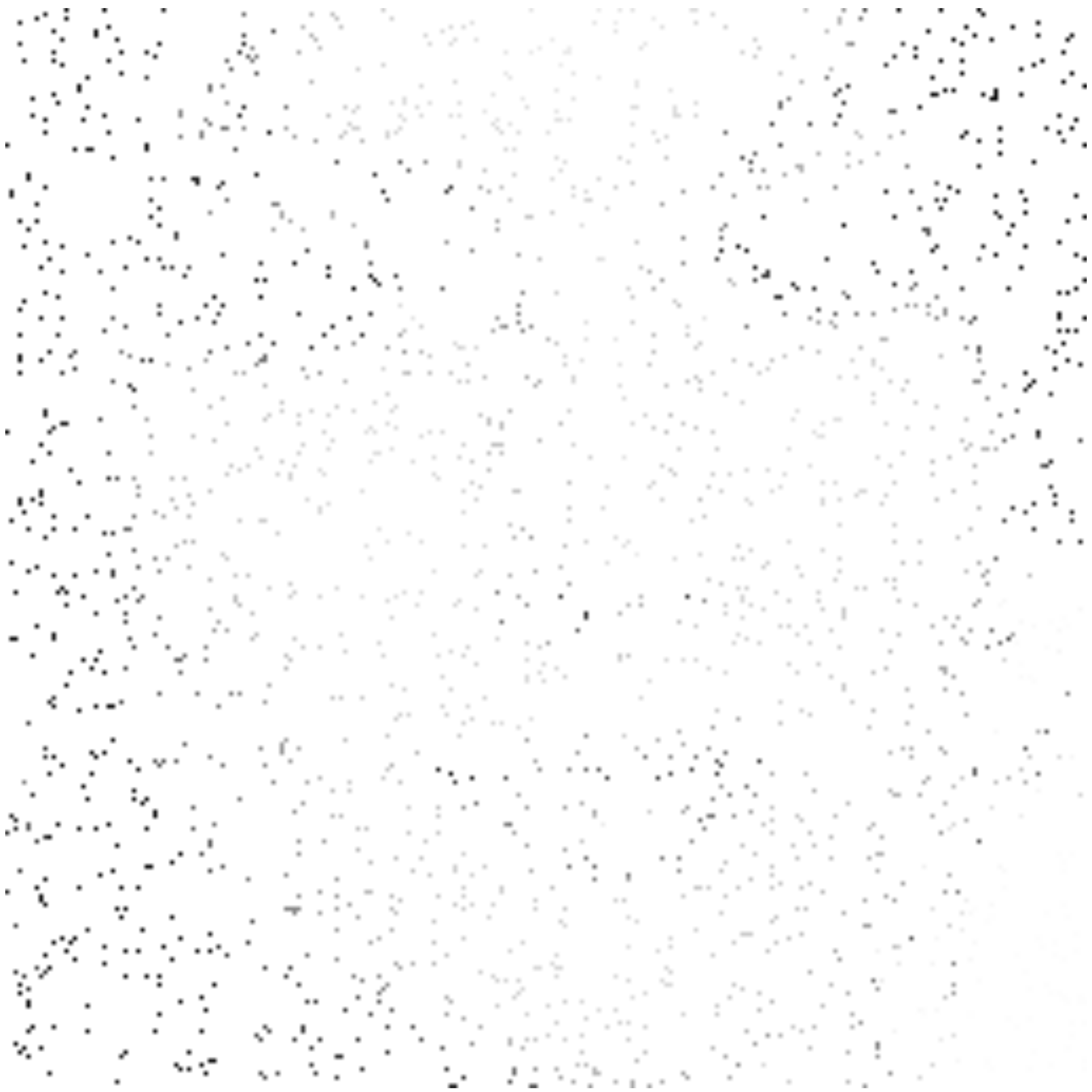


Image with 97% of missing pixels

AHE (AVERAGING AND HYPOELLIPTIC EVOLUTION)

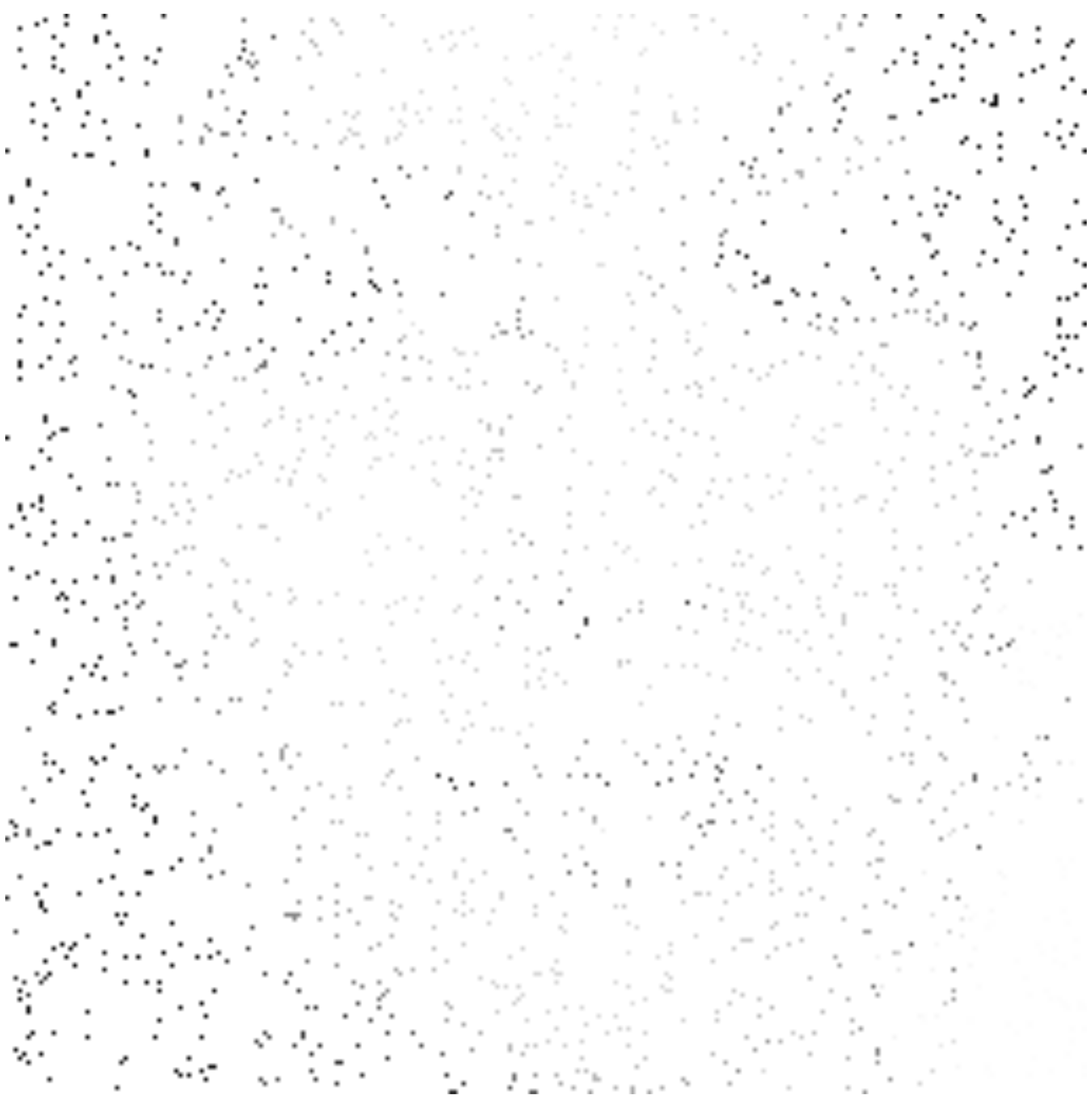
[Boscain *et al* '18]

1. Divide pixels in “good pixels” and “bad pixels”
2. Apply the hypoelliptic diffusion for a small time Δt
3. “Average” the values of the good pixels before and after the diffusion
4. The “bad pixels” near “good pixels” which are sufficiently reconstructed become “good pixels”
5. Repeat until there are no more bad pixels

AHE (AVERAGING AND HYPOELLIPTIC EVOLUTION)

[Boscain *et al* '18]

97% of random pixels missing



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[Boscain *et al* '18]

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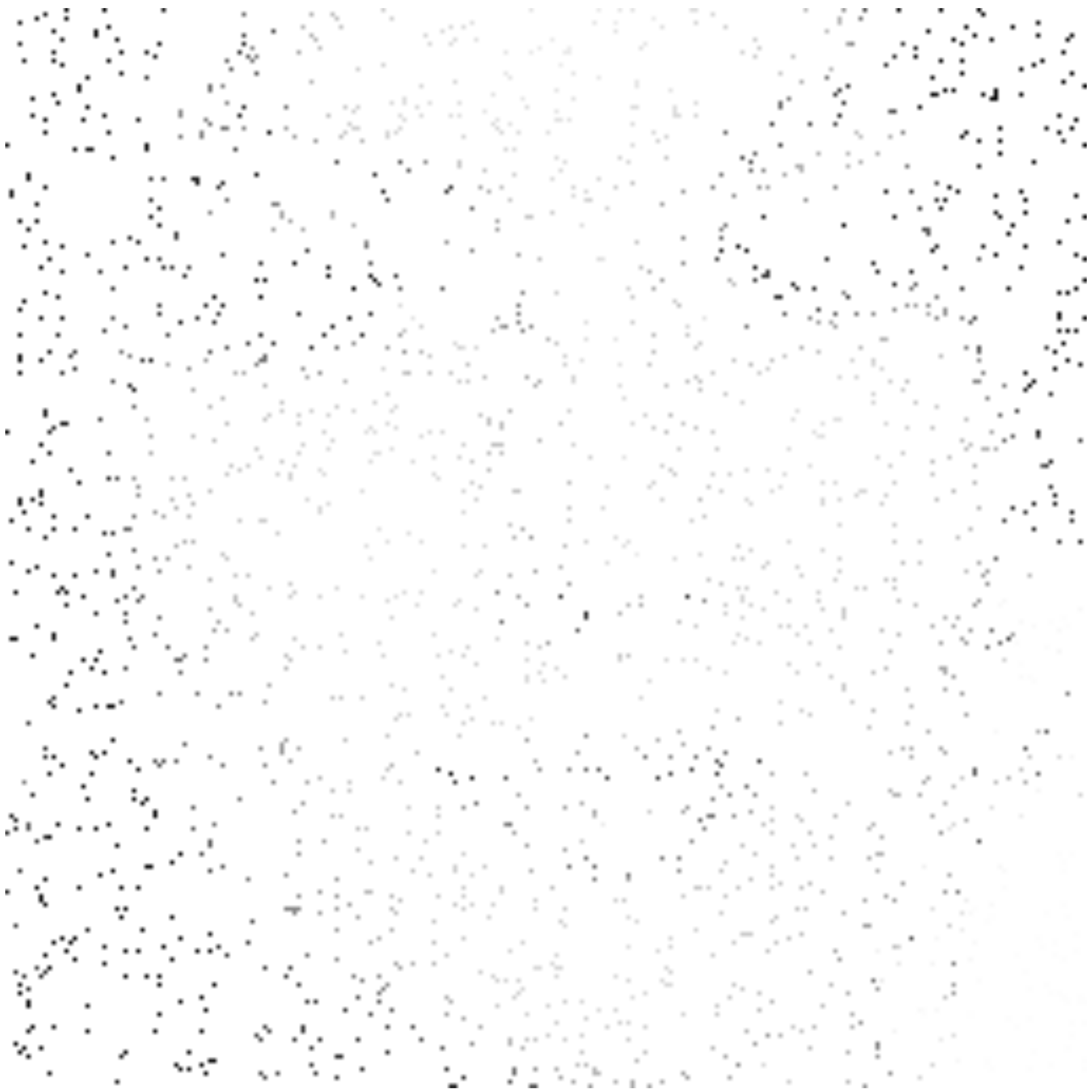

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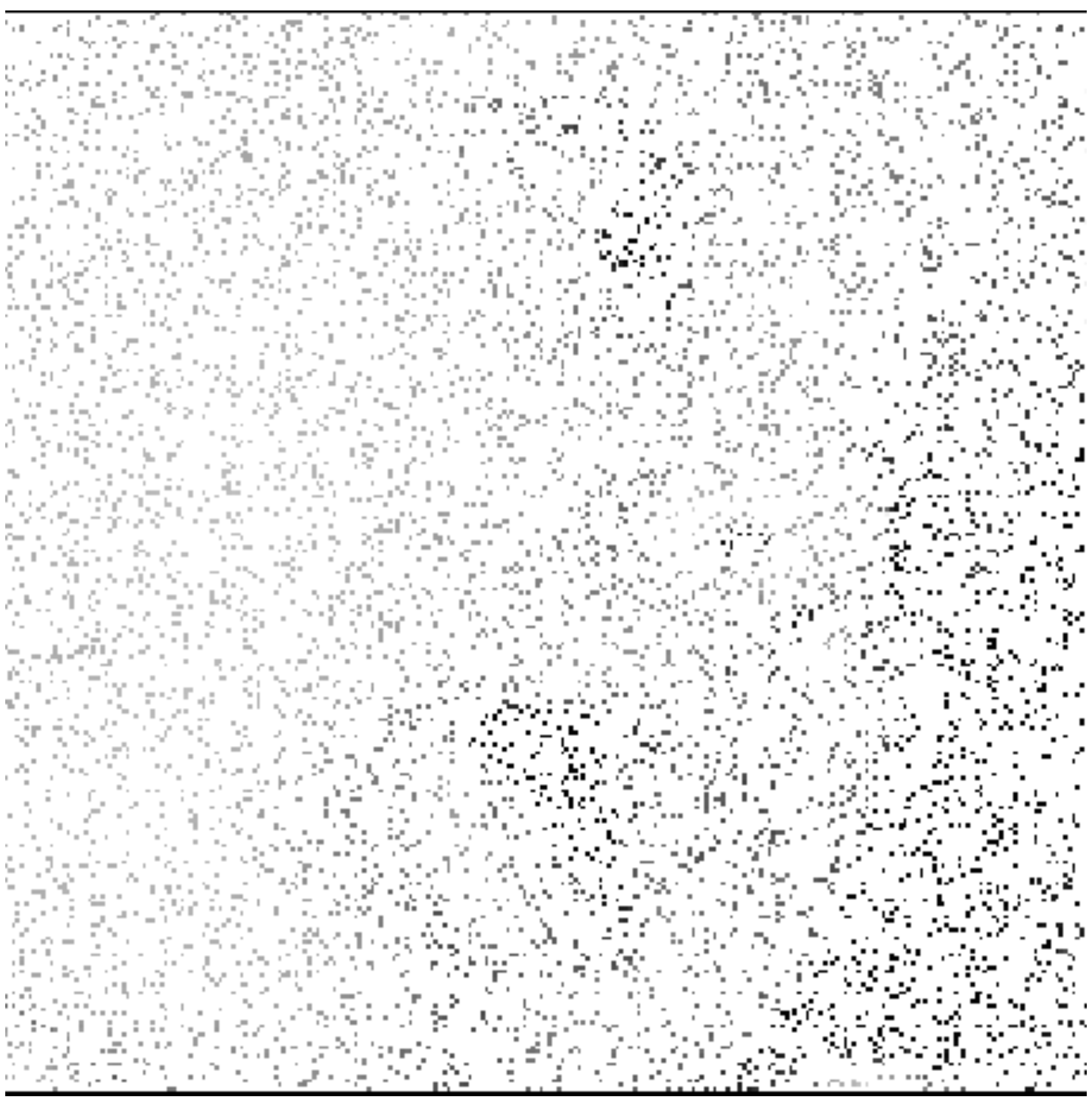

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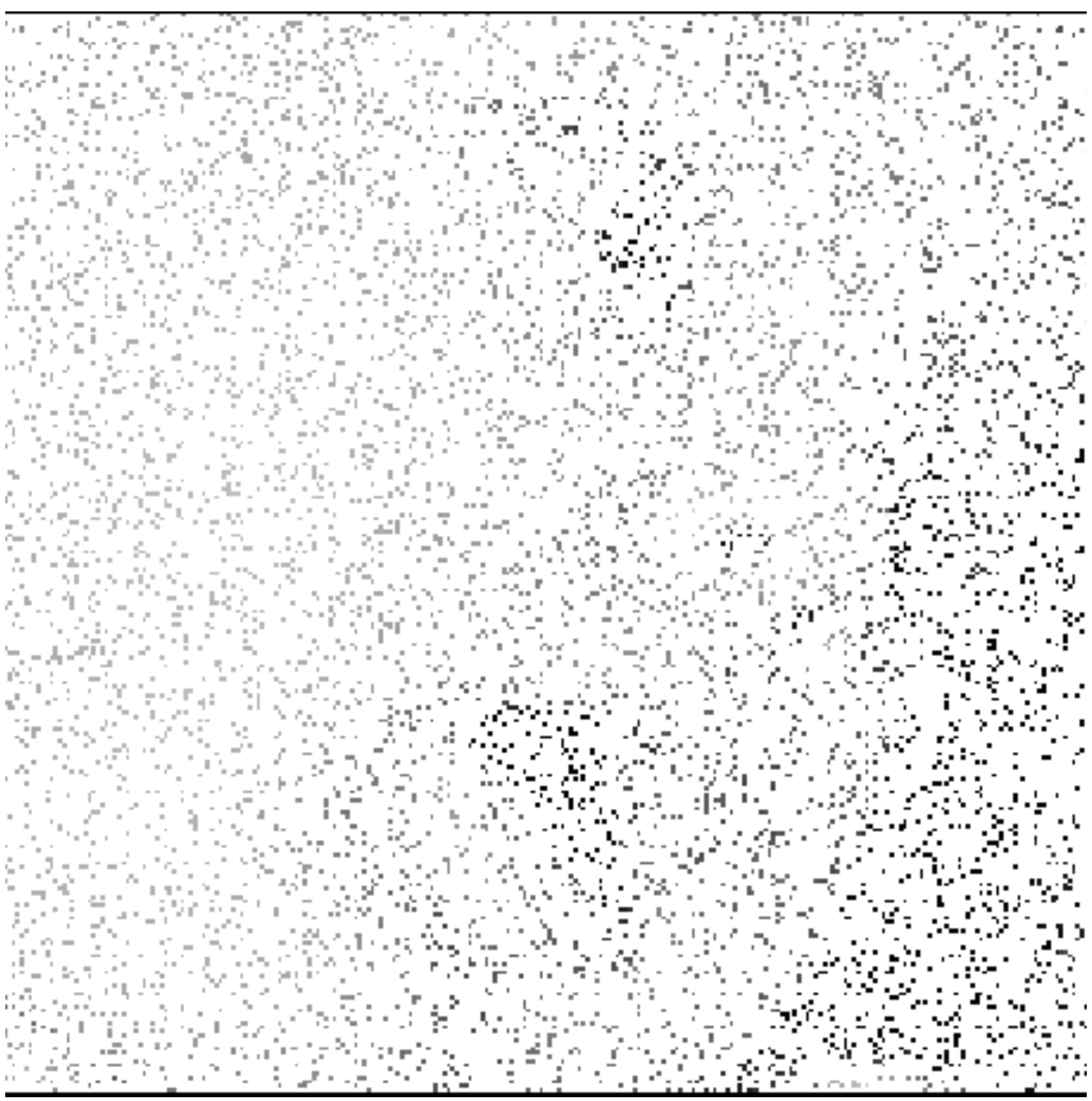
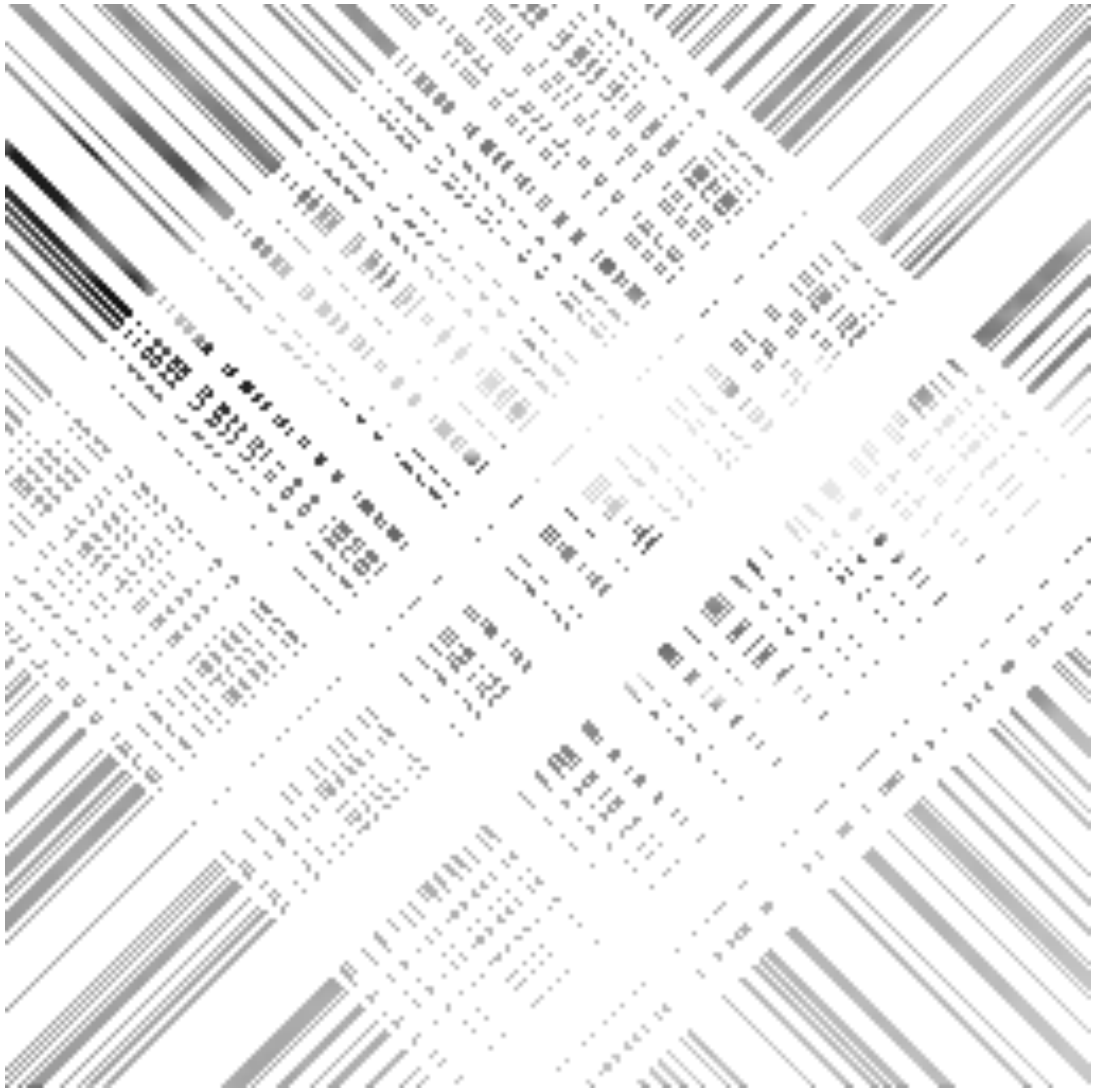


AHE (AVERAGING AND HYPOELLIPTIC EVOLUTION)

[Boscain *et al* '18]

95% of pixels missing along random diagonals



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[Boscain *et al* '18]

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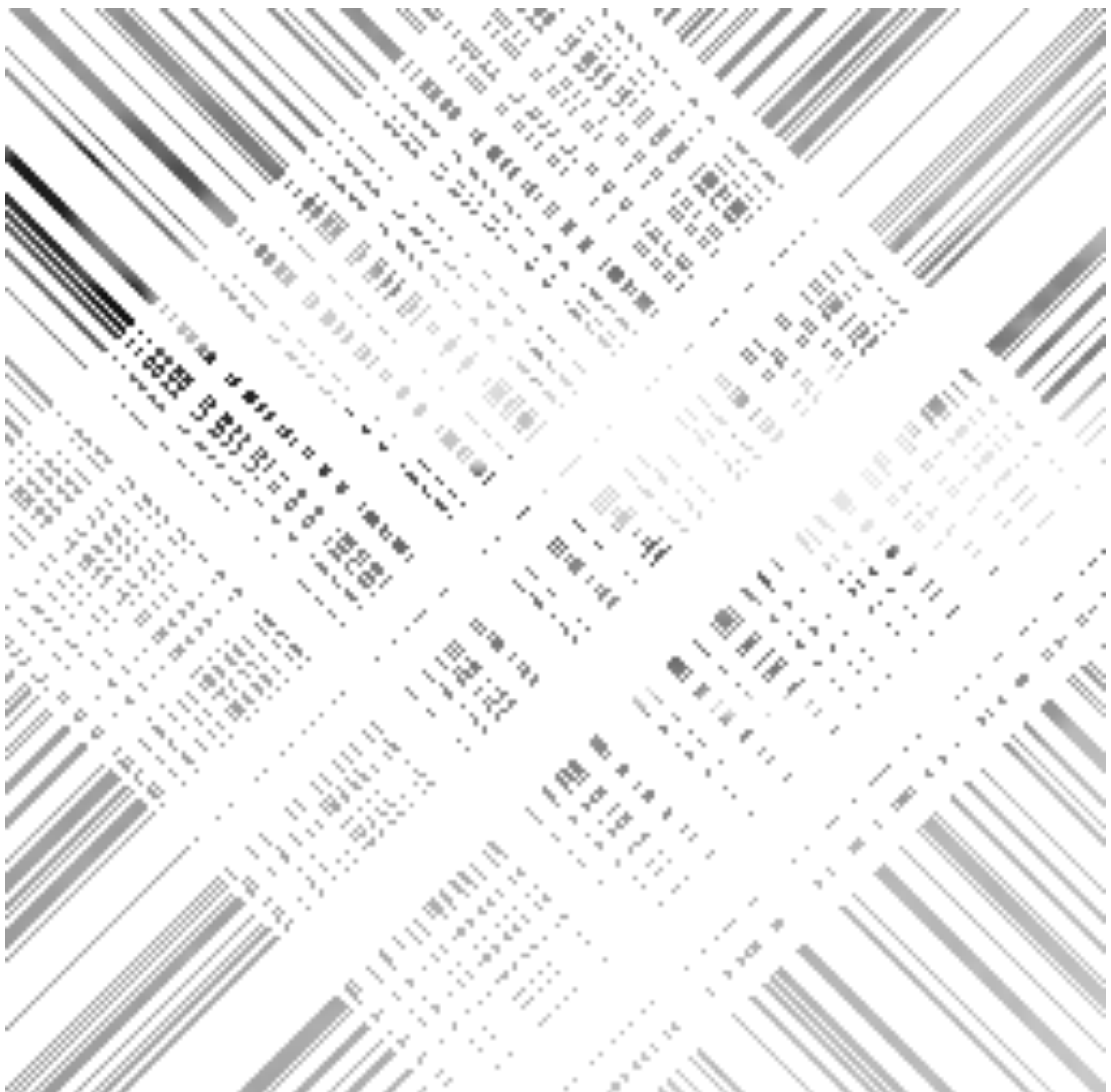

CONCLUSION

- Cortical-inspired image processing framework
- Introduction of anisotropy without non-linearities (but with additional dimension)
- Possibility to exploit the non-commutative group structure
- Good reconstructions with fast algorithms
- Although not presented: complete framework for image recognition via non-commutative Fourier descriptors

THANK YOU FOR YOUR ATTENTION

1. D. P., J.-P. Gauthier. *A semidiscrete version of the Petitot model as a plausible model for anthropomorphic image reconstruction and pattern recognition*. To appear on SpringerBriefs in Mathematics

2. U. Boscain, R. Chertovskih, J.P. Gauthier, D. P., A. Remizov,. Highly corrupted image inpainting through hypoelliptic diffusion. Journal of Math. Imag. And Vision, 2018.

3. D. P., U. Boscain, J.-P. Gauthier. *Image processing in the semidiscrete group of rototranslations.* Geometric Science of Information - Second International Conference, GSI 2015 -- October 28-30, 2015

4. U. Boscain, J.-P. Gauthier, D. P., A. Remizov. *Image reconstruction via non-isotropic diffusion in Dubins/Reed-Shepp-like control systems.* G53rd IEEE Conference on Decision and Control -- December 2014

5. U. Boscain, R. Chertovskih, J.P. Gauthier, A. Remizov. Hypoelliptic diffusion and human vision: a semi-discrete new twist. SIAM Journal on Imaging Sciences 2014, Vol. 7, No. 2, pp. 669-695