The Laplace-Beltrami operator on conic-type surfaces

Dario Prandi

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Geometrical Setting

In this talk we will consider the manifold

$$M = \big(\mathbb{R} \setminus \{0\}\big) \times \mathbb{S}^1,$$

endowed with the metric

$$\mathbf{g}_\alpha = dx^2 + |x|^{-2\alpha}\, d\theta^2 \qquad \alpha \in \mathbb{R}.$$

This metric has orthonormal basis

$$X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \Theta = \begin{pmatrix} 0 \\ |x|^\alpha \end{pmatrix}$$

and induces the measure

$$dV = |x|^{-\alpha}\, dx\, dy$$

Interpretation for $\alpha = 0$

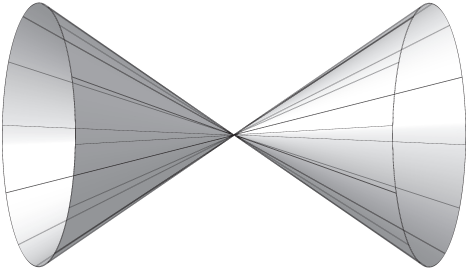
$$\mathbf{g}_0 = dx^2 + d\theta^2.$$

$$X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \Theta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad dV = dx\, dy$$

Interpretation for $\alpha = -1$

$$\mathbf{g}_{-1} = dx^2 + x^2 d\theta^2.$$

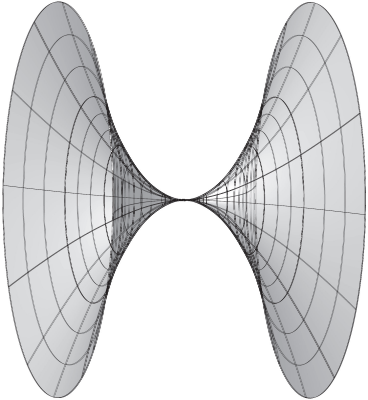
$$X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \Theta = \begin{pmatrix} 0 \\ \frac{1}{|x|} \end{pmatrix}, \qquad dV = |x|\, dx\, dy$$



Interpretation for $\alpha < -1$

$$\mathbf{g}_{-2} = dx^2 + x^4 d\theta^2.$$

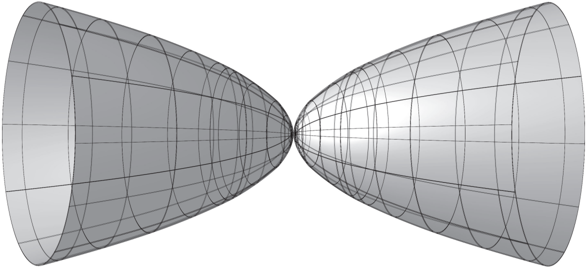
$$X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \Theta = \begin{pmatrix} 0 \\ \frac{1}{x^2} \end{pmatrix}, \qquad dV = |x|^2 \, dx \, dy$$



Interpretation for $\alpha \in (-1, 0)$

$$\mathbf{g}_{-\frac{1}{2}} = dx^2 + x d\theta^2.$$

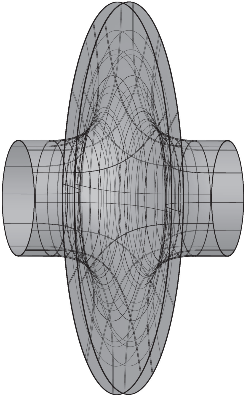
$$X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \Theta = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{|x|}} \end{pmatrix}, \qquad dV = \sqrt{|x|}\, dx\, dy$$



Interpretation for $\alpha > 0$

$$\mathbf{g}_1 = dx^2 + \frac{1}{x} d\theta^2.$$

$$X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \Theta = \begin{pmatrix} 0 \\ |x| \end{pmatrix}, \qquad dV = \frac{1}{|x|}\, dx\, dy$$



The Laplace-Beltrami operator

The Laplace-Beltrami operator is defined as

$$\Delta u = \partial_x^2 - \frac{\alpha}{x}\partial_x + |x|^{2\alpha}\partial_\theta^2.$$

- **Schrödinger equation for a free particle**:

$$-i\frac{\partial}{\partial t}u = \Delta u,$$

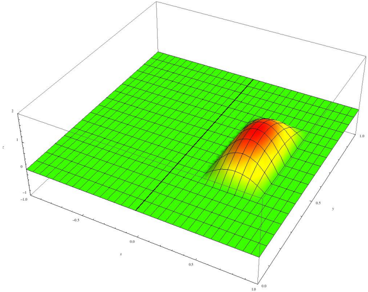
- **Heat equation**:

$$\frac{\partial}{\partial t}u = \Delta u,$$

Questions

We want to answer to the following questions:

1. Is it possible to transmit information from one side to the other of the singularity?
2. Does the heat flow through the singularity?



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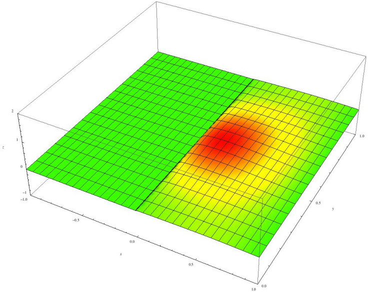

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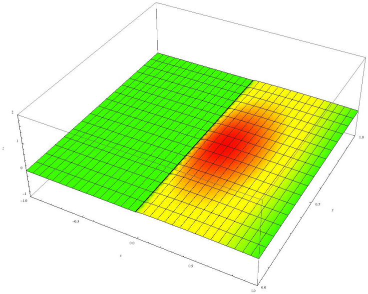

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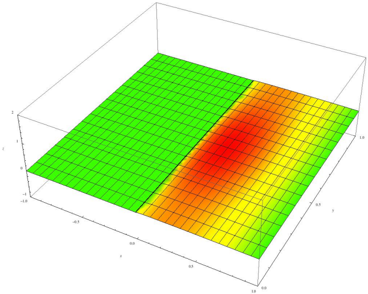

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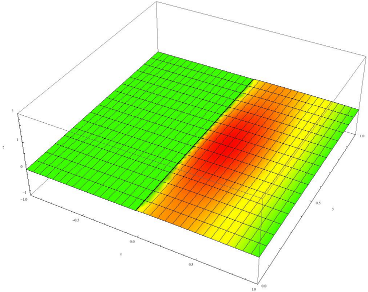

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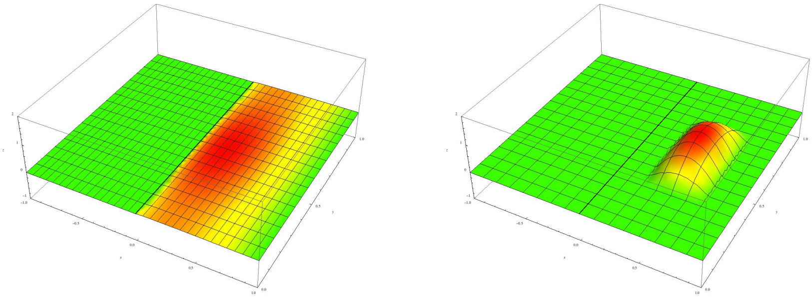

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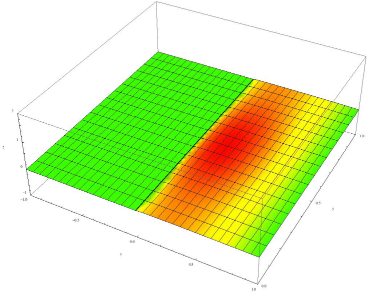

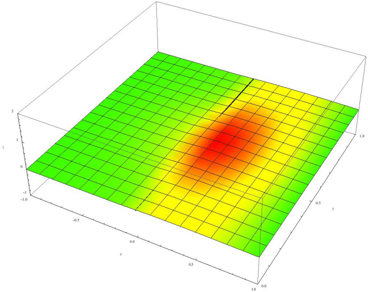

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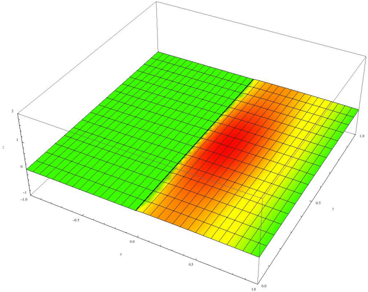

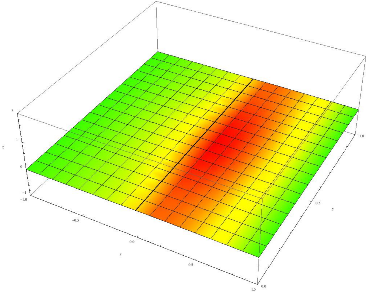

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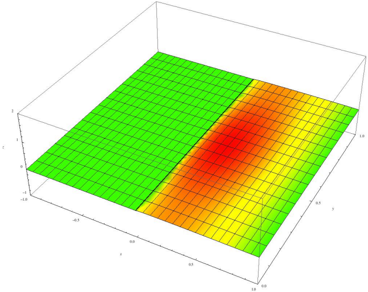

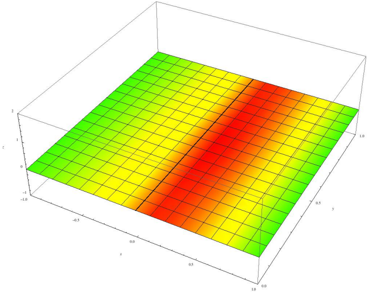

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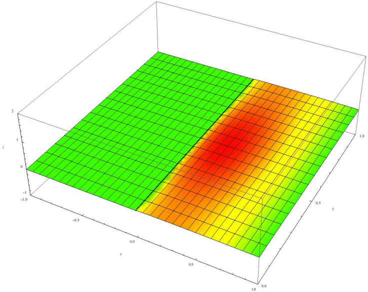

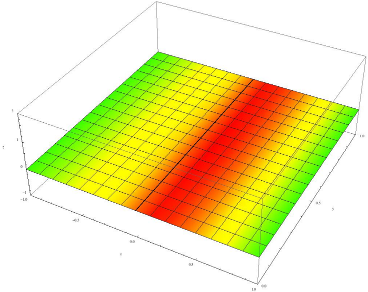

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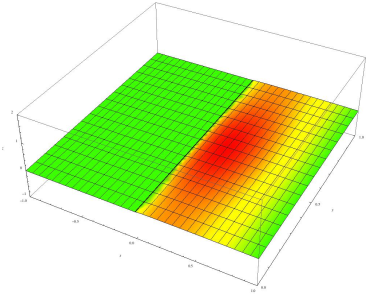

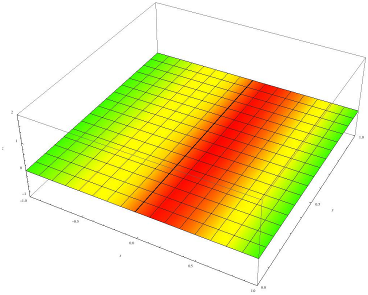

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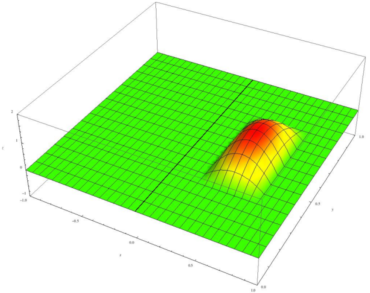

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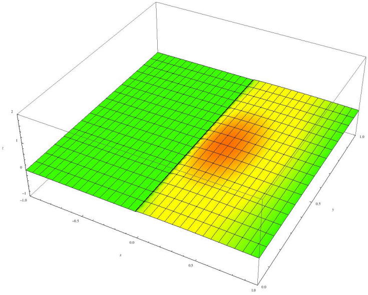

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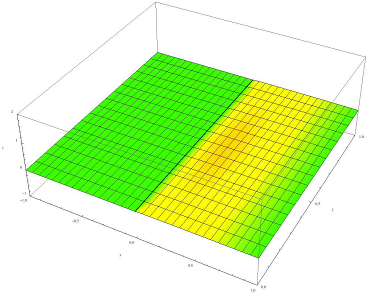

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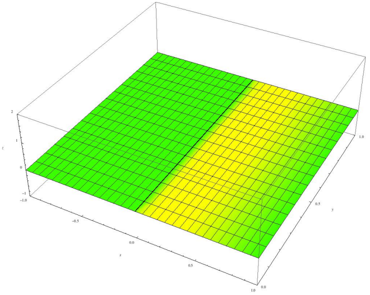

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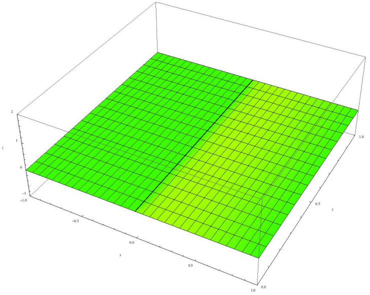

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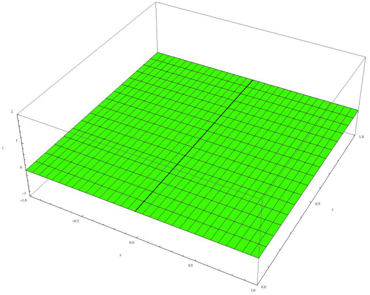

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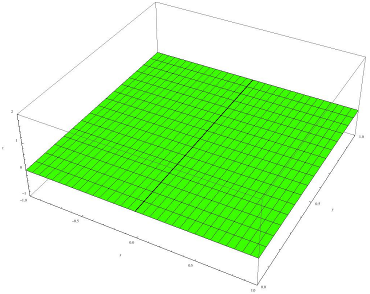

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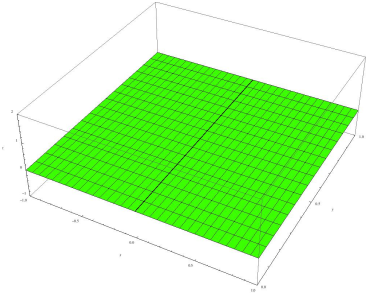

Evolution semigroups

Stone's theorem

On any Hilbert space there is a one-to-one correspondence,

A self-adjoint operator	$\longleftrightarrow$	e^{-itA} strongly continuous unitary semigroup

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The Laplace-Beltrami operator on $L^2(M, dV)$

Let $\Delta|_{C_c^\infty(M)} : C_c^\infty(M) \to L^2(M, dV)$.

By integration by parts,

$$(\Delta u, v)_{L^2(M,dV)} = (u, \Delta v)_{L^2(M,dV)} + \Big(\partial_x u\, v - u \partial_x v \Big)\Big|_{0^-}^{0^+}.$$

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$$\Longrightarrow \Delta|_{C_c^\infty(M)} : C_c^\infty(M) \to L^2(M, dV) \text{ is } \textbf{symmetric}.$$

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if $u \in C_c^\infty(M)$,

and $v, \Delta v \in L^2(M, dV)$

$$\implies D(\Delta^*) = \{v \in L^2(M, dV) \mid \Delta v \in L^2(M, dV)\}.$$

$$\implies \Delta|_{C_c^\infty(M)} : C_c^\infty(M) \to L^2(M, dV) \text{ is } \textbf{not self-adjoint}.$$

Self-adjoint extensions of $\Delta|_{C_c^\infty(M)}$

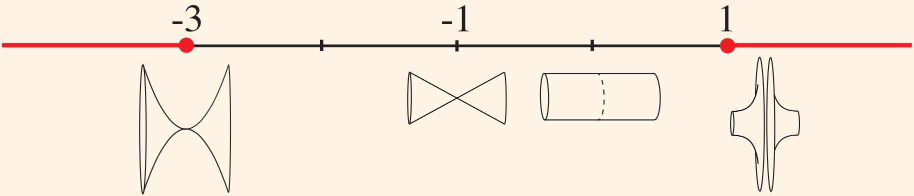
An operator A is a self-adjoint extension of $\Delta|_{C_c^\infty(M)}$ if

$$D(\Delta|_{C_c^\infty(M)}) \subset D(A) = D(A^*) \subset D(\Delta^*)$$
$$A^* u = \Delta^* u \quad \text{for any } u \in D(A).$$

The simplest case is when there exists only one such extension. In this case the operator is called **essentially self-adjoint**.

Theorem

The operator $\Delta|_{C_c^\infty(M)}$ is essentially self-adjoint if and only if $\alpha \notin (-3, 1)$.



Fourier decomposition

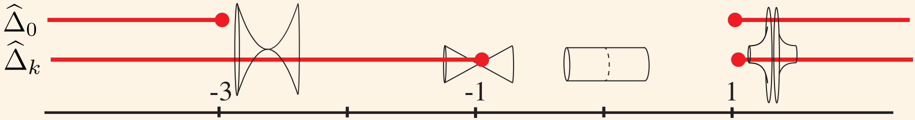
We make a Fourier decomposition in the θ variable

$$L^2(M, dV) = L^2(\mathbb{R}, |x|^{-\alpha}dx) \otimes L^2(\mathbb{S}^1, d\theta) \approx \bigoplus_{k\in\mathbb{Z}} H_k.$$

$$\Delta = \sum_{k\in\mathbb{Z}} \widehat{\Delta}_k, \quad \text{where } \widehat{\Delta}_k \text{ acts on } H_k \approx L^2(\mathbb{R}, |x|^{-\alpha}dx).$$

Theorem

The Fourier component $\widehat{\Delta}_0$ is essentially self-adjoint if and only if $\alpha \notin (-3, 1)$, while, for $k \neq 0$, the Fourier component $\widehat{\Delta}_k$ is essentially self-adjoint if and only if $\alpha \notin (-1, 1)$.



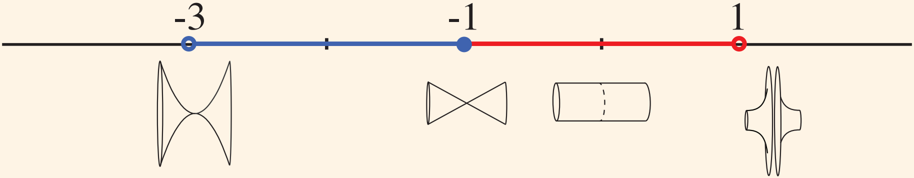
Transmission through the Schrödinger equation

For $u \in L^2(M, dV)$ the Fourier component on H_0 is

$$u_0(x) = \frac{1}{2\pi} \int_0^{2\pi} u(x, \theta)\, d\theta.$$

Answer to Question 1

For $\alpha \notin (-3, 1)$ it is not possible to transmit any information through the singularity via the Schrödinger equation. On the other hand, for $\alpha \in (-3, -1]$ it is possible to transmit only the mean value of the function, while for $\alpha \in (-1, 1)$ it is possible to have complete communication through the singularity.



Markovian extensions

In order to accept a self-adjoint extension of $\Delta|_{C_c^\infty(M)}$ as the operator of the heat equation we have to pose an additional condition.

In fact, we have to require that an initial temperature distribution, when evolving, does not lose its property of being a nonnegative and bounded function.

Definition

The positive self-adjoint operator A on $L^2(M, dV)$ is Markovian if and only if

$$u \in L^2(M, dV) \text{ s.t. } 0 \leq u \leq 1 \, a.e. \Longrightarrow 0 \leq e^{tA} u \leq 1 \, a.e.$$

Some Markovian extensions

We will consider the following three Markovian extensions.

- **Dirichlet (or Friederichs) extension Δ_D**

$$D(\Delta_D) = \{u \in H^1_0(M, dV) \mid \Delta u \in L^2(M, dV)\},$$

- **Bridging extension Δ_B**

$$D(\Delta_B) = H^2(\bar{M}, dV) = \{u \in L^2(\bar{M}, dV) \mid |\nabla u|, \Delta u \in L^2(\bar{M}, dV)\},$$

- **Neumann extension Δ_N**

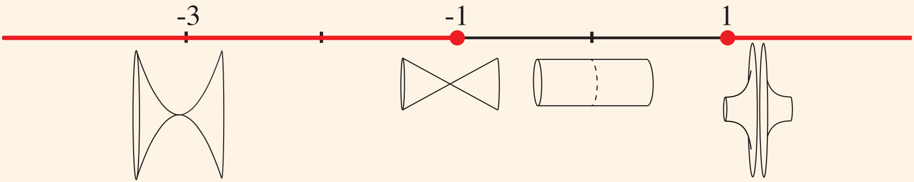
$$D(\Delta_N) = \{u \in H^1(M, dV) \mid (\Delta u, v)_{L^2(M,dV)} = (\nabla u, \nabla v)_{L^2(M,dV)} \text{ for any } v \in H^1(M, dV)\}.$$

Markov uniqueness of $\Delta_{C_c^\infty(M)}$ and transmission of heat

If the operator $\Delta|_{C_c^\infty(M)}$ has only one Markovian extension (i.e., if $\Delta_D = \Delta_B = \Delta_N$), it is said to be **Markov unique**.

Theorem

- *If $\alpha \leq -1$ or $\alpha \geq 1$, then $\Delta_{C_c^\infty(M)}$ is Markov unique;*
- *If $\alpha \in (-1, 1)$, then Δ_D, Δ_B and Δ_N are three different Markov extensions.*



This implies that we can have heat transmission only if $\alpha \in (-1, 1)$.

The conservation of heat

The Markov property allows to extend e^{tA} from $L^2(M, dV)$ to $L^\infty(M, dV)$.

Definition

The Markovian operator A is *stochastically complete* if

$$e^{tA}1 = 1 \qquad \text{for any } t \geq 0.$$

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Question 3'

Are the Markov extensions of $\Delta|_{C_c^\infty(M)}$ stochastically complete?

Connection with the Brownian motion

The operator A is stochastically complete	$\iff$	The Brownian motion associated with A has almost surely an infinite lifespan

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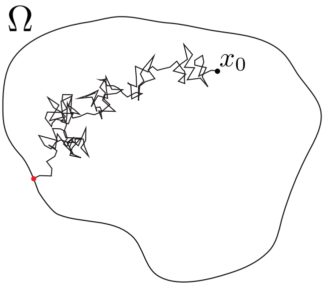
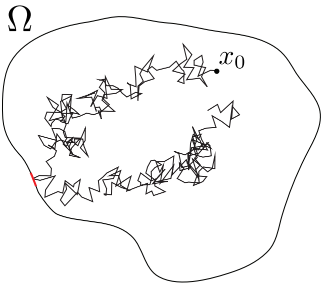
Brownian motion associated with $\Delta = \partial_x^2 + \partial_y^2$ on a bounded set $\Omega \subset \mathbb{R}^2$:



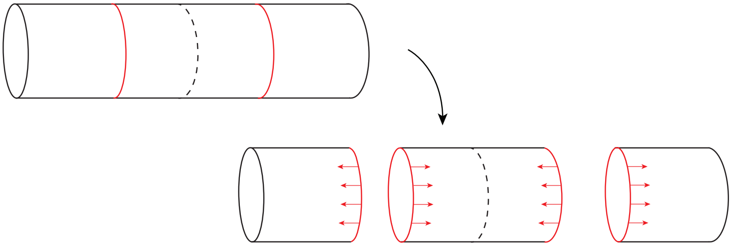
Dirichlet boundary conditions



Neumann boundary conditions

Connection with the Brownian motion

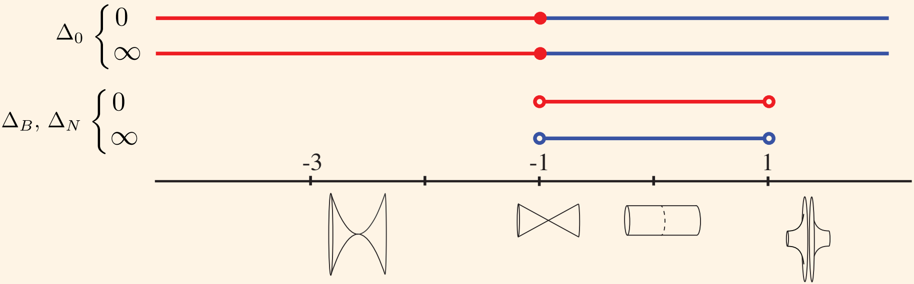
The operator A is stochastically complete	$\Longleftrightarrow$	The Brownian motion associated with A has almost surely an infinite lifespan



Stochastic completeness of Δ_D, Δ_B and Δ_N

Theorem

- *The Dirichlet extension Δ_D is stochastically complete (both at 0 and at ∞) if and only $\alpha \leq -1$.*
- *If $\alpha \in (-1, 1)$, then the bridging and the Neumann extensions, Δ_B and Δ_N, are stochastically complete at 0 and incomplete at ∞.*



Conclusions