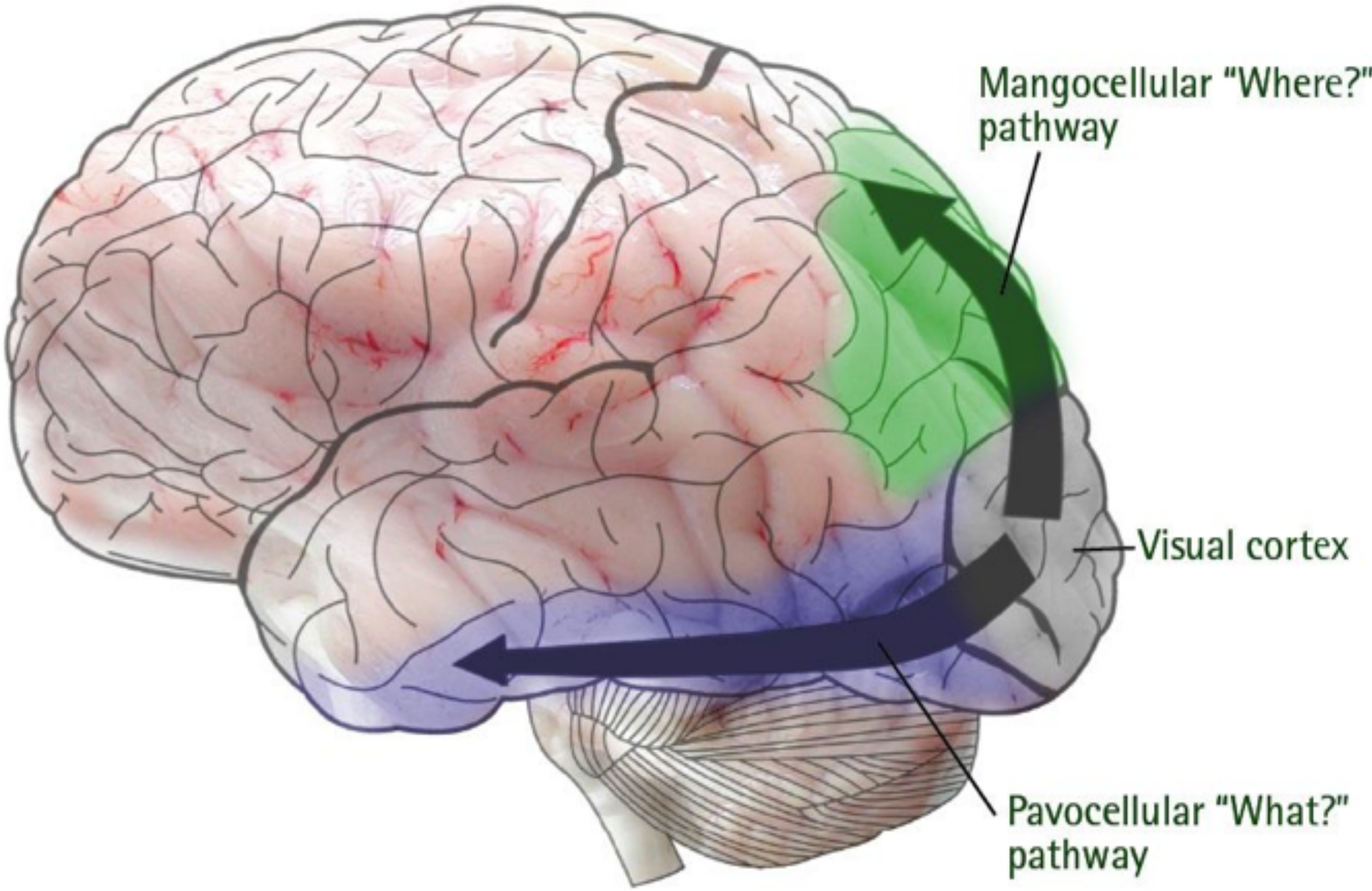
A VARIATIONAL FORMULATION OF THE SUB-RIEMANNIAN MODEL OF THE PRIMARY VISUAL CORTEX

DARIO PRANDI
(Univ. Paris-Sud Orsay)

Joint work with:
G. Peyre (Paris-Dauphine),
J.-M. Mirebeau (Paris-Sud Orsay), A. Sarti (EHESS, Paris)



CITTI–PETITOT–SARTI MODEL OF THE PRIMARY VISUAL CORTEX

ORIGIN OF THE MODEL

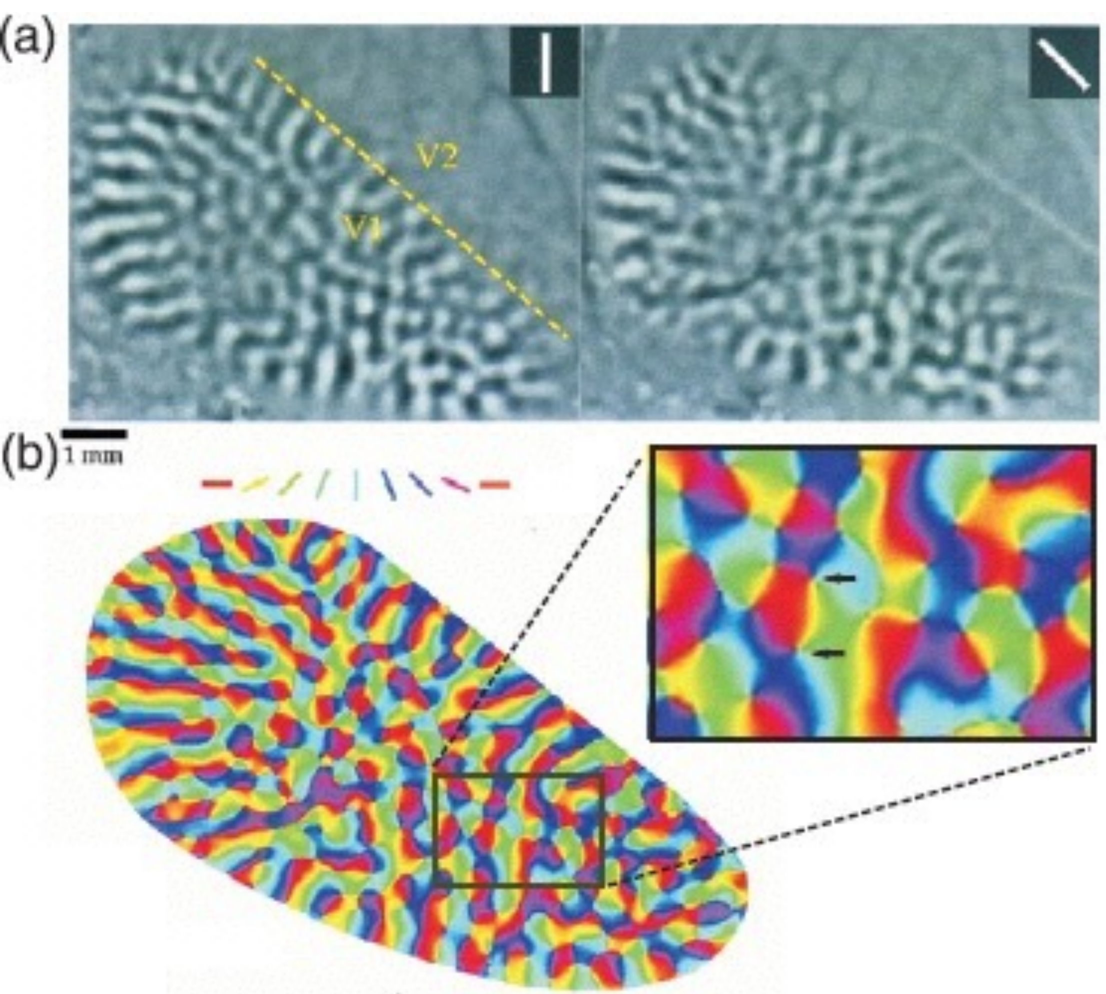
- Hoffman (1989): structure of a contact manifold
- Petitot (1999): structure of a sub-Riemannian manifold (Heisenberg group)

REFINEMENTS

- Citti and Sarti (2006): structure of the rototranslations of the plane $SE(2)$
- Agrachev, Boscain, Charlot, Gauthier, Rossi, D.P. (2010 – …)

ALSO STUDIED BY

- Sachkov (2010 – …)
- Duits (2009 – …)



Patterns of orientation columns of a tree shrew. Taken from Kaschube et al. (2008).

Neurons in V1 are sensible to positions and local orientations

Hubel and Wiesel (Nobel '81)

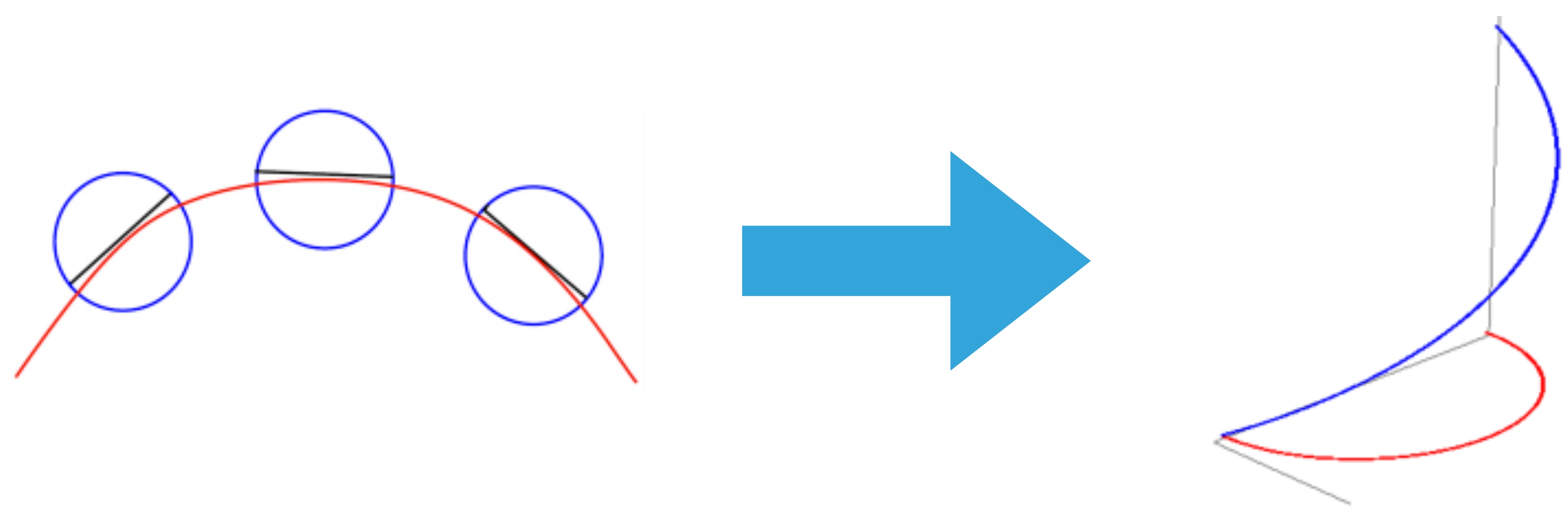
MODELING OF V1

- Neurons of V1 $\longleftrightarrow$ Points of the projective tangent bundle of $\mathbb{R}^2$

$$PT\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{P}^1$$

- Linear lift operator

$$\mathcal{L} : L^2(\mathbb{R}^2) \to L^2(PT\mathbb{R}^2)$$



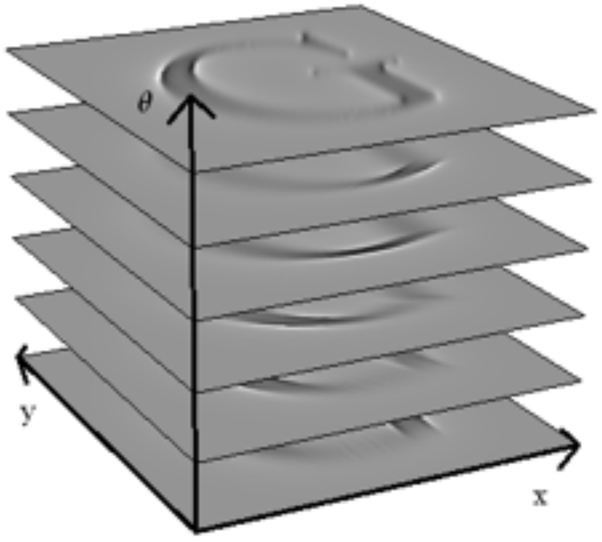
MODELING OF V1

- Neurons of V1 $\longleftrightarrow$ Points of the projective tangent bundle of $\mathbb{R}^2$

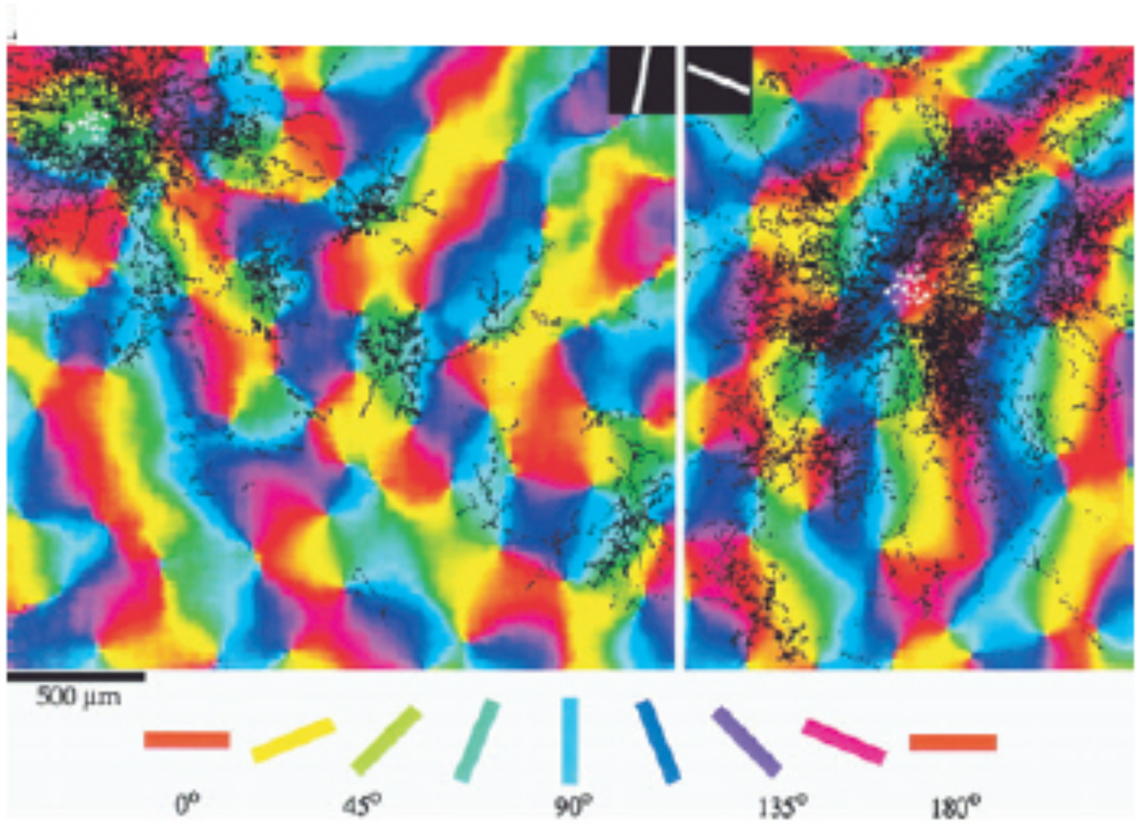
$$PT\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{P}^1$$

- Linear lift operator

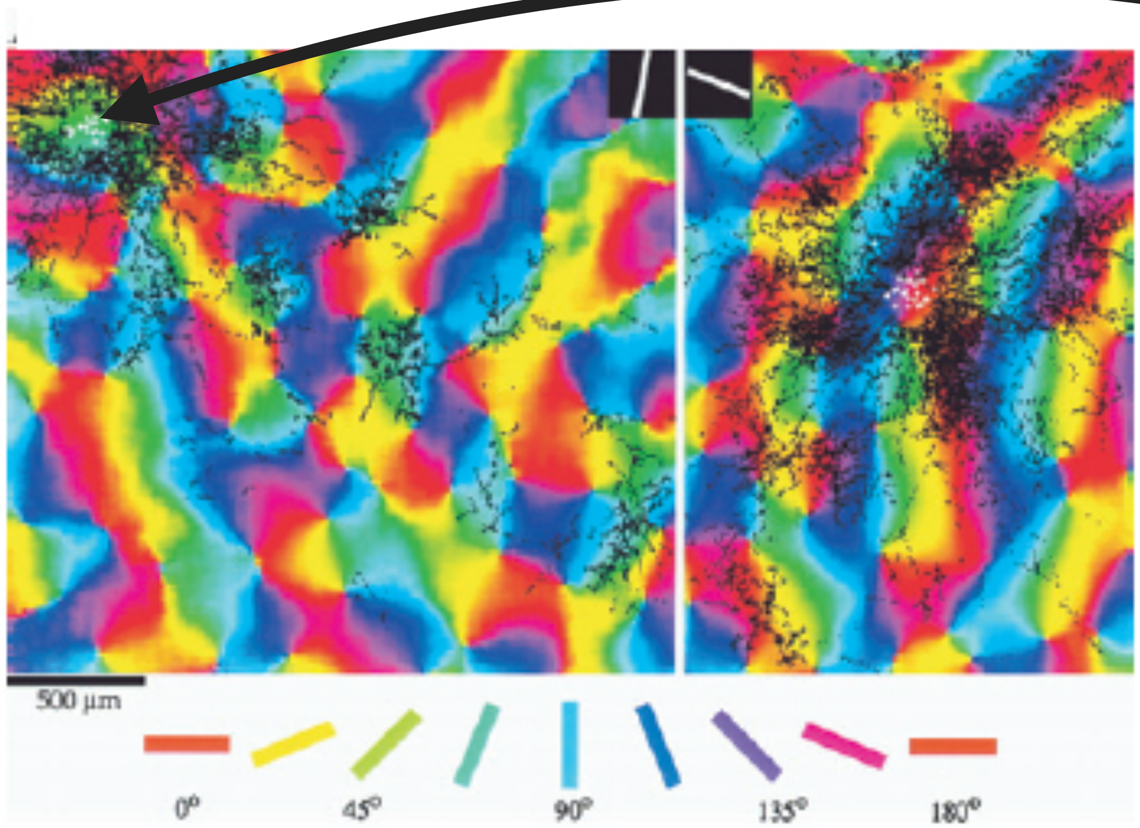
$$\mathcal{L} : L^2(\mathbb{R}^2) \to L^2(PT\mathbb{R}^2)$$



CONNECTIONS IN V1

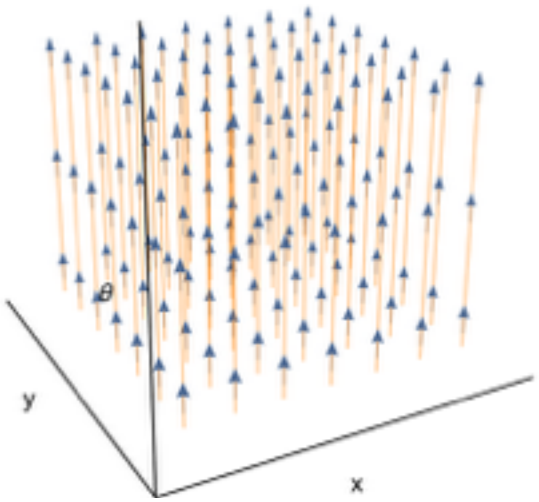


CONNECTIONS IN V1

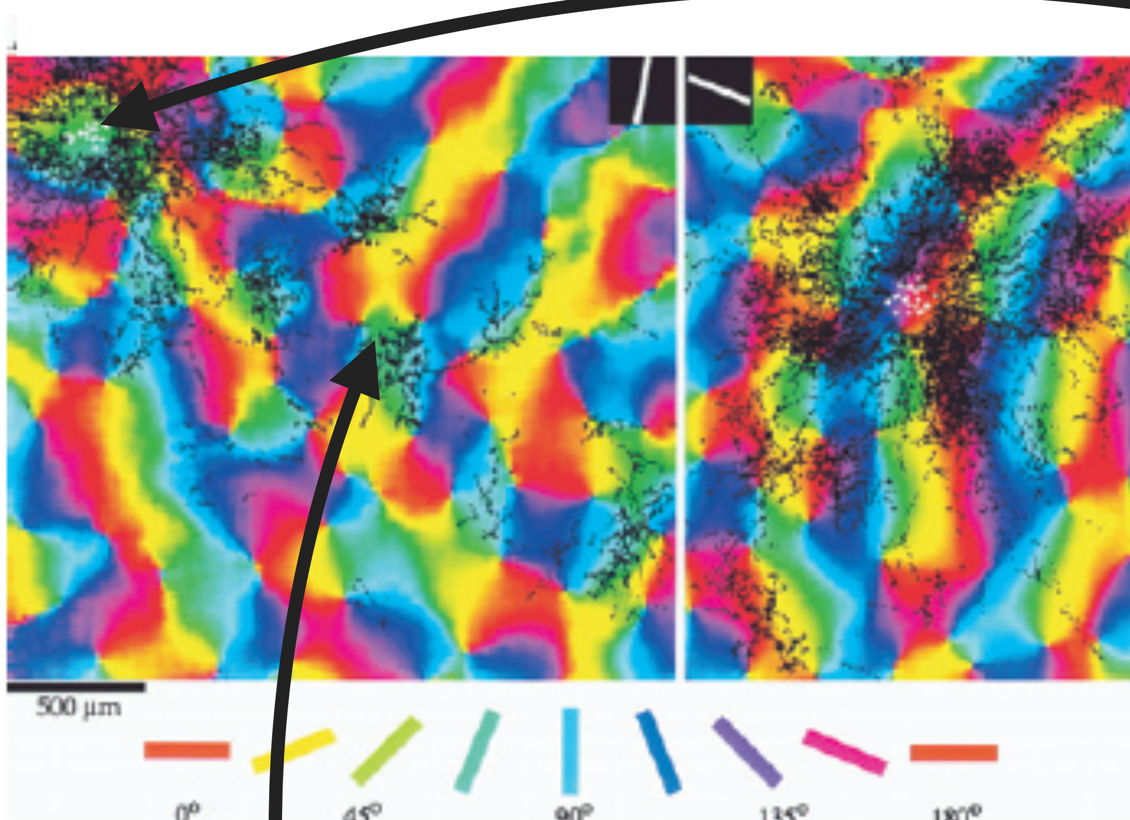


Local connections: Between neurons in the same hypercolumn.

$$\Theta(x,y,\theta) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

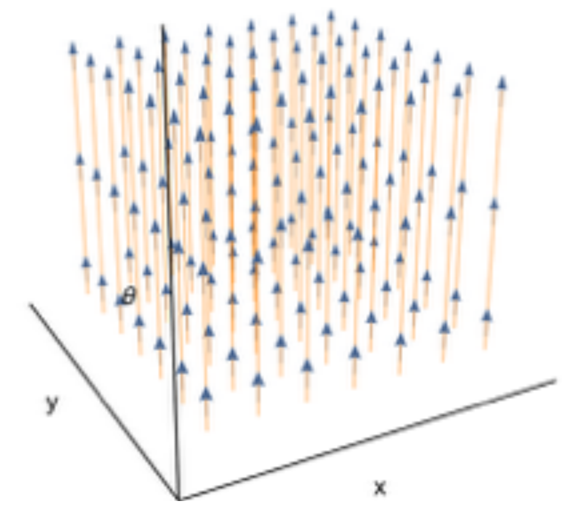


CONNECTIONS IN V1



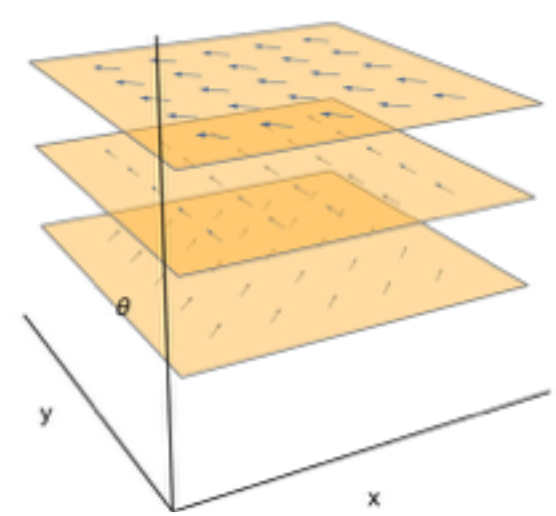
Local connections: Between neurons in the same hypercolumn.

$$\Theta(x, y, \theta) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



Lateral connections: Between iso-oriented neurons, in the direction of their orientation.

$$X(x, y, \theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$



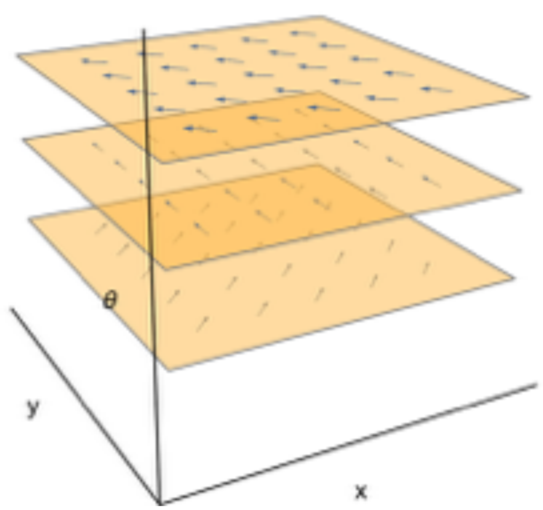
MAIN ASSUMPTION OF THE MODEL

The metric of neuronal connections in V1 is the sub-Riemannian metric associated with the orthonormal frame $\{X, \Theta\}$

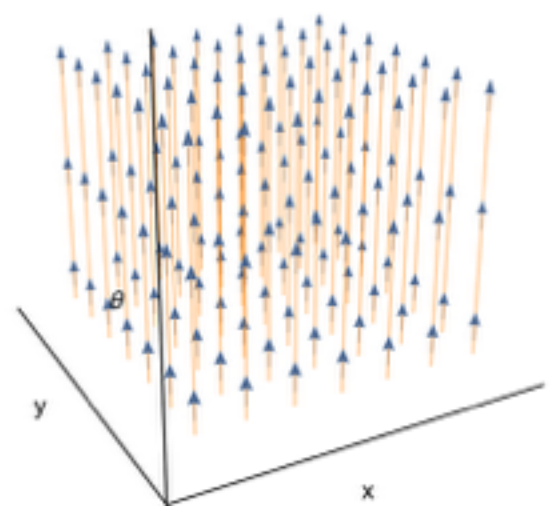
$$\frac{d}{dt}(x, y, \theta)^T = X(x, y, \theta)u + \Theta(x, y, \theta)v \qquad \int_0^T \sqrt{u^2 + v^2}\, dt \longrightarrow \min$$

$$X(x, y, \theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \qquad \Theta(x, y, \theta) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

X

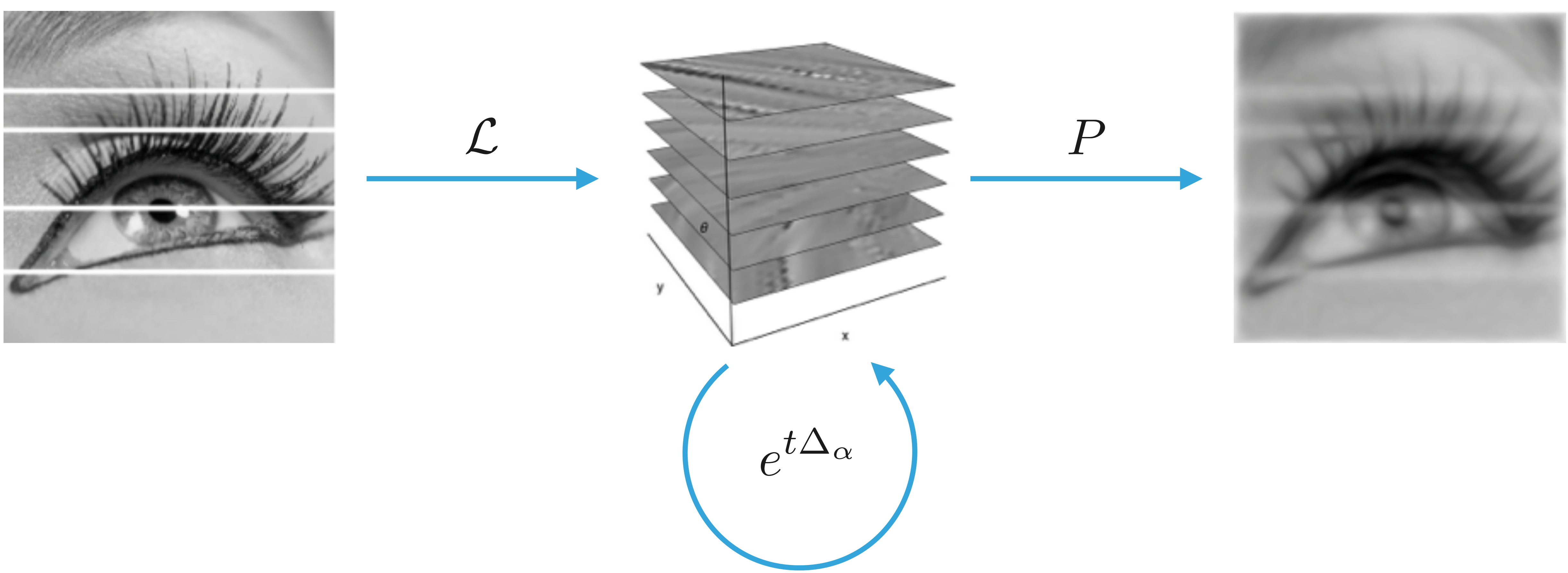


Θ



WE NEED TO DISCUSS:

- **Receptive fields:** How a greyscale image is lifted to a state on V1
- **Spontaneous evolution:** How it evolves via neuronal connections
- **Projection on the visual plane:** How it is interpreted as a 2D image

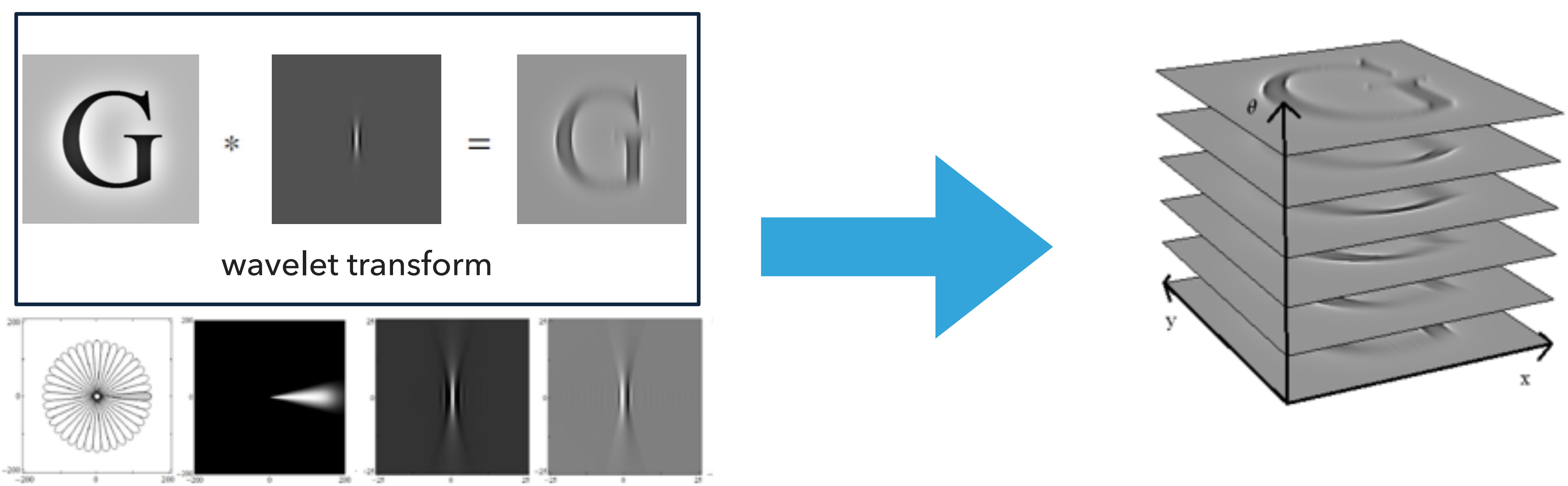


RECEPTIVE FIELDS

- A greyscale stimulus $f : \mathbb{R}^2 \to [0,1]$ feeds a V1 neuron $\xi = (x, y, \theta)$ with an extracellular voltage

$$\mathcal{L}f(x,y,\theta) = \langle f, \Psi_\xi \rangle_{L^2(\mathbb{R}^2)}$$

- A good fit is $\Psi_\xi(\cdot) = \Psi(\xi^{-1}\cdot) = \Psi(R_{-\theta}(\cdot - x))$ for some mother wavelet Ψ (usually a Gabor wavelet or similar)

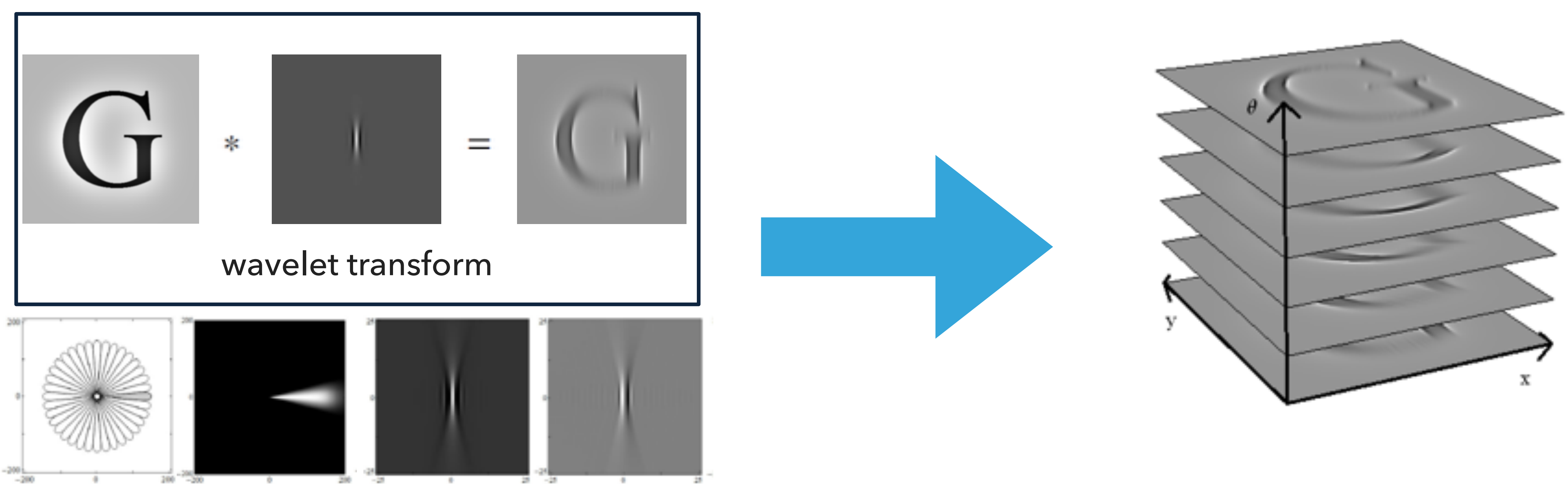


RECEPTIVE FIELDS

- A greyscale stimulus $f : \mathbb{R}^2 \to [0,1]$ feeds a V1 neuron $\xi = (x, y, \theta)$ with an extracellular voltage

$$\mathcal{L}f(x, y, \theta) = \langle f, \Psi_\xi \rangle_{L^2(\mathbb{R}^2)}$$

- A good fit is $\Psi_\xi(\cdot) = \Psi(\xi^{-1} \cdot) = \Psi(R_{-\theta}(\cdot - x))$ for some mother wavelet Ψ (usually a Gabor wavelet or similar)



EVOLUTION IN THE CPS MODEL

- The sub-Riemannian structure allows to define the energies

$$\mathcal{E}_\alpha(F) = \int_{\mathbb{R}^2}\int_{\mathbb{P}^1} |XF|^2 + \alpha^2|\Theta F|^2\, d\theta\, dx, \qquad \alpha > 0$$

- The associated Sobolev space is

$$H^1(PT\mathbb{R}^2) = \{F \in L^2(PT\mathbb{R}^2) \mid \mathcal{E}_\alpha(F) < +\infty\}$$

- A stimulus $F \in L^2(PT\mathbb{R}^2)$ evolves according to the sR heat equation

$$\frac{d}{dt}F(x,y,\theta) = \underbrace{\left((\cos\theta\partial_x + \sin\theta\partial_y)^2 + \alpha^2\partial_\theta^2\right)}_{=:\Delta_\alpha} F(x,y,\theta)$$

PROJECTION TO THE VISUAL PLANE

- The projection is done via the operator

$$PF(x,y) = \int_{\mathbb{P}^1} F(x,y,\theta)\,d\theta$$

PROJECTION TO THE VISUAL PLANE

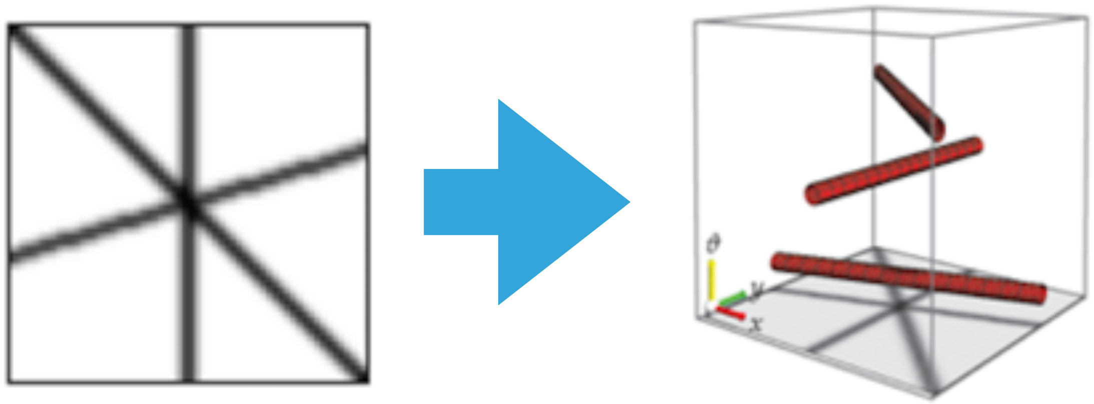
- The projection is done via the operator

$$PF(x,y) = \int_{\mathbb{P}^1} F(x,y,\theta)\, d\theta$$

REMARKS

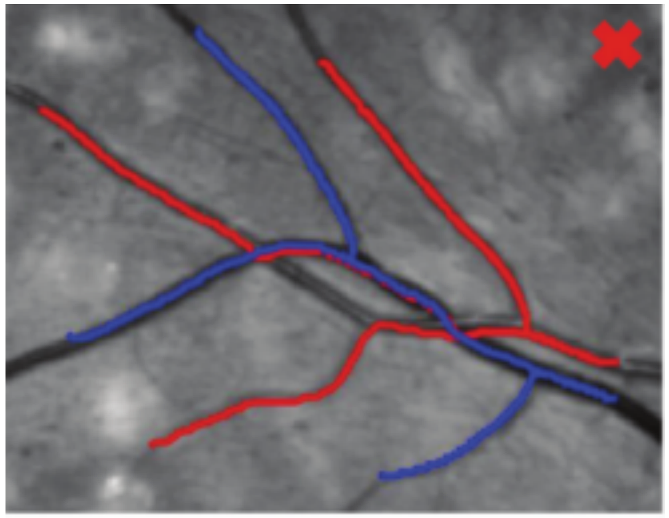
- Lift, evolution and projection are three independent operations
- In general evolving $F \in \operatorname{range} \mathcal{L}$ will **not yield** $G \in \operatorname{range} \mathcal{L}$

- Well adapted to treat images with crossings

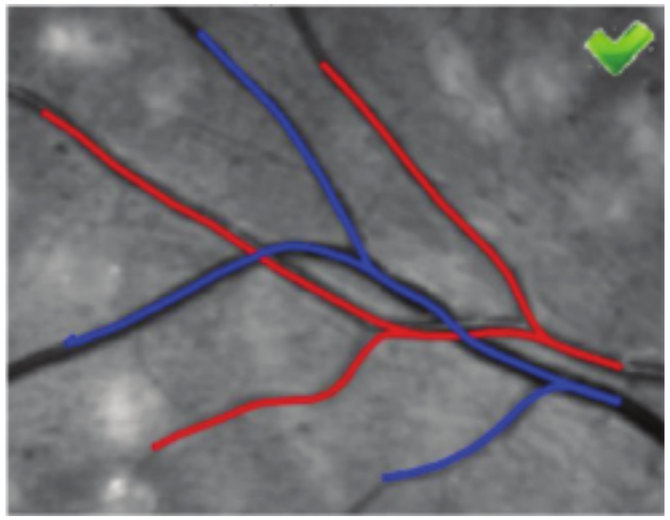


- Natural invariance under roto-translations
- Well-adapted for geodesic methods

Retinal vessel extraction
(Bekkers, Duits *et al.*, 2015)



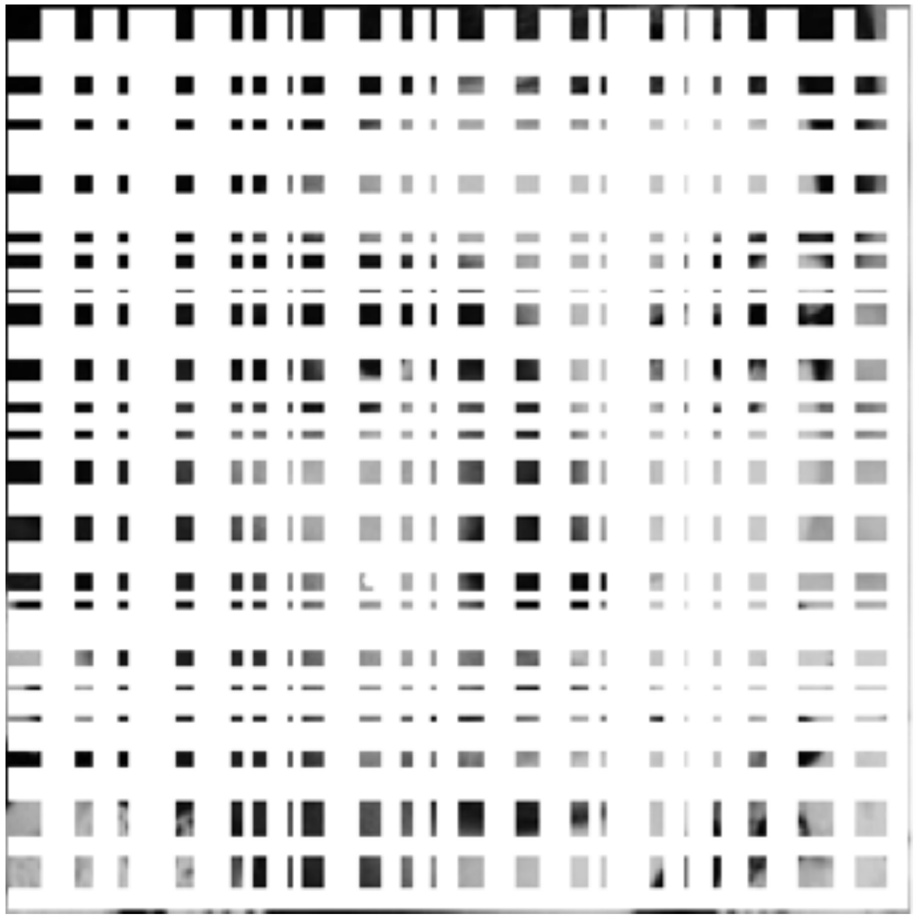
Standard $\mathbb{R}^2$ techniques



$SE(2)$ processing

Joint work with : U. Boscain (*École Polytechnique*),
J.-P. Gauthier (*LSIS Toulon*), A. Remizov (*São Paulo, Bresil*)

IMAGE INPAINTING



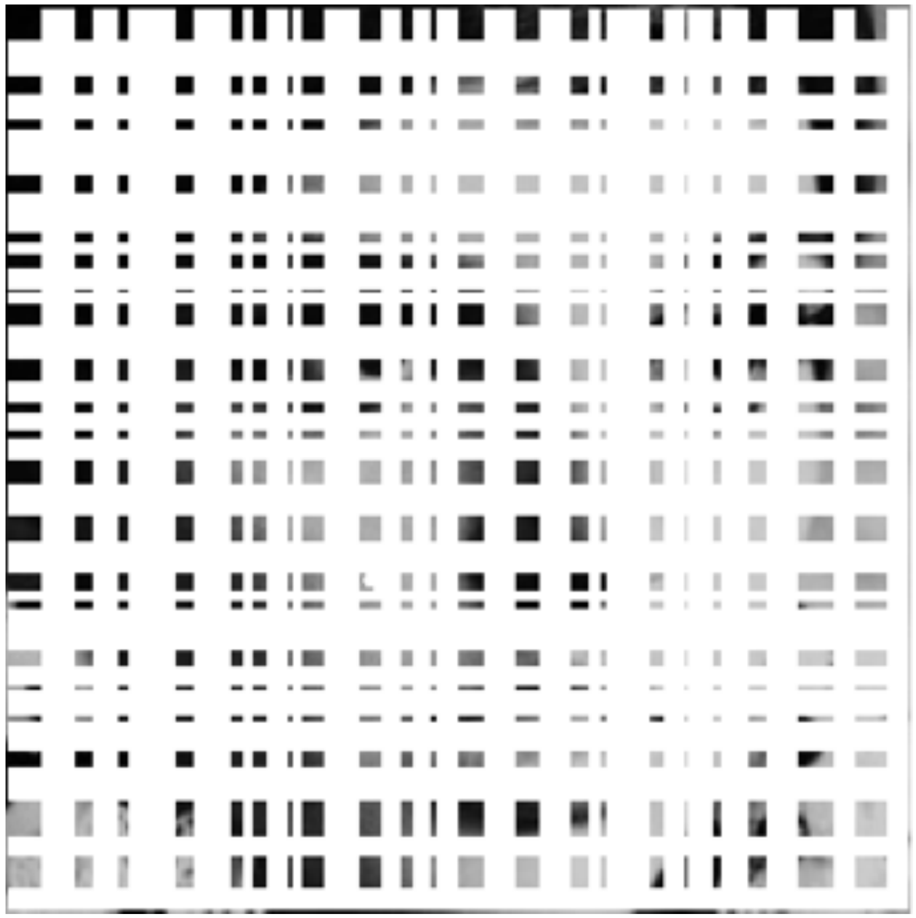
Spontaneous evolution on V1

$$\begin{cases} \partial_t \Psi = \big(\underbrace{X^2 + \alpha^2 \Theta^2}_{\Delta_\alpha} \big) \Psi \\ \Psi|_{t=0} = \mathcal{L} f \end{cases}$$

STRONGLY ANISOTROPIC EQUATION → **HARD NUMERICAL INTEGRATION**

Joint work with : U. Boscain (*École Polytechnique*), J.-P. Gauthier (*LSIS Toulon*), A. Remizov (*São Paulo, Bresil*)

IMAGE INPAINTING



Non-commutative Fourier analysis

Heuristic procedures

Spontaneous evolution on V1

$$\begin{cases} \partial_t \Psi = \big(\underbrace{X^2 + \alpha^2 \Theta^2}_{\Delta_\alpha}\big)\Psi \\ \Psi|_{t=0} = \mathcal{L}f \end{cases}$$

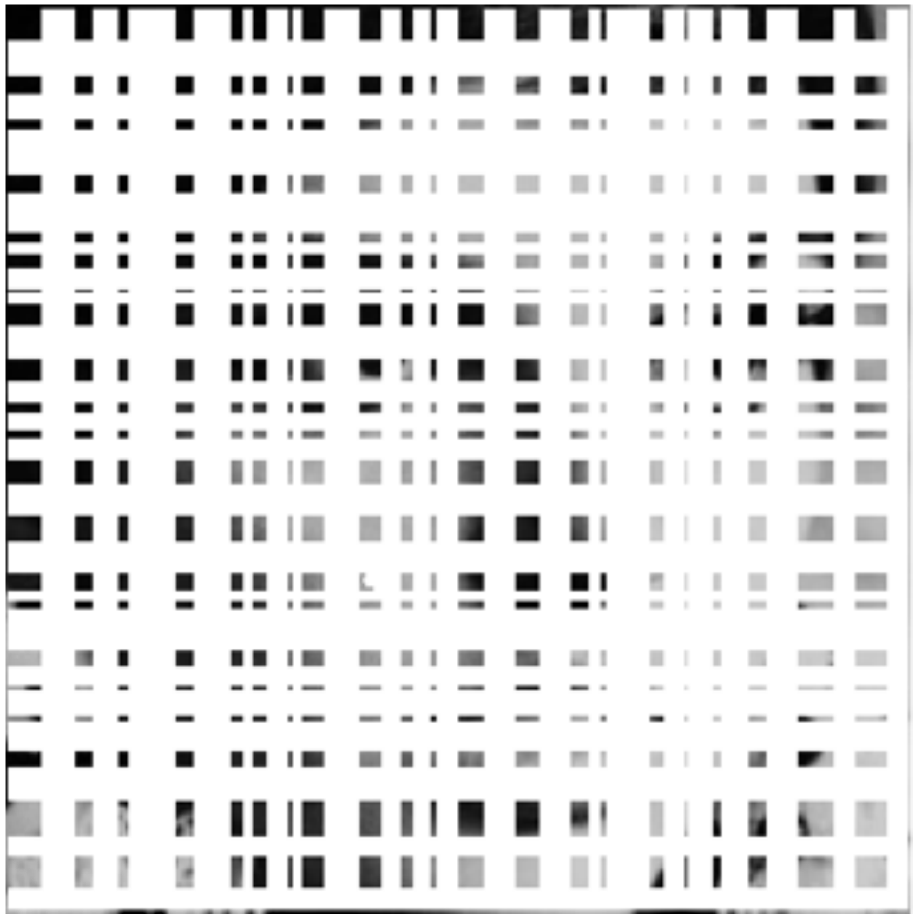
STRONGLY ANISOTROPIC EQUATION → HARD NUMERICAL INTEGRATION

Compared with state-of-the-art

- Very good reconstructions
- Reduced computational times (Strongly parallelizable, 5–30 s for 512x512px on GPU)

Joint work with : U. Boscain (*École Polytechnique*),
J.-P. Gauthier (*LSIS Toulon*), A. Remizov (*São Paulo, Bresil*)

IMAGE INPAINTING



Non-commutative Fourier analysis

Heuristic procedures

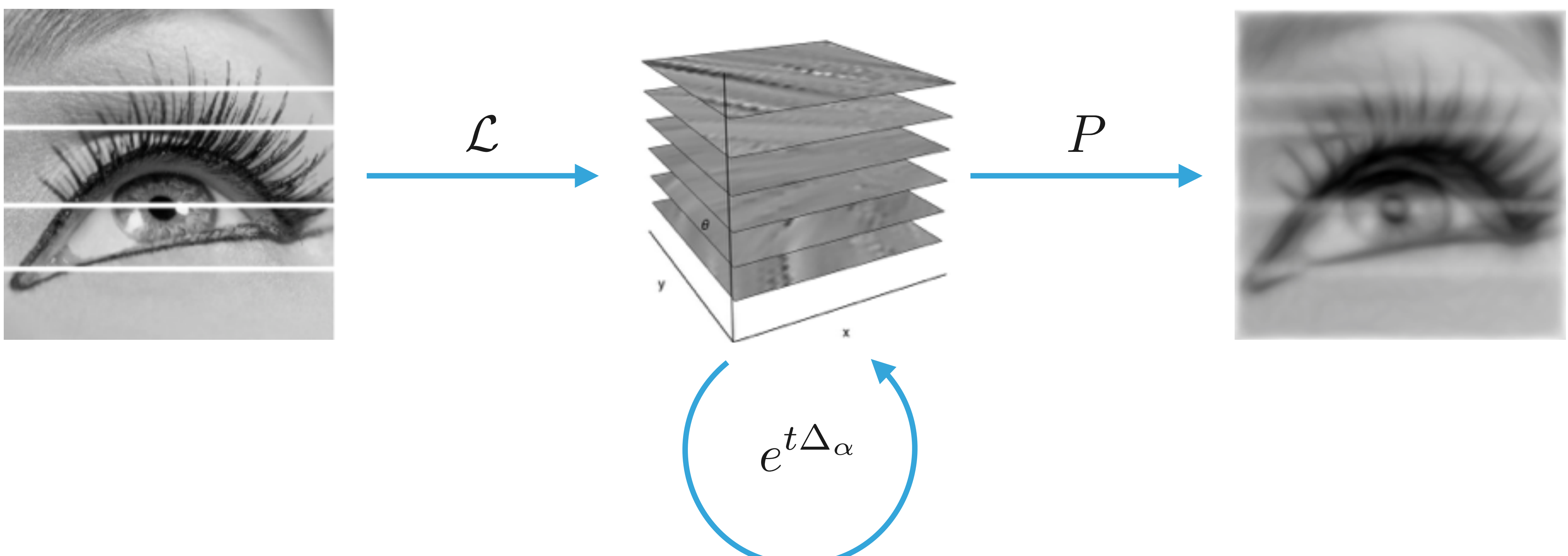
PROBLEMS

- No preservation of original image without introducing heuristic procedures
- No connection between the lift procedure and the evolution
- *Natural* lifts not sensible to high order features (e.g. curvature of level lines)

A VARIATIONAL ALTERNATIVE

MOTIVATIONS

- Effect of horizontal connectivity on receptive fields (extra-classical RF)
- Image processing techniques preserving the initial image
- Process low-regularity images via higher order procedures at the level of cortical stimuli

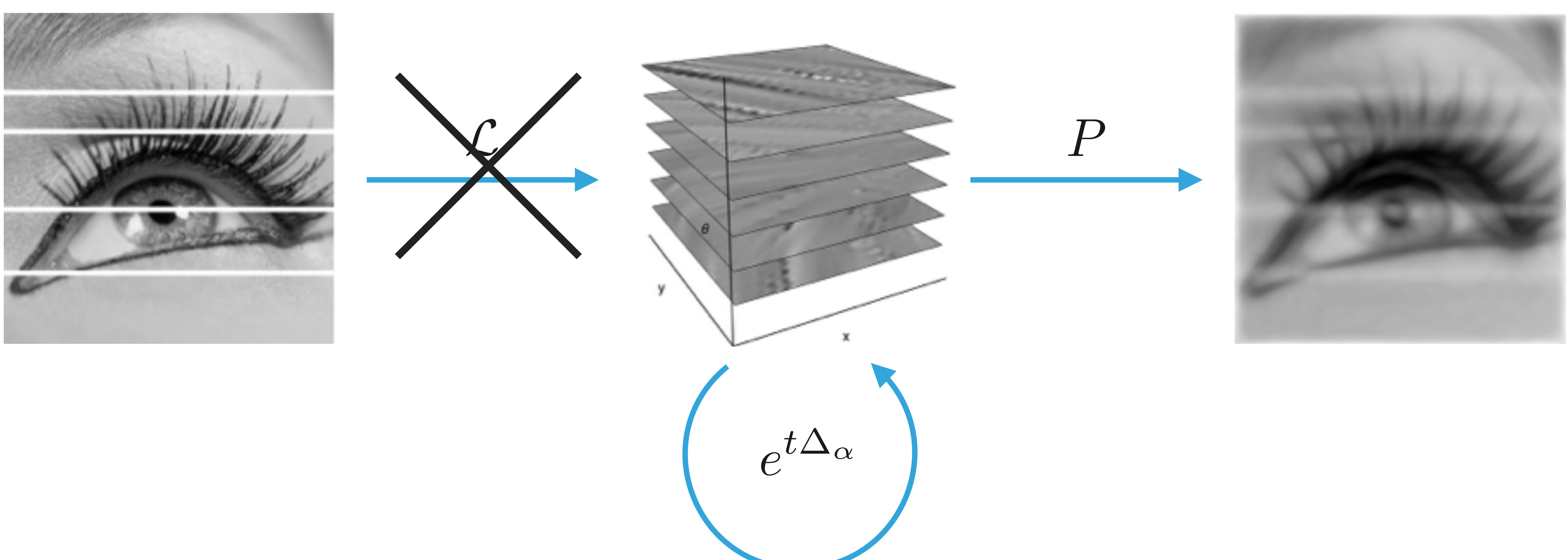


ASSUMPTIONS

- V1 is modeled as $PT\mathbb{R}^2$ endowed with the Citti-Petitot-Sarti sub-Riemannian structure

- Cortical activations are projected to visual stimuli via the projection

$$PF(x,y) = \int_{\mathbb{P}^1} F(x,y,\theta)\, d\theta$$



Recall that the sub-Riemannian energy on V1 is

$$\mathcal{E}_\alpha(F) = \int_{\mathbb{R}^2}\int_{\mathbb{P}^1} |XF|^2 + \alpha^2 |\Theta F|^2 \, d\theta \, dx, \qquad \alpha > 0$$

VARIATIONAL LIFT

- A visual stimulus $f \in L^2(\mathbb{R}^2)$ is lifted to the cortical activation

$$L_{\varepsilon,\alpha} f = \arg\min \left\{ \varepsilon^2 \|F\|^2_{L^2(PT\mathbb{R}^2)} + \mathcal{E}_\alpha(F) \mid PF = f \right\}$$

Recall that the sub-Riemannian energy on V1 is

$$\mathcal{E}_\alpha(F) = \int_{\mathbb{R}^2}\int_{\mathbb{P}^1} |XF|^2 + \alpha^2|\Theta F|^2 \, d\theta\, dx, \qquad \alpha > 0$$

VARIATIONAL LIFT

- A visual stimulus $f \in L^2(\mathbb{R}^2)$ is lifted to the cortical activation

$$L_{\varepsilon,\alpha} f = \arg\min \left\{ \varepsilon^2 \|F\|^2_{L^2(PT\mathbb{R}^2)} + \mathcal{E}_\alpha(F) \mid PF = f \right\}$$

Small regularizing parameter

sR energy

Coherence with initial image

Recall that the sub-Riemannian energy on V1 is

$$\mathcal{E}_\alpha(F) = \int_{\mathbb{R}^2} \int_{\mathbb{P}^1} |XF|^2 + \alpha^2 |\Theta F|^2 \, d\theta \, dx, \qquad \alpha > 0$$

VARIATIONAL LIFT

- A visual stimulus $f \in L^2(\mathbb{R}^2)$ is lifted to the cortical activation

$$L_{\varepsilon,\alpha} f = \arg\min \left\{ \varepsilon^2 \|F\|^2_{L^2(PT\mathbb{R}^2)} + \mathcal{E}_\alpha(F) \mid PF = f \right\}$$

Small regularizing parameter

sR energy

Coherence with initial image

- The cortical norm of a visual stimulus is

$$\|f\|_{\varepsilon,\alpha} = \varepsilon^2 \|\mathcal{L}_{\varepsilon,\alpha} f\|_{L^2(PT\mathbb{R}^2)} + \mathcal{E}_\alpha(\mathcal{L}_{\varepsilon,\alpha} f)$$

- The space of *admissible visual stimuli* is then

$$\mathcal{I} = \{ f \in L^2(\mathbb{R}^2) \mid \|f\|_{\varepsilon,\alpha} < +\infty \} \subset H^1(\mathbb{R}^2)$$

MAIN RESULT

Theorem. The following hold:

(i) There exists a distributional mother wavelet $\Psi_{\varepsilon,\alpha}$ such that

$$\mathcal{L}f(\xi) = \langle f, \Psi_{\varepsilon,\alpha}(\xi^{-1}\cdot)\rangle_{L^2(\mathbb{R}^2)}, \qquad \forall \xi \in PT\mathbb{R}^2$$

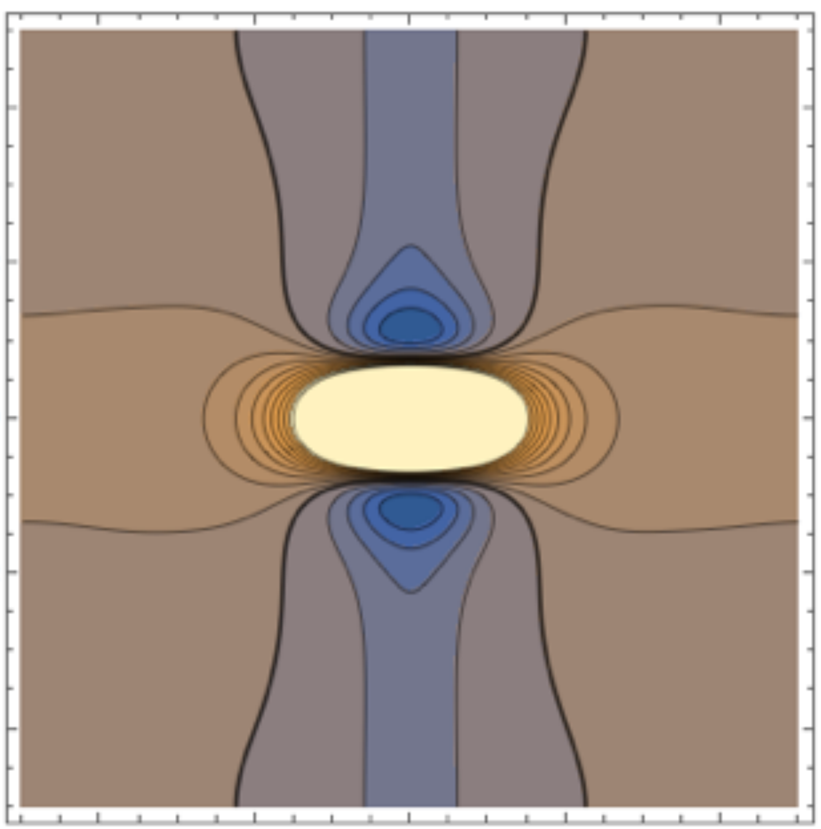
(ii) It holds that

$$\mathcal{I} = H^{\frac{3}{4}}(\mathbb{R}^2)$$

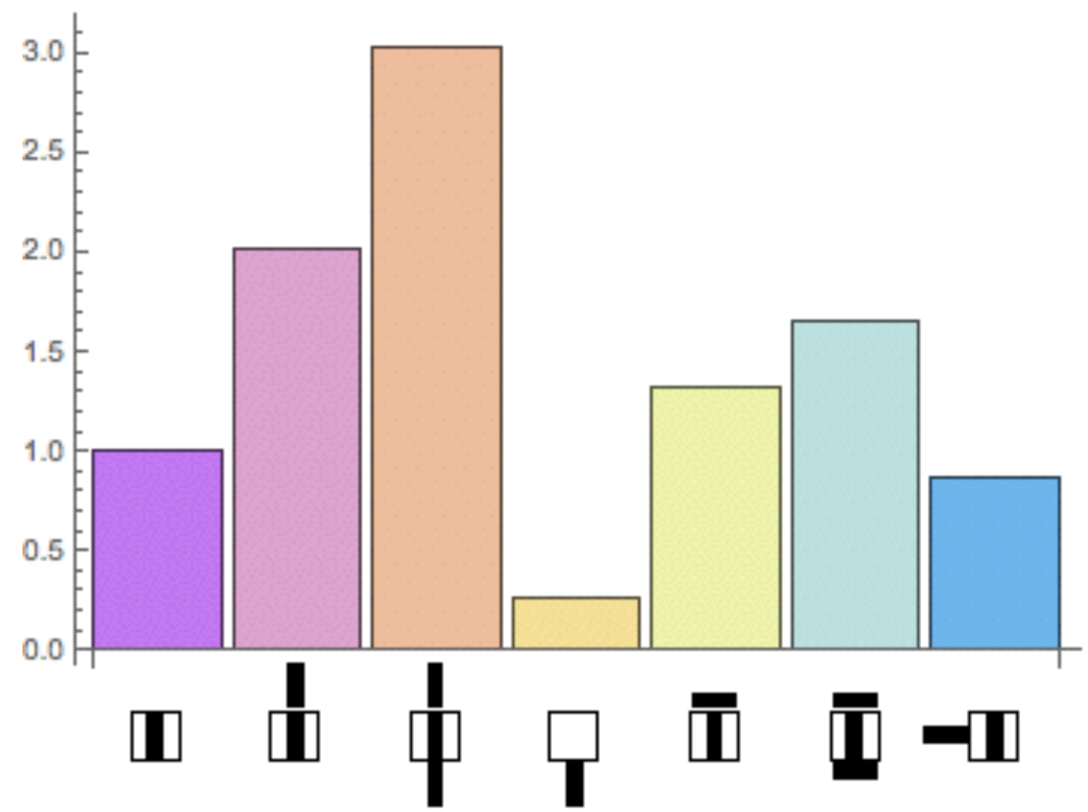
- Techniques of non-commutative Fourier analysis
- “Explicit” formula for $\Psi_{\varepsilon,\alpha}$ in terms of Mathieu functions
- Processing of non-$H^1(\mathbb{R}^2)$ images with linear operators in $H^1(PT\mathbb{R}^2)$

NEUROPHYSIOLOGICAL INTERPRETATION

After a Gaussian smoothing (operated by the retina) we observe numerically an inhibitory/excitatory behavior:



Receptive field $\Psi_{\varepsilon,\alpha} \star G_\sigma$



Responses of a neuron to simple stimuli

Connection with the extra-classical receptive fields observed in neuro-sciences (Gilbert *et al.*, 1999)

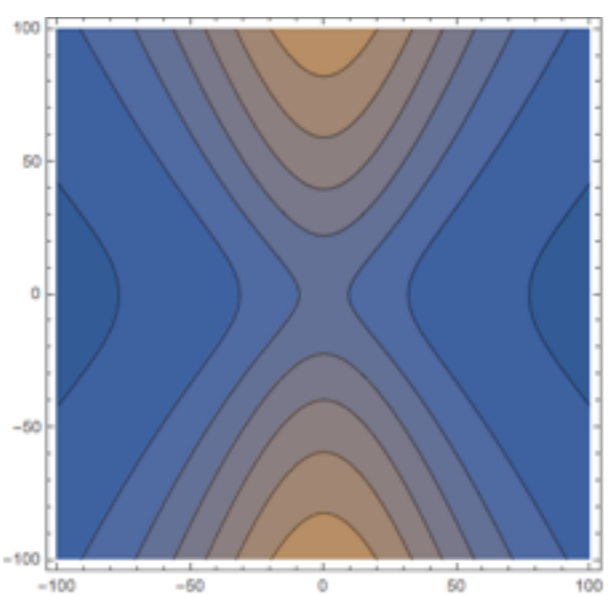
SENSITIVITY TO HIGHER ORDER FEATURES

Theorem. For $\varepsilon = \alpha$ we have that

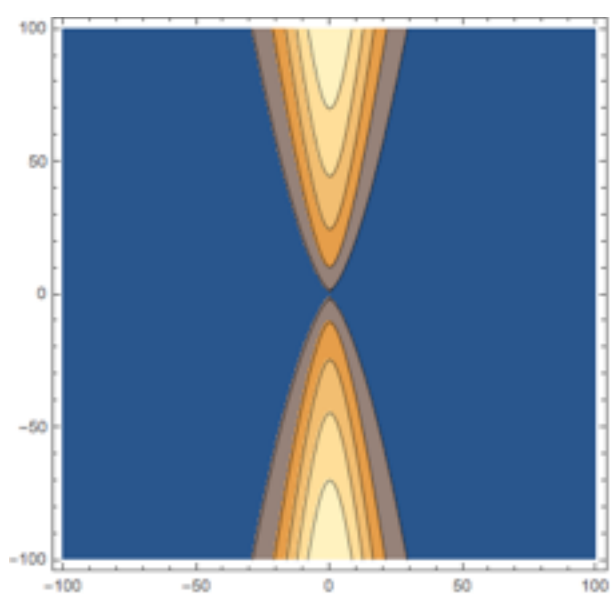
$$\lim_{\alpha \to 0} \Psi_{\alpha,\alpha} = \sqrt{-\partial_y^2} \star \delta_{y=0}$$

- In the non-limiting regime, a good approximation of $\Psi_{\varepsilon,\alpha}$ is given by a mollification of the above + a correction term

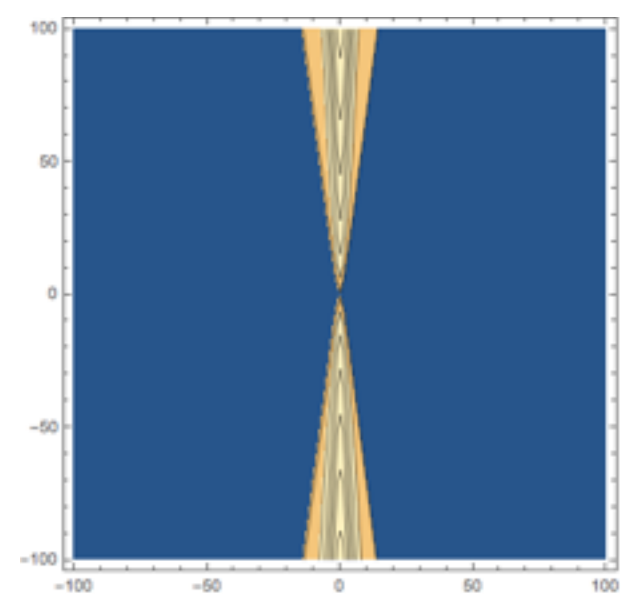
- Numerical approximations of the Fourier transform of $\Psi_{\varepsilon,\alpha}$



$\alpha = 10$



$\alpha = 1$



$\alpha = 0.1$

SENSITIVITY TO HIGHER ORDER FEATURES

Theorem. For $\varepsilon = \alpha$ we have that

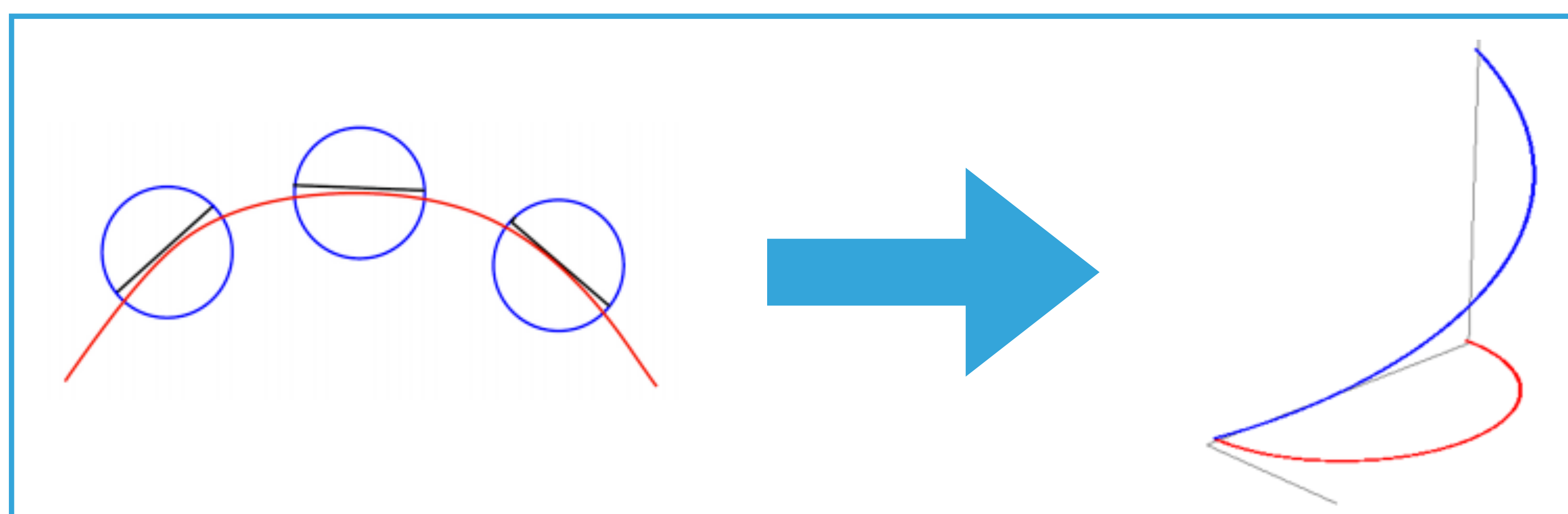
$$\lim_{\alpha \to 0} \Psi_{\alpha,\alpha} = \sqrt{-\partial_y^2} \star \delta_{y=0}$$

- In the non-limiting regime, a good approximation of $\Psi_{\varepsilon,\alpha}$ is given by a mollification of the above + a correction term

- This suggests that $\mathcal{L}_{\varepsilon,\alpha}$ be sensible to the curvature of level lines

Mother wavelet

$-\partial_y \star \delta_{y=0}$



NUMERICAL RESULTS

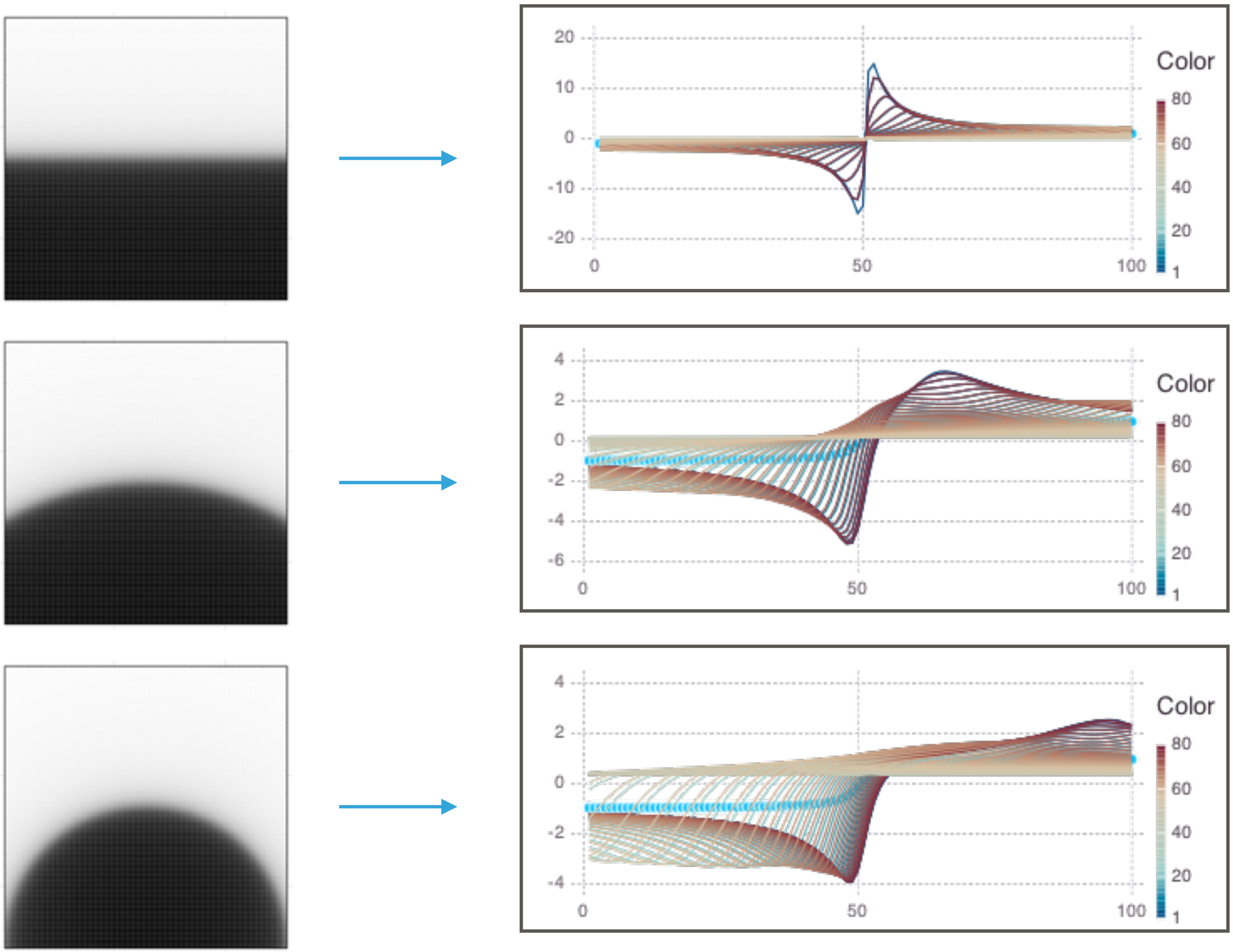
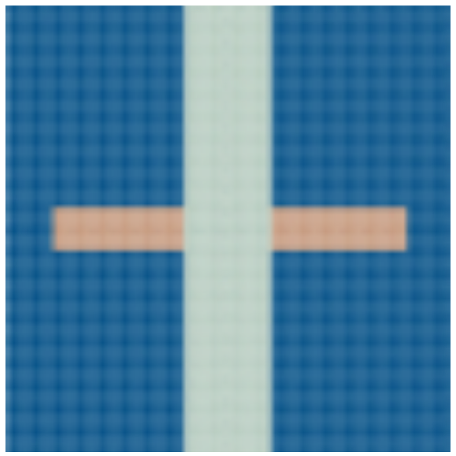
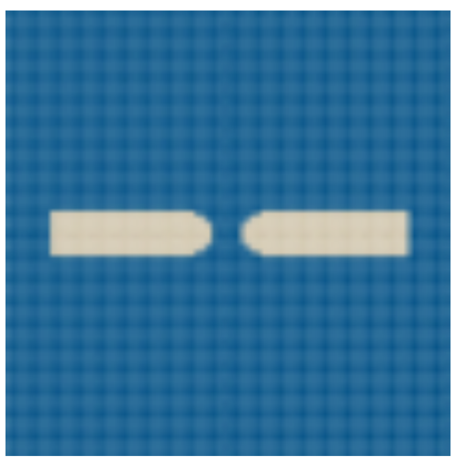


IMAGE INPAINTING

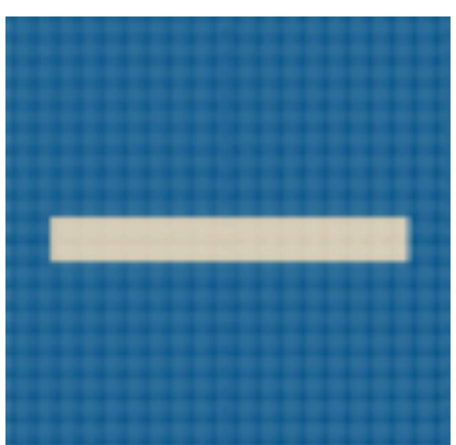


$f \in L^2(\mathbb{R}^2)$ corrupted outside of $\Omega \subset \mathbb{R}^2$



Standard isotropic inpainting:

$$f_{\text{iso}} = \arg \min_{\varphi|_\Omega = f|_\Omega} \int_{\mathbb{R}^2} |\nabla \varphi|^2 \, dx \, dy$$



Cortical inpainting:

$$f_{\text{an}} = P\tilde{F} \quad \text{where} \quad \tilde{F} = \arg \min_{PF|_\Omega = f|_\Omega} \mathcal{E}_\alpha(F)$$

MORE GENERAL CASES

A variational lift can be defined for any $p \geq 1$ as

$$\mathcal{L}f = \arg\min_{PF=f} \|F\|^p_{L^p(PT\mathbb{R}^2)} + \int_{\mathbb{R}^2}\int_{\mathbb{P}^1} \left(|XF|^2 + |\Theta F|^2\right)^{\frac{p}{2}} \, d\theta \, dx$$

Theorem. *Let* $f \in L^2(\mathbb{R}^2)$ be piecewise smooth with discontinuities along smooth lines. Then,

$$\|f\|_p < +\infty \iff p < \frac{3}{2}$$

Moreover, as $p \to 3/2$, we have that $\|f\|_p \to +\infty$ with logarithmic speed.

- Processing of discontinuous images via differentiable energies
- Numerically very hard to exploit

CONCLUSIONS

- We presented a novel variational model of the human visual cortex
- The lift obtained in this model is sensible to higher order features of the images than the Gabor lift of the standard CPS model
- The model accounts for well-known neurophysiological phenomena
- Promising framework for image processing
- Numerical difficulties: necessity of fine parametrization in θ