Weyl's law for singular Riemannian manifolds

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Classical Weyl's law

- (M, g) smooth, compact Riemannian manifold (with smooth boundary)
- Δ Laplace-Beltrami operator on $L^2(M, d\mu_g)$

$$\Delta = -\operatorname{div} \circ \nabla = -\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_i} \left(\sqrt{|g|} g^{ij} \frac{\partial}{\partial x_j} \right)$$

- Δ is self-adjoint with compact resolvent

$$\operatorname{spec}(\Delta) = \{0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \to \infty\}$$

- **Weyl's function**:

$$N(\lambda) := \#\{\text{eigenvalues } \lambda_k \leq \lambda\}$$

Theorem (Weyl's law)

$$N(\lambda) \sim c_n \operatorname{vol}(M) \lambda^{n/2}, \qquad \lambda \to \infty$$

The problem

Theorem (Weyl's law)

For a compact Riemannian manifold with boundary:

$$N(\lambda) \sim c_n \operatorname{vol}(M)\lambda^{n/2}, \qquad \lambda \to \infty$$

Remark: $\operatorname{vol}(M) < \infty$ is not necessary for discreteness of $\operatorname{spec}(\Delta)$

Problem: study the Weyl's law for singular Riemannian structures

A singular example (the Grushin sphere)

- $\mathbb{S}^2 \subset \mathbb{R}^3$. Let X, Y generators of rotations around x and y axes

$$X = y\partial_z - z\partial_y \qquad\qquad Y = x\partial_z - z\partial_x$$

- g metric s.t. X, Y are orthonormal. Singular at $\mathcal{S} = \{z = 0\}$
- close to the plane $z = 0$, in local coordinates:

$$g \sim dz^2 + \frac{1}{z^2} d\theta^2, \qquad \Delta \sim -\partial_z^2 + z^2 \partial_\theta^2 - \frac{1}{z}\partial_z, \qquad d\mu_g \sim \frac{1}{|z|} d\mu_{\mathbb{S}^2}$$

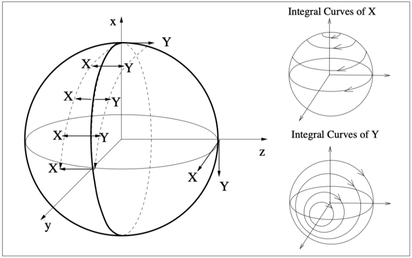

Figure 1: Taken from Agrachev, Boscain, Sigalotti 2008

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Proposition (Boscain, Laurent - 2009)

The Laplace-Beltrami operator with domain $C_c^\infty(\mathbb{S}^2 \setminus \mathcal{S})$ is essentially self-adjoint and it has compact resolvent.

Proposition (Boscain, P, Seri - 2014)

The Weyl's function has the following asymptotics

$$N(\lambda) \sim \tfrac{1}{4}\lambda \log \lambda, \qquad \lambda \to \infty$$

The setting

Singular Riemannian structures

- $(\mathcal{M}, g)$ non-complete Riemannian manifold
- metric, measure, curvature, ... explode at the metric boundary
- singularity $\subseteq$ metric boundary

Definition

The Laplace-Beltrami operator on $L^2(\mathcal{M}, d\mu_g)$ is the unique self-adjoint operator associated with the closure of the quadratic form

$$Q(u) = \int_{\mathcal{M}} \|\nabla u\|_g^2 \, d\mu_g, \qquad \forall u \in C_c^\infty(\mathcal{M})$$

- Friedrichs extension of Δ with domain $C_c^\infty(\mathcal{M})$
- On smooth functions $\Delta u = -\operatorname{div}(\nabla u)$

Assumptions on the singularity

Definition (Assumption Σ)

Non-complete Riemannian manifold $\mathcal{M}$ such that in a neighborhood of the metric boundary:

1. the distance from the metric boundary δ is smooth
2. $\mathrm{Hess}(\delta) \leq 0$ (convexity of the metric boundary)
3. there exists $C > 0$ such that

$$|\mathrm{Sec}| \leq \frac{C}{\delta^2}, \qquad \mathrm{inj} \geq \frac{\delta}{C},$$

- "unbounded geometry" but not too much
- cover "strongly regular" almost-Riemannian structures
- rules out conic singularities (see Cheeger)
- if convexity is *strict* $\Rightarrow$ $\mathrm{inj} \geq \frac{\delta}{C}$ (by Klingenberg-type arguments)

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Proposition (Chitour, P, Rizzi)

Assume that $\mathcal{M}$ has compact metric completion and satisfies Assumption Σ. Then Δ has compact resolvent and hence its spectrum is discrete.

- $\Rightarrow$ Weyl's function $N(\lambda)$ is well defined
- Only 1 and 2 are required for the Proposition

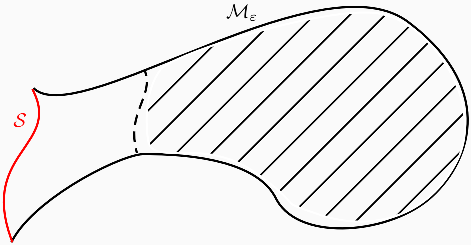
Results

Weyl's asymptotics for singular Riemannian manifolds

Theorem (Chitour, P, Rizzi)

Let $\mathcal{M}$ be a non-complete Riemannian manifold with compact metric completion satisfying assumption Σ. With $\mathcal{M}_\varepsilon = \{x \in \mathcal{M} \mid \delta(x) \geq \varepsilon\}$, we have

$$N(\lambda) \asymp \lambda^{n/2} \operatorname{vol}\left(\mathcal{M}_{1/\sqrt{\lambda}}\right), \qquad \lambda \to \infty$$



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- Consequence of quantitative heat kernel estimates and Karamata theory with remainder

Eigenfunctions $\{\phi_k\}_k$ concentrate at the singularity for high energy:

Theorem (Chitour, P, Rizzi)

If $\operatorname{vol} \mathcal{M} = +\infty$, then there exists a density one set $S \subseteq \mathbb{N}$ such that for any compact $K \subset \mathcal{M}$ it holds

$$\lim_{\substack{k \to \infty \\ k \in S}} \int_K |\phi_k|^2 d\mu_g = 0.$$

Exact Weyl's law for singular structures

Refinement yielding exact Weyl's law

Theorem (Chitour, P, Rizzi)

Let $\mathcal{M}$ be a non-complete Riemannian manifold with compact metric completion and satisfying assumption Σ. Assume also that

$$\upsilon(\lambda) = \mathrm{vol}(\mathcal{M}_{1/\sqrt{\lambda}})$$

is slowly varying[1] *(in the sense of Karamata). Then it holds*

$$N(\lambda) \sim c_n \lambda^{n/2} \upsilon(\lambda)$$

Reduces to the classical Weyl's law if $\mathrm{vol}(\mathcal{M}) < +\infty$

[1] That is, $\upsilon : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and

$$\lim_{\lambda\to\infty} \frac{\upsilon(a\lambda)}{\upsilon(\lambda)} = 1, \qquad \forall a > 0.$$

Examples include: $\log\lambda$, $\log_k \lambda = \log_{k-1}\log\lambda$, $\exp(\log\lambda/\log\log\lambda)$, etc...

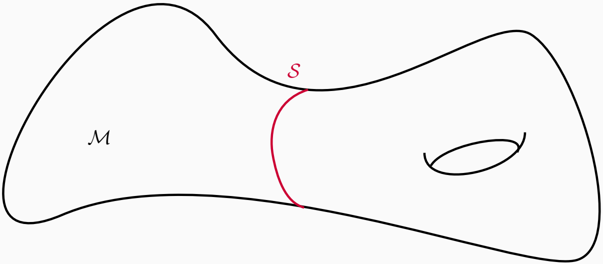
Application: Almost-Riemannian surfaces

Let $N = \mathcal{M} \sqcup \mathcal{S}$, $\dim N = 2$. The structure is an m-regular ARS, $m \in \mathbb{N}$, if locally near $\mathcal{S}$ we have

$$g = dx^2 + \frac{e^{\varphi(x,z)}}{x^{2m}} dz^2, \qquad \varphi \in C^\infty, \qquad \mathcal{S} = \{x = 0\}.$$

Equivalently: A local orthonormal frame is

$$X = \partial_x, \qquad Z = x^m e^{-\varphi/2} \partial_z$$



Generalize the Grushin sphere example

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Proposition (Chitour, P, Rizzi)

Strongly regular ARS satisfy the assumption Σ

- the boundaries of $\mathcal{M}_\varepsilon = \{x \in \mathcal{M} \mid \delta(x) > \varepsilon\}$ are strictly convex
- the curvature always explodes to $-\infty$
- false in presence of "tangency" points

Application: Almost-Riemannian surfaces

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Theorem (Chitour, P, Rizzi)

For an m-regular ARS on an compact surface it holds

i. *If* $m > 1$,

$$N(\lambda) \asymp \lambda^{(m+1)/2}.$$

ii. *If* $m = 1$,

$$N(\lambda) \sim \frac{\widehat{\sigma}(\mathcal{S})}{8\pi} \lambda \log \lambda, \qquad \widehat{\sigma}(\mathcal{S}) = \int_{\mathcal{S}} e^{\frac{\varphi}{2}} dz.$$

Coincides with the result by Colin de Verdière - Hillaret - Trélat, where the Riemannian measure is replaced by a smooth one.

Singular structures with prescribed Weyl's law

Example of inverse problem:

Theorem (Colin de Verdière - 1987)

Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$. Then, there exists a complete Riemannian manifold such that these are its first k eigenvalues

Question: Can we build (singular) Riemannian manifolds with prescribed asymptotic distribution of eigenvalues λ_k as $k \to \infty$?

- Equivalent to prescribe Weyl's law $N(\lambda)$ as $\lambda \to \infty$

Singular structures with prescribed Weyl's law

Theorem (Chitour, P, Rizzi)

For any compact manifold M of dimension $n \geq 2$ and non-decreasing slowly varying function $\upsilon : \mathbb{R}_+ \to \mathbb{R}_+$ there exists a singular Riemannian structure on M such that

$$N(\lambda) \sim c_n \lambda^{n/2} \upsilon(\lambda)$$

The singularity can be prescribed on a submanifold of any codimension

- The structure satisfies Σ
- The volume function satisfies $\mathrm{vol}(\mathcal{M}_{1/\sqrt{\lambda}}) \sim \upsilon(\lambda)$
- $\partial\mathcal{M}_\varepsilon$ are strictly convex
- The Laplace-Beltrami with domain $C_c^\infty(\mathcal{M})$ is essentially self-adjoint (follows from P, Rizzi, Seri 2017 + Nenciu, Nenciu 2009)

Strategy of proof

The variational method

Let $\Omega \subset \mathbb{R}^n$. Let $-\Delta$ with Dirichlet b.c. on $\partial\Omega$ (smooth).

1. Decompose $\Omega = \sqcup_i \Omega_i$, where Ω_i are cubes
2. Consider $-\Delta^{\pm}_{\Omega_i}$ with Dirichlet $(-)$ or Neumann $(+)$ conditions

Proposition (Dirichlet-Neumann bracketing)

$$\sum_i N^-_{\Omega_i}(\lambda) \leq N(\lambda) \leq \sum_i N^+_{\Omega_i}(\lambda)$$

3. Via an explicit computation:

$$N^{\pm}_{\Omega_i}(\lambda) = \frac{\omega_n}{(2\pi)^n} \operatorname{vol}(\Omega_i)\lambda^{n/2} + \textit{explicitly bounded remainder}$$

4. Limit for small cubes $\Rightarrow$ Weyl's law for smooth domains of $\mathbb{R}^n$ $\square$

The Tauberian method

Let Δ be the Laplace-Beltrami for a smooth manifold M

1. Consider the heat trace

$$Z(t) = \int_M p_t(x,x)\,d\mu_g(x) = \sum_{i=1}^{\infty} e^{-t\lambda_i} = \int_0^{\infty} e^{-t\lambda}\,dN(\lambda)$$

2. Small time behaviour of $Z(t) \leftrightarrow$ large eigenvalue behaviour of $N(\lambda)$

Theorem (Karamata)

$$Z(t) \sim ct^{-\alpha} \qquad \Rightarrow \qquad N(\lambda) \sim \tilde{c}\lambda^{\alpha}$$

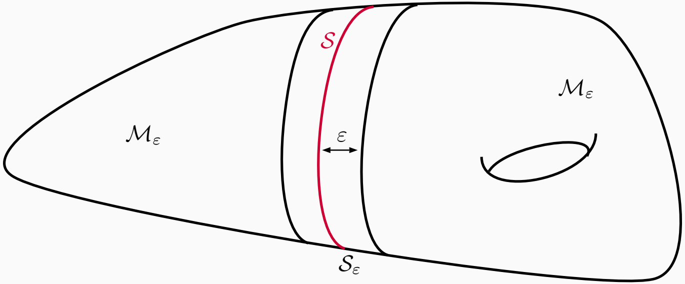
3. Minakshisundaram-Pleijel asymptotics

$$Z(t) \sim \mathrm{vol}(M)(4\pi t)^{-n/2} \qquad t \to 0$$

4. Weyl's law for smooth manifolds with boundary $\square$

Step 1: decomposition

- Decompose the manifold:



- Apply the variational method

$$N^-_{\mathcal{S}_\varepsilon}(\lambda) + N^-_{\mathcal{M}_\varepsilon}(\lambda) \le N(\lambda) \le N^+_{\mathcal{S}_\varepsilon}(\lambda) + N^+_{\mathcal{M}_\varepsilon}(\lambda)$$

- Need to estimate $N^\pm_{\mathcal{S}_\varepsilon}$ and $N^\pm_{\mathcal{M}_\varepsilon}(\lambda)$
- How to choose $\varepsilon = \varepsilon(\lambda)$?

Step 2: estimate close to the singularity

Lemma (Hardy inequality)

Thanks to the convexity assumption there exist $C_H > 0$ *such that*

$$\int_{\mathcal{S}_\varepsilon} \|\nabla u\|^2 d\mu_g \geq \frac{C_H}{\varepsilon^2} \int_{\mathcal{S}_\varepsilon} |u|^2 d\mu_g, \qquad \forall u \in H^1(\mathcal{S}_\varepsilon)$$

Implies a lower bound for the Dirichlet/Neumann spectrum close to $\mathcal{S}$

Corollary (Estimate close to the metric boundary)

$$N^{\pm}_{\mathcal{S}_\varepsilon}(\lambda) \equiv 0 \qquad \textit{if} \qquad \lambda < \frac{C_H}{\varepsilon^2}$$

Remark: As $\lambda \to \infty$ we have to let $\varepsilon \to 0$

Step 3: estimate for the truncation

For all $\varepsilon > 0$, $\mathcal{M}_\varepsilon$ is a smooth manifold with convex boundary

$$N^{\pm}_{\mathcal{M}_\varepsilon}(\lambda) = \frac{\omega_n}{(2\pi)^n} \operatorname{vol}(\mathcal{M}_\varepsilon)\, \lambda^{n/2} (1 + R_\varepsilon(\lambda))$$

Need to know how the remainder depends on ε!

Theorem (Ingham - 1960)

There exists a universal constant $C > 0$ (depending only on the dimension) such that, if as $t \to 0$

$$\int_M p_t(x,x) d\mu_g \sim ct^{-\alpha} (1 + \chi(t))$$

then, as $\lambda \to \infty$

$$N(\lambda) \sim \tilde{c}\lambda^\alpha (1 + R(\lambda)), \qquad |R(\lambda)| \leq \frac{C}{|\log \chi(\lambda^{-1})|}$$

Step 4: Quantitative remainder formula for heat trace

Theorem (Chitour, P, Rizzi)

Let (M, g) be a compact Riemannian manifold with convex ∂M. Let

$$|\,\mathrm{Sec}\,| \leq K, \qquad \mathrm{Hess}(\delta) \geq -H$$

Let χ be the remainder of the trace heat kernel asymptotics:

$$\int_M p_t^{\pm}(x,x)d\mu_g = \frac{\mathrm{vol}(M)}{(4\pi t)^{n/2}}\Big(1+\chi(t)\Big)$$

Then there exists $c > 0$ depending only on n s.t.

$$|\chi(t)| \leq c\left(\frac{t}{t_0}\right)^{1/2}, \qquad \forall t \leq t_0 = \min\left\{\mathrm{inj}_M, \mathrm{inj}_{\partial M}, \frac{\pi}{\sqrt{K}}, \frac{1}{H}\right\}$$

Sharp exponent and sharp constant! (also for the corresponding $N(\lambda)$)

Step 5: conclusion

E.g. for the upper bound:

1. By Neumann bracketing

$$N(\lambda) \leq N^{+}_{\mathcal{S}_\varepsilon}(\lambda) + N^{+}_{\mathcal{M}_\varepsilon}(\lambda)$$

2. Thanks to convexity/Hardy:

$$N^{+}_{\mathcal{S}_\varepsilon}(\lambda) = 0 \qquad \text{if} \qquad \varepsilon \lesssim \frac{1}{\sqrt{\lambda}}$$

3. Thanks to the remainder formula

$$N^{+}_{\mathcal{M}_\varepsilon}(\lambda) = c_n \operatorname{vol}(\mathcal{M}_\varepsilon)\, \lambda^{n/2} \left(1 + \mathcal{O}\left(\frac{1}{\lambda\varepsilon^2}\right)\right)$$

4. we must choose $\varepsilon \asymp \frac{1}{\sqrt{\lambda}}$ $\square$

$+$ finer decomposition $+$ detailed study of small slices $=$ sharp result

Conclusion

Under the assumption Σ the picture is quite clear:

- rough non-classical Weyl's asymptotics
- exact non-classical Weyl's law in the slowly varying case
- singular structures with prescribed Weyl's law
- concentration of eigenfunctions

Open questions:

- We do not have a (non slowly varying) example with

$$0 < \liminf_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{n/2} \upsilon(\lambda)} < \limsup_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{n/2} \upsilon(\lambda)} < +\infty$$

- What about more singular regimes?
- Singular magnetic field?

Thank you for your attention!

Y. Chitour, D. Prandi, L. Rizzi
Weyl's law for singular Riemannian manifolds
arXiv:1903.05639

(Generalized) Minkowski dimension

Tubular slice around the singularity: $\mathcal{M}_\varepsilon^{2\varepsilon} = \{\varepsilon \le \delta \le 2\varepsilon\}$

Definition

The singularity has **generalized Minkowski dimension** d if

$$0 < \liminf_{\varepsilon \to 0} \frac{\mathrm{vol}\left(\mathcal{M}_\varepsilon^{2\varepsilon}\right)}{\varepsilon^{n-d}} \le \limsup_{\varepsilon \to 0} \frac{\mathrm{vol}\left(\mathcal{M}_\varepsilon^{2\varepsilon}\right)}{\varepsilon^{n-d}} < \infty$$

Corollary (Chitour, P, Rizzi)

Assume furthermore that the singularity has Minkowski dimension d. Then

$$N(\lambda) \asymp \begin{cases} \lambda^{n/2} & d < n \\ \lambda^{n/2} \log \lambda & d = n \\ \lambda^{d/2} & d > n \end{cases} \qquad \lambda \to \infty$$

Question: Is d the Hausdorff dimension of the metric boundary?

Singular measure VS regular measure

What happens if g is singular, but we use a smooth measure ω?

Hilbert space	$L^2(\mathcal{M}, d\mu_g)$	$L^2(\mathcal{M}, d\omega)$
Quadratic form	$\int_{\mathcal{M}} \|\nabla u\|^2 d\mu_g$	$\int_{\mathcal{M}} \|\nabla u\|^2 d\omega$
Grushin example	$\partial_z^2 + z^2\partial_\theta^2 + \frac{1}{z}\partial_\theta$	$\partial_z^2 + z^2\partial_\theta^2$

- in the Riemannian case (no singularity) there is no difference
- no a priori reason to relate the two Weyl's asymptotics
- studied by Colin de Verdiére - Hillaret - Trélat (w.i.p., different ideas)
- for 2-dimensional regular ARS the results agree