Controllability of a 1d simplified fluid-structure system

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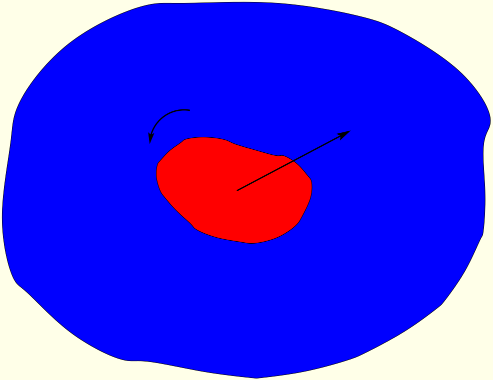
Presentation of the problem



- Fluid: viscous, incompressible
- Structure: rigid body

The 1d simplified model

$$
\begin{cases}
v_t - v_{xx} + v v_x = 0 \qquad (t \geqslant 0, \quad x \in [-1,1] \setminus \{h(t)\}), \\
v(t, h(t)) = \dot{h}(t) \qquad (t \geqslant 0), \\
m\ddot{h}(t) = [v_x](t, h(t)) \qquad (t \geqslant 0), \\
v(t,-1) = u(t), \quad v(t,1) = 0 \qquad (t \geqslant 0), \\
h(0) = h_0, \quad \dot{h}(0) = h_1, \\
v(0,x) = v_0(x) \quad x \in [-1,1] \setminus \{h_0\}.
\end{cases}
\tag{1}
$$

Notations

- $h = h(t)$ trajectory of the particle
- $v = v(t, x)$ fluid velocity, defined in the fluid domain $[-1, 1] \setminus \{h(t)\}$.
- $u = u(t)$ control of the system, velocity of the fluid at $x = -1$.

$$[f](x) = f(x^+) - f(x^-).$$

The main result

Theorem

Let $\tau > 0$. For inial data (position and velocities)

$$h_0 \in (-1,1), \quad h_1 \in \mathbb{R}, \quad v_0 \in H_0^1(-1,1), \quad v_0(h_0) = h_1,$$

small enough

$$|h_0| + |h_1| + \|v_0\|_{H_0^1(-1,1)} \leqslant r,$$

there exists $u \in C([0,\tau])$ such that

$$v \in L^2([0,\tau], H^2((-1,1)\backslash\{h(t)\})) \cap C([0,\tau], H^1(-1,1)), \quad h \in C^1[0,\tau],$$

and

$$\dot{h}(\tau) = 0, \quad v(\tau,\cdot) = 0 \quad \textit{and} \quad h(\tau) = 0.$$

Commentaries

- Well-posedness: Vázquez and Zuazua
- Previous controllability results: Doubova and Fernández-Cara, use of two controls, one at each boundary.
- Classical method for parabolic systems: we first prove the internal controllability and deduce from it the boundary controllability.

Our method

- Assume X Hilbert.
- Assume $A : D(A) \to X$, $A = A^*$ and A is a negative operator.
- Assume $B : U \to X$ is a bounded operator.

We consider the system

$$\begin{cases} \dot{z} = Az + Bu, \\ z(0) = z_0 \in X. \end{cases}$$

Definition

The pair (A, B) is said null-controllable in time τ if

$$\forall z_0 \in X, \quad \exists u \in L^2([0,\tau], U) \quad z(\tau) = 0.$$

Null-controllability with source term

Assume the pair (A, B) is null-controllable. Is it possible to obtain the null-controllability of

$$\begin{cases} \dot{z} = Az + Bu + f, \\ z(0) = z_0. \end{cases} \qquad ((A, B, f))$$

where $f : [0, \infty) \to X$?

More precisely, we would like to prove

$$\forall z_0 \in X, \quad \forall f \in \mathcal{F} \subsetneq L^2([0,\tau], X), \quad \exists u \in L^2([0,\tau], U) \quad z(\tau) = 0.$$

Functional spaces

$$\mathcal{F} = \left\{ f \in L^2([0,\tau], X) \;\middle|\; \frac{f}{\rho_{\mathcal{F}}} \in L^2([0,\tau], X) \right\},$$
$$\mathcal{U} = \left\{ u \in L^2([0,\tau], U) \;\middle|\; \frac{u}{\rho_0} \in L^2([0,\tau], U) \right\},$$
$$\mathcal{Z} = \left\{ z \in L^2([0,\tau], X) \;\middle|\; \frac{z}{\rho_0} \in L^2([0,\tau], X) \right\}.$$

$$\rho_{\mathcal{F}}, \rho_0 : [0,\tau] \to \mathbb{R}_+$$

with

$$\rho_{\mathcal{F}}, \rho_0 \;\; \text{decreasing and} \;\; \rho_{\mathcal{F}}(\tau) = \rho_0(\tau) = 0.$$

Our result

Proposition

Assume

1. *the pair* (A, B) *is null-controllable in any time* $t > 0$*;*
2. $\tau > 0$ *and let* $\rho_{\mathcal{F}}$ *and* ρ_0 *as above with some extra assumptions.*

Then

$$\forall z_0 \in X, \quad \forall f \in \mathcal{F}, \quad \exists u \in \mathcal{U} \quad \textit{such that} \quad z \in \mathcal{Z}.$$

Proposition (End)

Moreover, there exists a constant $C > 0$ such that

$$\left\| \frac{z}{\rho_0} \right\|_{C([0,\tau],X)} + \|u\|_{\mathcal{U}} \leqslant C\left(\|f\|_{\mathcal{F}} + \|z_0\|_X\right).$$

Remark

Since ρ_0 is a continuous function satisfying $\rho_0(\tau) = 0$,

$$\frac{z}{\rho_0} \in C([0,\tau],X) \Longrightarrow z(\tau) = 0.$$

Cost of the control

There exists a function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ such that for every $\tau > 0$, there exists a control driving the solution of

$$\begin{cases} \dot{z} = Az + Bu, \\ z(0) = z_0 \in X. \end{cases}$$

to rest in time τ such that

$$\|u\|_{L^2([0,\tau],U)} \leqslant \gamma(\tau)\|z_0\| \qquad (z_0 \in X). \tag{3}$$

The function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ can be chosen continuous and non-increasing function and it satisfies

$$\lim_{\tau \to 0^+} \gamma(\tau) = +\infty.$$

Proof of the proposition (1/6)

Take $q > 1$ and set

$$T_k := \tau - \frac{\tau}{q^k} \qquad (k \geqslant 0).$$

The sequence (T_k) is increasing and $\lim_{k\to\infty} T_k = \tau$.

- For $k \geqslant 0$, we consider the initial value problem:

$$\begin{cases} \dot{z}_1 = Az_1 + f & t \in (T_k, T_{k+1}), \\ z_1(T_k^+) = 0. \end{cases}$$

  We write

$$a_{k+1} = z_1(T_{k+1}^-) \quad \text{and} \quad a_0 = z_0.$$

Proof of the proposition (2/6)

- For $k \geqslant 0$, we consider the control problem:

$$\begin{cases} \dot{z}_2 = Az_2 + Bu_k & (t \in (T_k, T_{k+1})), \\ z_2(T_k^+) = a_k \in X. \end{cases}$$

Since (A, B) is null-controllable in any time $t > 0$, there exists $u_k \in L^2([T_k, T_{k+1}], U)$ is such that

$$z_2(T_{k+1}^-) = 0.$$

This control has a cost

$$\|u_k\|^2_{L^2([T_k,T_{k+1}],U)} \leqslant \gamma^2\,(T_{k+1} - T_k)\,\|a_k\|^2_X.$$

Proof of the proposition (3/6)

More precisely, energy estimates for z_1 yields

$$\|a_{k+1}\|_X^2 \leqslant C\,\|f\|^2_{L^2([T_k,T_{k+1}],X)} \qquad (k \geqslant 0).$$

Thus

$$\|u_{k+1}\|^2_{L^2([T_{k+1},T_{k+2}],U)} \leqslant C\,\underbrace{\gamma^2\left((q-1)\frac{\tau}{q^{k+2}}\right)\rho^2_{\mathcal{F}}(T_k)}_{\rho_0^2(T_{k+2})}\left\|\frac{f}{\rho_{\mathcal{F}}}\right\|^2_{L^2([T_k,T_{k+1}],X)}.$$

Proof of the proposition (4/6)

We define the control u by

$$u := \sum_{k=0}^{\infty} u_k \mathbf{1}_{[T_k, T_{k+1})}.$$

Then

$$\left\| \frac{u}{\rho_0} \right\|_{L^2([0,\tau],U)} \leqslant C \left(\left\| \frac{f}{\rho_{\mathcal{F}}} \right\|_{L^2([0,\tau],X)} + \|z_0\| \right) \qquad (z_0 \in X, \;\; f \in \mathcal{F}).$$

Proof of the proposition (5/6)

- What we have obtained: set

$$z = z_1 + z_2.$$

For every $k \geqslant 0$, we have

$$\begin{cases} \dot{z} = Az + Bu + f \qquad (t \in [T_k, T_{k+1}]), \\ z(T_k^+) = 0 + a_k. \end{cases} \tag{4}$$

Moreover,

$$z(T_{k+1}^-) = z_1(T_{k+1}^-) + z_2(T_{k+1}^-) = a_{k+1} + 0.$$

Therefore z is continuous at T_{k+1} for every $k \geqslant 0$.

Proof of the proposition (6/6)

We deduce that z is the solution of

$$\begin{cases} \dot{z} = Az + Bu + f & (t \in [0, \tau]), \\ z(0) = z_0. \end{cases}$$

Moreover, for all $k \geqslant 0$,

$$\|z\|^2_{C([T_k,T_{k+1}],X)} \leqslant \|a_k\|^2_X + C\,\|u\|^2_{L^2([T_k,T_{k+1}],X)} + C\,\|f\|^2_{L^2([T_k,T_{k+1}],X)}\,.$$

This allows to obtain

$$\left\|\frac{z}{\rho_0}\right\|_{C([0,\tau],X)} + \|u\|_{\mathcal{U}} \leqslant C\left(\|f\|_{\mathcal{F}} + \|z_0\|_X\right).$$

Consequence of the proposition

It is sufficient to have the null-controllability of (A, B):

$$\begin{cases} \dot{z} = Az + Bu, \\ z(0) = z_0 \in X. \end{cases}$$

For all $z_0 \in X$, find $u \in L^2([0, \tau], U)$ such that

$$z(\tau) = e^{\tau A} + \int_0^\tau e^{(\tau - t)A} Bu(t)\ dt = 0.$$

By duality, it is equivalent to prove the observability inequality

$$C \int_0^\tau \|B^* e^{tA^*} z_0\|_U^2 \mathrm{d}t \geqslant \|e^{\tau A^*} z_0\|^2 \qquad (z_0 \in X).$$

Classical method for parabolic systems: global Carleman estimates, local Carleman estimates.

A linear system

$$\begin{cases} v_t - v_{xx} = \mathbf{1}_I u \qquad (t \geqslant 0, \quad x \in [-1,1] \setminus \{0\}), \\ v(t,0) = \ell(t) \qquad (t \geqslant 0), \\ m\dot{\ell}(t) = [v_x](t,0) \qquad (t \geqslant 0), \\ v(t,-1) = 0, \quad v(t,1) = 0 \qquad (t \geqslant 0), \\ \ell(0) = \ell_0, \\ v(0,x) = v_0(x) \quad x \in [-1,1] \setminus \{0\}. \end{cases}$$

Control problem: find $u \in L^2([0,\tau]; L^2(I))$ such that

$$v(\tau,\cdot) = 0, \quad \ell(\tau) = 0.$$

The observability inequality

The adjoint system:

$$\begin{cases} \varphi_t - \varphi_{xx} = 0 \qquad (t \geqslant 0, \quad x \in [-1,1] \setminus \{0\}), \\ \varphi(t,0) = k(t) \qquad (t \geqslant 0), \\ m\dot{k}(t) = [\varphi_x](t,0) \qquad (t \geqslant 0), \\ \varphi(t,-1) = 0, \quad \varphi(t,1) = 0 \qquad (t \geqslant 0), \\ k(0) = k_0, \\ \varphi(0,x) = \varphi_0(x) \quad x \in [-1,1] \setminus \{0\}. \end{cases}$$

The observability inequality:

$$C \int_0^\tau \int_I |\varphi(t,x)| \ dx \ dt \geqslant \int_{-1}^1 |\varphi(\tau,x)| \ dx + |k(\tau)| \, .$$

To prove the observability inequality with Carleman estimates

Carleman estimates on $(-1, 0)$ and on $(0, 1)$:

$$\psi = e^{-s\theta}\varphi,$$

with

$$\theta(t, x) = \frac{e^{\lambda\overline{\beta}} - e^{\lambda\beta(x)}}{t(\tau - t)}.$$

β is a particular concave function on $(-1, 0)$ and on $(0, 1)$. We transform the equation of φ into

$$M_1\psi + M_2\psi = g$$

and then using λ and s big enough, we get the Carleman estimates.

To prove the observability inequality with Carleman estimates

- The interest of Carleman estimates: it allows to deal with lower terms, in particular a right-hand side is trivial to handle.

- The drawback of Carleman estimates: the weight function is “magical”, if it does not work, it is usually complicated to obtain anything.

- Here, Doubova and Fernández-Cara need $I = I_1 \cup I_2$, with $I_1 \subset (-1, 0)$ and $I_2 \subset (0, 1)$, both with non empty interior.

Spectral method

Result due to Fattorini and Russell:

Proposition

- *Assume that* A *admits an orthonormal basis of eigenvectors* $(\varphi_k)_{k\geqslant 1}$ *with the corresponding decreasing sequence of eigenvalues* $(-\lambda_k)_{k\geqslant 1}$.
- *Assume that*

$$\inf_{k\geqslant 0}(\lambda_{k+1}-\lambda_k)=s>0,$$

$$\lambda_k = rk^2 + O(1) \qquad (k\to\infty),$$

$$\|B^*\varphi_k\|_U \geqslant m \qquad (k\geqslant 1).$$

- *Assume that* U *is a separable Hilbert space.*

Then the pair (A,B) *is null-controllable in any time* $\tau>0$ *and we can take*

$$\gamma(\tau)=Ce^{\frac{M}{\tau}} \quad (\tau>0).$$

Proof of the proposition

- First
$$B^* e^{tA^*} z_0 = \sum_{k\geqslant 1} e^{-\lambda_k t} \langle z_0, \varphi_k \rangle B^* \varphi_k.$$
- U admits an orthonormal basis $(\psi_l)_{l\geqslant 1}$

$$\int_0^\tau \|B^* e^{tA^*} z_0\|_U^2 \mathrm{d}t = \sum_{l\geqslant 1} \int_0^\tau \left| \sum_{k\geqslant 1} e^{-\lambda_k t} \langle z_0, \varphi_k \rangle_X \langle B^* \varphi_k, \psi_l \rangle_U \right|^2 \mathrm{d}t$$

- Tenenbaum and Tucsnak

$$M_1 e^{\frac{M_2}{\tau}} \int_0^\tau \left| \sum_{k\geqslant 1} a_k e^{-\lambda_k t} \right|^2 \mathrm{d}t \geqslant \sum_{k\geqslant 1} |a_k|^2 e^{-2\lambda_k \tau}.$$

- From above

$$\begin{aligned} M_1 e^{\frac{M_2}{\tau}} \int_0^\tau & \|B^* e^{tA^*} z_0\|_U^2 \mathrm{d}t \\ & \geqslant \sum_{l\geqslant 1} \sum_{k\geqslant 1} e^{-2\lambda_k \tau} |\langle z_0, \varphi_k\rangle_X|^2 \; |\langle B^* \varphi_k, \psi_l\rangle_U|^2. \end{aligned}$$

The last inequality gives

$$M_1 e^{\frac{M_2}{\tau}} \int_0^\tau \|B^* e^{tA^*} z_0\|_U^2 \mathrm{d}t \geqslant m^2 \|e^{tA^*} z_0\|^2.$$

Back to the linear system

$$\begin{cases} \varphi_t - \varphi_{xx} = 0 \qquad (t \geqslant 0, \quad x \in [-1,1] \setminus \{0\}), \\ \varphi(t,0) = k(t) \qquad (t \geqslant 0), \\ m\dot{k}(t) = [\varphi_x](t,0) \qquad (t \geqslant 0), \\ \varphi(t,-1) = 0, \quad \varphi(t,1) = 0 \qquad (t \geqslant 0), \\ k(0) = k_0, \\ \varphi(0,x) = \varphi_0(x) \quad x \in [-1,1] \setminus \{0\}. \end{cases}$$

$$X = L^2(-1,1) \times \mathbb{R}, \quad U = L^2(-1,a) \quad \text{with } a \in (-1,1).$$

$$\mathcal{D}(A) = \left\{ z = \begin{bmatrix} \varphi \\ g \end{bmatrix} \in H_0^1(-1,1) \times \mathbb{R} \,\middle|\, \varphi(0) = g, \right.$$

$$\left. \varphi|_{(-1,0)} \in H^2(-1,0),\ \varphi|_{(0,1)} \in H^2(0,1) \right\},$$

$$A \begin{bmatrix} \varphi \\ g \end{bmatrix} = \begin{bmatrix} \varphi_{xx} \\ [\varphi_x](0) \end{bmatrix}. \tag{5}$$

Eigenproblem

$$\begin{cases} -\varphi_{xx} = \mu^2 \varphi(x) & x \in (-1,0) \cup (0,1), \\ \varphi(-1) = \varphi(1) = 0, \\ \varphi(0) = -\dfrac{1}{\mu^2}[\varphi_x](0). \end{cases} \tag{6}$$

From the first two equations we deduce that

$$\varphi(x) = \begin{cases} C_1 \sin\left(\mu(1+x)\right), & x \in (-1,0), \\ C_2 \sin\left(\mu(1-x)\right), & x \in (0,1), \end{cases}$$

Properties of the eigenelements

$$Bu = \begin{bmatrix} \mathbf{1}_{(-1,a)} u \\ 0 \end{bmatrix} \qquad (u \in U). \tag{7}$$

Proposition

The sequence $(-\lambda_k)_{k \geqslant 1}$ *is regular (i.e.,* $\inf_{n \geqslant 1}(\lambda_{n+1} - \lambda_n) > 0$*) and we have*

$$\lambda_k = \frac{k^2 \pi^2}{4} + O(1) \qquad (k \to \infty). \tag{8}$$

Furthermore, there exist two positive constants c_1 *and* c_2 *such that*

$$c_1 \leqslant \|B^* \varphi_k\|_U \leqslant c_2 \qquad (k \geqslant 1). \tag{9}$$

Consequences

Take

$$I = (-1, a), \quad \text{for some } a \in (-1, 1).$$

Then the system

$$\begin{cases} v_t - v_{xx} = \mathbf{1}_I u & (t \geqslant 0, \quad x \in [-1,1] \setminus \{0\}), \\ v(t,0) = \ell(t) & (t \geqslant 0), \\ m\dot{\ell}(t) = [v_x](t,0) & (t \geqslant 0), \\ v(t,-1) = 0, \quad v(t,1) = 0 & (t \geqslant 0), \\ \ell(0) = \ell_0, \\ v(0,x) = v_0(x) \quad x \in [-1,1] \setminus \{0\} \end{cases}$$

is null-controllable for any time $\tau > 0$.
For all (v_0, ℓ_0), there exists $u \in L^2([0,\tau]; L^2(I))$ such that

$$v(\tau, \cdot) = 0, \quad \ell(\tau) = 0.$$

Consequences

The system

$$\begin{cases} v_t - v_{xx} = \mathbf{1}_I u + f_1 & (t \geqslant 0, \quad x \in [-1,1] \setminus \{0\}), \\ v(t,0) = \ell(t) & (t \geqslant 0), \\ m\dot{\ell}(t) = [v_x](t,0) + f_2 & (t \geqslant 0), \\ v(t,-1) = 0, \quad v(t,1) = 0 & (t \geqslant 0), \\ \ell(0) = \ell_0, \\ v(0,x) = v_0(x) \quad x \in [-1,1] \setminus \{0\} \end{cases}$$

is null-controllable for any time $\tau > 0$.
For all (v_0, ℓ_0), for all $(f_1, f_2)/\rho_{\mathcal{F}} \in L^2(0,\tau; L^2(-1,1) \times \mathbb{R})$, there exists $u/\rho_0 \in L^2([0,\tau]; L^2(I))$ such that

$$\frac{(v,\ell)}{\rho_0} \in C([0,\tau]; L^2(-1,1) \times \mathbb{R})$$

and in particular

$$v(\tau, \cdot) = 0, \quad \ell(\tau) = 0.$$

Null-controllability Velocity-Position

The system

$$\begin{cases} v_t - v_{xx} = \mathbf{1}_I u + f_1 \qquad (t \geqslant 0, \quad x \in [-1,1] \setminus \{0\}), \\ v(t,0) = \ell(t) \qquad (t \geqslant 0), \\ m\dot{\ell}(t) = [v_x](t,0) + f_2 \qquad (t \geqslant 0), \\ v(t,-1) = 0, \quad v(t,1) = 0 \qquad (t \geqslant 0), \\ \dot{h}(t) = \ell(t) \qquad (t \geqslant 0), \\ h(0) = h_0, \quad \ell(0) = \ell_0, \\ v(0,x) = v_0(x) \quad x \in [-1,1] \setminus \{0\} \end{cases}$$

is null-controllable for any time $\tau > 0$.
For all (v_0, ℓ_0, h_0), for all $(f_1, f_2)/\rho_{\mathcal{F}} \in L^2(0,\tau; L^2(-1,1) \times \mathbb{R})$, there exists $u/\rho_0 \in L^2([0,\tau]; L^2(I))$ such that

$$\frac{(v,\ell)}{\rho_0} \in C([0,\tau]; L^2(-1,1) \times \mathbb{R}) \quad \text{and} \quad h(\tau) = 0.$$

Resolution of our fluid-structure problem

$$\begin{cases} v_t - v_{xx} + vv_x = \hat{u}(t,x) \qquad (t \geqslant 0, \quad x \in [-1,1] \setminus \{h(t)\}), \\ v(t,h(t)) = \dot{h}(t) \qquad (t \geqslant 0), \\ m\ddot{h}(t) = [v_x](t,h(t)) \qquad (t \geqslant 0), \\ v(t,-1) = 0, \quad v(t,1) = 0 \qquad (t \geqslant 0), \\ h(0) = h_0, \quad \dot{h}(0) = h_1, \\ v(0,x) = v_0(x) \quad x \in [-1,1] \setminus \{h_0\}. \end{cases}$$

Difficulties

- Nonlinear
- Moving Domain: $[-1,1] \setminus \{h(t)\}$
- Domain depending on the unknown h

Change of variables

$$y = \eta(t,x) = \begin{cases} \dfrac{(x-h(t))}{1-h(t)}, x > h(t), \\[2ex] \dfrac{(x-h(t))}{1+h(t)}, x < h(t). \end{cases}$$

We define a new function φ by setting

$$\varphi(t,\eta(t,x)) = v(t,x) \qquad (t \in [0,\tau],\ x \in (-1,h(t)) \cup (h(t),1)).$$

System after change of variables

$$
\left\{
\begin{aligned}
&\varphi_t - \left(\eta_x \circ \eta^{-1}\right)^2 \varphi_{yy} + (\eta_x \circ \eta^{-1})\varphi_y \varphi + (\eta_t \circ \eta^{-1})\varphi_y = u(y,t), \\
&\qquad\qquad\qquad\qquad\qquad\qquad (t \geqslant 0, \quad y \in [-1,1] \setminus \{0\}), \\
&m\dot{g}(t) = [(\eta_x \circ \eta^{-1})\varphi_y](0,t), \qquad (t \geqslant 0), \\
&\varphi(t,0) = g(t), \qquad (t \geqslant 0), \\
&\dot{h}(t) = g(t), \qquad (t \geqslant 0), \\
&\varphi(t,-1) = 0, \quad \varphi(t,1) = 0, \qquad (t \geqslant 0), \\
&h(0) = h_0, \quad g(0) = h_1, \\
&\varphi(0,y) = \varphi_0(y) \quad x \in [-1,1] \setminus \{0\},
\end{aligned}
\right.
$$

System after change of variables

$$\begin{cases} \varphi_t = \varphi_{yy} + f_1 + u \qquad (t \geqslant 0, \quad y \in [-1,1] \setminus \{0\}), \\ \dot{g}(t) = [\varphi_y](0,t) + f_2(t) \qquad (t \geqslant 0), \\ \varphi(0,t) = g(t) \qquad (t \geqslant 0), \\ \dot{h}(t) = g(t), \\ \varphi(-1,t) = 0, \quad \varphi(1,t) = 0 \qquad (t \geqslant 0), \\ h(0) = h_0, \quad g(0) = h_1, \\ \varphi(y,0) = \varphi_0(y) \qquad (y \in [-1,1] \setminus \{0\}), \end{cases}$$

where

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \frac{2h\,\mathrm{sign}(y) - h^2}{(1-h\,\mathrm{sign}(y))^2}\varphi_{yy} - \frac{1}{1-h\,\mathrm{sign}(y)}\varphi_y\varphi - \frac{g(1-|y|)}{1-h\,\mathrm{sign}(y)}\varphi_y \\ \frac{2h}{1-h^2}\varphi_y(0^+) + \left(\frac{1}{1+h} - 1\right)[\varphi_y](0) \end{bmatrix}.$$

Fixed Point Procedure (1/3)

Take r small enough and

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in \mathcal{F}, \quad \left\| \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right\|_{\mathcal{F}} \leqslant r.$$

Then there exists $u \in \mathcal{U}$ such that

$$\frac{(v,\ell)}{\rho} \in C([0,\tau]; H^1(-1,1) \times \mathbb{R}) \quad \text{and} \quad \frac{h}{\rho} \in C([0,\tau]; \mathbb{R}).$$

Fixed Point Procedure (2/3)

Set

$$\mathcal{N}\left(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}\right) := \left[\begin{array}{c} \frac{2h\,\mathrm{sign}(y)-h^2}{(1-h\,\mathrm{sign}(y))^2}\varphi_{yy} - \frac{1}{1-h\,\mathrm{sign}(y)}\varphi_y\varphi - \frac{g(1-|y|)}{1-h\,\mathrm{sign}(y)}\varphi_y \\ \frac{2h}{1-h^2}\varphi_y(0^+) + \left(\frac{1}{1+h} - 1\right)[\varphi_y](0) \end{array}\right].$$

Then

$$\begin{aligned} \left\|\mathcal{N}\left(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}\right)\right\|_{\mathcal{F}} \leqslant \left\|\frac{\rho^2}{\rho_{\mathcal{F}}}\right\|_{L^\infty(0,\tau)} & \left(\left\|\frac{g}{\rho}\right\|^2_{C[0,\tau]}\right. \\ & \left. + \left\|\frac{h}{\rho}\right\|_{C[0,\tau]}\left\|\frac{\varphi}{\rho}\right\|_{L^2([0,\tau],H^2)} + \left\|\frac{\varphi}{\rho}\right\|^2_{C([0,\tau],H^1)}\right) \end{aligned}$$

Fixed Point Procedure (3/3)

Thus

$$\left\| \mathcal{N}\left(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}\right) \right\|_{\mathcal{F}} \leqslant Cr^2$$

for

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in \mathcal{F}, \quad \left\| \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right\|_{\mathcal{F}} \leqslant r.$$

Similarly,

$$\|\mathcal{N}(f_1) - \mathcal{N}(f_2)\|_{\mathcal{F}} \leqslant Cr \left(\left\| \frac{g_1 - g_2}{\rho} \right\|_{C[0,\tau]} + \left\| \frac{\varphi_1 - \varphi_2}{\rho} \right\|_{L^2([0,\tau],H^2)} \right.$$
$$\left. + \left\| \frac{\varphi_1 - \varphi_2}{\rho} \right\|_{C([0,\tau],H^1)} \right). \quad (10)$$

Using Banach fixed point, we deduce the existence of a fixed point for small initial data.

From interior null-controllability to boundary null-controllability:

We want to control

$$\begin{cases} v_t - v_{xx} + vv_x = 0 \qquad (t \geqslant 0, \quad x \in [-1,1] \setminus \{h(t)\}), \\ v(t,h(t)) = \dot{h}(t) \qquad (t \geqslant 0), \\ m\ddot{h}(t) = [v_x](t,h(t)) \qquad (t \geqslant 0), \\ v(t,-1) = u(t), \quad v(t,1) = 0 \qquad (t \geqslant 0), \\ h(0) = h_0, \quad \dot{h}(0) = h_1, \\ v(0,x) = v_0(x) \quad x \in [-1,1] \setminus \{h_0\}. \end{cases}$$

We extend v_0 by 0 outside $(-1-\varepsilon,-1)$ and consider the control problem

$$\begin{cases} v_t - v_{xx} + vv_x = \hat{u}(t,x) \qquad (t \geqslant 0, \quad x \in [-1-\varepsilon,1] \setminus \{h(t)\}), \\ v(t,h(t)) = \dot{h}(t) \qquad (t \geqslant 0), \\ m\ddot{h}(t) = [v_x](t,h(t)) \qquad (t \geqslant 0), \\ v(t,-1-\varepsilon) = 0, \quad v(t,1) = 0 \qquad (t \geqslant 0), \\ h(0) = h_0, \quad \dot{h}(0) = h_1, \\ v(0,x) = v_0(x) \quad x \in [-1-\varepsilon,1] \setminus \{h_0\}. \end{cases}$$

From interior null-controllability to boundary null-controllability:

We can control the above system by using

$$\hat{u} \in L^2(0, \tau; L^2([-1, -1 - \varepsilon/2])).$$

We have

$$v \in L^2([0, \tau], H^2((-1 - \varepsilon, 1) \setminus \{h(t)\})) \cap C([0, \tau], H^1(-1 - \varepsilon, 1)),$$
$$h \in C^1[0, \tau],$$

and

$$\dot{h}(\tau) = 0, \quad v(\tau, \cdot) = 0 \quad \text{and} \quad h(\tau) = 0.$$

Then we can take $u(t) := v(t, -1)$, $u \in C([0, \tau])$.

Conclusion

- General method to control problem of the form

$$\begin{cases} \dot{z} = Az + Bu + f, \\ z(0) = z_0, \end{cases}$$

  with $A = A^* < 0$.
- We obtain the controllability of a fluid-structure problem in 1d, with only one control.
- This general method can be applied for other parabolic systems, the important hypothesis is the null-controllability of (A, B).

Collaborators

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