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Minimal Bit Rates and Entropy for Stabilization

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The stabilization problem

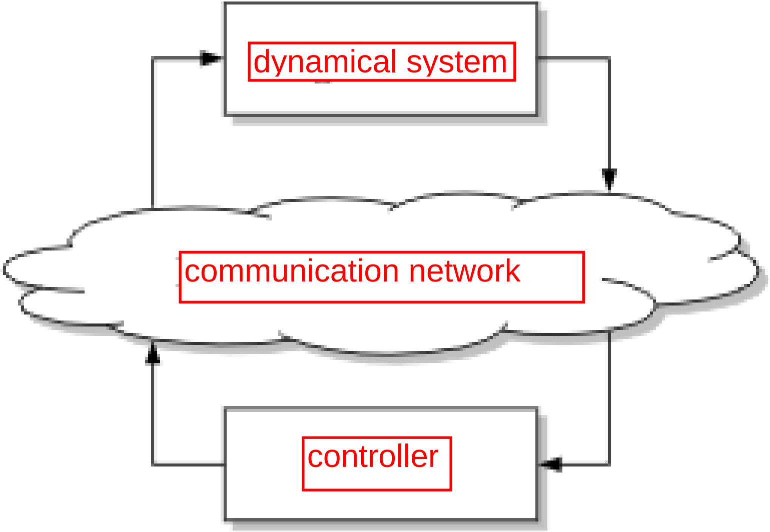
Consider for

$$\dot{x}(t) = f(x(t), u(t)), u \in \mathcal{U} := \{u : [0, \infty) \to U \subset \mathbb{R}^m\}$$

stabilization about the equilibrium

$$0 = f(0, 0),\ \ 0 \in U.$$

A stabilizing feedback $u = F(x)$ requires continual measurements of the state (or of observed values).
For a (noiseless) digital communication channel, **minimal bit rates** are of interest.
Nair/Evans/Mareels/Moran: topological feedback entropy (IEEE TAC '04)
Kawan: invariance entropy (SICON '09, '11)


dynamical system
communication network
controller

Overview

- Digitally connected systems and feedback

- Exponential stabilization

- Strict minimal bit rates and strict entropy

- Estimates for the stabilization entropy

- A formula in the linear case

- Minimal bit rates

Digitally connected systems

- **Various information patterns possible, in particular**

- quantization
- event based control
- symbolic controllers
- model predictive control

For example: The controller computes (via optimal control) at time t_i a control function $u(t)$, which is used on $[t_i, t_{i+1}]$ until at time t_{i+1} new information is available (adaptive sampling allowed).

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- **In any case:**

The controller generates a set of control functions on $[\mathbf{0}, \infty)$. We may count the bits necessary to distinguish these controls!

Tatikonda/Mitter 2004

Feedbacks

A stabilizing feedback $u = F(x)$ generates controls (depending on x_0)

$$u(t) := F(x(t, x_0)), x_0 \in \mathbb{R}^n,$$

where $x(t, x_0)$ solves

$$\dot{x}(t, x_0) = f(x(t, x_0), F(x(t, x_0))),\ x(0, x_0) = x_0.$$

However, with a digital connection, only finitely many controls can be generated on finite time intervals.

Exponential stabilization

Let $K \subset \mathbb{R}^n$ be a bounded set of initial states with $0 \in \mathrm{int} K$.

Assumption: There are $M, \alpha > 0$ such that for all $x_0 \in K$ there is $u \in \mathcal{U}$ with

$$\boxed{\|x(t, x_0, u)\| < M e^{-\alpha t} \|x_0\| \text{ for all } t \geq 0.}$$

For stable systems it is also of interest to increase the exponential decay rate α.

Two ways to count bits I

(i) Consider a set $\mathcal{R}$ of control functions such that for every $x_0 \in K$ there is $u \in \mathcal{R}$ with $\|x(t, x_0, u)\| < Me^{-\alpha t}\|x_0\|$ for all $\mathbf{t} \geq \mathbf{0}$. With

$$\mathcal{R}_T := \{u_{|[0,T]} \mid u \in \mathcal{R}\}, \;\; T > 0,$$

the bits of $\mathcal{R}_T$ are $\log_2 \#\mathcal{R}_T$ and the bit rate of $\mathcal{R}$ is

$$b(\mathcal{R}) := \overline{\lim_{T\to\infty}} \frac{1}{T} \log_2 \#\mathcal{R}_T.$$

The (strict) **minimal bit rate** is

$$b^*(\alpha, M) := \inf_{\mathcal{R}} b(\mathcal{R}).$$

Two ways to count bits II

(ii) For $T > 0$ consider a set $\mathcal{S}(T)$ of control functions on $[\mathbf{0}, \mathbf{T}]$ such that for every $x_0 \in K$ there is $u \in \mathcal{S}(T)$ with

$$\|x(t, x_0, u)\| < Me^{-\alpha t} \|x_0\| \text{ for all } t \in [0, T].$$

The bits for $\mathcal{S}(T)$ are $\log_2 \#\mathcal{S}(T)$ and the minimal number of bits on $[0, T]$ is

$$\inf_{\mathcal{S}(T)} \log_2 \#\mathcal{S}(T).$$

The (strict) **stabilization entropy** is

$$h^*(\alpha, M) := \overline{\lim_{T\to\infty}} \frac{1}{T} \left[\inf_{\mathcal{S}(T)} \log_2 \#\mathcal{S}(T) \right].$$

Comparison

Fact: The (strict) stabilization bit rate and the (strict) stabilization entropy coincide,

$$b^*(\alpha, M) = h^*(\alpha, M) \leq \infty.$$

However:

There are no general criteria guaranteeing that either of them is finite!

A weakened version of stabilization entropy

- **For** $\varepsilon, T > 0$ **consider sets** $\mathcal{S}(\varepsilon, T)$ **of control functions on** $[\mathbf{0}, \mathbf{T}]$ **such that for every** $x_0 \in K$ **there is** $u \in \mathcal{S}(\varepsilon, T)$ **with**

$$\|x(t, x_0, u)\| < e^{-\alpha t}\left[\varepsilon + M\|x_0\|\right] \text{ for all } t \in [0, T].$$

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- **The stabilization entropy is**

$$h(\alpha, M) := \lim_{\varepsilon \searrow 0} \overline{\lim_{T \to \infty}} \frac{1}{T}\left[\inf_{\mathcal{S}(\varepsilon, T)} \ln \#\mathcal{S}(\varepsilon, T)\right].$$

Properties of stabilization entropy

Let $K \subset \mathbb{R}^n$ be a bounded set of initial states with $0 \in \operatorname{int} K$.
Assumption: There are $M^*, \alpha^* > 0$ such that for all $x_0 \in K$ there is $u \in \mathcal{U}$ with

$$\boxed{\left\| x(t, x_0, u) \right\| < M^* e^{-\alpha^* t} \left\| x_0 \right\| \text{ for all } t \geq 0.}$$

Then the stabilization entropy

$$h(\alpha, M) = \lim_{\varepsilon \searrow 0} \overline{\lim_{T \to \infty}} \frac{1}{T} \left[\inf_{\mathcal{S}(\varepsilon, T)} \ln \# \mathcal{S}(\varepsilon, T) \right] \leq (L + \alpha)\, n.$$

In particular, it is finite.
(Proof based on methods from topological theory of dynamical systems)

A formula in the linear case

- **Consider**

$$\dot{x}(t) = Ax(t) + Bu(t), \;\; u(t) \in U.$$

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- **Theorem. Let $\alpha \in (0, \alpha^*)$ such that $\alpha \neq \operatorname{Re}\lambda$ for all eigenvalues λ of A. Then the stabilization entropy is**

$$\boxed{h(\alpha, M) = \sum_{\operatorname{Re}\lambda > -\alpha} (\alpha + \operatorname{Re}\lambda),}$$

where summation is over all eigenvalues λ of A with $\operatorname{Re}\lambda > -\alpha$.

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where summation is over all eigenvalues λ of A with $\operatorname{Re} \lambda > -\alpha$.
- **Note that this also of interest if** $\operatorname{Re} \lambda < 0$ **for all eigenvalues** λ**.**

A weakened version of minimal bit rates

- For $\varepsilon > 0$ consider a set $\mathcal{R}(\varepsilon)$ of control functions such that for every $x_0 \in K$ there is $u \in \mathcal{R}(\varepsilon)$ with

$$\|x(t, x_0, u)\| < e^{-\alpha t}\left[\varepsilon + M\|x_0\|\right] \text{ for all } t \geq 0.$$

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- **With**

$$\mathcal{R}(\varepsilon)_T := \{u_{|[0,T]} \mid u \in \mathcal{R}(\varepsilon)\}, \ \ T > 0,$$

the minimal bit rate is

$$b(\alpha, M) := \lim_{\varepsilon \searrow 0} \inf_{\mathcal{R}(\varepsilon)} \overline{\lim_{T \to \infty}} \frac{1}{T} \ln \#\mathcal{R}(\varepsilon)_T.$$

Comparison

Assumption: There are $M, \alpha^* > 0$ such that for all $x_0 \in K$ there is $u \in \mathcal{U}$ with

$$\boxed{\|x(t, x_0, u)\| < Me^{-\alpha^* t} \|x_0\| \text{ for all } t \geq 0.}$$

Then there is $\alpha \in (0, \alpha^*)$ such that

$$h_{\mathrm{stab}}(\alpha, M) \leq b_{\mathrm{stab}}(\alpha, M) \leq 2 \cdot h_{\mathrm{stab}}(\alpha^*, M).$$

The linear case again

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$$\|x(t, x_0, u)\| < M^* e^{-\alpha^* t} \|x_0\| \text{ for all } t \geq 0.$$
- **Then there is** $\alpha \in (0, \alpha^*)$ **such that**
$$\sum_{\operatorname{Re}\lambda > 0} \operatorname{Re}\lambda \leq b_{\mathrm{stab}}(\alpha, M) \leq 2 \cdot \sum_{\operatorname{Re}\lambda > 0} \operatorname{Re}\lambda.$$

Conclusions

We have considered two concepts in order to count bit rates for stabilization considering time dependent controls.
For linear systems, a formula has been given, which also takes into account stable eigenvalues.

Extension (joint work with Girish Nair, Melbourne): A differential game approach for perturbed problems
Some open questions:
- a formula in the nonlinear case
- generalization to stabilization under deterministic/random perturbations
- generalizations for distributed control systems