Geometry of the limit sets of linear switched systems

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Linear Switched Systems

$$\dot{x} = B_{u(t)}x$$

where

- $B_1, \ldots, B_p \in \mathcal{M}(d; \mathbb{R})$
- The input u: $[0, +\infty[\longmapsto \{1, \ldots, p\}$ is piecewise constant and right continuous.

Assumption

The matrices $B_1, \ldots, B_p$ share a common quadratic Lyapunov function P (i.e. P is a symmetric, positive, definite matrix), **non strict in general**:

$$B_i^T P + PB_i \leq 0 \qquad i = 1, \ldots, p.$$

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The system under consideration II

We can assume

$$P = Id$$

hence

$$B_i^T + B_i \leq 0 \qquad i = 1, \ldots, p.$$

The set $\mathcal{V}_i$

We set $\mathcal{V}_i = \{x \in \mathbb{R}^d;\ \forall t \in \mathbb{R}\ \left\| e^{tB_i} x \right\| = \|x\|\}$

Lemma

For each i, $\mathbb{R}^d = \mathcal{V}_i \oplus \mathcal{V}_i^{\perp}$ and

- $\mathcal{V}_i$ and $\mathcal{V}_i^{\perp}$ are B_i-invariant subspaces.
- $B_i \mid_{\mathcal{V}_i}$ is skew-symmetric (in a suitable orthonormal basis)
- $B_i \mid_{\mathcal{V}_i^{\perp}}$ is Hurwitz

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Examples

In $\mathbb{R}^3$ with $\{x, y, z\}$ coordinates

$$B_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \mathcal{V}_1 = \{x = 0\}.$$

$$B_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{V}_2 = \{x = y = 0\}.$$

In the paper: *On the convergence of linear switched systems* (2011), U. Serres, J.C. Vivalda, and P. Riedinger have studied the case where

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Another important set

$$\mathcal{K}_i = \ker(B_i^T + B_i) = \{x;\ \frac{d}{dt}_{|t=0} \left\| e^{tB_i} x \right\|^2 = 0\}$$

$\mathcal{K}_i$ **is not** B_i**-invariant in general**.

The set $\mathcal{V}_i$ is related to $\mathcal{K}_i$ by:

$$\mathcal{V}_i \subseteq \mathcal{K}_i$$

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Example

B Hurwitz does not imply $\mathcal{K} = \{0\}$.

$$B = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{is Hurwitz.}$$

$$B^T + B = \begin{pmatrix} -4 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathcal{K} = \{x = 0\}$$

So for

$$B = \begin{pmatrix} -2 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix} \text{ we have } \begin{cases} \mathcal{V} = \{x_1 = x_2 = 0\} \\ \mathcal{K} = \{x_1 = 0\} \end{cases}$$

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The switching law

The **input**, or **switching signal** $u(t)$ is characterized by the switching times $(a_n)_{n\geq 0}$ and the values u_n it takes on the intervals $[a_n, a_{n+1}[$.

The solution to the equation

$$\dot{x} = B_{u(t)}x$$

is

$$t \longmapsto \Phi_u(t)x$$

where

$$\Phi_u(t) = e^{tB_{u_n}} e^{(a_n - a_{n-1})B_{u_{n-1}}} \dots e^{(a_1 - a_0)B_{u_0}}$$

The ω-limit sets

$\Omega_u(x)$ stands for the set of ω-limit points of the trajectory

$$t \longmapsto \Phi_u(t)x$$

Proposition

The ω-limit set $\Omega(x)$ is a compact and **connected** subset of a sphere

$$\mathcal{S}_r = \{x;\ \|x\| = r\}$$

for some $r \geq 0$.

Two features

1. If for $t \in [a_n, a_{n+1}[$ we have

$$\|\Phi_u(t)x\| = \|\Phi_u(a_n)x\|$$

then $\Phi_u(t)x \in \mathcal{V}_{u_n}$ for $t \in [a_n, a_{n+1}[$.

2. If $\Omega_u(x)$ meets $\mathcal{V}_1$ and $\mathcal{V}_2$ on a sphere $\mathcal{S}_r$ and does not leave $\mathcal{V}_1 \cup \mathcal{V}_2$, then $\Omega_u(x)$ is a connected component of

$$(\mathcal{V}_1 \cup \mathcal{V}_2) \bigcap \mathcal{S}_r$$

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Non Chaotic inputs

Definition

The input u is said to be **chaotic** if there exists a sequence $[t_k, t_k + s]$ of intervals s.t.

1. $t_k \longrightarrow_{k \to +\infty} +\infty$ and $s > 0$.
2. For all $\epsilon > 0$ there exists k_0 such that for all $k \geq k_0$, the input u is constant on no subinterval of $[t_k, t_k + s]_{k \geq 0}$ of length greater than or equal to ϵ.

If not u is called a **non chaotic input**.

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$$\forall x \in \mathbb{R}^d \qquad \Omega_u(x) \subseteq \bigcup_{i=1}^{p} \mathcal{V}_i$$

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Regular inputs

Definition

The input $u = (a_n, u_n)_{n\geq 0}$ satisfies the assumption $H(i)$ if there exist a subsequence $(a_{n_k})_{k\geq 0}$ and $\delta > 0$ such that

$$\forall k \geq 0 \qquad u_{n_k} = i \;\text{ and }\; a_{n_{k+1}} - a_{n_k} \geq \delta.$$

Proposition

If the input u satisfies the assumption $H(i)$ then:

$$\forall x \in \mathbb{R}^d \qquad \Omega_u x \bigcap \mathcal{V}_i \neq \emptyset.$$

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An input u is said to be **regular** if it is non chaotic and satisfies the assumption $H(i)$ for $i = 1, \ldots, p$.

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Main result I

A geometric condition

(C) The sets $\mathcal{V}_i$ are said to satisfy the condition (C) if for $r > 0$, no connected component of the set $(\bigcup_{i=1}^{p} \mathcal{V}_i) \bigcap \mathcal{S}(r)$ intersects all the $\mathcal{V}_i$'s.

Theorem

If Condition (C) is satisfied, then for every regular input u the switched system is asymptotically stable.

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Theorem

If Condition (C) is satisfied, then for every regular input u the switched system is asymptotically stable.

Main result II

Theorem

Under the hypothesis

$$\bigcap_{i=1}^{p} \mathcal{V}_i = \{0\},$$

the switched system is asymptotically stable for every regular input as soon as one of the following conditions hold:

1. there exists i such that $\dim \mathcal{V}_i = 0$
2. there exists i such that $\dim \mathcal{V}_i = 1$, and $\mathcal{V}_i \subseteq \mathcal{V}_j \Longrightarrow \mathcal{V}_i = \mathcal{V}_j$
3. $p = 2$
4. $p > 2$, and $\dim(\sum_{i=1}^{p} \mathcal{V}_i) > \sum_{i=1}^{p} \dim(\mathcal{V}_i) - p + 1$

In particular in the plane, that is for $d = 2$, at least one of these conditions is satisfied as soon as $\mathcal{V}_i \neq \mathbb{R}^2$ for $i = 1, \ldots, p$.

Chaotic inputs

For general inputs consider

- $\mathcal{K}_i = \ker(B_i^T + B_i) = \{x;\ \frac{d}{dt}_{|t=0} \left\| e^{tB_i} x \right\|^2 = 0\}$
- $J_u \subseteq \{1, 2 \ldots, p\}$

$$i \in J_u \Longleftrightarrow m\{t \geq 0;\ u(t) = i\} = +\infty.$$

- $F_u = \bigcup_{i \in J_u} \mathcal{K}_i$.

Proposition

For all $x \in \mathbb{R}^d$, the set $\Omega_u(x)$ is included in F_u and moreover verifies

$$\forall i \in J_u \qquad \Omega_u(x) \bigcap \mathcal{K}_i \neq \emptyset$$

Theorem 4

With the previous notations, it is assumed that $\bigcap_{i\in J_u} \mathcal{K}_i = \{0\}$.

Then the switched system is asymptotically stable, as soon as for $r > 0$ no connected component of

$$(\cup_{i\in J_u}\mathcal{K}_i)\bigcap\mathcal{S}(r)$$

intersects all the $\mathcal{K}_i$'s for $i \in J_u$. This condition is verified if

1. the cardinality $|J_u|$ of J_u is 2;
2. $|J_u| > 2$, and $\dim(\sum_{i\in J_u} \mathcal{K}_i) > \sum_{i\in J_u} \dim(\mathcal{K}_i) - q + 1$;
3. there exists $i \in J_u$ such that $\dim \mathcal{K}_i = 0$;
4. there exists $i \in J_u$ such that $\dim \mathcal{K}_i = 1$, and for $j \in J_u$, $\mathcal{K}_i \subseteq \mathcal{K}_j \Longrightarrow \mathcal{K}_i = \mathcal{K}_j$.

In particular for $d = 2$, at least one of these conditions is satisfied as soon as $\mathcal{K}_i \neq \mathbb{R}^2$ for all $i \in J_u$.

Stability of pairs of Hurwitz matrices

Theorem 5

Let B_1 and B_2 be two $d \times d$ Hurwitz matrices, assumed to share a common, but not necessarily strict, Lyapunov matrix P. Then the switched system is asymptotically stable for any input as soon as

$$\mathcal{K}_1 \bigcap \mathcal{K}_2 = \{0\}$$

where $\mathcal{K}_i = \ker\left((QB_iQ^{-1})^T + QB_iQ^{-1}\right)$ for $i = 1, 2$, and Q is the symmetric positive square root of P.

The lift of the problem

- The solution to the equation

$$\dot{x} = B_{u(t)}x \qquad \text{is} \qquad t \longmapsto \Phi_u(t)x$$

where

$$\Phi_u(t) = e^{tB_{u_n}} e^{(a_n - a_{n-1})B_{u_{n-1}}} \dots e^{(a_1 - a_0)B_{u_0}}$$

- We can consider the ω-limit set Ω_u of the matrix trajectory

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The lift of the problem II

Proposition

- The set Ω_u is a compact and connected subset of $K = B'(0,1) \subset \mathcal{M}(d;\mathbb{R})$, and for all $x \in \mathbb{R}^d$ the set $\Omega_u(x)$ is equal to:

$$\Omega_u(x) = \{Mx; \quad M \in \Omega_u\} = \Omega_u x.$$

- Moreover there exists a symmetric nonnegative matrix S_u

$$\forall M, N \in \Omega_u \qquad M^T M = N^T N = S_u^2.$$

- The switched system is asymptotically stable if and only if $S_u = 0$

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The lift of the problem III

Theorem

- Let $S(t)$ be the symmetric positive square root of $\Phi_u(t)^T\Phi_u(t)$. Then

$$\Phi_u(t)^T\Phi_u(t) = S(t)^2 \searrow_{t\mapsto+\infty} S_u^2$$

- There exists a compact and connected subset of $\mathcal{O}_u$ of SO_d such that

$$\Omega_u = \mathcal{O}_u S_u$$

The matrix S_u can be computed using a convergent subsequence of $\Phi_u(t)$.

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Example

$$B_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \; B_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; B_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Here

$$\bigcap_i \mathcal{V}_i = \{0\} \quad \text{and} \quad \bigcup_i \mathcal{V}_i = \{x = 0\}$$

The input $u = (a_n, u_n)_{n \geq 0}$ is defined by $a_n = n\frac{\pi}{2}$ and

$$u_{4k} = u_{4k+2} = 1 \qquad u_{4k+1} = 2 \qquad u_{4k+3} = 3.$$

Example

$\Phi_u(4k\pi)$ tends to $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ as k tends to $+\infty$.

The matrix S_u is therefore equal to $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

We know that $\Omega_u = \mathcal{O}_u S_u$ for a certain subset $\mathcal{O}_u$ of SO_d, so

$$\Omega_u(x) = \{0\} \iff x_2 = 0$$

Convergence on intervals

Consider

- $(t_k)_{k\geq 0}$ with $t_k > 0$ and $t_k \upuparrows_{k\mapsto+\infty} +\infty$
- $s > 0$
- The sequence $(\Phi_k)_{k\geq 0}$ of functions from $[0, s]$ into $\mathcal{M}(d, \mathbb{R})$ defined by

$$\Phi_k(t) = \Phi_u(t_k + t).$$

Proposition

There exists a subsequence of $(\Phi_k)_{k\geq 0}$ that converges uniformly to a continuous function

$$t \longmapsto \Psi(t)$$

from $[0, s]$ into Ω_u.

Proof: use Ascoli's Theorem

Open problems

- Necessary and sufficient condition of stability for all inputs for pairs of Hurwitz matrices.
- Rate of convergence.
- Extension to nonlinear switched systems

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