Lecture
Notes

2025

timisation, Control and D

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Contents

Part I

Optimisation

Chapter 1

Convex Analysis

This chapter closely follows chapter 5 of the lecture notes [1]. For most of the proof in this chapter we refer to [2].

1.1 Convex sets

Definition 1.1.1
A set $K \subset \mathbb{R}^d$ is *convex* if

$$tx + (1-t)y \in K \qquad \forall x, y \in K,\ t \in [0,1] \tag{1.1}$$

We have the following fact.

Proposition 1.1.1
The set $K \subset \mathbb{R}^d$ is convex if and only if for any $n \in \mathbb{N}$, and $t_1, \ldots, t_n \geq 0$ such that $\sum_{i=1}^n t_i = 1$, it holds

$$x_1, \ldots, x_n \in K \implies \sum_{i=1}^n t_i x_i \in K. \tag{1.2}$$

Proof: Property (1.2) with $n = 2$ is exactly the definition of K is convex. The statement then follows by induction on n. ■

Definition 1.1.2
The convex hull $\operatorname{conv}(\Omega)$ of $\Omega \subset \mathbb{R}^d$ is the smallest convex set K containing Ω.

By Proposition 1.1.1, it is immediate to observe that

$$\operatorname{conv}(\Omega) = \left\{ \sum_{i=1}^n t_i x_i \mid t_i \geq 0, \sum_{i=1}^n t_i = 1,\ x_i \in \Omega \right\}. \tag{1.3}$$

Example 1.1.1 (Convex sets)

- Unit ball w.r.t. any norm.
- Vector subspaces.
- Hyperplanes, i.e., for any $v \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$,

$$H_{v,\lambda} := \{x \in \mathbb{R}^d \mid \langle v, x\rangle \geq \lambda\}. \tag{1.4}$$

An important result (that we will not prove) on convex sets is the following.

Theorem 1.1.1 Separation theorem

Let $K_1, K_2 \subset \mathbb{R}^d$ be two convex sets such with disjoint interior. Then there exists $v \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$ such that

$$K_1 \subset H_{v,\lambda}, \qquad K_2 \subset \mathbb{R}^d \setminus H_{v,\lambda}. \tag{1.5}$$

Here, $H_{v,\lambda}$ is defined in (1.4).

When K is a convex polygon, it is natural to expect it to be determined by its vertices. In order to formalize this intuition we need the following.

Definition 1.1.3

Let $K \subset \mathbb{R}^d$ be a convex set. A point $x \in K$ is an extremum of K if for any $y, z \in K$ and $t \in (0,1)$ we have that

$$x = ty + (1-t)z \implies x = y = z. \tag{1.6}$$

The set of exterma of K is denoted by $\mathrm{extr}(K)$.

In particular, for a convex polygon $\mathrm{extr}(K)$ is the set of its vertices.

Proposition 1.1.2

Let $K \subset \mathbb{R}^d$ be a convex set that is compact. Then,

$$\mathrm{conv}(K) = \mathrm{conv}\left(\mathrm{extr}(K)\right). \tag{1.7}$$

1.2 Cones

Definition 1.2.1

A set $K \subset \mathbb{R}^d$ is a cone if

$$tx \in K \qquad \forall x \in K,\ t \geq 0. \tag{1.8}$$

Observe that every cone contains the origin.

Example 1.2.1 (Cones)

- The second order cone
$$C = \{x = (x', x_n) \in \mathbb{R}^d \times \mathbb{R} \mid \|x'\|_2 \leq x_n\}.$$
- Positive orthant $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d \mid x_i \geq 0, \quad \forall i \in [\![1,d]\!]\}$.
- The set of positive semidefinite matrices $\mathrm{Sym}_+(\mathbb{R}^d)$.

Definition 1.2.2

The conic hull $\mathrm{cone}(\Omega)$ of a set $\Omega \subset \mathbb{R}^d$ is the smallest cone containing Ω. Namely,

$$\mathrm{cone}(\Omega) = \left\{\sum_{i=1}^n t_i x_i \mid t_i \geq 0 \text{ and } x_i \in \Omega \text{ for any } i \in [\![1,n]\!]\right\}. \tag{1.9}$$

Figure 1.1: Two examples of polar cone.

Definition 1.2.3
The polar cone K^* of a cone $K \subset \mathbb{R}^d$ is the set

$$K^* := \left\{ y \in \mathbb{R}^d \ |\rangle x, y\langle \geq 0 \quad \forall x \in K \right\}. \tag{1.10}$$

We have the following properties for the polar cone.

Proposition 1.2.1
The polar cone K^* is a closed, convex cone. If, moreover, the cone K is closed, then $K^*{*} = K$.

1.3 Convex functions

We will work with extended functions $F : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$. The domain of an extended function is

$$\mathrm{dom}(F) = \{x \in \mathbb{R}^d \mid F(x) < +\infty\}. \tag{1.11}$$

Given a standard function $F : \Omega \to \mathbb{R}$, we can identify it with the extended function $\bar{F} : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\bar{F}(x) = \begin{cases} F(x) & \text{if } x \in \Omega, \\ +\infty & \text{if } x \in \mathbb{R}^d \setminus \Omega. \end{cases} \tag{1.12}$$

Definition 1.3.1 Convex functions
Let $F : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be an extended function. Then,

- F is convex if
$$F(tx + (1-t)y) \leq tF(x) + (1-t)F(y) \qquad \forall x, y \in \mathbb{R}^d,\ t \in [0,1]. \tag{1.13}$$

- F is strictly convex if
$$F(tx + (1-t)y) < tF(x) + (1-t)F(y) \qquad \forall x, y \in \mathbb{R}^d,\ x \neq y,\ t \in [0,1]. \tag{1.14}$$

- F is strongly convex if there exists $\gamma > 0$ such that
$$F(tx + (1-t)y) \leq tF(x) + (1-t)F(y) - \frac{\gamma}{2} t(t-1)\|x-y\|_2^2 \qquad \forall x, y \in \mathbb{R}^d,\ t \in [0,1]. \tag{1.15}$$

We say that F is *concave* if $-F$ is convex.

Observe that it holds

$$\text{convex} \impliedby \text{strongly convex} \impliedby \text{strictly convex} \tag{1.16}$$

We say that a standard function $F : K \to \mathbb{R}$ is convex, strictly convex, strongly convex, or concave, if the same is true for its extension $\bar{F}$. Observe that this requires K to be convex.

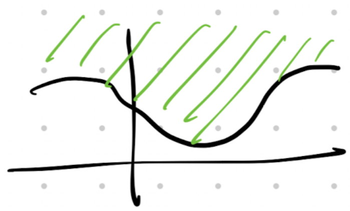

Figure 1.2: Epigraph of a function.

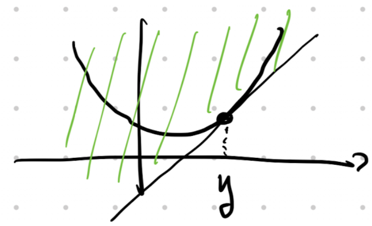

Figure 1.3: Graphical representation of Proposition 1.3.2.

Example 1.3.1

- The prototypical convex function, used in the definition of strongly convex, is the quadratic function

$$F(x) = \frac{\|x\|_2^2}{2} = \frac{1}{2}\sum_{i=1}^{d} |x_i|^2. \tag{1.17}$$

- More generally, every norm is convex.
- The norm ℓ_p is strictly convex if and only if $p \in (1, +\infty)$.
- $F(x) = x^\top Ax$ is convex if A is positive semidefinite (i.e., $A \in \mathrm{Sym}_{\geq 0}(\mathbb{R}^d)$), and strongly convex if A is positive definite.

Proposition 1.3.1

A function $F : K \to \mathbb{R}$ is convex if and only if its epigraph $\mathrm{epi}(F) \subset \mathbb{R}^{d+1}$ is convex. Here, we let

$$\mathrm{epi}(F) = \{(x, r) \mid r \geq F(x)\}. \tag{1.18}$$

Proof: Assume F is convex and let $(x, r), (y, s) \in \mathrm{epi}(F)$. In particular, $r \geq F(x)$ and $s \geq F(y)$. Let $t \in [0, 1]$ and observe that

$$tr + (1-t)s \geq tF(x) + (1-t)F(y) \geq F(tx + (1-t)y). \tag{1.19}$$

Hence, $t(x, r) + (1-t)(y, s) \in \mathrm{epi}(F)$. A similar reasoning proves the opposite implication. ■

Proposition 1.3.2 Differential characterisations of convexity

Let $F : \mathbb{R}^d \to \mathbb{R}$ be an everywhere differentiable function. Then,

- F is convex if and only if

$$F(y) \geq F(x) + \langle \nabla F(x), y - x\rangle, \qquad \forall x, y \in \mathbb{R}^d. \tag{1.20}$$

- F is strongly convex with parameter $\gamma > 0$ if and only if

$$F(y) \geq F(x) + \langle \nabla F(x), y - x\rangle + \frac{\gamma}{2}\|x - y\|_2^2, \qquad \forall x, y \in \mathbb{R}^d. \tag{1.21}$$

- If F is everywhere twice differentiable, then it is convex if and only if

$$\mathrm{Hess}\, F(x) \geq 0 \qquad \forall x \in \mathbb{R}^d. \tag{1.22}$$

  Here, we denoted by $\mathrm{Hess}\, F(x)$ the Hessian of F at x.

Proposition 1.3.3

Let $F : K \longrightarrow \mathbb{R}$ be convex. Then, F is continuous on the interior of K.

Proof: Let $x_0 \in \mathrm{int}(K)$ and consider $r > 0$ such that $B(x_0, r) \subset K$. Without loss of generality, we assume $x_0 = 0$ (otherwise, replace the function F by its translation $G(x) = F(x) - F(x_0)$).

Convexity will allow to bound the difference $F(y) - F(0)$ with the values of F on the sphere $\partial B(0, r)$. However, without continuity, the function F need not be bounded on the compact set $\partial B(0, r)$, and hence we need some additional care.

Pick $d + 1$ linearly independent points $v_0, \dots v_{d+1} \in \partial B(0, r)$, and consider the corresponding symplex

$$\Delta = \mathrm{conv}\left(\{v_0, \dots, v_{d+1}\}\right) = \left\{\sum_{i=1}^{d+1} t_i v_i \mid t_i \geq 0, \sum_i t_i = 1\right\} \subset B(0, r). \tag{1.23}$$

Then, letting $M = \max_{i \in [\![1, d+1]\!]} F(v_i)$, the fact that F is convex yields that for any $x = \sum_{i=1}^{d+1} t_i v_i \in \Delta$ it holds

$$F(x) \leq \sum_{i=1}^{d+1} t_i F(v_i) \leq M. \tag{1.24}$$

In particular, we can fix a radius $r' < r$ such that $B(0, r') \subset \Delta$ where F is bounded.

We now proceed to bound the difference $F(x) - F(0)$. Let $x \in U \subset B(0, r')$ and set $t = \|x\|/r'$. In particular, $t \in [0, 1]$ and the ray $\{sx \mid s \geq 0$ meets the sphere $\partial B(x_0, r')$ at the point

$$y = \frac{r'}{\|x\|}(x). \tag{1.25}$$

In particular, $x = (1 - t)0 + ty$. By convexity and (1.24), we have

$$F(x) \leq (1 - t)F(0) + tF(y) \leq .(1 - t)F(0) + tM \implies F(x) - F(0) \leq t(M - F(0)). \tag{1.26}$$

To derive a bound from below, we proceed similarly, considering

$$z = \frac{r'}{\|x\| - r'} x. \tag{1.27}$$

Indeed, we then have $0 = (1 - t)x + tx$, where $t = \|x\|/r'$ as above. Then, convexity and the fact that $z \in B(0, r')$ yield

$$F(0) \leq (1 - t)F(x) + tM \implies F(x) - F(0) \geq -\frac{t}{1 - t}(M - F(0)). \tag{1.28}$$

Combining (1.26) and (1.28), we obtain

$$-\frac{t}{1 - t}(M - F(0)) \leq F(x) - F(0) \leq t(M - F(0)), \qquad t = \frac{\|x\|}{r'} \tag{1.29}$$

Since x was arbitrary in $B(0, r')$ we can take the limit as $x \to 0$, which implies $t \to 0$ and thus that

$$\lim_{x \to 0} |F(x) - F(0)| = 0, \tag{1.30}$$

concluding the proof. ■

The following result is at the core of the relation between optimisation and convexity.

Theorem 1.3.1

Let $F : \mathbb{R}^d \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a convex extended function. Then,

- Any local minimum of F is global.
- The set of minima of F is convex.

- If F is strictly convex and admits a minimum, this minimum is unique.
- If F is real-valued and strongly convex, then it has a unique minimum.

Proof: Assume that $x^\star$ is a local minimum, i.e., there exists $r > 0$ such that $F(x^\star) \le F(x)$ for any $x \in B(0, r)$. Let $y \in \mathbb{R}^d$ and consider a point on the ray starting at $x^\star$ and passing through y:

$$z = x^\star + s(y - x^\star) = (1 - s)x^\star + sy \qquad s \ge . \tag{1.31}$$

Taking $s < \min\{1, r'/\|x^\star - y\|\}$ we have that $z \in B(0, r)$. Hence, by local minimality of $x^\star$ and convexity of F we have

$$F(x^\star) \le F(z) \le (1 - s)F(x^\star) + sF(y) \implies F(x^\star) \le F(y). \tag{1.32}$$

This concludes the proof of the first point.

Assume now that x_1, x_2 are minima for F. This clearly implies that $F(x_1) = F(x_2) =: m$, and thus, by convexity of F, for any $t \in [0, 1]$ we have

$$m \le F(tx_1 + (1 - t)x_2) \le tF(x_1) + (1 - t)F(x_2) = m \implies F(tx_1 + (1 - t)x_2). \tag{1.33}$$

This implies that $tx_1 + (1 - t)x_2$ is a minimum for any $t \in [0, 1]$, thus proving the second point.

The same argument as above in the case of a strictly convex function yields to

$$m \le F(tx_1 + (1 - t)x_2) < m \qquad \text{if } x_1 \ne x_2. \tag{1.34}$$

This implies immediately that the minimum is unique.

Assume, finally, that F is strongly convex. Since it is strictly convex, we just need to prove the existence of a minimum. By Proposition 1.3.3 we have that F is continuos, and thus it suffices to prove its coercivity: $F(x) \to +\infty$ if $\|x\| \to +\infty$. We provide a proof of this fact in the case where F is differentiable. In this case, by Proposition 1.3.2 we have that

$$F(y) \ge F(0) + \langle \nabla F(0), y \rangle + \frac{\gamma}{2}\|y\|_2^2 \qquad \forall y \in \mathbb{R}^d. \tag{1.35}$$

Since $\langle \nabla F(0), y \rangle \le \|y\|_2$, the quadratic term on the right-hand side of the above equation, implies that the limit as $\|y\|_2 \to +\infty$ is $+\infty$. ■

Remark: Strict convexity is not enough to ensure the existence of a minimum. Consider, for example, $F(x) = e^x$.

Chapter 2

Second Chapter

2.1 Section 1

Bibliography

[1] Massimo Fornasier. “Foundations of Data Analysis”.

[2] Ralph Tyrell Rockafellar. *Convex Analysis*. Princeton Landmarks in Mathematics and Physics. Princeton: Princeton University Press, 2015. 470 pp. ISBN: 978-0-691-01586-6.