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COMPLEXITY OF CONTROL-AFFINE MOTION PLANNING*

F. JEAN† AND D. PRANDI‡

Abstract. In this paper we study the complexity of the motion planning problem for control-affine systems. Such complexities are already defined and rather well understood in the particular case of nonholonomic (or sub-Riemannian) systems. Our aim is to generalize these notions and results to systems with a drift. Accordingly, we present various definitions of complexity, as functions of the curve that is approximated, and of the precision of the approximation. Due to the lack of time-rescaling invariance of these systems, we consider geometric and parametrized curves separately. Then, we give some asymptotic estimates for these quantities. As a byproduct, we are able to treat the long time local controllability problem, giving quantitative estimates on the cost of stabilizing the system near a nonequilibrium point of the drift.

Key words. control-affine systems, sub-Riemannian geometry, motion planning, complexity

AMS subject classifications. 53C17, 93C15

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1. Introduction.

The concept of complexity was first developed for the nonholonomic motion planning problem in robotics. Given a control system on a manifold M, the motion planning problem consists in finding an admissible trajectory connecting two points, usually under further requirements such as obstacle avoidance. If a cost function is given, it makes sense to try to find the trajectory costing the least.

Different approaches are possible to solve this problem (see [23]). Here we focus on those based on the following scheme:

1. find any (usually nonadmissible) curve or path solving the problem,
2. approximate it with admissible trajectories.

The first step is independent of the control system, since it depends only on the topology of the manifold and of the obstacles, and it is already well understood (see [27]). Here, we are interested in the second step, which depends only on the local nature of the control system near the path. The goal of the paper is to understand how to measure the complexity of the approximation task. By complexity we mean a function of the nonadmissible curve $\Gamma \subset M$ (or path $\gamma : [0,T] \to M$) and of the precision of the approximation, quantifying the difficulty of the latter by means of the cost function.

1.1. Control theoretical setting.

We consider a control-affine system on a smooth manifold M of the form

$$\text{(D)} \qquad \dot{q}(t) = f_0(q(t)) + \sum_{i=1}^{m} u_i(t)\, f_i(q(t)), \qquad \text{a.e. } t \in [0,T],$$

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†ENSTA ParisTech, 91762 Palaiseau, France, and Team GECO, INRIA Saclay – Île-de-France, 91120 Palaiseau, France (frederic.jean@ensta-paristech.fr). This author's work was supported by iCODE (Institute for Control and Decision), research project of the IDEX Paris - Saclay, and by the Commission of the European Communities under the 7th Framework Programme Marie Curie Initial Training Network (FP7-PEOPLE-2010-ITN), project SADCO, contract 264735.

‡Team GECO, Centre de Recherche, INRIA Saclay – Île-de-France, 91120 Palaiseau, France, and SISSA, Via Bonomea, 265, 34136 Trieste, Italy. Current address: Laboratoire LSIS, Université de Toulon, 83957 La Garde, France (prandi@univ-tln.fr).

where $u : [0, T] \to \mathbb{R}^m$ is an integrable control function and $f_0, f_1, \ldots, f_m$ are smooth vector fields. The uncontrolled vector field f_0 is called the *drift*. These kinds of systems appear in many applications. As examples we cite mechanical systems with controls on the acceleration (see, e.g., [9, 3]), where the drift is the velocity; quantum control (see, e.g., [10, 8]), where the drift is the free Hamiltonian; or the swimming of microscopic organisms (see, e.g., [26]).

When removing f_0 in (D) we obtain the driftless control-affine system associated with (D), that is,

$$\text{(SR)} \qquad \dot{q}(t) = \sum_{i=1}^{m} u_i(t)\, f_i(q(t)), \qquad \text{a.e. } t \in [0, T].$$

Such a system is commonly referred to as a sub-Riemannian control system.

Consider system (D), and let Δ be the C^∞-module generated by the control vector fields $\{f_1, \ldots, f_m\}$ (in particular, it is closed under multiplication by $\mathrm{C}^\infty(M)$ functions and summation). Let $\Delta^0 = \{0\}$, and define recursively $\Delta^s = \Delta^{s-1} + [\Delta^{s-1}, \Delta]$ for every $s \in \mathbb{N}$. Due to the Jacobi identity, Δ^s is the C^∞-module of linear combinations of all commutators of $f_1, \ldots, f_m$ with length $\leq s$. In the rest of the paper we will always assume the following hypotheses to be satisfied.

(H1) *Equiregularity:* for any $s \in \mathbb{N}$, $\dim \Delta^s(q)$ does not depend on $q \in M$.
(H2) *Strong Hörmander condition:* there exists $r \in \mathbb{N}$ such that $\Delta^r(q) = T_qM$ for any $q \in M$.

Families of vector fields $\{f_1, \ldots, f_m\}$ satisfying these assumptions are commonly referred to as equiregular sub-Riemannian structures.

Hypothesis (H1) is made for technical reasons and to lighten the notation. It would be possible to avoid it through a desingularization procedure similar to the one in [20]. On the other hand, (H2) is essential to applying our methods. Indeed, this guarantees the small time local controllability of system (D), which is essential to being able to approximate any given curve with nearby trajectories, and of system (SR). We remark that (H2) is generically satisfied by systems with at least two controls, e.g., by finite-dimensional quantum control, such as those studied in [7, 6, 11].

Given a sub-Riemannian control system in the form (SR), a natural choice for the cost is the L^1-norm of the controls. Indeed, due to the linearity and the reversibility in time of such a system, the associated value function is in fact a distance d_{SR}, the sub-Riemannian distance, that endows M with a metric space structure. In the case of control-affine systems, we will focus on the following cost functions:

$$(1) \qquad \mathcal{J}(u, T) = \int_0^T \sqrt{\sum_{j=1}^{m} u_j(t)^2}\, dt \quad \text{and} \quad \mathcal{I}(u, T) = \int_0^T \sqrt{1 + \sum_{j=1}^{m} u_j(t)^2}\, dt.$$

Namely, $\mathcal{J}(u, T) = \|u\|_{\mathrm{L}^1([0,T],\mathbb{R}^m)}$, and $\mathcal{I}(u, T) = \|(1, u)\|_{\mathrm{L}^1([0,T],\mathbb{R}^{m+1})}$. Let $q \in M$, and define $q_u : [0, T] \to M$ as the trajectory associated with a control $u \in \mathrm{L}^1([0, T], \mathbb{R}^m)$ such that $q_u(0) = q$. The cost $\mathcal{J}$, measuring the L^1-norm of the control, quantifies the cost spent by the controller to steer the system (D) along q_u. On the other hand, the cost $\mathcal{I}$ measures the sub-Riemannian length of q_u w.r.t. the sub-Riemannian structure defined by the vector fields $f_0, f_1, \ldots, f_m$.

Fix a time $\mathcal{T} > 0$, and consider the two value functions $\mathrm{V}^{\mathcal{J}}(q, q')$ and $\mathrm{V}^{\mathcal{I}}(q, q')$ as the infima of the costs $\mathcal{J}$ and $\mathcal{I}$, respectively, over all controls u in the set $\mathcal{U}_{\mathcal{T}} = \bigcup_{0<T\leq\mathcal{T}} \mathrm{L}^1([0, T], \mathbb{R}^m)$ steering the system from q to q'. Contrary to what happens

for sub-Riemmanian control systems with the sub-Riemannian distance, these value functions are not symmetric and hence do not induce a metric space structure on M. In fact, system (D) is not reversible—i.e., changing orientation to an admissible trajectory does not yield an admissible trajectory.

Note that we consider controls defined on $T \leq \mathcal{T}$ since we are interested in the local behavior of system (D). Indeed, without an upper bound for the time of definition of the controls, the reachable sets $\mathcal{R}^{f_0}(q,\varepsilon) = \{q' \in M \mid \mathrm{V}^{\mathcal{J}}(q,q') \leq \varepsilon\}$ are in general noncompact for any $q \in M$ and $\varepsilon > 0$. As a byproduct of this choice, by taking $\mathcal{T}$ sufficiently small, it is possible to prevent any exploitation of the geometry of the orbits of the drift (that could be, for example, closed).

Let us also remark that, since controls can be defined on arbitrarily small times, it is possible to approximate admissible trajectories for system (D) via trajectories for the associated sub-Riemannian system rescaled on small intervals. Namely, given a control $u \in \mathrm{L}^1([0,T],\mathbb{R}^m)$, for any $\varepsilon > 0$ we can define $u_\varepsilon \in \mathrm{L}^1([0,\varepsilon T],\mathbb{R}^m)$ as $u_\varepsilon(t) := \varepsilon^{-1}u(\varepsilon^{-1}t)$. Observe that $\mathcal{J}(u,T) = \mathcal{J}(u_\varepsilon,\varepsilon T)$ and $\mathcal{I}(u,T) \geq \mathcal{I}(u_\varepsilon,\varepsilon T)$. Moreover, as $\varepsilon \downarrow 0$, the associated trajectories q_ε converge to the trajectory of u in system (SR). From this it is easy to prove that existence of minimizers holds for neither $\mathcal{J}$ nor $\mathcal{I}$ (see [24, Example 4.2]). Nevertheless, the value functions associated with these costs turn out to be continuous [24] and are thus more appropriate for our purposes than, e.g., quadratic costs which although admitting minimizers do not define continuous value functions in general [28].

1.2. Complexities. Heuristically, the complexity of a curve Γ (or path $\gamma : [0,T] \to M$) at precision ε is defined as the ratio

$$\frac{\text{“cost” to track } \Gamma \text{ at precision } \varepsilon}{\text{“cost” of an elementary } \varepsilon\text{-piece}}. \tag{2}$$

In order to obtain a precise definition of complexity, we need to give meaning to the notions appearing above. Namely, we have to specify what we mean by “cost,”[1] tracking at precision ε, and elementary ε-piece. Indeed, these choices will depend on the type of motion planning problem at hand.

First, we classify motion planning problems as *time-dependent* or *static*, depending on whether the constraints depend on time or not. The typical example of a static motion planning problem is the obstacle avoidance problem with fixed obstacles. On the other hand, the same problem where the position of the obstacles depends on time and rendezvous problems are examples of time-dependent motion planning problems.

For static motion planning problems, the solution of the first step of the motion planning scheme (introduced at the beginning of the paper) is usually given as a curve, i.e., a one-dimensional connected submanifold of $\Gamma \subset M$ diffeomorphic to a closed interval. On the other hand, in time-dependent problems we have to keep track of the time. Thus, for this type of problem, the solution of the first step is a path, i.e., a smooth function $\gamma : [0,T] \to M$. As a consequence, when computing the complexity of paths we will require the approximating trajectories to respect also the parametrization of the path, not only its geometry. While in the sub-Riemannian case, due to the time-rescaling properties of the control system, these concepts coincide, this is not the case for control-affine systems.

In this paper, we consider four distinct notions of complexity: two for curves (static problems) and two for paths (time-dependent problems). In both cases, one

[1] The cost appearing in (2) is not necessarily related to the cost function ($\mathcal{J}$ or $\mathcal{I}$) that is taken into account. This is the reason for the quotation marks.

of the two will be based on the interpolation of the given curve or path, while the other will consider trajectories that stay near the curve or path w.r.t. some fixed distance. In this work we will consider only the sub-Riemannian distance of the associated sub-Riemannian control system (SR), and we will denote by $\mathrm{B}_{\mathrm{SR}}(q,\varepsilon)$ the balls centered at $q \in M$ and of radius $\varepsilon > 0$ w.r.t. this distance. Moreover, we denote with $\mathrm{Tube}(\Gamma,\varepsilon)$ the tubular neighborhood of radius ε around the curve Γ, i.e., $\mathrm{Tube}(\Gamma,\varepsilon) = \bigcup_{p\in\Gamma} \mathrm{B}_{\mathrm{SR}}(q,\varepsilon)$.

The two complexities for curves that we consider have been introduced in the sub-Riemannian context in [18, 19]. This is also true for what we call the neighboring approximation complexity of a path, since in the sub-Riemannian case it coincides with the tubular approximation complexity. On the other hand, what we call the interpolation by time complexity has never appeared in the literature, to the best of our knowledge. Here we give the definitions for a generic cost $J : \mathcal{U}_{\mathcal{T}} \to [0,+\infty)$ ($J = \mathcal{J}$ or $\mathcal{I}$).

Fix a curve Γ and, for any $\varepsilon > 0$, define the following complexities for Γ.

- *Interpolation by cost complexity* Σ_{int} (see Figure 1). For $\varepsilon > 0$, we call *ε-cost interpolation* of Γ any control $u \in \mathcal{U}_{\mathcal{T}}$ such that there exist $0 = t_0 < t_1 < \cdots < t_N = T \le \mathcal{T}$ for which the trajectory q_u with initial condition $q_u(0) = x$ satisfies $q_u(T) = y$, $q_u(t_i) \in \Gamma$ and $J(u|_{[t_{i-1},t_i)}, t_i - t_{i-1}) \le \varepsilon$ for any $i = 1,\dots,N$. Then, we set
$$\Sigma_{\mathrm{int}}(\Gamma,\varepsilon) = \frac{1}{\varepsilon} \inf \big\{ J(u,T) \mid u \text{ is an } \varepsilon\text{-cost interpolation of } \Gamma \big\}.$$
This function measures the number of admissible curves of cost ε necessary to interpolate Γ. Namely, following a trajectory given by a control admissible for $\Sigma_{\mathrm{int}}(\Gamma,\varepsilon)$, at any given moment it is possible to go back to Γ with a cost less than ε.
- *Tubular approximation complexity* Σ_{app} (see Figure 2). Set
$$\Sigma_{\mathrm{app}}(\Gamma,\varepsilon) = \frac{1}{\varepsilon} \inf \left\{ J(u,T) \;\middle|\; \begin{array}{l} 0 < T \le \mathcal{T}, \\ q_u(0) = x,\ q_u(T) = y, \\ q_u\big([0,T]\big) \subset \mathrm{Tube}(\Gamma,\varepsilon) \end{array} \right\}.$$
This complexity measures the number of pieces of admissible curve of cost ε necessary to go from x to y without leaving the sub-Riemannian tube $\mathrm{Tube}(\Gamma,\varepsilon)$. Such a property is especially useful for motion planning with obstacle avoidance. In fact, if the sub-Riemannian distance of Γ from the obstacles is at least $\varepsilon_0 > 0$, then trajectories obtained from controls admissible for $\Sigma_{\mathrm{app}}(\Gamma,\varepsilon)$, $\varepsilon < \varepsilon_0$, will avoid such obstacles.

We then define the following complexities for a path $\gamma : [0,T] \to M$ at precision $\varepsilon > 0$.

- *Interpolation by time complexity* σ_{int} (see Figure 3). Define a δ-time interpolation of γ to be any control $u \in \mathrm{L}^1([0,T],\mathbb{R}^m)$ such that its trajectory $q_u : [0,T] \to M$ in (D) with $q_u(0) = \gamma(0)$ is such that $q_u(T) = \gamma(T)$ and that, for any interval $[t_0,t_1] \subset [0,T]$ of length $t_1 - t_0 \le \delta$, there exists $t \in [t_0,t_1]$ with $q_u(t) = \gamma(t)$. Then, fix a $\delta_0 > 0$ and let
$$\sigma_{\mathrm{int}}(\gamma,\varepsilon) = \inf \left\{ \frac{T}{\delta} \;\middle|\; \begin{array}{c} \delta \in (0,\delta_0) \text{ and there exists } u \in \mathrm{L}^1([0,T],\mathbb{R}^m), \\ \delta\text{-time interpolation of } \gamma \text{ s.t. } \frac{\delta}{T} J(u,T) \le \varepsilon \end{array} \right\}.$$
Here, to lighten the notation, we omit the dependence on δ_0, in the same way as we omit the dependence on $\mathcal{T}$. Controls admissible for this complexity

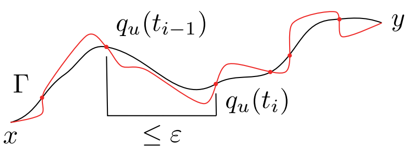

FIG. 1. *Interpolation by cost complexity.*

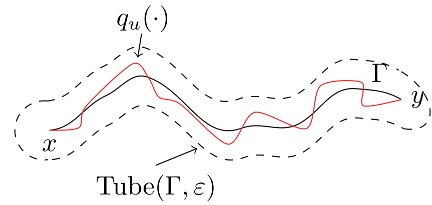

FIG. 2. *Tubular approximation complexity.*

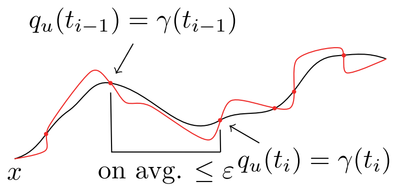

FIG. 3. *Time interpolation complexity.*

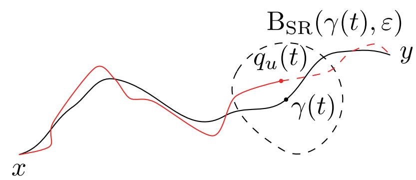

FIG. 4. *Neighboring approximation complexity.*

define trajectories such that the minimal average cost between any two consecutive times such that $\gamma(t) = q_u(t)$ is less than ε (see Remark 1.1 below). It is thus well suited for time-dependent applications where one is interested in minimizing the number of interpolation points, e.g., motion planning in rendezvous problems.

- *Neighboring approximation complexity* σ_{app} (see Figure 4). Set

$$\sigma_{\mathrm{app}}(\gamma, \varepsilon) = \frac{1}{\varepsilon} \inf \left\{ J(u,T) \,\middle|\, \begin{array}{l} q_u(0) = x,\ q_u(T) = y, \\ q_u(t) \in \mathrm{B}_{\mathrm{SR}}(\gamma(t), \varepsilon)\ \forall t \in [0,T] \end{array} \right\}.$$

This complexity measures the number of pieces of admissible curve of cost ε necessary to go from x to y following a trajectory that at each instant $t \in [0,T]$ remains inside the sub-Riemannian ball $\mathrm{B}_{\mathrm{SR}}(\gamma(t), \varepsilon)$. Such complexity can be applied to motion planning in rendezvous problems where it is sufficient to attain the rendezvous only approximately, or to motion planning with obstacle avoidance where the obstacles are moving.

Whenever we will need to specify w.r.t. which cost a complexity is measured, we will write the cost function as an apex—e.g., we will denote the interpolation by cost complexity w.r.t. $\mathcal{J}$ as $\Sigma_{\mathrm{int}}^{\mathcal{J}}$.

Observe that for the interpolation by time complexity the "cost" in (2) is the time, while for all other complexities it is the cost function associated with the system.

Remark 1.1. In order to better understand the definition of the interpolation by time complexity, let us introduce the following function:

$$\omega(\gamma, \delta) = \delta \inf \big\{ J(u,T) \big|\, u \text{ is a } \delta\text{-time interpolation of } \gamma \big\}.$$

Controls admissible for the above infimum define trajectories touching γ at intervals of time of length at most δ, so that the function $\omega(\gamma, \delta)$ measures the minimal average cost of these intervals. Then, the interpolation by time complexity can be expressed

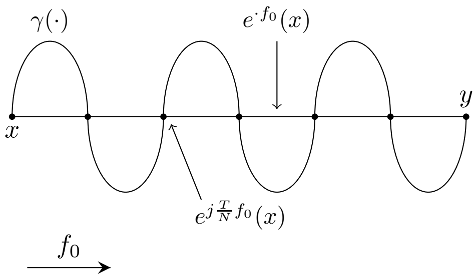

FIG. 5. *An example of a curve for which, when δ_0 is sufficiently big and the drift is rectified, we can uniformly bound from above the interpolation by time complexity.*

as

$$\sigma_{\mathrm{int}}(\gamma,\varepsilon) = \inf_{\delta\leq\delta_0}\left\{\frac{T}{\delta}\,\middle|\,\omega(\gamma,\delta)\leq\varepsilon\right\} = \sup_{\delta\leq\delta_0}\left\{\frac{T}{\delta}\,\middle|\,\omega(\gamma,\delta')\geq\varepsilon \text{ for any } \delta'\geq\delta\right\}, \tag{3}$$

that is, the maximal (resp., minimal) number of interpolating pieces such that their average cost is smaller (resp., bigger) than ε.

We also stress that the only purpose of the bound δ_0 is to prevent our interpolating curves from taking advantage of particular configurations of the path w.r.t. the drift. For example, consider the cost $\mathcal{J}(u,T) = \|u\|_{\mathrm{L}^1([0,T],\mathbb{R}^m)}$ and a curve such that, for some $N\in\mathbb{N}$, it holds that $\gamma(jT/N) = e^{j(T/N)f_0}(\gamma(0))$ for any $j=1,\dots,T/N$ (see, e.g., Figure 5). In this case, the null control is a (T/N)-time interpolation of γ, with $\mathcal{J}(0,T)=0$. In particular, if $\delta_0 > T/N$, it holds that $\sigma_{\mathrm{int}}(\gamma,\varepsilon)\leq N$.

In this paper we are mostly interested in the asymptotic behavior of the above-defined complexities. As we will see later, this justifies the omission of the dependence on $\mathcal{T}$ and δ_0 in the notation for the complexities. Indeed, whenever $\mathcal{T}$ and δ_0 are sufficiently small, the asymptotic behavior of the complexities is independent of them.

Two functions $f(\varepsilon)$ and $g(\varepsilon)$ are *weakly equivalent* for $\varepsilon\downarrow 0$ if both $f(\varepsilon)/g(\varepsilon)$ and $g(\varepsilon)/f(\varepsilon)$ are bounded when $\varepsilon\downarrow 0$. In this case we will write $f(\varepsilon)\asymp g(\varepsilon)$. When $f(\varepsilon)/g(\varepsilon)$ is bounded, we will write $f(\varepsilon)\preccurlyeq g(\varepsilon)$. In the sub-Riemannian context, the complexities are always measured w.r.t. the L^1-cost of the control, $\mathcal{J}$. Then, for any curve $\Gamma\subset M$ and path $\gamma:[0,T]\to M$ such that $\gamma([0,T])=\Gamma$, it holds that $\Sigma^{\mathcal{J}}_{\mathrm{int}}(\Gamma,\varepsilon)\asymp\Sigma^{\mathcal{J}}_{\mathrm{app}}(\Gamma,\varepsilon)\asymp\sigma^{\mathcal{J}}_{\mathrm{app}}(\gamma,\varepsilon)$.

A complete characterization of weak asymptotic equivalence of sub-Riemannian complexities is obtained in [21]. Here we state such characterization in the special case where $\{f_1,\dots,f_m\}$ defines an equiregular structure.

THEOREM 1.2. *Assume that $\{f_1,\dots,f_m\}$ defines an equiregular sub-Riemannian structure. Let $\Gamma\subset M$ be a curve and $\gamma:[0,T]\to M$ be a path such that $\gamma([0,T])=\Gamma$. Then, if there exists an integer $k\geq 1$ such that $T_q\Gamma\subset\Delta^k(q)\setminus\Delta^{k-1}(q)$ for any $q\in\Gamma$, it holds that*

$$\Sigma_{\mathrm{int}}(\Gamma,\varepsilon)\asymp\Sigma_{\mathrm{app}}(\Gamma,\varepsilon)\asymp\sigma_{\mathrm{app}}(\gamma,\varepsilon)\asymp\frac{1}{\varepsilon^k}.$$

Here the complexities are measured w.r.t. the cost $\mathcal{J}(u,T)=\|u\|_{L^1([0,T],\mathbb{R}^m)}$.

We mention also that for a restricted set of sub-Riemannian systems, i.e., one-step bracket generating or with two controls and dimension not larger than 6, strong asymptotic estimates and explicit asymptotic optimal syntheses are obtained in the series of papers [25, 13, 14, 15, 16, 17, 12] (see [5] for a review).

1.3. Main results. Our first result regards the sub-Riemannian weak asymptotic estimates for the interpolation by time complexity, completing the description started by Theorem 1.2. It is proved in section 4.

THEOREM 1.3. *Assume that $\{f_1,\dots,f_m\}$ defines an equiregular sub-Riemannian structure, and let $\gamma:[0,T]\to M$ be a path. Then, if there exists an integer $k\geq 1$ such that $\dot\gamma(t)\in\Delta^k(\gamma(t))\setminus\Delta^{k-1}(\gamma(t))$ for any $t\in[0,T]$, it holds that*

$$\sigma_{\mathrm{int}}(\gamma,\varepsilon)\asymp\frac{1}{\varepsilon^k}.$$

Here the complexity is measured w.r.t. the cost $\mathcal{J}(u,T)=\|u\|_{L^1([0,T],\mathbb{R}^m)}$.

The main result of the paper is then a weak asymptotic equivalence of the above defined complexities in control-affine systems, generalizing Theorems 1.2 and 1.3.

THEOREM 1.4. *Assume that $\{f_1,\dots,f_m\}$ defines an equiregular sub-Riemannian structure and that $f_0\subset\Delta^s\setminus\Delta^{s-1}$ for some $s\geq 2$. Also, assume that the complexities are measured w.r.t. the cost function $\mathcal{J}(u,T)=\|u\|_{L^1([0,T],\mathbb{R}^m)}$ or $\mathcal{I}(u,T)=\|(1,u)\|_{L^1([0,T],\mathbb{R}^{m+1})}$. We then have the following.*

(i) *Let $\Gamma\subset M$ be a curve, and define $\kappa=\max\{k\colon\ T_p\Gamma\in\Delta^k(p)\setminus\Delta^{k-1}(p)$ for some $p\in\Gamma\}$. Then, whenever the maximal time of definition of the controls $\mathcal{T}$ is sufficiently small, it holds that*

$$\Sigma_{\mathrm{int}}(\Gamma,\varepsilon)\asymp\Sigma_{\mathrm{app}}(\Gamma,\varepsilon)\asymp\frac{1}{\varepsilon^\kappa}.$$

(ii) *On the other hand, let $\gamma:[0,T]\to M$ be a path such that $f_0(\gamma(t))\neq\dot\gamma(t)\mod\Delta^{s-1}(\gamma(t))$ for any $t\in[0,T]$, and define $\kappa=\max\{k\colon\ \gamma(t)\in\Delta^k(\gamma(t))\setminus\Delta^{k-1}(\gamma(t))$ for some $t\in[0,T]\}$. Then, it holds that*

$$\sigma_{\mathrm{int}}(\gamma,\varepsilon)\asymp\sigma_{\mathrm{app}}(\gamma,\varepsilon)\asymp\frac{1}{\varepsilon^{\max\{\kappa,s\}}},$$

where the first asymptotic equivalence is true only when δ_0, i.e., the maximal time-step in $\sigma_{\mathrm{int}}(\gamma,\varepsilon)$, is sufficiently small.

We point out that in (ii) we use the convention $\Delta^{-1}=\varnothing$. Thus the theorem also contains the case $\kappa=0$ which corresponds to $\dot\gamma\equiv 0$ (see section 1.4). On the other hand, in (i) it is clear that $k\geq 1$ since Γ is a one-dimensional submanifold of M.

This theorem shows that, asymptotically, the complexity of curves is not influenced by the drift and depends only on the underlying sub-Riemannian system, while that of paths depends also on how "bad" the drift is w.r.t. this system. Observe that for the neighboring approximation complexity, σ_{app}, it is not necessary to have an a priori bound on $\mathcal{T}$.

We remark that the approximation complexities depend on the choice of the sub-Riemannian distance, which in turn depends on the choice of the control vector fields $f_1,\dots,f_m$ and not only on the choice of the module $\Delta\subset TM$. However, Theorem 1.4 shows that at the level of the asymptotic equivalence only the relation between the curve or path and the flag $\Delta\subset\cdots\subset\Delta^r$ is important. Thus, weak asymptotics of the complexities are invariant by feedback equivalence.

The proof of Theorem 1.4 is quite technical, but heuristically it relies on the following observations. In the case of curves we are allowed to concentrate the controls on small time intervals, thus being able to approximate the associated sub-Riemannian system, which essentially allows us to reduce the $\preccurlyeq$ part to Theorem 1.2. The hard part of the proof is then to show that asymptotically this is always the best strategy, i.e., that the drift never does help. In particular, this explains why the maximal time $\mathcal{T}$ does not play any role in the asymptotics.[2] On the other hand, in the case of paths we are obliged to follow the parametrization and hence to fight the drift, since our path is never an integral curve of the drift. Proving this, however, requires fine estimates for both of the inequalities, which are new even in the sub-Riemannian case. In any case, our arguments heavily rely on the results from [24] recalled in section 2.3.

1.4. Long time local controllability. As an application of the above theorem, let us briefly mention the problem of *long time local controllability* (henceforth simply *LTLC*), i.e., the problem of staying near some point for a long period of time $T > 0$. This is essentially a stabilization problem around a nonequilibrium point.

Since the system (D) satisfies the strong Hörmander condition, it is always possible to satisfy some form of LTLC. Hence, it makes sense to quantify the minimal cost needed by posing the following. Let $T > 0$, $q_0 \in M$, and $\gamma_{q_0} : [0, T] \to M$, $\gamma_{q_0}(\cdot) \equiv q_0$.

- *LTLC complexity by time:*

$$\mathrm{LTLC}_{\mathrm{time}}(q_0, T, \varepsilon) = \sigma_{\mathrm{int}}(\gamma_{q_0}, \varepsilon).$$

  Here, we require trajectories defined by admissible controls to pass through q_0 at intervals of time such that the minimal average cost between each passage is less than ε.

- *LTLC complexity by cost:*

$$\mathrm{LTLC}_{\mathrm{cost}}(q_0, T, \varepsilon) = \sigma_{\mathrm{app}}(\gamma_{q_0}, \varepsilon).$$

  Admissible controls for this complexity will always be contained in the sub-Riemannian ball of radius ε centered at q_0.

Clearly, if $f_0(q_0) = 0$, then $\mathrm{LTLC}_{\mathrm{time}}(q_0, T, \varepsilon) = \mathrm{LTLC}_{\mathrm{cost}}(q_0, T, \varepsilon) = 0$ for any $\varepsilon, T > 0$. Hence, as a corollary of Theorem 1.4 we get the following asymptotic estimate for the LTLC complexities.

COROLLARY 1.5. *Assume that* $\{f_1, \ldots, f_m\}$ *defines an equiregular sub-Riemannian structure and that* $f_0 \subset \Delta^s \setminus \Delta^{s-1}$ *for some* $s \geq 2$. *Also, assume that the complexities are measured w.r.t. the cost function* $\mathcal{J}(u, T) = \|u\|_{L^1([0,T],\mathbb{R}^m)}$ *or* $\mathcal{I}(u, T) = \|(1, u)\|_{L^1([0,T],\mathbb{R}^{m+1})}$. *Then, for any* $q_0 \in M$ *and* $T > 0$ *it holds that*

$$\mathrm{LTLC}_{\mathrm{time}}(q_0, T, \varepsilon) \asymp \mathrm{LTLC}_{\mathrm{cost}}(q_0, T, \varepsilon) \asymp \frac{1}{\varepsilon^s}.$$

1.5. Structure of the paper. In section 2 we introduce all the preliminaries needed in the proofs of Theorems 1.2 and 1.4: in section 2.1 we recall some important notions of sub-Riemannian geometry, in section 2.2 we present some technical results regarding families of coordinates depending continuously on the base point, and in section 2.3 we have collected some useful properties of the costs $\mathcal{J}$ and $\mathcal{I}$, proved mainly in [24]. The proof of the main results is then contained in sections 3 and 4 for curves and paths, respectively. Finally, in the appendix we have collected some results regarding the behavior of the complexities, which are interesting but not needed in the proofs of the main theorems.

[2] As already remarked, we have to assume $\mathcal{T}$ to be sufficiently small in order to be able to consider only the local behavior of the drift and not its global structure.

2. Preliminaries. Throughout this paper, M is an n-dimensional connected smooth manifold. Let f_0 and $\{f_1,\dots,f_m\}$ be smooth vector fields on M, and, for some $\mathcal{T} > 0$, define $\mathcal{U}_\mathcal{T} = \bigcup_{0<T<\mathcal{T}} \mathrm{L}^1([0,T],\mathbb{R}^m)$. We consider the control-affine control system

$$\text{(D)} \qquad \dot q(t) = f_0(q(t)) + \sum_{j=1}^m u_j\, f_j(q(t)), \quad u \in \mathcal{U}_\mathcal{T}.$$

An absolutely continuous curve $\gamma : [0,T] \to M$ is a trajectory of (D) if there exists some control $u \in \mathrm{L}^1([0,T],\mathbb{R}^m)$ such that γ solves (D) for a.e. $t \in [0,T]$. Observe that if $f_0 \neq 0$, the admissibility for (D) is not invariant under time reparametrization, e.g., a time reversal. In particular, the value function associated with the costs (1) (or to any "reasonable" cost), being nonsymmetric and not satisfying the triangle inequality, does not define a distance.

2.1. Sub-Riemannian control systems. The sub-Riemannian (or nonholonomic) control system associated with the control-affine system (D) is the control-linear system obtained by putting $f_0 = 0$. Namely, it is a control system in the form

$$\text{(SR)} \qquad \dot q(t) = \sum_{i=1}^m u_i(t)\, f_i(q(t)), \qquad \text{a.e. } t \in [0,T].$$

We let $f_u = \sum_{i=1}^m u_i\, f_i$. We now recall some basic notions on sub-Riemannian control systems. For more details, we refer the reader to [1, 4, 22].

The value function $\mathrm{d_{SR}}$ of system (SR) associated with the L^1-cost is a distance, called the *sub-Riemannian distance*. Namely, for any $q, q' \in M$,

$$\mathrm{d_{SR}}(q,q') = \inf \int_0^T \sqrt{\sum_{j=1}^m u_j(t)^2}\, dt,$$

where the infimum is taken among all the controls $u \in \mathrm{L}^1([0,T],\mathbb{R}^m)$, for some $T > 0$, such that its trajectory in (SR) is such that $q_u(0) = q$ and $q_u(T) = q'$.

By the Chow–Rashevsky theorem, the hypothesis of connectedness of M and the Hörmander condition guarantee the finiteness and continuity of $\mathrm{d_{SR}}$ w.r.t. the topology of M. Hence, the sub-Riemannian distance induces on M a metric space structure. The open balls centered at $q \in M$ and of radius $\varepsilon > 0$ w.r.t. $\mathrm{d_{SR}}$ are denoted by $\mathrm{B_{SR}}(q,\varepsilon)$.

We say that a control $u \in \mathrm{L}^1([0,T],\mathbb{R}^m)$, $T > 0$, is a minimizer of the sub-Riemannian distance between $q, q' \in M$ if the associated trajectory q_u with $q_u(0) = q$ is such that $q_u(T) = q'$ and $\|u\|_{\mathrm{L}^1([0,T],\mathbb{R}^m)} = \mathrm{d_{SR}}(q,q')$. Equivalently, u is a minimizer between $q, q' \in M$ if it is a solution of the free-time optimal control problem, associated with (SR),

$$\text{(4)} \quad \|u\|_{\mathrm{L}^1(0,T)} = \int_0^T \sqrt{\sum_{j=1}^m u_j^2(t)}\, dt \to \min, \qquad q_u(0) = q, \quad q_u(T) = q', \quad T > 0.$$

It is a classical result that, for any couple of points $q, q' \in M$ sufficiently close, there exists at least one minimizer.

By the Hörmander condition, the values of the sets Δ^s at q form a flag of subspaces of T_qM,

$$\Delta^1(q) \subset \Delta^2(q) \subset \cdots \subset \Delta^r(q) = T_qM.$$

The integer r, which is the minimum number of brackets required to recover the whole T_qM, is called the *degree of nonholonomy* (or *step*) of the family $\{f_1, \ldots, f_m\}$ at q. The degree of nonholonomy is independent of q since we assume the family $\{f_1, \ldots, f_m\}$ to define an equiregular sub-Riemannian structure. Let $n_s = \dim \Delta^s(q)$ for any $q \in M$. The integer list $(n_1, \ldots, n_r)$ is called the *growth vector* associated with (SR). Finally, let $w_1 \leq \cdots \leq w_n$ be the *weights* associated with the flag, defined by $w_i = s$ if $n_{s-1} < i \leq n_s$, setting $n_0 = 0$.

We now introduce the notion of privileged coordinates, which is essential in obtaining estimates on the anisotropic local behavior of the sub-Riemannian distance. For any smooth vector field f, we denote its action, as a derivation on smooth functions, by $f : a \in \mathrm{C}^\infty(M) \mapsto fa \in \mathrm{C}^\infty(M)$. For any smooth function a and every vector field f with $f \not\equiv 0$ near q, their *(nonholonomic) order* at q is

$$\begin{aligned}\mathrm{ord}_q(a) &= \min\{s \in \mathbb{N}\colon\ \exists i_1, \ldots, i_s \in \{1, \ldots, m\} \text{ s.t. } (f_{i_1} \ldots f_{i_s}\, a)(q) \neq 0\},\\ \mathrm{ord}_q(f) &= \max\{\sigma \in \mathbb{Z}\colon\ \mathrm{ord}_q(fa) \geq \sigma + \mathrm{ord}_q(a) \text{ for any } a \in \mathrm{C}^\infty(M)\}.\end{aligned}$$

In particular it can be proved that $\mathrm{ord}_q(a) \geq s$ if and only if $a(q') = \mathcal{O}(\mathrm{d_{SR}}(q', q)^s)$.

DEFINITION 2.1. *A system of privileged coordinates at q for $\{f_1, \ldots, f_m\}$ is a system of local coordinates $z = (z_1, \ldots, z_n)$ centered at q and such that* $\mathrm{ord}_q(z_i) = w_i$, *$1 \leq i \leq n$.*

Let $q \in M$. A set of vector fields $\{f_1, \ldots, f_n\}$ such that

$$\{f_1(q), \ldots, f_n(q)\} \text{ is a basis of } T_qM \quad \text{and} \quad f_i \in \Delta^{w_i} \text{ for } i = 1, \ldots, n \tag{5}$$

is called an adapted frame at q. We remark that with any system of privileged coordinates z at q is associated a (nonunique) adapted frame at q such that $\partial_{z_i} = z_* f_i(q)$ (i.e., privileged coordinates are always linearly adapted to the flag). Here we denote by $z_* : TM \to T\mathbb{R}^n$ the pull-back associated with coordinates z.

For any ordering $\{i_1, \ldots, i_n\}$, the inverse of each of the local diffeomorphisms

$$(z_1, \ldots, z_n) \mapsto e^{z_{i_1} f_{i_1} + \cdots + z_{i_n} f_{i_n}}(q), \qquad (z_1, \ldots, z_n) \mapsto e^{z_{i_n} f_{i_n}} \circ \cdots \circ e^{z_{i_1} f_{i_1}}(q)$$

defines privileged coordinates at q, called *canonical coordinates of the first kind* and of the *second kind*, respectively. We remark that, for the canonical coordinates of the second kind, it holds that $z_* f_{i_n}(z) \equiv \partial_{z_{i_n}}$.

We recall the celebrated ball-box theorem, which gives a rough description of the shape of small sub-Riemannian balls.

THEOREM 2.2 (ball-box theorem). *Let $z = (z_1, \ldots, z_n)$ be a system of privileged coordinates at $q \in M$ for $\{f_1, \ldots, f_m\}$. Then there exist $C, \varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, it holds that*

$$\mathrm{Box}\left(\frac{\varepsilon}{C}\right) \subset \mathrm{B_{SR}}(q, \varepsilon) \subset \mathrm{Box}\,(C\varepsilon)\,,$$

where $\mathrm{B_{SR}}(q, \varepsilon)$ is identified with its coordinate representation $z(\mathrm{B_{SR}}(q, \varepsilon))$ and, for any $\eta > 0$, we let

$$\mathrm{Box}\,(\eta) = \{z \in \mathbb{R}^n\colon\ |z_i| \leq \eta^{w_i}\}. \tag{6}$$

Remark 2.3. Let $N \subset M$ be compact, and let $\{z^q\}_{q\in N}$ be a family of systems of privileged coordinates at q depending continuously on q. Then there exist uniform constants $C, \varepsilon_0 > 0$ such that the ball-box theorem holds for any $q \in N$ in the system z^q (see [20]).

2.2. Continuous families of coordinates. In this section we consider properties of families of coordinates depending continuously on the points of the curve or path, in order to be able to exploit Remark 2.3.

From the definition of privileged coordinates, we immediately get the following.

PROPOSITION 2.4. *Let $\gamma : [0,T] \to M$ be a path. Let $t > 0$, and let z be a system of privileged coordinates at $\gamma(t)$ for $\{f_1, \dots, f_m\}$. Then, there exists $C > 0$ such that*

$$|z_j(\gamma(t+\xi))| \le C|\xi| \quad \text{for any } j = 1, \dots, n \text{ and any } t+\xi \in [0,T]. \tag{7}$$

Moreover, if for $k \in \mathbb{N}$ it holds that $\dot\gamma(t) \notin \Delta^{k-1}(\gamma(t))$, then there exist $C_1, C_2, \xi_0 > 0$ and a coordinate z_α, of weight $\ge k$, such that for any $t \in [0,T]$ and any $|\xi| \le \xi_0$ with $t + \xi \in [0,T]$ it holds that

$$C_1\xi \le z_\alpha(\gamma(t+\xi)) \le C_2\xi. \tag{8}$$

Finally, if $\dot\gamma(t) \in \Delta^k(\gamma(t)) \setminus \Delta^{k-1}(\gamma(t))$, the coordinate z_α can be chosen to be of weight k.

Proof. By the smoothness of γ, there exists a constant $C > 0$ such that $|(z_j)_*\dot\gamma(t+\xi)| \le C$ for any $j = 1, \dots, n$ and any $t+\xi \in [0,T]$. Thus, we obtain

$$|z_j(\gamma(t+\xi))| \le \left| \int_t^{t+\xi} |(z_j)_*\dot\gamma(t+\eta)|\, d\eta \right| \le C\,|\xi|.$$

Let us prove (8). Let $\{f_1, \dots, f_n\}$ be an adapted basis associated with the system of coordinates z. In particular it holds that $z_* f_i(\gamma(t)) = \partial_{z_i}$. Moreover, let $k' \ge k$ be such that $\dot\gamma(t) \in \Delta^{k'}(\gamma(t)) \setminus \Delta^{k'-1}(\gamma(t))$, and write $\dot\gamma(t) = \sum_{w_i \le k'} a_i(t) f_i(\gamma(t))$ for some $a_i \in \mathrm{C}^\infty([0,T])$. Hence

$$z_*\dot\gamma(t) = \sum_{w_i \le k'} a_i(t)\, z_* f_i(\gamma(t)) = \sum_{w_i \le k'} a_i(t)\, \partial_{z_i}.$$

Since there exists i with $w_i = k'$ such that $a_i(t) \ne 0$, this implies that $(z_i)_*\dot\gamma(t) \ne 0$. Since $k' \ge k$, we have then proved (7). ∎

As already observed in Remark 2.3, in order to apply the estimates of Theorem 2.2 uniformly on γ we need to consider a continuous family of coordinates $\{z^t\}_{t\in[0,T]}$ such that each z^t is privileged at $\gamma(t)$ for $\{f_1, \dots, f_m\}$. We will call such a family a *continuous coordinate family for γ*.

Let us remark that, for any basis $\{f_1, \dots, f_n\}$ adapted to the flag in a neighborhood of $\gamma([0,T])$, the inverse z^t of the diffeomorphism

$$(z_1, \dots, z_n) \mapsto e^{z_1 f_1} \circ \dots \circ e^{z_n f_n}(\gamma(t)) \tag{9}$$

defines a continuous coordinate family for γ.

The following proposition gives more precise estimates w.r.t. those in Proposition 2.4.

PROPOSITION 2.5. *Let $\gamma : [0,T] \to M$ be a path, and let $k \in \mathbb{N}$ such that $\dot\gamma(s) \in \Delta^k(\gamma(s))$ for any $t \in [0,T]$. Then, for any continuous coordinate family*

$\{z^t\}_{t\in[0,T]}$ for γ there exist constants $C, \xi_0 > 0$ such that for any $t \in [0,T]$ and $0 \le \xi \le \xi_0$ it holds that

$$|z^t_j(\gamma(t+\xi))| \le C\xi \text{ if } w_j \le k \quad \text{and} \quad |z^t_j(\gamma(t+\xi))| \le C\xi^{\frac{w_j}{k}} \text{ if } w_j > k. \tag{10}$$

Proof. Fix $t \in [0,T]$, and let $\{f_1, \dots, f_n\}$ be an adapted basis associated with the privileged coordinate system z^t. To lighten the notation, we do not explicitly write the dependence on time of such a basis. Writing $z^t_* f_i(z) = \sum_{j=1}^n f_i^j(z)\partial_{z^t_j}$, it holds that f_i^j is of weighted order $\ge w_j - w_i$, and hence there exists a constant $C > 0$ such that

$$|f_i^j(z)| \le C\|z\|^{(w_j-w_i)^+}. \tag{11}$$

Here $\|z\|$ is the pseudonorm $|z_1|^{\frac{1}{w_1}} + \cdots + |z_n|^{\frac{1}{w_n}}$, and $h^+ = \max\{0,h\}$ for any $h \in \mathbb{R}$. Due to the compactness of $[0,T]$, the constant C can be chosen to be uniform w.r.t. the time.

Since $\dot\gamma(\xi) \in \Delta^k(\gamma(\xi))$ for $\xi > 0$, there exist functions $a_i \in \mathrm{C}^\infty([0,T])$ such that

$$\dot\gamma(\xi) = \sum_{w_i \le k} a_i(\xi) f_i(\gamma(\xi)) \quad \text{for any } \xi \in [0,T]. \tag{12}$$

Observe that, for any $t \in [0,T]$, it holds that

$$\frac{1}{\xi}\int_t^{t+\xi} |a_i(\eta)|\, d\eta = |a_i(t)| + \mathcal{O}(\xi) \quad \text{as } \xi \downarrow 0, \tag{13}$$

where $\mathcal{O}(\xi)$ is uniform w.r.t. t. In particular, for any ξ sufficiently small, this integral is bounded.

By (12), for any $t \in [0,T]$ we get

$$z^t_j(\gamma(t+\xi)) = \sum_{w_i\le k}\int_t^{t+\xi} a_i(\eta) f_i^j(z^t(\gamma(\eta)))\, d\eta \quad \text{for any } t+\xi \in [0,T]. \tag{14}$$

Then, applying (11) we obtain

$$\begin{aligned} \max_{\rho\in[0,\xi]} |z^t_j(\gamma(t+\rho))| &\le \sum_{w_i \le k} \int_t^{t+\xi} |a_i(\eta)|\, |f_i^j(z^t(\gamma(\eta)))|\, d\eta \\ &\le C\left(\max_{\rho\in[0,\xi]} \|z^t(\gamma(t+\rho))\|\right)^{(w_j-k)^+} \sum_{w_i\le k}\int_t^{t+\xi} |a_i(\eta)|\, d\eta. \end{aligned} \tag{15}$$

Consider the previous equation evaluated at ξ^k, and observe that $\max_{\rho\in[0,\xi^k]} |z^t_j(\gamma(t+\rho))| = \max_{\rho\in[0,\xi]} |z^t_j(\gamma(t+\rho^k))|$. Then, up to enlarging the constant C, (15) and (13) yield

$$\begin{aligned} &\frac{\max_{\rho\in[0,\xi]} |z^t_j(\gamma(t+\rho^k))|}{\xi^{w_j}} \\ &\le C\left(\frac{\max_{\rho\in[0,\xi]} \|z^t(\gamma(t+\varrho^k))\|}{\xi}\right)^{(w_j-k)^+} \sum_{w_i\le k}\frac{1}{\xi^k}\int_t^{t+\xi^k} |a_i(\eta)|\, d\eta \\ &\le C\left(\frac{\max_{\rho\in[0,\xi]} \|z^t(\gamma(t+\varrho^k))\|}{\xi}\right)^{(w_j-k)^+}. \end{aligned} \tag{16}$$

Clearly, if $\max_{\rho\in[0,\xi]} \|z^t(\gamma(t+\rho^k))\|/\xi \le C$ uniformly in t, inequality (16) proves (10). Then, let us assume by contradiction that $\max_{\rho\in[0,\xi]} \|z^t(\gamma(t+\rho^k))\|/\xi$ is unbounded as $\xi \downarrow 0$. For any ξ, we let $\bar\xi \in [0,\xi]$ be such that $\|z^t(\gamma(t+\bar\xi^k))\| = \max_{\rho\in[0,\xi]} \|z^t(\gamma(t+\rho^k))\|$. Then, there exists a sequence $\xi_\nu \to +\infty$ such that

$$b_\nu = \frac{|z_j^t(\gamma(t+\bar\xi_\nu^k))|}{\xi_\nu^{w_j}} \longrightarrow +\infty \quad \text{and} \quad \frac{1}{n}\frac{\|z^t(\gamma(t+\bar\xi_\nu^k))\|}{\xi_\nu} \le b_\nu^{\frac{1}{w_j}} \le \frac{\|z^t(\gamma(t+\bar\xi_\nu^k))\|}{\xi_\nu}.$$

Moreover, by (16), it has to hold that $w_j > k$. Then, again by (16), it follows that

$$b_\nu \le Cn\, b_\nu^{1-\frac{k}{w_j}} \longrightarrow 0 \quad \text{as } \nu \to +\infty.$$

This contradicts the fact that $b_\nu \to +\infty$ and proves that there exists $\xi_0 > 0$, a priori depending on t, such that $\|z^t(\gamma(t+\bar\xi^k))\|/\xi \le C$ for any $\xi < \xi_0$. Since $[0,T]$ is compact, both constants ξ_0, C are uniform for $t \in [0,T]$, thus completing the proof of (10) and of the proposition. ∎

We now focus on coordinate systems adapted to the drift. In particular, if for some $s \in \mathbb{N}$ it holds that $f_0 \subset \Delta^s \setminus \Delta^{s-1}$, it makes sense to consider the following definition.

Definition 2.6. *A* privileged coordinate system adapted to f_0 at q *is a system of privileged coordinates z at q for $\{f_1,\dots,f_m\}$ such that there exists a coordinate z_ℓ such that $z_* f_0 \equiv \partial_{z_\ell}$.*

Observe that completing f_0 to an adapted basis $\{f_1,\dots,f_0,\dots,f_n\}$ allows us to consider the coordinate system adapted to f_0 at q, given by the inverse of the diffeomorphism

$$(z_1,\dots,z_n) \mapsto e^{z_\ell f_0} \circ \cdots \circ e^{z_n f_n}(q). \tag{17}$$

The following definition combines continuous coordinate families for a path $\gamma : [0,T] \to M$ with coordinate systems adapted to a drift.

Definition 2.7. *A* continuous coordinate family for γ adapted to f_0 *is a continuous coordinate family $\{z^t\}_{t\in[0,T]}$ for γ such that each z^t is a privileged coordinate system adapted to f_0 at $\gamma(t)$.*

Such coordinate systems can be built as in (17), letting the point q vary on the curve.

We end this section by observing that when the path is well behaved w.r.t. the sub-Riemannian structure, it is possible to construct a very special continuous coordinate family, rectifying both γ and f_0 at the same time.

Proposition 2.8. *Let $\gamma : [0,T] \to M$ be a path and $k \in \mathbb{N}$ be such that $\dot\gamma(t) \in \Delta^k(\gamma(t)) \setminus \Delta^{k-1}(\gamma(t))$ for any $t \in [0,T]$; then there exists a continuous coordinate family $\{z^t\}_{[0,T]}$ for γ adapted such that*

1. *there exists a coordinate z_α of weight k such that $z^t_* \dot\gamma \equiv \partial_{z_\alpha}$;*
2. *for any ξ, $t \in [0,T]$ it holds that $z_\alpha^t = z_\alpha^{t-\xi} + \xi$ and $z_i^t = z_i^\xi$ if $i \ne \alpha$.*

Moreover, if there exists $s \in \mathbb{N}$ such that $f_0 \subset \Delta^s \setminus \Delta^{s-1}$ and such that $f_0(\gamma(t)) \ne \dot\gamma(t)$ mod *$\Delta^{s-1}(\gamma(t))$ for any $t \in [0,T]$ whenever $s = k$, such a family can be chosen adapted to f_0.*

Proof. By the assumptions on $\dot\gamma$, it is possible to choose $f_\alpha \subset \Delta^k \setminus \Delta^{k-1}$ such that $\dot\gamma(t) = f_\alpha(\gamma(t))$. Then let $\{f_1,\dots,f_n\}$ be the adapted basis obtained by completing f_α and f_0. Finally, to complete the proof it is enough to consider the family of coordinates given by the inverse of the diffeomorphisms

$$(z_1,\dots,z_n) \mapsto e^{z_\ell f_0} \circ \cdots \circ e^{z_\alpha f_\alpha}(\gamma(t)). \qquad ∎$$

2.3. Cost functions. In this section we focus on properties of the cost functions defined in (1) and of the associated value functions, respectively, denoted by $\mathrm{V}^{\mathcal{J}}(\cdot,\cdot)$ and $\mathrm{V}^{\mathcal{I}}(\cdot,\cdot)$. For $\mathcal{J}$ such a function is defined by

$$\mathrm{V}^{\mathcal{J}}(q,q') = \inf\left\{\mathcal{J}(u,T) \middle|\, 0<T<\mathcal{T},\, q_u(0)=q,\, q_u(T)=q'\right\}. \tag{18}$$

The definition of $\mathrm{V}^{\mathcal{I}}$ is analogous.

The following result, in the case of $\mathcal{J}$, is contained in [24, Proposition 4.1]. The proof can easily be extended to $\mathcal{I}$.

THEOREM 2.9. *For any $\mathcal{T}>0$, the functions $\mathrm{V}^{\mathcal{J}}$ and $\mathrm{V}^{\mathcal{I}}$ are continuous from $M\times M\to[0,+\infty)$ (in particular, they are finite). Moreover, for any $q,q'\in M$ it holds that*

$$\mathrm{V}^{\mathcal{J}}(q,q') \le \min_{0\le t\le\mathcal{T}} \mathrm{d}_{\mathrm{SR}}(e^{tf_0}q,q'),$$
$$\mathrm{V}^{\mathcal{I}}(q,q') \le \min_{0\le t\le\mathcal{T}} \big(t+\mathrm{d}_{\mathrm{SR}}(e^{tf_0}q,q')\big).$$

Here e^{tf_0} denotes the flow of f_0 at time t, and d_{SR} denotes the sub-Riemannian distance w.r.t. the sub-Riemannian system (SR) *associated with* (D).

We remark that this fact follows from the following formula (obtained by adapting [24, Lemma 3.6] to control-affine systems). For any fixed $T>0$ sufficiently small and for any $q_0,q_1\in M$, it holds that

$$\inf\{\mathcal{J}(u,T)\mid q_u(0)=q_0,\, q_u(T)=q_1\}\le \mathrm{d}_{\mathrm{SR}}(q_0,q_1). \tag{19}$$

An immediate consequence of the fact that $\mathcal{J}\le\mathcal{I}$ and of Theorem 2.9 is the following.

COROLLARY 2.10. *Let $\Gamma\subset M$ be a curve and $\gamma:[0,T]\to M$ be a path.*

(i) *Any complexity relative to the cost $\mathcal{J}$ is smaller than the same complexity relative to $\mathcal{I}$. Namely, for any $\varepsilon>0$, it holds that*

$$\Sigma^{\mathcal{J}}_{\mathrm{int}}(\Gamma,\varepsilon)\le\Sigma^{\mathcal{I}}_{\mathrm{int}}(\Gamma,\varepsilon),\quad \Sigma^{\mathcal{J}}_{\mathrm{app}}(\Gamma,\varepsilon)\le\Sigma^{\mathcal{I}}_{\mathrm{app}}(\Gamma,\varepsilon),$$
$$\sigma^{\mathcal{J}}_{\mathrm{int}}(\gamma,\varepsilon)\le\sigma^{\mathcal{I}}_{\mathrm{int}}(\gamma,\varepsilon),\quad \sigma^{\mathcal{J}}_{\mathrm{app}}(\gamma,\varepsilon)\le\sigma^{\mathcal{I}}_{\mathrm{app}}(\gamma,\varepsilon).$$

(ii) *In the case of curves, the complexities relative to the cost $\mathcal{I}$ are always smaller than the same complexities computed for the associated sub-Riemannian system. Namely, for any $\varepsilon>0$, it holds that*

$$\Sigma^{\mathcal{I}}_{\mathrm{int}}(\Gamma,\varepsilon)\le\Sigma^{SR\text{-}s}_{\mathrm{int}}(\Gamma,\varepsilon),\quad \Sigma^{\mathcal{I}}_{\mathrm{app}}(\Gamma,\varepsilon)\le\Sigma^{SR\text{-}s}_{\mathrm{app}}(\Gamma,\varepsilon).$$

We denote the reachable set from the point $q\in M$ with cost $\mathcal{J}$ less than $\varepsilon>0$ as

$$\mathcal{R}^{f_0}_T(q,\varepsilon)=\left\{p\in M\mid \mathrm{V}^{\mathcal{J}}(q,p)\le\varepsilon\right\}. \tag{20}$$

For any $\{z_i\}_{i=1}^n$, a privileged coordinate system adapted to f_0 in the sense of Definition 2.6, and for any $\eta>0$ and $T>0$, we let

$$\Xi_T(\eta)=\bigcup_{0\le\xi\le T}\Big(\xi\partial_{z_\ell}+\mathrm{Box}\,(\eta)\Big),$$

$$\Pi_T(\eta)=\mathrm{Box}\,(\eta)\cup\bigcup_{0<\xi\le T}\{z\in\mathbb{R}^n:\ 0\le z_\ell-\xi\le\eta^s,\, |z_i|\le\eta^{w_i}+\eta\xi^{\frac{w_i}{s}} \text{ for } w_i\le s, i\ne\ell,$$
$$\text{and } |z_i|\le\eta(\eta+\xi^{\frac{1}{s}})^{w_i-1} \text{ for } w_i>s\},$$

where Box(η) has been defined as in (6).

A more general version of the following result, in the same spirit as Theorem 2.2, is proved in [24].

THEOREM 2.11. *Assume that there exists $s \in \mathbb{N}$ such that $f_0 \subset \Delta^s \setminus \Delta^{s-1}$. Assume, moreover, that $z = (z_1, \dots, z_n)$ is a privileged coordinate system adapted to f_0, i.e., such that $z_* f_0 = \partial_{z_\ell}$. Then, there exist $C, \varepsilon_0, T_0 > 0$ such that*

$$\Xi_T\left(\frac{1}{C}\varepsilon\right) \subset \mathcal{R}_T^{f_0}(q,\varepsilon) \subset \Pi_T(C\varepsilon) \qquad \text{for } \varepsilon < \varepsilon_0 \text{ and } T < T_0. \tag{21}$$

Here, with abuse of notation, we denote by $\mathcal{R}_T^{f_0}(q,\varepsilon)$ the coordinate representation of the reachable set. In particular,

$$\mathrm{Box}\left(\frac{1}{C}\varepsilon\right) \cap \{z_\ell \leq 0\} \subset \mathcal{R}_T^{f_0}(q,\varepsilon) \cap \{z_\ell \leq 0\} \subset \mathrm{Box}\,(C\varepsilon) \cap \{z_\ell \leq 0\}. \tag{22}$$

Remark 2.12. Let $N \subset M$ be compact, and let $\{z^q\}_{q\in N}$ be a family of systems of privileged coordinates at q depending continuously on q. Then, as for Theorem 2.2 (see Remark 2.3), there exist uniform constants $C, \varepsilon_0, T_0 > 0$ such that Theorem 2.11 holds for any $q \in N$ in the system z^q.

We also notice that, since [24, Example 4.2] is easily extendable to $\mathcal{I}$, it follows that the existence of minimizers is assured for neither $\mathcal{J}$ nor $\mathcal{I}$.

We now present some results regarding the behavior of the optimal control problem w.r.t. specific points. First, we show the existence of the minimizer for $\mathcal{J}$ and $\mathcal{I}$ between points on the same integral curve of the drift.

PROPOSITION 2.13. *Assume that there exists $s \in \mathbb{N}$ such that $f_0 \subset \Delta^s \setminus \Delta^{s-1}$. For any $0 < t < \mathcal{T}$, the unique minimizer between any $q_0 \in M$ and $e^{tf_0}q_0$ for the cost $\mathcal{J}$ is the null control on $[0,t]$. Moreover, if $f_0 \notin \Delta(q_0)$, i.e., $s \geq 2$, and the maximal time of definition of the controls $\mathcal{T}$ is sufficiently small, the same is true for $\mathcal{I}$.*

Proof. Since, for $t \in [0, \mathcal{T}]$, we have that $\mathrm{V}^{\mathcal{J}}(q, e^{tf_0}q) = 0$, the first statement is trivial.

To prove the second part of the statement we proceed by contradiction. Namely, we assume that there exists a sequence $\mathcal{T}_n \longrightarrow 0$ such that for any $n \in \mathbb{N}$ there exists a control $v_n \in \mathrm{L}^1([0,t_n],\mathbb{R}^m) \subset \mathcal{U}_{\mathcal{T}_n}$, $v_n \not\equiv 0$, steering the system from q_0 to $e^{\mathcal{T}_n f_0}(q_0)$ and such that

$$t_n + \|v_n\|_{\mathrm{L}^1([0,t_n],\mathbb{R}^m)} = \mathcal{I}(v_n, t_n) \leq \mathcal{I}(0, \mathcal{T}_n) = \mathcal{T}_n\,. \tag{23}$$

Let $z = (z_1, \dots, z_n)$ be a privileged coordinate system adapted to f_0 at q, as per Definition 2.6. Thus, by Theorem 2.11, it holds that

$$|z_\ell(e^{\mathcal{T}_n f_0}(q_0))| \leq t_n + C\|v_n\|^2_{\mathrm{L}^1([0,t_n],\mathbb{R}^m)}. \tag{24}$$

Since $z_\ell(e^{\mathcal{T}_n f_0}(q_0)) = \mathcal{T}_n$, putting together (23) and (24) yields $\|v_n\|_{\mathrm{L}^1([0,t_n],\mathbb{R}^m)} \leq C\|v_n\|^2_{\mathrm{L}^1([0,t_n],\mathbb{R}^m)}$ for any $n \in \mathbb{N}$. Since by the continuity of $\mathrm{V}^{\mathcal{I}}$ we have that $\|v_n\|_{\mathrm{L}^1([0,t_n],\mathbb{R}^m)} \to 0$, this is a contradiction. ∎

We remark that, in the case of $\mathcal{I}$, the assumption $f_0 \notin \Delta(q_0)$ of Proposition 2.13 is essential. In particular, in the following example we show that, when $f_0 \subset \Delta$, even if a minimizer between q_0 and $e^{tf_0}(q_0)$ exists, it does not necessarily coincide with an integral curve of the drift.

Example 2.14. Consider the control-affine system on $\mathbb{R}^2$,

$$\frac{d}{dt}x = f_0(x) + u_1 f_0(x) + u_2 f(x), \tag{25}$$

where $f_0 = (1,0)$ and $f = (\phi_1, \phi_2)$ for some $\phi_1, \phi_2 : \mathbb{R}^2 \to \mathbb{R}$, with $\phi_2 \neq 0$ and $\partial_x(\phi_1/\phi_2)|_{(0,0)} \neq 0$. Since f_0 and f are always linearly independent, the underlying small sub-Riemannian system is indeed Riemannian with metric

$$g = \begin{pmatrix} 1 & -\phi_1/\phi_2 \\ -\phi_1/\phi_2 & \frac{1-\phi_1^2}{\phi_2^2} \end{pmatrix}.$$

Let us now prove that the curve $\gamma : [0,1] \to \mathbb{R}^2$, $\gamma(t) = (t\,T, 0)$ is not a minimizer of the Riemannian distance between $(0,0)$ and $(T,0)$. In particular, it is enough to prove that γ is not a geodesic for small $T > 0$. For γ the geodesic equation reads as

$$\begin{cases} t^2\Gamma^1_{11}(\gamma(t)) = 0 \\ t^2\Gamma^2_{11}(\gamma(t)) = 0 \end{cases} \qquad \text{for any } t \in [0,1] \qquad \iff \qquad \Gamma^1_{11}(\cdot,0) = \Gamma^2_{11}(\cdot,0) = 0 \text{ near } 0.$$

Here, $\Gamma^i_{k\ell}$ are the Christoffel numbers of the second kind associated with g. A simple computation shows that

$$\Gamma^1_{11} = \frac{\phi_1}{\phi_2}\,\partial_{x_1}\left(\frac{\phi_1}{\phi_2}\right), \qquad \Gamma^2_{11} = \partial_{x_1}\left(\frac{\phi_1}{\phi_2}\right).$$

Thus, if $\partial_{x_1}(\phi_1/\phi_2)|_{(0,0)} \neq 0$, then $\Gamma^2_{11}(0,0) \neq 0$, showing that γ is not a geodesic.

We now show that this fact implies that for any minimizing sequence $u_n = (u_n^1, u_n^2) \in \mathrm{L}^1([0,t_n], \mathbb{R}^2)$ for $\mathrm{V}^{\mathcal{I}}$ between $(0,0)$ and $e^{Tf_0}((0,0)) = (T,0)$, such that $J(u_{n+1}, t_{n+1}) \leq J(u_n, t_n)$, it holds that $u_n^2 \neq 0$ for sufficiently big n. To this aim, fix any $t_n \to 0$, and let $u_n(s) = u(s/t_n)$ and $q_n(\cdot)$ be the trajectory associated with u_n in system (25). Moreover, let $v = (v_1, 0) \in \mathrm{L}^1([0,S], \mathbb{R}^2)$ be the minimizer of $\mathcal{I}$ between $(0,0)$ and $(T,0)$ in the system $\dot{x}_1 = 1 + v_1$. Since the trajectory of v is exactly γ, by rescaling it holds that $\text{length}(\gamma) = \mathcal{I}(v,S)$. Then, by standard results in the theory of ordinary differential equations, it follows that $q_n(t_n) \to (T,0)$, and the fact that γ is not a Riemannian minimizing curve implies that

$$\|u_n\|_{\mathrm{L}^1} = \|u\|_{\mathrm{L}^1} < \text{length}(\gamma) = \mathcal{I}(v,S).$$

Hence, for sufficiently big n it holds that $\mathcal{I}(u_n, t_n) < \mathcal{I}(v,S)$, proving the claim.

Finally, we conclude the section by showing that when we consider two points on different integral curves of the drift, the two costs $\mathcal{J}$ and $\mathcal{I}$ are equivalent, as proved in the following.

PROPOSITION 2.15. *Assume that there exists $s \in \mathbb{N}$ such that $f_0 \subset \Delta^s \setminus \Delta^{s-1}$. Let $q \in M$ be such that there exists a set of privileged coordinates adapted to f_0 at q. Then, for any $q' \in M$ there exist $C, \varepsilon_0, \mathcal{T} > 0$ such that if for some $u \in \mathcal{U}_{\mathcal{T}}$ and $T < \mathcal{T}$ it holds that $q_u(T) = q'$ and $\mathcal{J}(u,T) < \varepsilon_0$, then*

$$\mathcal{J}(u,T) \leq \mathcal{I}(u,T) \leq C\mathcal{J}(u,T).$$

The proof of this fact relies on the following particular case of [24, Lemma 25], which we will need in what follows.

LEMMA 2.16. *Under the assumptions of Proposition 2.15, there exist $C, \varepsilon_0, \mathcal{T} > 0$ such that, for any $u \in \mathcal{U}_{\mathcal{T}}$, with $\mathcal{J}(u,T) < \varepsilon_0$ for some $T < \mathcal{T}$, it holds that*

$$T \leq C\big(\mathcal{J}(u,T)^s + z_\ell(q_u(T))^+\big).$$

Here, we let $\xi^+ = \max\{\xi, 0\}$.

This lemma is crucial, since it allows us to bound the time of definition of any control by its cost. We now prove Proposition 2.15.

Proof of Proposition 2.15. The first inequality is trivial. The second follows by applying Lemma 2.16 and computing

$$\mathcal{I}(u,T) \le T + \mathcal{J}(u,T) \le (C\varepsilon_0^{s-1}+1)\mathcal{J}(u,T). \qquad \square$$

3. Complexity of curves. This section is devoted to proving the statement on curves of Theorem 1.4. Namely, we will prove the following.

THEOREM 3.1. *Assume that there exists $s \ge 2$ such that $f_0 \subset \Delta^s \setminus \Delta^{s-1}$. Let $\Gamma \subset M$ be a curve, and define $\kappa = \max\{k\colon\ T_p\Gamma \in \Delta^k(p) \setminus \Delta^{k-1}(p)$ for some $p \in \Gamma\}$. Then, if the maximal time of definition of the controls $\mathcal{T}$ is small enough,*

$$\Sigma^{\mathcal{J}}_{\text{int}}(\Gamma,\varepsilon) \asymp \Sigma^{\mathcal{I}}_{\text{int}}(\Gamma,\varepsilon) \asymp \Sigma^{\mathcal{J}}_{\text{app}}(\Gamma,\varepsilon) \asymp \Sigma^{\mathcal{I}}_{\text{app}}(\Gamma,\varepsilon) \asymp \frac{1}{\varepsilon^{\kappa}}.$$

In order to prove this result, we will treat separately the $\preccurlyeq$ and the $\succcurlyeq$. Due to the fact that the value functions associated with the costs $\mathcal{J}$ and $\mathcal{I}$ are always smaller than the sub-Riemannian distance associated with system (SR), the $\preccurlyeq$ immediately follows from the results in [21]. As already remarked in the introduction, the difficult part of the proof is the $\succcurlyeq$. Namely, one has to show that exploiting the drift does not help (in an asymptotic sense) to track the curve.

In order to prove $\succcurlyeq$ we will exploit a subadditivity property of the complexities. This property holds only away from certain badly behaving points, called cusps, near which the value function behaves like the Euclidean distance does near algebraic cusps (e.g., $(0,0)$ for the curve $y = \sqrt{|x|}$ in $\mathbb{R}^2$). In the sub-Riemannian context they have been introduced in [21].

DEFINITION 3.2. *The point $q \in \Gamma$ is a* cusp *for the cost J if it is not an endpoint of Γ and if, for every $c, \eta > 0$, there exist two points $q_1, q_2 \in \Gamma$ such that q lies between q_1 and q_2, with q_1 before q and q_2 after q w.r.t. the orientation of Γ (in particular, $q \ne q_1, q_2$), $\mathrm{V}^J(q_1,q_2) \le \eta$, and $\mathrm{V}^J(q,q_2) \ge c\,\mathrm{V}(q_1,q_2)$.*

We now give the proof of Theorem 3.1, which relies on three lemmas whose exact statement and proof we postpone until after the following proof.

Proof of Theorem 3.1. We start by proving the $\preccurlyeq$ part. By (i) in Corollary 2.10, it follows that we only have to prove the upper bound for the complexities relative to the cost $\mathcal{I}$. Moreover, by the same proposition, the fact that $T\Gamma \subset \Delta^\kappa$, and [21, Theorem 3.14], it follows immediately that $\Sigma^{\mathcal{I}}_{\text{int}}(\Gamma,\varepsilon)$ and $\Sigma^{\mathcal{I}}_{\text{app}}(\Gamma,\varepsilon) \preccurlyeq \varepsilon^{-k}$, completing the proof of the first part of the statement.

By Corollary 2.10, to prove the $\succcurlyeq$ part it suffices to prove that $\Sigma^{\mathcal{J}}_{\text{int}}(\Gamma,\varepsilon)$ and $\Sigma^{\mathcal{J}}_{\text{app}}(\Gamma,\varepsilon) \succcurlyeq \varepsilon^{-\kappa}$. The proof is then divided into the following steps:

1. Identify sufficient conditions guaranteeing the absence of cusps (Lemma 3.3).
2. Prove that the complexities are subadditive w.r.t. subcurves $\Gamma' \subset \Gamma$ whose endpoints are not cusps in Γ for the cost $\mathcal{J}$ (Lemma 3.5).
3. Exploiting the subadditivity, prove the $\succcurlyeq$ part of the statement for curves such that $T_p\Gamma \subset \Delta^k(p) \setminus \Delta^{k-1}(p)$ (Lemma 3.6).
4. Using step 1, show that if $T_p\Gamma \subset \Delta^k(p)$, then there exists $\Gamma' \subset \Gamma$ whose endpoints are not cusps and such that $T_p\Gamma' \subset \Delta^k(p) \setminus \Delta^{k-1}(p)$.
5. Conclude by applying step 3 to the curve Γ' found in the previous point and then using the subadditivity obtained in step 2.

To conclude the proof, we need only prove step 4. By smoothness of Γ, the set $A = \{p \in \Gamma \mid T_p\Gamma \in \Delta^\kappa(p) \setminus \Delta^{\kappa-1}(p)\}$ has a nonempty interior. Then let $\Gamma' \subset A$

be a nontrivial curve such that either $f_0(p) \notin T_p\Gamma' \oplus \Delta^{s-1}(p)$ for any $p \in \Gamma'$ or $f_0|_{\Gamma'} \subset T\Gamma' \oplus \Delta^{s-1}$. Then, by Lemma 3.3 we can choose Γ' such that it does not contain any cusps. ∎

We now prove Lemmas 3.3, 3.5, and 3.6.

LEMMA 3.3. *Assume that there exists $s \geq 2$ such that $f_0 \subset \Delta^s \setminus \Delta^{s-1}$. Let $\Gamma \subset M$ be a curve such that $T\Gamma \subset \Delta^k \setminus \Delta^{k-1}$. Moreover, if $s = k$, let Γ be such that either $f_0(p) \notin T_p\Gamma \oplus \Delta^{s-1}(p)$ for any $p \in \Gamma$ or $f_0|_\Gamma \subset T\Gamma \oplus \Delta^{s-1}$. Then Γ has no cusps for the cost $\mathrm{V}^{\mathcal{J}}$.*

Proof. If $f_0|_\Gamma \subset T_p\Gamma \oplus \Delta^{s-1}(p)$, the statement is a consequence of Proposition 2.13. Hence, we assume that $f_0(p) \notin T_p\Gamma \oplus \Delta^{s-1}(p)$ for any $p \in \Gamma$. Let $\gamma : [0, \mathfrak{T}] \to M$ be a path parametrizing Γ, and consider the continuous coordinate family $\{z^t\}_{t\in[0,\mathfrak{T}]}$ adapted to f_0 given by Proposition 2.8. In particular, it holds that $z^t_*\dot\gamma(\cdot) \equiv \partial_{z_\alpha}$ for some coordinate z_α of weight k and for any $t \in [0, \mathfrak{T}]$. We now fix any $t_0 \in (0, \mathfrak{T})$ and prove that $\gamma(t_0)$ is not a cusp. In fact, letting $\eta > 0$ be sufficiently small, by Theorem 2.11 and the fact that $z^t_\ell(\gamma(\cdot)) \equiv 0$ we get

$$\begin{aligned} V^{\mathcal{J}}(\gamma(t_0), \gamma(t_0+\eta)) &\leq C \sum_{j=1}^n |z_j^{t_0}(\gamma(t_0+\eta))|^{\frac{1}{w_j}} = C|z_\alpha^{t_0}\gamma(t_0+\eta)|^{\frac{1}{k}} \\ &= 2C|z_\alpha^{t_0-\eta}(\gamma(t_0+\eta))|^{\frac{1}{k}} \leq CV(\gamma(t_0-\eta), \gamma(t_0+\eta)). \end{aligned}$$

Letting $t_1 = t_0 - \eta$ and $t_2 = t_0 + \eta$, this proves that $V^{\mathcal{J}}(\gamma(t_0), \gamma(t_2)) \leq V^{\mathcal{J}}(\gamma(t_1), \gamma(t_2))$. By definition, this implies that $\gamma(t_0)$ is not a cusp, completing the proof of the proposition. ∎

The above lemma shows that cusps can appear only if the drift becomes tangent to the curve at isolated points. Indeed, the following example shows that this condition is essential. Moreover it shows that, while cusps cannot appear in equiregular sub-Riemannian structure [21], this condition alone is not sufficient for control-affine systems.

Example 3.4. Consider the following vector fields on $\mathbb{R}^3$, with coordinates (x, y, z):

$$f_1(x, y, z) = \partial_x, \qquad f_2(x, y, z) = \partial_y + x\partial_z.$$

Since $[f_1, f_2] = \partial_z$, $\{f_1, f_2\}$ is a bracket-generating family of vector fields. The sub-Riemannian control system associated with $\{f_1, f_2\}$ on $\mathbb{R}^3$ corresponds to the Heisenberg group.

Now let $f_0 = \partial_z \subset \Delta^2 \setminus \Delta$ be the drift, and let us consider the curve $\Gamma = \{(t^2, 0, t) \mid t \in (-\eta, \eta)\}$. Let $q = (0, 0, 0)$. Since $T_q\Gamma \notin \Delta(q)$, by smoothness of Γ and Δ, for η sufficiently small $T\Gamma \subset \Delta^2 \setminus \Delta$. We now show that the point q is indeed a cusp for the cost $\mathcal{J}$. In fact, for any $\xi > 0$ such that $2\xi < \mathcal{T}$, it holds that the null control defined over time $[0, 2\xi]$ steers the control-affine system from $q_1 = (\xi^2, 0, -\xi) \in \Gamma$ to $q_2 = (\xi^2, 0, \xi) \in \Gamma$. Hence, $V^{\mathcal{J}}(q_1, q_2) = 0$. Moreover, since q and q_2 are not on the same integral curve of the drift, $V^{\mathcal{J}}(q, q_2) > 0 = V^{\mathcal{J}}(q_1, q_2)$. This proves that q is a cusp for $\mathcal{J}$.

In the following we prove the subadditivity of the curve complexities.

LEMMA 3.5. *Let $\Gamma' \subset \Gamma$ be two curves in M. Then, if the endpoints of Γ' are not cusps for the cost $\mathcal{J}$, there exists a constant $C > 0$ such that for sufficiently small $\mathcal{T}$ it holds that*

$$\Sigma^{\mathcal{J}}_{\mathrm{int}}(\Gamma', \varepsilon) \preccurlyeq \Sigma^{\mathcal{J}}_{\mathrm{int}}(\Gamma, \varepsilon), \qquad \Sigma^{\mathcal{J}}_{\mathrm{app}}(\Gamma', \varepsilon) \preccurlyeq \Sigma^{\mathcal{J}}_{\mathrm{app}}(\Gamma, \varepsilon).$$

Proof. Cost interpolation complexity. Let $u \in \mathrm{L}^1([0,T],\mathbb{R}^m)$ be a control admissible for $\Sigma_{\mathrm{int}}^{\mathcal{J}}(\Gamma,\varepsilon)$, and let $0 = t_1 < \cdots < t_N = T$ be such that $\|u\|_{\mathrm{L}^1([t_{i-1},t_i])} \leq \varepsilon$. Recall that by Theorem 2.9, $V^{\mathcal{J}}$ is a continuous function. Since for small $\mathcal{T} > 0$, for any $\varepsilon > 0$, and for any $q_0 \in M$ the reachable set $\mathcal{R}_{\mathcal{T}}(q,\varepsilon)$ is bounded, it holds that $\mathcal{R}_{\mathcal{T}}(q,\varepsilon) \searrow \{e^{tf_0}(q_0) \mid t \in [0,\mathcal{T}]\}$ as $\varepsilon \downarrow 0$, in the sense of pointwise convergence of characteristic functions. From this follows that, for ε and $\mathcal{T}$ sufficiently small, there exist $i_1 \neq i_2$ such that $q_u(t_i) \in \Gamma'$ for any $i \in \{i_1,\ldots,i_2\}$ and $q_u(t_i) \notin \Gamma'$ for any $i \notin \{i_1,\ldots,i_2\}$. Since x' and y' are not cusps, there exists $c > 0$ such that, letting x' and y' be the endpoints of Γ', it holds that $\mathrm{V}^{\mathcal{J}}(x', q_u(t_{i_1})) \leq c\,\mathrm{V}^{\mathcal{I}}(q_u(t_{i_1-1}, q_u(t_{i_1})) \leq \varepsilon$ and $\mathrm{V}^{\mathcal{J}}(q_u(t_{i_2}), y') \leq \mathrm{V}^{\mathcal{J}}(q_u(t_{i_2}), q_u(t_{i_2+1})) \leq c\varepsilon$. Thus, there exists a constant $C > 0$ such that

$$\Sigma_{\mathrm{int}}^{\mathcal{J}}(\Gamma',\varepsilon) \leq \frac{\mathcal{J}(u|_{[t_{i_1},t_{i_2}]})}{\varepsilon} + 2c \leq C\frac{\mathcal{J}(u|_{[t_{i_1-1},t_{i_2+1}]})}{\varepsilon} \leq C\frac{\mathcal{J}(u)}{\varepsilon}.$$

Taking the infimum over all controls u admissible for $\Sigma_{\mathrm{int}}^{\mathcal{J}}(\Gamma,\varepsilon)$ completes the proof.

Tubular approximation complexity. Let $u \in \mathrm{L}^1([0,T],\mathbb{R}^m)$ be a control admissible for $\Sigma_{\mathrm{app}}^{\mathcal{J}}(\Gamma,\varepsilon)$. Then, letting q_u be its trajectory such that $q_u(0) = x$, there exist two times t_1 and t_2 such that $q_u(t_1) \in \mathrm{B}_{\mathrm{SR}}(x', C\varepsilon)$ and $q_u(t_2) \in \mathrm{B}_{\mathrm{SR}}(y', C\varepsilon)$. Then, since $V^{\mathcal{J}} \leq \mathrm{d}_{\mathrm{SR}}$ by Theorem 2.9, the same argument as above applies. ◻

Finally, exploiting the subadditivity, we prove the $\succcurlyeq$ part of Theorem 3.1 in the case where the curve is always tangent to the same stratum $\Delta^k \setminus \Delta^{k-1}$.

LEMMA 3.6. *Assume that there exists $s \in \mathbb{N}$ such that $f_0 \subset \Delta^s \setminus \Delta^{s-1}$. Let $\Gamma \subset M$ be a curve such that there exists $k \in \mathbb{N}$ for which $T_p\Gamma \in \Delta^k(p) \setminus \Delta^{k-1}(p)$ for any $p \in \Gamma$. Then, for sufficiently small time $\mathcal{T}$, it holds that*

$$\Sigma_{\mathrm{int}}^{\mathcal{I}}(\Gamma,\varepsilon) \succcurlyeq \Sigma_{\mathrm{int}}^{\mathcal{J}}(\Gamma,\varepsilon) \succcurlyeq \frac{1}{\varepsilon^k}, \qquad \Sigma_{\mathrm{app}}^{\mathcal{I}}(\Gamma,\varepsilon) \succcurlyeq \Sigma_{\mathrm{app}}^{\mathcal{J}}(\Gamma,\varepsilon) \succcurlyeq \frac{1}{\varepsilon^k}.$$

Proof. By Corollary 2.10, $\Sigma_{\mathrm{int}}^{\mathcal{I}}(\Gamma,\varepsilon) \succcurlyeq \Sigma_{\mathrm{int}}^{\mathcal{J}}(\Gamma,\varepsilon)$ and $\Sigma_{\mathrm{app}}^{\mathcal{I}}(\Gamma,\varepsilon) \succcurlyeq \Sigma_{\mathrm{app}}^{\mathcal{J}}(\Gamma,\varepsilon)$. We will prove only that $\Sigma_{\mathrm{int}}^{\mathcal{J}}(\Gamma,\varepsilon) \succcurlyeq \varepsilon^{-k}$, since the same arguments apply to $\Sigma_{\mathrm{app}}^{\mathcal{J}}(\Gamma,\varepsilon)$.

Let $\gamma : [0,\mathfrak{T}] \to M$ be a path parametrizing Γ. We will distinguish three cases.

Case 1: $f_0(p) \notin \Delta^{s-1}(p) \oplus T_p\Gamma$ *for any* $p \in \Gamma$. Fix $\eta > 0$, and consider a control $u \in \mathrm{L}^1([0,T],\mathbb{R}^m)$, admissible for $\Sigma_{\mathrm{int}}(\Gamma,\varepsilon)$ such that

$$\frac{\|u\|_{\mathrm{L}^1}}{\varepsilon} \leq \Sigma_{\mathrm{int}}(\Gamma,\varepsilon) + \eta. \tag{26}$$

Let $u_i = u|_{[t_{i-1},t_i]}$, $i = 1,\ldots,N = \lceil \frac{\|u\|_{\mathrm{L}^1}}{\varepsilon} \rceil$, be such that $\|u_i\|_{\mathrm{L}^1} = \varepsilon$ for any $1 \leq i < N$, $\|u_N\|_{\mathrm{L}^1} \leq \varepsilon$. Moreover, let s_i be the times such that $\gamma(s_i) = q_u(t_i)$.

By (26), it holds that $N \leq \lceil \Sigma_{\mathrm{int}}(\Gamma,\varepsilon) + \eta + 1 \rceil$. However, we can assume without loss of generality (w.l.o.g.) that $N \leq \lceil \Sigma_{\mathrm{int}}(\Gamma,\varepsilon) + \eta \rceil$. In fact, $N > \lceil \Sigma_{\mathrm{int}}(\Gamma,\varepsilon) + \eta \rceil$ only if $\|u_N\| < \varepsilon$. In this case we can simply restrict ourselves to computing $\Sigma_{\mathrm{int}}(\tilde{\Gamma},\varepsilon)$, where $\tilde{\Gamma}$ is the segment of Γ of extrema x and $q_u(t_{N-1})$. Indeed, by Lemmas 3.3 and 3.5, it follows that $\Sigma_{\mathrm{int}}(\tilde{\Gamma},\varepsilon) \preccurlyeq \Sigma_{\mathrm{int}}(\Gamma,\varepsilon)$.

We now assume that ε and $\mathcal{T}$ are sufficiently small in order to satisfy the hypotheses of Theorem 2.11 at any point of Γ. Moreover, let $\{z^t\}_{t\in[0,\mathfrak{T}]}$ be the continuous coordinate family for Γ adapted to f_0 given by Proposition 2.8.

Then, it holds that

(27)
$$\mathfrak{T} = \sum_{i=1}^{N}(s_i - s_{i-1}) = \sum_{i=1}^{N} |z_\alpha^{s_{i-1}}(\gamma(s_i))| = \sum_{i=1}^{N} |z_\alpha^{s_{i-1}}(q_u(t_i))| \le C(\Sigma_{\text{int}}(\Gamma,\varepsilon)+\eta)\varepsilon^k.$$

In the last inequality we applied Theorem 2.11 and the fact that $z_\ell^{s_i-1}(q_u(t_i)) = 0$ by Proposition 2.8. Finally, letting $\eta \downarrow 0$ in (27), we get that for any ε sufficiently small it holds that $\Sigma_{\text{int}}(\Gamma,\varepsilon) \ge C\mathfrak{T}\,\varepsilon^{-k}$. This completes the proof in this case.

Case 2: *$s = k$ and $f_0(p) \in \Delta^{s-1}(p) \oplus T_p\Gamma$ for any $p \in \Gamma$.* Let $\{z^t\}_{t\in[0,\mathfrak{T}]}$ be a continuous coordinate family for γ adapted to f_0. In this case, since $(z_\ell^t)_* f_0 = 1$, it holds that $(z_\ell^t)_*\dot\gamma(\cdot) \neq 0$. Hence, there exist C_1, $C_2 > 0$ such that for any $t, \xi \in [0,T]$

$$C_1(t-\xi) \le z_\ell^t(\gamma(\xi)) \le C_2(t-\xi) \quad \text{if } (z_\ell^t)_*\dot\gamma(\cdot) > 0. \tag{28}$$

$$C_1(t-\xi) \le -z_\ell^t(\gamma(\xi)) \le C_2(t-\xi) \quad \text{if } (z_\ell^t)_*\dot\gamma(\cdot) < 0. \tag{29}$$

If (29) holds, then we can proceed as in Case 1 with $\alpha = \ell$. In fact, $|z_\ell^{s_{i-1}}(q_u(t_i)| \le C\varepsilon^s$ by Theorem 2.11. On the other hand, if (28) holds, by applying Theorem 2.11 we get

$$\begin{aligned}\mathfrak{T} = \sum_{i=1}^{N}(s_i - s_{i-1}) &\le \frac{1}{C_1}\sum_{i=1}^{N} |z_\ell^{s_{i-1}}(\gamma(s_i))| = \frac{1}{C_1}\sum_{i=1}^{N} |z_\ell^{s_{i-1}}(q_u(t_i))| \\ &\le \frac{1}{C_1}\sum_{i=1}^{N}(C\varepsilon^s + t_i - t_{i-1}) \le C\big(\Sigma_{\text{int}}^{\mathcal{J}}(\Gamma,\varepsilon)+\eta\big)\varepsilon^s + T.\end{aligned}$$

By taking $\mathcal{T}$ sufficiently small, it holds that $T \le \mathcal{T} < \mathfrak{T}$. Then, letting $\eta \downarrow 0$, this proves that $\Sigma_{\text{int}}^{\mathcal{J}}(\Gamma,\varepsilon) \ge ((T-\mathcal{T})/C)\varepsilon^{-s} \succcurlyeq \varepsilon^{-s}$. This completes the proof of Case 2.

Case 3: *$s = k$ and $f_0(p) \in \Delta^{s-1}(p) \oplus T_p\Gamma$ for some $p \in \Gamma$.* In this case, there exists an open interval $(t_1,t_2) \subset [0,\mathfrak{T}]$ such that $f_0(\gamma(t)) \neq \dot\gamma(t) \mod \Delta^{s-1}(\gamma(t))$ for any $t \in (t_1,t_2)$. Thus, $\Gamma' = \gamma((t_1,t_2))$ satisfies the assumption of Case 1 and hence $\Sigma_{\text{int}}^{\mathcal{J}}(\Gamma',\varepsilon) \succcurlyeq \varepsilon^{-k}$. Moreover, by Lemma 3.3, we can assume that $\gamma(t_1)$ and $\gamma(t_2)$ are not cusps. Then, by Lemma 3.5 we get

$$\frac{1}{\varepsilon^k} \preccurlyeq \Sigma_{\text{int}}^{\mathcal{J}}(\Gamma',\varepsilon) \preccurlyeq \Sigma_{\text{int}}^{\mathcal{J}}(\Gamma,\varepsilon),$$

completing the proof of Lemma 3.6. ◻

4. Complexity of paths. In this section we will prove the statement on paths of Theorems 1.3 and 1.4. In the proofs we will use the quantity $\omega(\gamma,\delta)$ introduced in section 1.2. Observe that, by (3), this quantity is related to the interpolation by time by the following for any $k \in \mathbb{N}$:

(30)
$$\sigma_{\text{int}}(\gamma,\varepsilon) \preccurlyeq \varepsilon^{-k} \iff \omega(\gamma,\delta) \preccurlyeq \delta^{\frac{1}{k}} \quad \text{and} \quad \sigma_{\text{int}}(\gamma,\varepsilon) \succcurlyeq \varepsilon^{-k} \iff \omega(\gamma,\delta) \succcurlyeq \delta^{\frac{1}{k}}.$$

Exploiting this fact, we are able to prove Theorem 1.3.

Proof of Theorem 1.3. Let $\{z^t\}_{t\in[0,T]}$ be the continuous family of coordinates for γ given by Proposition 2.8. We start by proving that $\omega(\gamma,\delta) \preccurlyeq \delta^{\frac{1}{k}}$, which, by (30), will imply $\sigma_{\rm int}^{\rm SR\text{-}s}(\gamma,\varepsilon)\preccurlyeq\varepsilon^{-k}$. Fix any partition $0=t_0<t_1<\cdots<t_N=T$ such that $\delta/2\le t_i-t_{i-1}\le\delta$. If δ is sufficiently small, from Theorem 2.2 it follows that there exists a constant $C>0$ such that for any $i=0,\dots,N$ in the coordinate system z^{t_i} it holds that $\mathrm{Box}(\gamma(t_i),C\delta^{\frac1k})\subset \mathrm{B}_{\rm SR}(\gamma(t_i),\delta^{\frac1k})$. Observe that $z_\alpha^{t_{i-1}}(\gamma(t_i))=t_i-t_{i-1}$ and that $z_j^{t_{i-1}}(\gamma(t_i))=0$ for any $j\neq\alpha$. Hence, since $N\le\lceil 2T/\delta\rceil\le CT/\delta$, we get

$$\begin{aligned}\omega(\gamma,\delta)&\le\delta\sum_{i=1}^N \mathrm{d}_{\rm SR}(\gamma(t_{i-1}),\gamma(t_i))\\&\le C\delta\sum_{i=1}^N\sum_{j=1}^n|z_j^{t_{i-1}}(\gamma(t_i))|^{\frac{1}{w_j}}=C\delta\sum_{i=1}^N(t_i-t_{i-1})^{\frac1k}\le CT\delta^{\frac1k}.\end{aligned}$$

This completes the proof of the first part of the theorem.

Conversely, to prove that $\sigma_{\rm int}(\gamma,\varepsilon)\preccurlyeq\varepsilon^{-k}$, we need to show that $\omega(\gamma,\delta)\succcurlyeq\delta^{\frac1k}$. To this aim, let $\eta>0$, and let $u\in\mathrm{L}^1$ be a control admissible for $\omega(\gamma,\delta)$ such that

$$\|u\|_{\mathrm{L}^1([t_{i-1},t_i])}\le\frac{\omega(\gamma,\delta)}{\delta}+\eta.$$

Let $0=t_0<t_1<\cdots<t_N=T$ be times such that $q_u(t_i)=\gamma(t_i)$, $i=0,\dots,N$, $0<t_i-t_{i-1}\le\delta$. Moreover, let $u_i\in\mathrm{L}^1([t_{i-1},t_i])$ be the restriction of u between t_{i-1} and t_i. Observe that, up to removing some t_i, we can assume that $t_i-t_{i-1}\in\big(\frac{\delta}{2},\frac32\delta\big]$. This implies that $\lceil 2T/(3\delta)\rceil\le N\le\lceil 2T/\delta\rceil$.

To complete the proof it suffices to show that $\|u_i\|_{\mathrm{L}^1([t_{i-1},t_i])}\ge C\delta^{\frac1k}$. In fact, for any $\eta>0$, this yields

$$\frac{\omega(\gamma,\delta)}{\delta}\ge\|u\|_{\mathrm{L}^1([0,T],\mathbb{R}^m)}-\eta=\sum_{i=1}^N\|u_i\|_{\mathrm{L}^1([t_{i-1},t_i])}-\eta\ge C\sum_{i=1}^N\delta^{\frac1k}-\eta\ge C\,\frac{2T}{3\delta}\delta^{\frac1k}-\eta.$$

Letting $\eta\downarrow0$, this will prove that $\omega(\gamma,\delta)\succcurlyeq\delta^{\frac1k}$, completing the proof.

Observe that, by Theorem 2.2, for any $i=1,\dots,N$ in the coordinate system $z^{t_{i-1}}$ it holds that $\mathrm{B}_{\rm SR}(\gamma(t_i),\|u_i\|_{\mathrm{L}^1([t_{i-1},t_i])})\subset\mathrm{Box}\big(\gamma(t_i),C\|u_i\|_{\mathrm{L}^1([t_{i-1},t_i])}\big)$. Since $z_\alpha^{t_{i-1}}(t_i)=t_i-t_{i-1}$, this implies that

$$\frac{\delta}{2}\le t_i-t_{i-1}=|z_\alpha^{t_{i-1}}(\gamma(t_i))|\le C\,\|u_i\|^k_{\mathrm{L}^1([t_{i-1},t_i])},$$

proving that $\|u_i\|_{\mathrm{L}^1([t_{i-t},t_i])}\ge C\delta^{\frac1k}$ and thus the theorem. ∎

The rest of the section will be devoted to the proof of the statement on paths of Theorem 1.4. Namely, we will prove the following.

THEOREM 4.1. *Assume that there exists $s\ge2$ such that $f_0\subset\Delta^s\setminus\Delta^{s-1}$. Let $\gamma:[0,T]\to M$ be a path such that $f_0(\gamma(t))\neq\dot\gamma(t)\mod\Delta^{s-1}(\gamma(t))$ for any $t\in[0,T]$ and define $\kappa=\max\{k\colon\ \gamma(t)\in\Delta^k(\gamma(t))\setminus\Delta^{k-1}(\gamma(t))$ for any t in an open subset of $[0,T]\}$. Then, it holds that*

$$\sigma_{\rm int}^{\mathcal J}(\gamma,\varepsilon)\asymp\sigma_{\rm int}^{\mathcal I}(\gamma,\varepsilon)\asymp\sigma_{\rm app}^{\mathcal J}(\gamma,\varepsilon)\asymp\sigma_{\rm app}^{\mathcal I}(\gamma,\varepsilon)\asymp\frac{1}{\varepsilon^{\max\{\kappa,s\}}},$$

where the asymptotic equivalences regarding the interpolation by time complexity hold only when δ_0, i.e., the maximal time-step in $\sigma_{\rm int}(\gamma,\varepsilon)$, is sufficiently small.

As for Theorem 3.1, we will prove separately the $\preccurlyeq$ and the $\succcurlyeq$ in Theorem 4.1.

Contrary to what happened for curves, the $\preccurlyeq$ part of Theorem 4.1, contained in Lemma 4.2, does not immediately follow from the estimates of sub-Riemannian complexities but requires additional care: when estimating these complexities from above we cannot simply use controls concentrated in very small time, since we are forced to follow the parametrization of the path and thus to fight the drift. This is the reason for the exponent $\max\{k, s\}$.

On the other hand, the $\succcurlyeq$ part can be proved using a scheme similar to that used in the proof of Theorem 3.1. Namely, we will first prove in Lemma 4.3 a subadditivity property of the complexities that will allow us to reduce the proof of the $\succcurlyeq$ part to curves such that $T_p\Gamma \subset \Delta^k(p) \setminus \Delta^{k-1}(p)$ for any $p \in \Gamma$. This is done in Lemma 4.4, using the technical result of Lemma 4.5. We remark that, although the proof of this estimate is much more technical than in the case of curves, the subadditivity property for path complexity holds without any assumption on cusps, due to the requirement to follow the parametrization.

LEMMA 4.2. *Assume that there exists $s \in \mathbb{N}$ such that $f_0 \subset \Delta^s \setminus \Delta^{s-1}$. Let $\gamma : [0,T] \to M$ be a path such that $\dot\gamma(t) \in \Delta^k(\gamma(t))$. Then, it holds that*

$$\sigma_{\mathrm{int}}^{\mathcal{J}}(\gamma,\varepsilon) \preccurlyeq \sigma_{\mathrm{int}}^{\mathcal{I}}(\gamma,\varepsilon) \preccurlyeq \frac{1}{\varepsilon^{\max\{s,k\}}}, \qquad \sigma_{\mathrm{app}}^{\mathcal{J}}(\Gamma,\varepsilon) \preccurlyeq \sigma_{\mathrm{app}}^{\mathcal{I}}(\Gamma,\varepsilon) \preccurlyeq \frac{1}{\varepsilon^{\max\{s,k\}}}. \tag{31}$$

Proof. By (i) in Corollary 2.10, it follows that we only have to prove the upper bound for the complexities relative to the cost $\mathcal{I}$. We will start by proving (31) for $\sigma_{\mathrm{int}}^{\mathcal{I}}$. In particular, by (30) it will suffice to prove $\omega^{\mathcal{I}}(\gamma,\delta) \preccurlyeq \delta^{\frac{1}{k}}$.

For any $u \in \mathrm{L}^1([0,T],\mathbb{R}^m)$, by the variation formula (see [2]), it holds that

$$\overrightarrow{\exp}\int_0^T \left(f_0 + \sum_{i=1}^m u_i(t)\, f_i\right) dt = e^{Tf_0} \circ \overrightarrow{\exp}\int_0^T \sum_{i=1}^m u_i(t)\,(e^{-tf_0})_* f_i\, dt. \tag{32}$$

Here, we denoted by $\overrightarrow{\exp}\int_0^T g^t\,dt$ the flow of the time-dependent vector field g^t. This shows that a control steering system (D) from $p \in M$ to $q \in M$ in time $T > 0$ steers from p to $e^{-Tf_0}q$ the time-dependent control system

$$\dot q(t) = \sum_{j=1}^m u_j(t)\,(e^{-tf_0})_* f_j(q(t)). \tag{TD}$$

Let $\{z^t\}_{t\in[0,T]}$ be a continuous coordinate family for γ adapted to f_0. Let $\tilde\gamma_t(\xi) = e^{-(\xi-t)f_0}(\gamma(\xi))$. Then, since $z^t_* f_0 = \partial_{z_\ell}$, it holds that

$$z_\ell^t(\tilde\gamma_t(\xi)) = z_\ell^t(\gamma(\xi)) - (\xi - t), \qquad z_i^t(\tilde\gamma_t(\xi)) = z_i^t(\gamma(\xi)) \quad \text{for any } i \neq \ell. \tag{33}$$

Fix $\xi > 0$ sufficiently small for Proposition 2.5 to hold, and choose a partition $0 < t_1 < \cdots < t_N = T$ such that $\delta/2 \le t_i - t_{i-1} \le \delta$. In particular, $N \le \lceil 2T/\delta \rceil$. We then select a control $u \in \mathrm{L}^1([0,T],\mathbb{R}^m)$ such that its trajectory q_u in (D), with $q_u(0) = x$, satisfies $q_u(t_i) = \gamma(t_i)$ for any $i = 1,\dots,N$ as follows. For each i, we choose $u_i \in \mathrm{L}^1([t_{i-1},t_i],\mathbb{R}^m)$ to be a control steering system (TD) from $\gamma(t_{i-1}) = \tilde\gamma_{t_{i-1}}(t_{i-1})$ to $\tilde\gamma_{t_{i-1}}(t_i)$. Then, by (32) and the definition of $\tilde\gamma_{t_{i-1}}$, the control u_i steers system (D) from $\gamma(t_{i-1})$ to $\gamma(t_i)$.

Since by [24, Theorem 8] it holds that $\mathrm{V}_{\mathrm{TD}}^{\mathcal{I}} \le \mathrm{d}_{\mathrm{SR}}$, by (33), Proposition 2.5, and Theorem 2.2, if δ is sufficiently small, we can choose u_i such that there exists $C > 0$

for which

$$\begin{aligned}\mathcal{I}(u_i, t_i - t_{i-1}) &\le C \sum_{j=1}^{n} |z_j^{t_{i-1}}(\tilde{\gamma}_{t_{i-1}}(t_i))|^{\frac{1}{w_j}} \le C \sum_{j=1}^{n} |z_j^{t_{i-1}}(\gamma(t_i))|^{\frac{1}{w_j}} + \delta^{\frac{1}{s}} \\ &\le C \left(\sum_{w_j \le k} \delta^{\frac{1}{w_j}} + \delta^{\frac{1}{s}} + \sum_{w_j > k} \delta^{\frac{1}{k}} \right) \le C \delta^{\frac{1}{\max\{k,s\}}}.\end{aligned} \tag{34}$$

Hence, we obtain that

$$\mathcal{I}(u, T) \le N\, \mathcal{I}(u_i, t_i - t_{i-1}) \le 3C \frac{T}{\delta} \delta^{\frac{1}{\max\{k,s\}}}. \tag{35}$$

Since the control u is admissible for $\omega^{\mathcal{I}}(\gamma, \delta)$, this implies that $\omega^{\mathcal{I}}(\gamma, \delta) \preccurlyeq \delta^{\frac{1}{\max\{k,s\}}}$. This proves the part of the statement regarding σ_{int}.

To complete the proof for $\sigma_{\text{app}}(\gamma, \varepsilon)$, let $\delta = \varepsilon^{\max\{k,s\}}$. Then, by Theorems 2.2 and 2.11, there exists a constant $C > 0$ such that $\mathcal{R}_\delta^{f_0}(\gamma(t), \varepsilon) \subset \mathrm{B}_{\text{SR}}(\gamma(t), C\varepsilon)$ for any $t \in [0, T]$. In particular, $\mathrm{d}_{\text{SR}}(\gamma(t_i), q_u(t)) \le C\varepsilon$ for any $t \in [t_{i-1}, t_i]$. Moreover, again by Theorem 2.2, Proposition 2.5, and the fact that $\dot{\gamma}(\cdot) \in \Delta^k(\gamma(\cdot))$, this choice of δ also implies that $\mathrm{d}_{\text{SR}}(\gamma(t_{i-1}), \gamma(t)) \le C\varepsilon$ for any $t \in [t_{i-1}, t_i]$. Hence, for any $t \in [t_{i-1}, t_i]$, we get

$$\mathrm{d}_{\text{SR}}(\gamma(t), q_u(t)) \le \mathrm{d}_{\text{SR}}(\gamma(t_{i-1}), q_u(t)) + \mathrm{d}_{\text{SR}}(\gamma(t_{i-1}), \gamma(t)) \le 2C\varepsilon.$$

Thus, u is admissible for $\sigma_{\text{app}}^{\mathcal{I}}(\gamma, C\varepsilon)$. Finally, from (35) we get that $\sigma_{\text{app}}^{\mathcal{I}}(\gamma, C\varepsilon) \le \varepsilon^{-1}\mathcal{I}(u, T) \le 3CT\varepsilon^{-\max\{k,s\}}$, proving that $\sigma_{\text{app}}(\gamma, \varepsilon) \preccurlyeq \varepsilon^{-\max\{k,s\}}$. This completes the proof. $\square$

As for the case of curves, we will need the following subadditivity property.

LEMMA 4.3. *Let $\gamma : [0, T] \to M$ be a path, and let $t_1, t_2 \subset [0, T]$.*

(i) *If there exists $k \in \mathbb{N}$ such that $\sigma_{\text{int}}^{\mathcal{J}}(\gamma|_{[t_1,t_2]}, \varepsilon) \succcurlyeq \varepsilon^{-k}$, then $\sigma_{\text{int}}^{\mathcal{J}}(\gamma, \varepsilon) \succcurlyeq \varepsilon^{-k}$.*

(ii) *$\sigma_{\text{app}}^{\mathcal{J}}(\gamma|_{[t_1,t_2]}, \varepsilon) \preccurlyeq \sigma_{\text{app}}^{\mathcal{J}}(\gamma, \varepsilon)$.*

Proof.

Time interpolation complexity. By (30), it suffices to prove that $\omega^{\mathcal{J}}(\gamma|_{[t_1,t_2]}, \delta) \preccurlyeq \omega^{\mathcal{J}}(\gamma, \delta)$. Let $u \in \mathrm{L}^1([0, T], \mathbb{R}^m)$ be a control admissible for $\Sigma_{\text{int}}^{\mathcal{J}}(\Gamma, \varepsilon)$, and let $0 = \xi_1 < \cdots < \xi_N = T$ be the times where $q_u(\xi_i) = \gamma(\xi_i)$. Let $i_1 \ne i_2$ such that $t_1 \le \xi_i \le t_2$ for any $i \in \{i_1, \ldots, i_2\}$. Observe that, by Theorems 2.2 and 2.9, we have $\mathrm{V}^{\mathcal{J}}(\gamma(t_1), \gamma(\xi_{i_1})) \le \mathrm{d}_{\text{SR}}(\gamma(t_1), \gamma(\xi_{i_1})) \le C\delta^{\frac{1}{r}}$ and $\mathrm{V}^{\mathcal{J}}(\gamma(\xi_{i_2}), \gamma(t_2)) \le \mathrm{d}_{\text{SR}}(\gamma(\xi_{i_2}), \gamma(t_2)) \le C\delta^{\frac{1}{r}}$, where δ is sufficiently small, C is independent of δ, and r is the nonholonomic degree of the distribution. Thus, assuming w.l.o.g. that $C \ge 1$,

$$\omega^{\mathcal{J}}(\gamma|_{[t_1,t_2]}, \delta) \le \delta \mathcal{J}(u|_{[t_{i_1}, t_{i_2}]}) + 2C\delta^{1+\frac{1}{r}} \le C\delta\mathcal{J}(u) + C\delta^{1+\frac{1}{r}}.$$

Taking the infimum over all controls u admissible for $\omega^{\mathcal{J}}(\gamma, \delta)$ and recalling that, by Lemma 4.2, it holds that $\omega^{\mathcal{J}}(\gamma, \delta) \preccurlyeq \delta^{\frac{1}{r}}$, we complete the proof.

Neighboring approximation complexity. In this case, the proof is identical to that of Lemma 3.5 for the tubular approximation complexity. The sole difference is that here, by definition of $\sigma_{\text{app}}^{\mathcal{J}}$, we do not need to assume the absence of cusps. $\square$

Now, we prove the $\succcurlyeq$ part of the statement in the case where $\dot{\gamma}$ is always contained in the same stratum $\Delta^k \setminus \Delta^{k-1}$.

LEMMA 4.4. *Assume that there exists $s \ge 2$ such that $f_0 \subset \Delta^s \setminus \Delta^{s-1}$. Let $\gamma : [0, T] \to M$ be a path such that $\dot{\gamma}(t) \in \Delta^k(\gamma(t)) \setminus \Delta^{k-1}(\gamma(t))$ for any $t \in [0, T]$.*

Moreover, if $s = k$*, assume that* $f_0(\gamma(t)) \neq \dot\gamma(t) \mod \Delta^{s-1}$ *for any* $t \in [0, T]$*. Then, it holds that*

$$\sigma^{\mathcal{I}}_{\rm int}(\gamma,\varepsilon) \succcurlyeq \sigma^{\mathcal{J}}_{\rm int}(\gamma,\varepsilon) \succcurlyeq \frac{1}{\varepsilon^{\max\{s,k\}}}, \qquad \sigma^{\mathcal{I}}_{\rm app}(\gamma,\varepsilon) \succcurlyeq \sigma^{\mathcal{J}}_{\rm app}(\gamma,\varepsilon) \succcurlyeq \frac{1}{\varepsilon^{\max\{s,k\}}}.$$

To prove this lemma, we need the following technical result.

LEMMA 4.5. *Under the assumptions of Lemma* 4.4 *there exist a continuous coordinate family* $\{z^t\}_{t\in[0,T]}$ *for* γ *adapted to* f_0*, constants* $\xi_0, \rho, m > 0$*, and a coordinate* $\alpha \neq \ell$ *of weight* s *such that, for any* $t \in [0, T]$ *and* $0 \le \xi \le \xi_0$*, it holds that*

$$(z^t_\ell)_*\dot\gamma(t+\xi) \le 1-\rho \quad \text{if } t \in E_1 = \{\varphi_\ell < 1-2\rho\}, \tag{36}$$

$$(z^t_\alpha)_*\dot\gamma(t+\xi) \ge m \quad \text{if } t \in E_2 = \{1-2\rho \le \varphi_\ell \le 1+2\rho\}, \tag{37}$$

$$(z^t_\ell)_*\dot\gamma(t+\xi) \ge 1+\rho \quad \text{if } t \in E_3 = \{\varphi_\ell > 1+2\rho\}. \tag{38}$$

In particular, it holds that $E_1 \cup E_2 \cup E_3 = [0, T]$*.*

Proof. By the assumptions of f_0 and $\dot\gamma$ there exist $f_\alpha \subset \Delta^s \setminus \Delta^{s-1}$ and two functions $\varphi_\ell, \varphi_\alpha \in C^\infty([0, T])$, $\varphi_\alpha \ge 0$, such that

$$\dot\gamma(t) \mod \Delta^{s-1}(\gamma(t)) = \varphi_\ell(t) f_0(\gamma(t)) + \varphi_\alpha(t) f_\alpha(\gamma(t)).$$

Moreover, by the assumption $f_0(\gamma(t)) \neq \dot\gamma(t) \mod \Delta^{s-1}(\gamma(t))$, if $\varphi_\ell(t) = 1$, then $\varphi_\alpha(t) > 0$. Then, using f_α as an element of the adapted basis used to define a continuous coordinate family for γ adapted to f_0 via (17) with $q = \gamma(t)$, it holds that $(z^t_i)_*\dot\gamma(t) = \varphi_i(t)$ for $i = \alpha, \ell$ and any $t \in [0, T]$.

Since $\varphi_\alpha > 0$ on $\varphi_\ell^{-1}(1)$, by continuity of φ_ℓ and φ_α there exists $\rho > 0$ such that $\varphi_\alpha > 0$ on $\varphi_\ell^{-1}([1-2\rho, 1+2\rho])$. Since $E_2 = \varphi_\ell^{-1}([1-2\rho, 1+2\rho])$ is closed, letting $2m = \min_{E_2} \varphi_\alpha > 0$, property (37) follows by the uniform continuity of $(t, \xi) \mapsto (z^t_\alpha)_*\dot\gamma(t+\xi)$ on $E_2 \times [0, \xi_0]$ for sufficiently small ξ_0. Finally, the uniform continuity of $(t, \xi) \mapsto (z^t_\ell)_*\dot\gamma(t+\xi)$ over $\overline{E_1} \times [0, \xi_0]$ and $\overline{E_3} \times [0, \xi_0]$ yields (36) and (38). $\square$

Proof of Lemma 4.4. By Corollary 2.10, $\sigma^{\mathcal{J}}_{\rm int}(\gamma,\varepsilon) \preccurlyeq \sigma^{\mathcal{I}}_{\rm int}(\gamma,\varepsilon)$ and $\sigma^{\mathcal{J}}_{\rm app}(\gamma,\varepsilon) \preccurlyeq \sigma^{\mathcal{I}}_{\rm app}(\gamma,\varepsilon)$. Hence, to complete the proof it suffices to prove the asymptotic lower bound for $\sigma^{\mathcal{J}}_{\rm int}(\gamma,\varepsilon)$ and $\sigma^{\mathcal{J}}_{\rm app}(\gamma,\varepsilon)$. In the following, to lighten the notation, we write $\sigma_{\rm int}$ and $\sigma_{\rm app}$ instead of $\sigma^{\mathcal{J}}_{\rm int}$ and $\sigma^{\mathcal{J}}_{\rm app}$.

Interpolation by time complexity. By (30), it suffices to prove that $\omega(\gamma,\delta) \succcurlyeq \delta^{\frac{1}{\max\{k,s\}}}$. Let $\eta > 0$, and let $u \in \mathrm{L}^1([0, T], \mathbb{R}^m)$ be a control admissible for $\omega(\gamma,\delta)$ such that

$$\mathcal{J}(u, T) = \|u\|_{\mathrm{L}^1([0,T],\mathbb{R}^m)} \le \frac{\omega(\gamma,\delta)}{\delta} + \eta. \tag{39}$$

Let $N = \lceil T/\delta \rceil$ and $0 = t_0 < t_1 < \cdots < t_N = T$ be times such that $q_u(t_i) = \gamma(t_i)$, $i = 0, \dots, N$, and $0 < t_i - t_{i-1} \le \delta$. Observe that, up to removing some t_i, we can always assume $\delta/2 \le t_i - t_{i-1} \le (3/2)\delta$ and $N \ge \lceil (2T)/(3\delta) \rceil$. Moreover, let $u_i = u|_{[t_{i-1},t_i]}$. Proceeding as in the proof of Theorem 1.3 in section 4, we get that in order to show that $\omega(\gamma,\delta) \succcurlyeq \delta^{\frac{1}{\max\{k,s\}}}$ it suffices to prove

$$\|u_i\|_{\mathrm{L}^1([t_{i-1},t_i])} \ge C\delta^{\frac{1}{\max\{s,k\}}}, \qquad i = 1, \dots, N. \tag{40}$$

We distinguish three cases.

Case 1: $k > s$. Let $\{z^t\}$ be the continuous coordinate family for γ adapted to f_0 given by Proposition 2.8. Then, since $z^t_\ell(\gamma(\cdot)) = 0$ and $z^t_\alpha(\gamma(\xi)) = \xi - t$, by Theorem 2.11 it holds that

$$\frac{\delta}{2} \leq (t_i - t_{i-1}) = |z_\alpha^{t_{i-1}}(\gamma(t_i))| \leq C\|u_i\|^k_{\mathrm{L}^1([t_{i-1},t_i])}. \tag{41}$$

This proves (40).

Case 2: $k < s$. Also in this case, let $\{z^t\}$ be the continuous coordinate family for γ adapted to f_0 given by Proposition 2.8. Then, by Lemma 2.16, we get

$$\frac{\delta}{2} \leq t_i - t_{i-1} \leq C\|u_i\|^s_{\mathrm{L}^1([t_{i-1},t_i])},$$

which immediately proves (40).

Case 3: $k = s$. Let $\{z^t\}_{t\in[0,T]}$ be a continuous coordinate family for γ adapted to f_0. By the mean value theorem there exists $\xi \in [t_{i-1}, t_i]$ such that

$$z_\ell^{t_{i-1}}(\gamma(t_i)) = \int_{t_{i-1}}^{t_i} (z^t_\ell)_*\dot\gamma(t)\,dt = \big((z_\ell^{t_{i-1}})_*\dot\gamma(\xi)\big)(t_i - t_{i-1}). \tag{42}$$

Consider the partition $\{E_1, E_2, E_3\}$ of $[0, T]$ given by Lemma 4.5, and let $\delta \leq \delta_0$. Then, depending on which E_j contains t_{i-1}, we proceed differently.

(a) $t_{i-1} \in E_1$: By Lemma 2.16 and (42), we get

$$\begin{aligned} t_i - t_{i-1} &\leq C\|u_i\|^s_{\mathrm{L}^1([t_{i-1},t_i])} + z_\ell^{t_{i-1}}(\gamma(t_i))^+ \\ &= C\|u_i\|^s_{\mathrm{L}^1([t_{i-1},t_i])} + \big((z_\ell^{t_{i-1}})_*\dot\gamma(\xi)\big)(t_i - t_{i-1}). \end{aligned}$$

Then, by (36) of Lemma 4.5, we get

$$\|u_i\|_{\mathrm{L}^1([t_{i-1},t_i])} \geq \left(\frac{1 - (z_\ell^{t_{i-1}})_*\dot\gamma(\xi)}{C}\right)^{\frac1s} (t_i - t_{i-1})^{\frac1s} \geq \left(\frac{\rho}{C}\right)^{\frac1s} \delta^{\frac1s}.$$

This proves (40).

(b) $t_{i-1} \in E_2$: By (37) of Lemma 4.5, (42), and Theorem 2.11, we get

$$\begin{aligned} m(t_i - t_{i-1}) &\leq |z_\alpha^{t_{i-1}}(\gamma(t_i))| \\ &\leq C\left(\|u_i\|^s_{\mathrm{L}^1([t_{i-1},t_i])} + \|u_i\|_{\mathrm{L}^1([t_{i-1},t_i])}|z_\ell^{t_{i-1}}(\gamma(t_i))|\right). \end{aligned}$$

Reasoning as in (34) yields that we can assume $\|u_i\|_{\mathrm{L}^1([t_{i-1},t_i])} \leq C\delta^{\frac1s}$. Then, by (42) and letting $\delta \leq (m/(2+4\rho))^s$, we get

$$\|u_i\|_{\mathrm{L}^1([t_{i-1},t_i])} \geq (m - \delta^{\frac1s}(1+2\rho))^{\frac1s}(t_i - t_{i-1})^{\frac1s} \geq \left(\frac{m}{2}\right)^{\frac1s}\delta^{\frac1s},$$

proving (40).

(c) $t_{i-1} \in E_3$: By Theorem 2.11 it follows that

$$|z_\ell^{t_{i-1}}(\gamma(t_i))| \leq C\|u_i\|^s_{\mathrm{L}^1([t_{i-1},t_i])} + (t_i - t_{i-1}). \tag{43}$$

Then, by (42) and (43) we obtain

$$\|u_i\|_{\mathrm{L}^1([t_{i-1},t_i])} \geq \left(\frac{(z_\ell^{t_{i-1}})_*\dot\gamma(\xi) - 1}{C}\right)^{\frac1s} (t_i - t_{i-1})^{\frac1s} \geq \left(\frac{\rho}{C}\right)^{\frac1s}\delta^{\frac1s}.$$

The last inequality follows from (38) of Lemma 4.5. This proves (40).

Neighboring approximation complexity. Fix $\eta > 0$, and let $u \in \mathrm{L}^1([0,T],\mathbb{R}^m)$ be admissible for $\sigma_{\mathrm{app}}(\gamma,\varepsilon)$ and such that $\|u\|_{\mathrm{L}^1([0,T],\mathbb{R}^m)} \le \sigma_{\mathrm{app}}(\gamma,\varepsilon)+\eta$. Let $q_u : [0,T] \to M$ be the trajectory of u with $q_u(0)=\gamma(0)$. Then let $N = \lceil \sigma_{\mathrm{app}}(\gamma,\varepsilon)+\eta \rceil$ and $0=t_0<t_1<\cdots<t_N=T$ be such that $\|u\|_{\mathrm{L}^1([t_{i-1},t_i])} \le \varepsilon$ for any $i=1,\ldots,N$. By (19) and the fact that $q_u(t) \in \mathrm{B}_{\mathrm{SR}}(\gamma(t),\varepsilon)$ for any $t\in[0,T]$, we can build a new control, still denoted by u, such that $q_u(t_i)=\gamma(t_i)$, $i=1,\ldots,N$, and $\|u\|_{\mathrm{L}^1([t_{i-1},t_i])} \le 3\varepsilon$.

Fixing a $\delta_0 > 0$, w.l.o.g. we can assume that $t_i - t_{i-1} \le \delta_0$. In fact, we can split each interval $[t_{i-1},t_i]$ not satisfying this property as $t_{i-1}=\xi_1<\cdots<\xi_M=t_i$, with $\xi_\nu - \xi_{\nu-1} \le \delta_0$. Then, as above, it is possible to modify the control u so that $q_u(\xi_\nu)=\gamma(\xi_\nu)$ for any $\nu=1,\ldots,M$. Since $M \le \lceil T/\delta_0\rceil$ and $q_u(\cdot)\in \mathrm{B}_{\mathrm{SR}}(\gamma(\cdot),\varepsilon)$, we have $\|u\|_{\mathrm{L}^1([\xi_i,\xi_{i-1}])} \le 5\varepsilon$, and the new total number of intervals is $\le (1+\lceil T/\delta_0\rceil)\lceil \sigma_{\mathrm{app}}(\gamma,\varepsilon)+\eta\rceil \le C(\sigma_{\mathrm{app}}(\gamma,\varepsilon)+\eta)$.

We claim that to prove $\sigma_{\mathrm{app}}(\gamma,\varepsilon) \succcurlyeq \varepsilon^{-\max\{s,k\}}$, it suffices to show that there exists a constant $C>0$, independent of u, such that

$$t_i - t_{i-1} \le C\varepsilon^{\max\{s,k\}} \qquad \text{for any } i=1,\ldots,N. \tag{44}$$

In fact, since $N \le C(\sigma_{\mathrm{app}}(\gamma,\varepsilon)+\eta)$, this will imply that

$$T = \sum_{i=1}^N t_i - t_{i-1} \le C(\sigma_{\mathrm{app}}(\gamma,\varepsilon)+\eta)\varepsilon^{\max\{s,k\}}.$$

Letting $\eta \downarrow 0$, we get that $\sigma_{\mathrm{app}}(\gamma,\varepsilon) \succcurlyeq \varepsilon^{-\max\{s,k\}}$, proving the claim.

We now let δ_0 be sufficiently small in order to apply Lemma 4.5, Theorem 2.11, and Lemma 2.16. As before, we distinguish three cases.

Case 1: $k>s$. Let $\{z^t\}$ be the continuous coordinate family for γ adapted to f_0 given by Proposition 2.8. By Theorem 2.11, using the fact that $\gamma(t_i)=q_u(t_i)$ for $i=1,\ldots,N$, we get

$$(t_i - t_{i-1}) = |z_\alpha^{t_{i-1}}(\gamma(t_i))| \le C\varepsilon^k. \tag{45}$$

This proves (44).

Case 2: $k<s$. Again, let $\{z^t\}$ be the continuous coordinate family for γ adapted to f_0 given by Proposition 2.8. As for the interpolation by time complexity, by Lemma 2.16 and the fact that $q_u(t_i)=\gamma(t_i)$, we get

$$(t_i - t_{i-1}) \le C\varepsilon^s,$$

thus proving (44).

Case 3: $k=s$. Let $\{z^t\}_{t\in[0,T]}$ be a continuous coordinate family for γ adapted to f_0. Consider the partition $\{E_1,E_2,E_3\}$ of $[0,T]$ given by Lemma 4.5, and recall (42). We distinguish three cases.

(a) $t_{i-1}\in E_1$: By Lemma 2.16 and (42) we get

$$t_i - t_{i-1} \le C\varepsilon^s + z_\ell^{t_{i-1}}(\gamma(t_i)) = 2C\varepsilon^s + \big((z_\ell^{t_{i-1}})_*\dot\gamma(\xi)\big)(t_i-t_{i-1}).$$

By (36) of Lemma 4.5, this implies

$$t_i - t_{i-1} \le \left(\frac{2C}{1-(z_\ell^{t_{i-1}})_*\dot\gamma(\xi)}\right)\varepsilon^s \le \frac{2C}{\rho}\varepsilon^s.$$

Hence, (44) is proved.

(b) $t_{i-1} \in E_2$: By (37) of Lemma 4.5, (42), and Theorem 2.11, we get

$$m(t_i - t_{i-1}) \leq |z_\alpha^{t_{i-1}}(\gamma(t_i))| \leq C\left(\varepsilon^s + \varepsilon |z_\ell^{t_{i-1}}(\gamma(t_i))|\right) \leq C\left(\varepsilon^s + \varepsilon^{s+1} + \varepsilon(t_i - t_{i-1})\right).$$

This, by taking ε sufficiently small and enlarging C, implies (44).

(c) $t_{i-1} \in E_3$: By Theorem 2.11 it follows that

$$|z_\ell^{t_{i-1}}(\gamma(t_i))| \leq C\varepsilon^s + (t_i - t_{i-1}). \tag{46}$$

Then, by (42) and (46) we obtain

$$t_i - t_{i-1} \leq \frac{C}{(z_\ell^{t_{i-1}})_* \dot\gamma(\xi) - 1}\varepsilon^s \leq \frac{C}{\rho}\varepsilon^s.$$

The last inequality follows from (38) of Lemma 4.5 and proves (44). ∎

We can now complete the proof of Theorem 1.4 by proving Theorem 4.1.

Proof. The proof is analogous to that of Theorem 3.1, using Lemmas 4.2, 4.4, and 4.3. ∎

Appendix A. Complements. In this appendix we collect two simple results on the general behavior of the complexities which are not needed for the proofs of Theorems 3.1 and 4.1.

We first prove a general result on the behavior of complexities as $\varepsilon \to 0$. For all the complexities under consideration, except the interpolation by time complexity, this result holds w.r.t. a general cost function $J : \mathcal{U}_\mathcal{T} \to [0, +\infty)$, satisfying some weak hypotheses.

PROPOSITION A.1. *Assume that for any $q_1 \in M$ and any $q_2 \notin \{e^{tf_0}q_1\}_{t\in[0,\mathcal{T}]}$, it holds that $\mathrm{V}^J(q_1, q_2) > 0$. Then, the following hold.*

(i) *For any curve $\Gamma \subset M$ we have the following.*
  (a) *If the maximal time of definition of the controls, $\mathcal{T}$, is sufficiently small, then $\lim_{\varepsilon\downarrow 0} \Sigma_{\mathrm{int}}(\Gamma, \varepsilon) = \lim_{\varepsilon\downarrow 0} \Sigma_{\mathrm{app}}(\Gamma, \varepsilon) = +\infty$.*
  (b) *If Γ is an admissible curve for (D), then $\varepsilon\, \Sigma_{\mathrm{int}}(\Gamma, \varepsilon)$ and $\varepsilon\, \Sigma_{\mathrm{app}}(\Gamma, \varepsilon)$ are bounded from above for any $\varepsilon > 0$.*

(ii) *For any path $\gamma : [0, T] \to M$ we have the following.*
  (a) *If γ is not a solution of (D), then $\lim_{\varepsilon\downarrow 0} \sigma_{\mathrm{app}}(\gamma, \varepsilon) = +\infty$.*
  (b) *If the cost is either $\mathcal{J}$ or $\mathcal{I}$, $f_0 \subset \Delta^s \setminus \Delta^{s-1}$, $\dot\gamma(t) \subset \Delta^k(\gamma(t)) \setminus \Delta^{k-1}(\gamma(t))$, and $f_0(\gamma(t)) \neq \dot\gamma(t) \mod \Delta^{s-1}(\gamma(t))$ for any $t \in [0, T]$, then $\lim_{\varepsilon\downarrow 0} \sigma_{\mathrm{int}}(\gamma, \varepsilon) = +\infty$ whenever δ_0 is sufficiently small.*
  (c) *If γ is an admissible curve for (D), then $\varepsilon\sigma_{\mathrm{int}}(\gamma, \varepsilon)$ and $\varepsilon\, \sigma_{\mathrm{app}}(\gamma, \varepsilon)$ are bounded by above for any $\delta, \varepsilon > 0$.*

Proof. The last statement for curves and paths follows simply by considering the control whose trajectory is the curve or the path itself, which is always admissible regardless of ε.

We now prove the first statement for the interpolation by cost complexity of a curve Γ. The same reasoning will hold for Σ_{app} and σ_{app}. Let x, y be the two endpoints of Γ, and assume $\mathcal{T}$ to be sufficiently small so that $\mathrm{V}^{\mathrm{J}}(x, y) > 0$. Then, the first statement follows from

$$\lim_{\varepsilon\downarrow 0} \Sigma_{\mathrm{int}}(\Gamma, \varepsilon) \geq V(x, y) \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} = +\infty.$$

Consider now the interpolation by time complexity, and proceed by contradiction. Namely, let us assume that there exists a constant $C > 0$ such that $\sigma_{\mathrm{int}}(\gamma, \varepsilon) \le C$ for any $\varepsilon > 0$. Then, by definition of σ_{int}, this implies that for any $\varepsilon > 0$ there exist $\delta_\varepsilon \in [CT/2, \delta_0)$ and a δ_ε-time interpolation $u_\varepsilon \in \mathrm{L}^1([0,T], \mathbb{R}^m)$ such that $\frac{\delta_\varepsilon}{T}\mathrm{J}(u_\varepsilon, T) \le \varepsilon$.

First, observe that by Lemma 2.16 and the assumptions on f_0 and $\dot\gamma$, we obtain that $\mathrm{V}(\gamma(0), \gamma(t)) \to 0$ if and only if $t \to 0$. For any ε, let $0 = t_0^\varepsilon < t_1^\varepsilon < \cdots < t_{N_\varepsilon}^\varepsilon = T$ be a partition of $[0,T]$ such that $q_{u_\varepsilon}(t_i^\varepsilon) = \gamma(t_i^\varepsilon)$ for any $i \in \{0, N_\varepsilon\}$ and $t_i^\varepsilon - t_{i-1}^\varepsilon \le \delta_\varepsilon$. Thus, for any ε there exists i_ε such that $\delta_\varepsilon/2 \le t_{i_\varepsilon} \le 2\delta_\varepsilon$. By the definition of u_ε and δ_ε it follows that $J(u_\varepsilon, T) \to 0$. Then, $\mathrm{V}^{\mathrm{J}}(\gamma(0), \gamma(\tau_\varepsilon)) \to 0$ as $\varepsilon \downarrow 0$, and we obtain a contradiction. In fact, this is equivalent to

$$0 = \lim_{\varepsilon \downarrow 0} t_{i_\varepsilon}^\varepsilon \ge \frac{CT}{4} > 0. \qquad \square$$

Remark A.2. Result (ii)(b), regarding the interpolation by time complexity, holds for any cost satisfying the assumptions of Proposition A.1 such that for any path γ we have

$$V\big(\gamma(0), \gamma(t_h)\big) \longrightarrow 0 \text{ as } h \to +\infty \implies t_h \longrightarrow 0 \text{ as } h \to +\infty$$

and that, for any $u \in \mathrm{L}^1([0,T], \mathbb{R}^m)$, there exists a constant such that if $t_1, t_2 \in [0,T]$, $t_1 < t_2$, then

$$\mathrm{J}(u|_{[t_1,t_2]}(\cdot + t_1), t_2 - t_1) \le C\mathrm{J}(u, T).$$

As a consequence of Proposition 2.13, we can prove the following property for the complexities w.r.t. the costs $\mathcal{J}$ and $\mathcal{I}$. It generalizes to the control-affine setting the trivial minimality of the sub-Riemannian complexity on the path $\Gamma = \{q\}$.

COROLLARY A.3. *Assume that there exists $s \ge 2$ such that $f_0 \subset \Delta^s \setminus \Delta^{s-1}$. Let $x \in M$ and $y = e^{Tf_0}x$ for some $0 < T < \mathcal{T}$. Then, for any $\varepsilon > 0$, the minimum over all curves $\Gamma \subset M$ (resp., paths $\gamma : [0,T] \to M$) connecting x and y of $\Sigma_{\mathrm{int}}^{\mathcal{J}}(\cdot, \varepsilon)$ and $\Sigma_{\mathrm{app}}^{\mathcal{J}}(\cdot, \varepsilon)$ (resp., $\sigma_{\mathrm{int}}^{\mathcal{J}}(\cdot, \delta)$ and $\sigma_{\mathrm{app}}^{\mathcal{J}}(\cdot, \varepsilon)$) is attained at $\Gamma = \{e^{tf_0}\}_{t\in[0,T]}$ (resp., at $\gamma(t) = e^{tf_0}x$). Moreover, the same is true for the cost $\mathcal{I}$, whenever $\mathcal{T}$ is sufficiently small.*

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