Spectral analysis and the Aharonov-Bohm effect on certain almost-Riemannian manifolds

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We study spectral properties of the Laplace-Beltrami operator on two relevant almost-Riemannian manifolds, namely the Grushin structures on the cylinder and on the sphere. As for general almost-Riemannian structures (under certain technical hypothesis), the singular set acts as a barrier for the evolution of the heat and of a quantum particle, although geodesics can cross it. This is a consequence of the self-adjointness of the Laplace-Beltrami operator on each connected component of the manifolds without the singular set.

We get explicit descriptions of the spectrum, of the eigenfunctions and their properties. In particular in both cases we get a Weyl law with dominant term $E \log E$. We then study the effect of an Aharonov-Bohm non-apophantic magnetic potential that has a drastic effect on the spectral properties.

Other generalized Riemannian structures including conic and anti-conic type manifolds are also studied. In this case, the Aharonov-Bohm magnetic potential may affect the self-adjointness of the Laplace-Beltrami operator.

1. Introduction

A 2-dimensional almost-Riemannian structure is a generalized Riemannian structure that can be locally defined by a pair of smooth vector fields on a 2-dimensional manifold, satisfying the Hörmander condition (see for instance [2, 8]). These vector fields play the role of an orthonormal frame.

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Where the linear span of the two vector fields is 2-dimensional, the corresponding metric is Riemannian. Where it is 1-dimensional, the corresponding Riemannian metric is not well-defined. Generically this happens on a 1-dimensional submanifold denoted by $\mathcal{Z}$. However, thanks to the Hörmander condition, one can still define a distance between two points (called Carnot-Carathéodory distance), which happens to be finite and continuous.

Almost-Riemannian structures were introduced in the context of hypoelliptic operators [6, 21, 23]. They appeared in problems of population transfer in quantum systems [15, 16, 17] and have applications to orbital transfer in space mechanics [9].

These structures present interesting phenomena. For instance, there may be a conjugate locus even if the Gaussian curvature is always negative, where it is defined (see [3]). Moreover generically the singular set acts as a barrier for the heat flow and for a quantum particle, even though geodesics can pass through the singular set without singularities, see [13]. See also [11, 12].

Not much is known about spectral properties of the Laplace-Beltrami operator on almost-Riemannian structures. The only exception is the discreteness of the spectrum under some general assumptions in the compact case, proved in [13]. We remark that this result is not trivial since the considered structures, when genuinely almost-Riemannian, have always infinite volume.

In this paper we study the spectral properties of the Laplace-Beltrami operator Δ in two relevant almost-Riemannian structures with infinite volume, one of which is non-compact. In particular, we obtain an explicit description of the spectrum, the eigenfunctions and their properties. This allows us to compute the asymptotic behavior of the eigenvalues counting function for $-\Delta$,

$$N(E) := \#\{\lambda \in \sigma_{\mathrm{d}}(-\Delta) \mid \lambda \leq E\}. \tag{1.1}$$

We obtain $E\log(E)$ as leading order, which is fairly unusual for the Laplace-Beltrami operators on 2-dimensional Riemannian manifolds [10, 22, 24, 30, 32].

In these settings, we also investigate the effects of hidden magnetic fluxes on this spectrum. To this purpose, we introduce a magnetic vector potential for the zero magnetic field whose flux is non zero. In the classical Euclidean cases, it is well understood that these fluxes, despite being classically invisible, affect the wavefunctions by introducing a change of phase. This is known as Aharonov-Bohm effect [1, 20] and requires the space to be non simply connected (one can imagine these hidden fluxes as passing through the points removed from the space).

If in the Euclidean case the effect of the hidden magnetic fields is surprising but somewhat simple, the same cannot be said for what concerns asymptotically hyperbolic manifolds with cusps. In such cases, as proved in [22], a change in vector potentials (that preserves the magnetic fields, including the zero field) can drastically modify the spectral properties of the operator, e.g. by destroying the absolutely continuous component of the spectrum. This phenomenon can be useful for counting eigenvalues embedded in the absolutely continuous spectrum in non-separable problems [22, 28].

In this work we show that the aforementioned drastic effect on the spectrum is present in almost-Riemannian manifolds. Additionally, we show that the degeneracy of the eigenvalues is extremely sensitive to the vector potential.

Due to the explicit nature of these examples, we are able to define a continuous parametrization of the Ahronov-Bohm vector potentials. This enable us to follow closely the mechanisms of decompactification of the spectrum in the limit in which a trapping vector potential (i.e. a vector potential for which the spectrum is purely discrete) becomes non-trapping (i.e. a vector potential for which the absolutely continuous spectrum is present). In this regime we can show that the Ahronov-Bohm perturbation strongly affects the structure of a subset of the eigenfunctions and makes it degenerate into the set of generalized eigenfunctions. Up to our knowledge this is the first result of this kind.

With the same techniques we are also able to study spectral properties of the Laplace-Beltrami operator on conic and anti-conic surfaces of the type considered in [14]. Here, the Aharonov-Bohm effect affects not only the spectrum but also the self-adjointness properties of the operator.

1.1. Almost-Riemannian structures under consideration

In this paper we consider only *trivializable* 2-dimensional almost-Riemannian structures i.e., structures globally defined by a pair of smooth vector fields (X_1, X_2).

Definition 1.1. *Let (X_1, X_2) be a pair of smooth vector fields on a two-dimensional smooth manifold M. Let Lie(X_1, X_2) be the Lie algebra generated by X_1 and X_2. i.e. the smallest Lie sub-algebra of Vec(M) containing X_1 and X_2. Fix $q \in M$ and let*

$$\text{Lie}_q(X_1, X_2) = \{X(q) \mid X \in \text{Lie}(X_1, X_2)\}.$$

We say that (X_1, X_2) satisfies the Hörmander condition if Lie$_q(X_1, X_2) = T_qM$ for every $q \in M$. In this case we call the triple (M, X_1, X_2) a trivializable almost-Riemannian structure*. The pair (X_1, X_2) is called the* generating frame *of the almost-Riemannian structure.*

Trivializable almost-Riemannian structures are particular cases of rank-varying sub-Riemannian structures. These can be defined in terms of Euclidean bundles or in terms of sub-moduli of the modulus of vector fields, see for instance [2, 3, 4]. These are more general definitions than 1.1, since they permit to study structures that are defined by a pair of vector fields only locally, and more intrinsic, because they permit to avoid fixing the generating frame. However for the purpose of this paper it is more simple to work with structures defined as in Definition 1.1.

Definition 1.2. *Let $\blacktriangle(q) = span\{X_1(q), X_2(q)\}$. For every $v \in \blacktriangle(q)$ we define*

$$\|v\| = \min_{u_1,u_2\in\mathbb{R}} \left\{ \sqrt{u_1^2 + u_2^2} \,\middle|\, v = u_1X_1(q) + u_2X_2(q) \right\}.$$

Notice that, thanks to the Hörmander condition, $\blacktriangle(q)$ is either one or two dimensional. The set where $\blacktriangle(q)$ is one dimensional is called the *singular set* and it is denoted by $\mathcal{Z}$. Generically the singular set is a one dimensional sub-manifold of M and beside isolated points $\blacktriangle(q)$, $q \in \mathcal{Z}$, is not tangent to $\mathcal{Z}$, see [11]. Notice that the presence of the *min* is necessary since, where $\blacktriangle$ is one dimensional, there is more than one choice for u_1 and u_2 such that $v = u_1X_1(q) + u_2X_2(q)$.

Definition 1.3. *A Lipschitz curve $q(\cdot) : [0, T] \to M$ is said to be* admissible *for the almost-Riemannian structure if there exist two functions $u_1(\cdot), u_1(\cdot) \in L^\infty([0, T], \mathbb{R})$ such that $\dot{q}(t) = u_1(t)X_1(q(t)) + u_2(t)X_2(q(t))$ for a.e. $t \in [0, T]$. By definition, the* length *of an admissible curve is*

$$\ell(q(\cdot)) = \int_0^T \|\dot{q}(t)\| dt.$$

One can prove that the length of an admissible curve is well-defined since the map $t \mapsto \|\dot{q}(t)\|$ is measurable [2].

Definition 1.4. *Given $q_1, q_2 \in M$ we define their Carnot-Carathéodory distance as*

$$d(q_1, q_2) = \inf \Big\{ \ell(q(\cdot)) \mid q(0) = q_1, q(T) = q_2, q(\cdot) : [0, T] \to M \text{ admissible} \Big\}.$$

In this definition T can be fixed or not, since the length $\ell(q(\cdot))$ is invariant by reparametrization of the curve.

It is well-known that the distance d is well-defined (Chow Theorem [18]) and endows the manifold with a metric structure compatible with its original topology, see [2, 8, 27]. When $\mathcal{Z}$ is empty, the Carnot-Carathéodory distance is the Riemannian distance for which (X_1, X_2) is an orthonormal frame.

Notice that if (M, X_1, X_2) is an almost-Riemannian structure, then for every smooth function $\theta : M \to \mathbb{S}^1$ the triple (M, Y_1, Y_2), where

$$\begin{cases} Y_1(q) = \cos\theta(q) X_1(q) - \sin\theta(q) X_2(q), \\ Y_2(q) = \sin\theta(q) X_1(q) + \cos\theta(q) X_2(q), \end{cases}$$

defines an equivalent almost-Riemannian structure. Here, by equivalent we mean that (X_1, X_2) and (Y_1, Y_2) define the same $\blacktriangle(q)$, the same norm of vectors in $\blacktriangle(q)$, and hence the same distance.

Given an almost-Riemannian structure (M, X_1, X_2) and a point $q_0 \in M$ it is always possible to find a neighborhood U of q_0 such that $(M, X_1, X_2)|_U$ is equivalent to (U, Y_1, Y_2), where, in coordinates (x, y),

$$Y_1(x, y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad Y_2(x, y) = \begin{pmatrix} 0 \\ f(x, y) \end{pmatrix},$$

for some smooth function f.

With this choice, the singular set $\mathcal{Z}$ is the zero-level set of f. On $U \setminus \mathcal{Z}$ the Riemannian metric g, the area element $d\omega$, and the Gaussian curvature K are

$$g(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{f(x,y)^2} \end{pmatrix},$$

$$d\omega = \frac{1}{|f(x, y)|} dx\, dy,$$

$$K(x, y) = \frac{f(x, y)\partial_x^2 f(x, y) - 2\partial_x f(x, y)^2}{f(x, y)^2}.$$

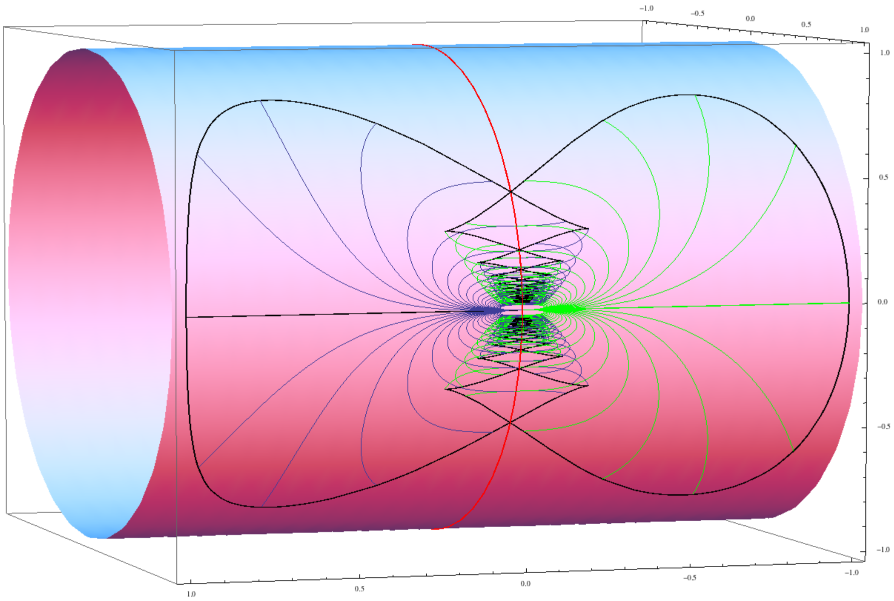

Figure 1.1: The geodesics for the Grushin cylinder starting from the singular set (red circle), for $t \in [0,1]$. The black (self-intersecting) curve line is the wave front (i.e., the end point of all geodesics at time 1). For the explicit expression of these geodesics see for instance [13].

The gradient of a smooth function φ is

$$\operatorname{grad}\varphi = \partial_x\varphi + f^2\partial_y\varphi,$$

and the divergence of a vector field $V = (V^1, V^2)$ is

$$\operatorname{div}(V) = \partial_x V^1 + \partial_y V^2 - \frac{\partial_x f}{f}V^1 - \frac{\partial_y f}{f}V^2.$$

As a consequence, the Laplace-Beltrami operator is

$$\Delta\varphi := \operatorname{div}\operatorname{grad}\varphi = \partial_x^2\varphi + f^2\partial_y^2\varphi - \frac{\partial_x f}{f}\partial_x\varphi + f(\partial_y f)\partial_y\varphi.$$

Notice that this operator is not well-defined on $\mathcal{Z}$ as a consequence of the divergence of the volume.

In the following we introduce the two main structures studied in this paper and we describe some of their properties.

1.1.1. Grushin almost-Riemannian structure on the cylinder

The generating frame on $M = \mathbb{R} \times \mathbb{S}^1$

$$X_1(x,\theta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad X_2(x,\theta) = \begin{pmatrix} 0 \\ x \end{pmatrix},$$

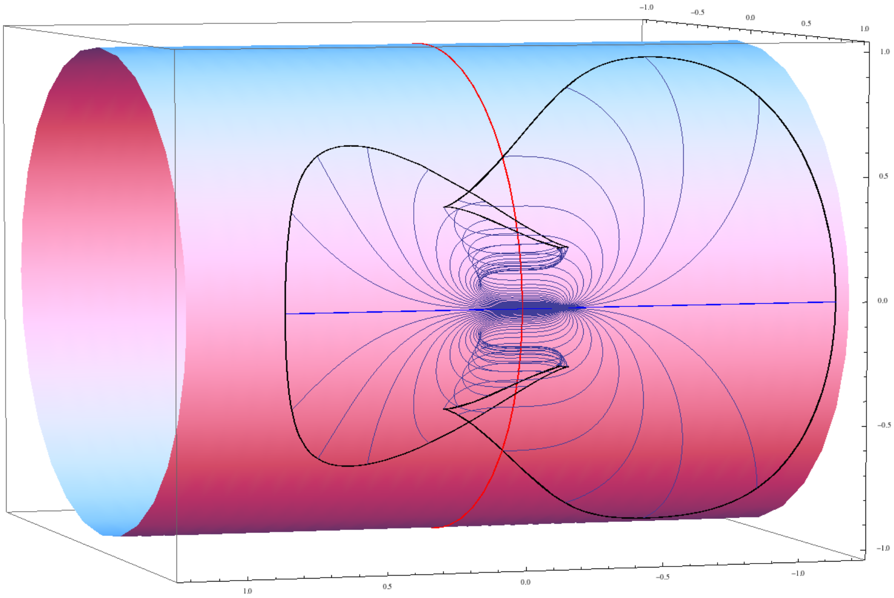

Figure 1.2: The geodesics for the Grushin cylinder starting from the the point $(-0.15, 0)$, for $t \in [0, 1]$. Notice that they cross the singular set (red circle) with no singularities. For the explicit expression of these geodesics see for instance [13].

defines the so-called Grushin almost-Riemannian structure on the cylinder. For this structure the singular set is $\mathcal{Z} = \{0\} \times \mathbb{S}^1$. On $M \setminus \mathcal{Z}$ the Riemannian metric g, the area element $d\omega$, and the Gaussian curvature K are

$$g(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{x^2} \end{pmatrix},$$

$$d\omega = \frac{1}{|x|} dx\, dy,$$

$$K(x, y) = -\frac{2}{x^2}.$$

Notice that $d\omega$ is not integrable on any open set intersecting the θ axis.

The associated Laplace-Beltrami operator on $L^2(M \setminus \mathcal{Z})$ is

$$\Delta\, u = \partial_x^2 u - \frac{1}{x} \partial_x u + x^2 \partial_\theta^2 u. \tag{1.2}$$

With domain $\mathrm{C}_\mathrm{c}^\infty(M \setminus \mathcal{Z})$, this operator is essentially self-adjoint in $L^2(M, d\omega)$. Hence as shown in [13] it separates in the direct sum of its restrictions to $M_\pm = \mathbb{R}_\pm \times \mathbb{S}^1$. Therefore, w.l.o.g. we will focus on Δ on M_+.

Notice that the Grushin metric restricted to M_+ is not geodesically complete, since geodesics can exit M_+ in finite time. Moreover, even if the heat does not flow from M_+ to M_-, due to the self-adjointness of Δ restricted to M_+ we have that M_+ is not stochastically complete (roughly speaking, the L^1 norm of solutions decreases in time). The same happens for M_-. For further details see [13, 14].

1.1.2. Grushin almost-Riemannian structure on the sphere

The Grushin almost-Riemannian structure on the sphere is the trivializable almost-Riemannian structure obtained by taking as generating frame two unitary rotations along two orthogonal axis. More precisely, $\mathbb{S}^2 = \{y_1^2 + y_2^2 + y_3^2 = 1\}$ and

$$X_1 = \begin{pmatrix} 0 \\ -y_3 \\ y_2 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} -y_3 \\ 0 \\ y_1 \end{pmatrix}.$$

Passing in spherical coordinates $(y_1, y_2, y_3) = (\cos x \cos \phi, \cos x \sin \phi, \sin x)$ and rotating X_1, X_2 by

$$\begin{cases} Y_1(q) = \cos(\phi - \pi/2) X_1(q) - \sin(\phi - \pi/2) X_2(q), \\ Y_2(q) = \sin(\phi - \pi/2) X_1(q) + \cos(\phi - \pi/2) X_2(q), \end{cases}$$

we obtain the generating frame

$$Y_1(x, \phi) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad Y_2(x, \phi) = \begin{pmatrix} 0 \\ \tan x \end{pmatrix}.$$

For this structure the singular set is $\mathcal{Z} = \{0\} \times \mathbb{S}^1$, while the singularities for $x = \pm\pi/2$ are apparent and due to the choice of the coordinates. On $\mathbb{S}^2 \setminus \mathcal{Z}$ the Riemannian metric g, the area element $d\omega$, and the Gaussian curvature K are

$$g(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\tan(x)^2} \end{pmatrix},$$
$$d\omega = \frac{1}{|\tan(x)|}\, dx\, d\phi,$$
$$K(x, y) = -\frac{2}{\sin(x)^2}.$$

Notice that $d\omega$ is not integrable on $\mathbb{S}^2$.

The associated Laplace-Beltrami operator on $L^2(\mathbb{S}^2 \setminus \mathcal{Z})$ is

$$\Delta u = \partial_x^2 u - \frac{1}{\sin(x)\cos(x)}\, \partial_x u + (\tan x)^2 \partial_\phi^2 u. \tag{1.3}$$

This operator is essentially self-adjoint with domain $\mathrm{C}_c^\infty(\mathbb{S}^2 \setminus \mathcal{Z})$ in $L^2(\mathbb{S}^2, d\omega)$ and its spectrum is purely discrete as shown in [13]. Similarly to the cylinder case, this operator separates in the direct sum of its restrictions to the north and south hemispheres $S_\pm$, cutted at the equatorial singularity. Thus, we will restrict to consider Δ on the north hemisphere S_+.

This almost-Riemannian metric has been first defined in [17].

1.2. Main results

We now state the results, proved in the rest of the paper, regarding spectral properties of the Laplace-Beltrami operator associated with the Grushin metric on the cyilinder and on the sphere, both without and with an Aharonov-Bohm magnetic field.

Note that we use the convention $0 \notin \mathbb{N}$. When needed, we will denote $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$

1.2.1. Spectra of the Laplace-Beltrami operators

In the following theorem we describe explicitly the spectrum of the Laplace-Beltrami operator on the Grushin cyilinder.

Theorem 1.1 (Grushin cylinder case)**.** *The operator $-\Delta$ on $L^2(M_+)$, defined in (1.2), has absolutely continuous spectrum $\sigma(-\Delta) = [0, \infty)$ with embedded discrete spectrum*

$$\sigma_{\mathrm{d}}(-\Delta) = \{\lambda_{n,k} = 4|k|n \mid n \in \mathbb{N},\, k \in \mathbb{Z} \setminus \{0\}\}\,.$$

The corresponding eigenfunctions are given by

$$\psi_{n,k}(x, \theta) = e^{ik\theta}\frac{1}{x}W_{n,\frac{1}{2}}(|k|x^2),$$

where $W_{\nu,\mu}$ is the Whittaker W-function of parameters ν and μ.

Through the above explicit description, it is then possible to calculate the Weyl law for the Grushin cylinder.

Corollary 1.2 (Grushin cylinder case)**.** *The Weyl law with remainder as $E \to +\infty$ is*

$$N(E) = \frac{E}{2}\log(E) + (\gamma - 2\log(2))\,\frac{E}{2} + O(1),$$

where γ is the Euler-Mascheroni constant.

A similar kind of results can be obtained also for the Laplace-Beltrami operator of the Grushin sphere.

Theorem 1.3 (Grushin sphere case)**.** *The operator $-\Delta$ on $L^2(S_+)$, defined in (1.3), has purely discrete spectrum*

$$\sigma(-\Delta) := \{\lambda_{n,k} := 4n(n + |k|) \mid n \in \mathbb{N},\, k \in \mathbb{Z}\}\,.$$

The corresponding eigenfunctions are given by

$$\psi_{n,k}(x, \phi) = e^{ik\left(\phi+\frac{\pi}{2}\right)}\cos(x)^k F\left(-(n+1),\; n+k+1;\; 1+k;\; \cos(x)^2\right)$$

where $F(a, b; c; x)$ is the Gauss Hypergeometric function with parameters a, b, c.

Corollary 1.4 (Grushin sphere case)**.** *The Weyl law with remainder as $E \to +\infty$ is*

$$N(E) = \frac{E}{4}\log(E) + \left(\gamma - \log(2) - \frac{1}{2}\right)\frac{E}{2} + O(\sqrt{E}),$$

where γ is the Euler-Mascheroni constant.

1.2.2. Spectra of the Aharonov-Bohm perturbed Laplace-Beltrami operator

To mimic the Ahronov-Bohm effect for the Laplace-Beltrami operator in the Grushin cylinder we consider the connection (e.g. a one-form) $\omega^b = -ib\ d\theta,\ b \in$. A computation like the one described in Section A.1 leads to the following expression for the associated magnetic Laplace-Beltrami operator on the Grushin cylinder,

$$\Delta^b = \partial_x^2 - \frac{1}{x}\partial_x + |x|^2(\partial_\theta^2 - 2ib\ \partial_\theta - b^2).$$

As expected, for $b = 0$ this coincides with Δ.

As first results, we obtain the following explicit description of the spectrum of the operator Δ^b, which will allow to compute the corresponding Weyl law.

Theorem 1.5 (Grushin cylinder case)**.** *The operator $-\Delta^b$ on $L^2(M_+)$ has a non-empty discrete spectral component*

$$\sigma_{\mathrm{d}}(-\Delta^b) = \left\{\lambda_{n,k}^b := 4n|k-b| \mid n \in \mathbb{N},\ k \in \mathbb{Z} \setminus \{b\}\right\}. \tag{1.4}$$

When $b \in \mathbb{Z}$ the operator has in addition absolutely continuous spectrum $[0, +\infty)$. When $b \notin \mathbb{Z}$ the spectrum has no absolutely continuous part. In any case, the eigenfunctions are

$$\psi_{n,k}^b(x,\theta) = e^{ik\theta}\frac{1}{x}W_{n,\frac{1}{2}}(|k-b|x^2).$$

Corollary 1.6. *If $b \in \mathbb{Z}$, the Weyl law is the one of Corollary 1.2. If $b \notin \mathbb{Z}$, let $\kappa \in \mathbb{Z}$ be the closest integer to b. Then, the Weyl law with remainder as $E \to +\infty$ is*

$$N(E) = \frac{E}{2}\log(E) + \frac{E}{2}\left(\frac{1}{2|\kappa-b|} + \gamma - 2\log(2) - \frac{\psi(1-|\kappa-b|) + \psi(1+|\kappa-b|)}{2}\right) + O(1),$$

where γ is the Euler-Mascheroni constant and $\psi(x)$ is the digamma function. Here, the $O(1)$ is uniformly bounded with respect to b.

Remark 1.1. *Notice that $N(E)$ diverges for $b \to \kappa$ since, in this limit, part of the discrete spectrum degenerates and gives rise to an absolutely continuous one.*

For this operator, we can also explicitly describe the degeneracy of the spectrum, depending on the value of b.

Theorem 1.7 (Degeneracy of the spectrum in the Grushin cylinder case)**.** *Let $d(n)$ denote the number of divisors of n. Then,*

- *If $b \in \mathbb{R} \setminus \mathbb{Q}$, the spectrum is simple.*
- *If $b \in \mathbb{Q}$, the discrete spectrum is degenerate in the following sense: each eigenvalue λ has multiplicity bounded from above by $2d(\lambda/4)$.*

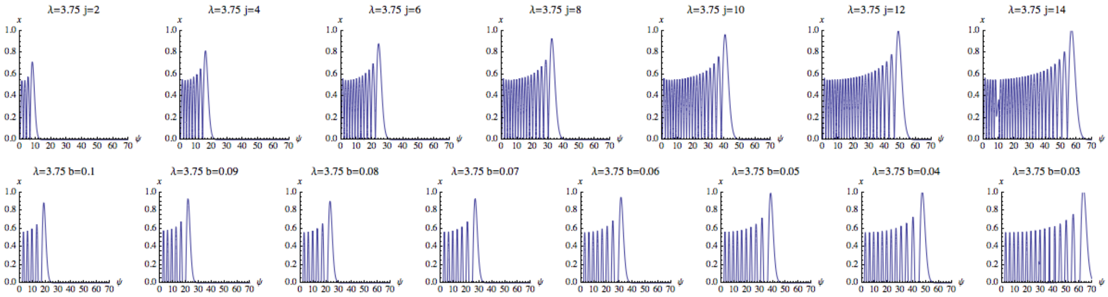

Figure 1.3: The first row shows the spreading of the projection onto $\theta = 0$ of $\psi^{b_j}_{n_j,0}(x)$ as j increases for $\lambda = 3.75$. The second row shows the spreading of the projection onto $\theta = 0$ of $\psi^b_{n(b),0}(x)$ as $b \to 0$ for $\lambda = 3.75$. See Theorem 1.9 and Remark 1.3

- *If $b \in \mathbb{Z}$, the eigenvalues achieve the maximal degeneracy and the multiplicity is exactly*

$$\begin{cases} 2d(\lambda/4), & \text{if } \lambda/4 \text{ is odd}, \\ 2d(\lambda/4) - 2, & \text{if } \lambda/4 \text{ is even}, \end{cases}$$

(in particular it is bounded below by 2).

Remark 1.2. *A direct consequence of the previous theorem is that the maximal multiplicity of the eigenvalues has very slow growth. In fact, it is well known [5] that as $n \to \infty$*

$$d(n) = o(n^\epsilon), \qquad \text{for any } \epsilon > 0.$$

Finally, we can give deeper information on the decompactification of the spectrum in the limit $b \to k$ and the corresponding degeneration of the eigenfuctions. As an immediate consequence of Theorem 1.5 we have the following.

Corollary 1.8 (Decompactification of the spectrum on the Grushin cylinder)**.** *Fix $k \in \mathbb{Z}$. Then, for every $n \in \mathbb{N}$, the spacing between the eigenvalues*

$$|\lambda^b_{n,k} - \lambda^b_{n-1,k}| \to 0 \text{ as } b \to k.$$

Moreover, for any fixed interval $I = [x_1, x_2] \subset [0, \infty)$ and any $N \in \mathbb{N}$

$$\#\{n \in \mathbb{N} \mid \lambda^b_{n,k} \in I\} \geq N \text{ as } b \to k.$$

Theorem 1.9 (Degeneration of the eigenfunctions on the Grushin cylinder)**.** *Fix $k \in \mathbb{Z}$. Then for any $\lambda \in \mathbb{Q}$, $\lambda > 0$, there exist a sequence of pairs $(b_j, n_j) \in \left(k - \frac{1}{2}, k + \frac{1}{2}\right) \times \mathbb{N}$, with $b_j \to k$ and $n_j \to \infty$, such that*

$$\psi^{b_j}_{n_j,k}(x,\theta) \to e^{ik\theta} \frac{\sqrt{\lambda}}{2} J_1(\sqrt{\lambda} x) \tag{1.5}$$

uniformly on compact sets, where $J_\nu(z)$ is the Bessel function of the first kind of order ν. The limit function on the r.h.s. is the generalized eigenfunction of Δ^b with generalized eigenvalue λ (see Remark 2.1).

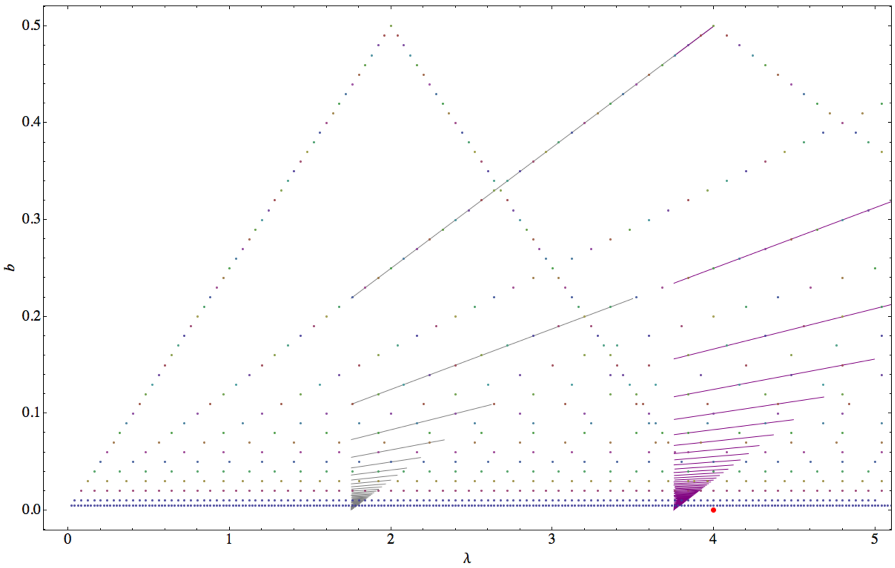

Figure 1.4: The dots correspond to the eigenvalues up to energy 5 for some values of b as it gets closer to $\kappa = 0$. The thick red dot represents the only embedded eigenvalue $\lambda = 4$ of the operator with $b = 0$ up to energy 5. The grey line is the the curve $\lambda^b_{n(b),\kappa}$ (see Remark 1.3) converging to 1.75 as $b \to \kappa$. The purple one is the curve $\lambda^b_{n(b),\kappa}$ converging to 3.75.

Remark 1.3. *Theorem 1.9 can be rewritten as follows. For every $\lambda > 0$, let*

$$n(b) := 2\left\lceil \frac{\lambda}{8|b-k|} \right\rceil. \tag{1.6}$$

Then

$$\lim_{b\to k} \psi^b_{n(b),k}(x,\theta) = e^{ik\theta}\frac{\sqrt{\lambda}}{2}J_1(\sqrt{\lambda}x)$$

uniformly on compact sets. The proof is similar to the one of Theorem 1.9 with n_j replaced by $n(b)$.

Remark 1.4. *Figure 1.3 shows the collapse of the eigenfunctions to the generalized eigenfunctions, while Figure 1.4 shows the collapse of the eigenvalues to the continuous spectrum.*

On the Grushin sphere we consider the magnetic Laplace-Beltrami operator induced by the magnetic vector potential $\omega_b = -ib\ d\phi$ on the north emisphere of $\mathbb{S}^2$ with removed north pole S°_+ and Dirichlet boundary conditions. See Section 3.3 for more details. The corresponding operator is

$$\Delta^b = \partial_x^2 - \frac{1}{\sin(x)\cos(x)}\partial_x + \tan(x)^2\left(\partial_\phi^2 - 2ib\partial_\phi - b^2\right). \tag{1.7}$$

We then have the description of the spectrum and the corresponding Weyl law, depending on b.

Theorem 1.10 (Grushin sphere case)**.** *The operator $-\Delta^b$ defined in (1.7) and acting on $L^2(S_+^\circ)$, has purely discrete spectrum*

$$\sigma(-\Delta^b) = \{\lambda_{n,k} = 4n(n + |k-b|) \mid n \in \mathbb{N}, k \in \mathbb{Z}\}.$$

The corresponding eigenfunctions are given by

$$\psi_{n,k}(x,\phi) = e^{ik\phi} e^{i(k-b)\frac{\pi}{2}} \cos(x)^{k-b} F\left(-(n+1), n+k-b+1; 1+k-b; cos(x)^2\right).$$

As a consequence of the explicit description of the spectrum, we obtain the following.

Corollary 1.11. *The Weyl law with remainder as $E \to +\infty$ is*

$$N(E) = \frac{E}{4}\log(E) + \left(\gamma - \log(2) - \frac{1}{2}\right) E + O(\sqrt{E}),$$

where γ is the Euler-Mascheroni constant, and the big O is uniformly bounded w.r.t. b.

The degeneracy of the spectrum for the Laplace-Beltrami operator on Grushin sphere is less explicit than the one in Theorem 1.8. Indeed, we can only obtain the following result. A brief but more detailed discussion of the topic can be found in Section 3.

Corollary 1.12 (Degeneracy of the spectrum in the Grushin sphere case)**.** *If $b \in \mathbb{R} \setminus \mathbb{Q}$ the spectrum is simple, if $b \in \mathbb{Q}$ the spectrum is finitely degenerate.*

1.2.3. More general results on conic and anti-conic type surfaces

The effect of the Aharonov-Bohm perturbation can be even stronger than the one we saw in almost-Riemannian geometry when considering the more general structures studied in [14]. Consider on $M := (\mathbb{R} \setminus \{0\}) \times \mathbb{S}^1$ the orthonormal frame

$$X_1(x,\theta) := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad X_2(x,\theta) := \begin{pmatrix} 0 \\ |x|^\alpha \end{pmatrix}.$$

Here $(x,\theta) \in M$ and $\alpha \in \mathbb{R}$ is a parameter. In other words we are defining a Riemannian manifold (M, g_α) where the metric in matrix notation is given by

$$g_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & |x|^{-2\alpha} \end{pmatrix}.$$

The corresponding volume form on M is $d\omega_\alpha = |x|^{-\alpha} dx d\theta$

For any $\alpha \geq 0$, this metric can be completed in $\mathbb{R} \times \mathbb{S}^1$ in such a way that the corresponding distance induces the topology of a cylinder. In particular, when α is a positive integer this is a trivializable almost-Riemannian structure in the sense of Definition 1.1.

When $\alpha < 0$ this metric can be extended in $\mathbb{R} \times \mathbb{S}^1 / \sim$, where $p \sim q$ if $p = q$ or $p, q \in \{0\} \times \mathbb{S}^1$. The corresponding distance induces on $\mathbb{R} \times \mathbb{S}^1 / \sim$ the topology of a cone.

The Laplace-Beltrami operator on $L^2(M, d\omega_\alpha)$ is

$$\Delta_\alpha = \partial_x^2 + |x|^{2\alpha}\partial_\theta^2 - \frac{\alpha}{x}\partial_x.$$

As in [14] we consider the Fourier decomposition $L^2(M, d\omega_\alpha) = \bigoplus_{k=0}^{\infty} H_k \simeq L^2(\mathbb{R} \setminus \{0\}, |x|^{-\alpha}dx)$ that yields on each H_k the operator

$$\hat{\Delta}_{\alpha,k} = \partial_x^2 - \frac{\alpha}{x}\partial_x - |x|^{2\alpha}k^2. \tag{1.8}$$

Proposition 1.13 ([14]). *The operator Δ_α on $L^2(M, d\omega_\alpha)$ with domain $\mathrm{C}_\mathrm{c}^\infty(M)$ is essentially self-adjoint if and only if $\alpha \geq 1$ or $\alpha \leq -3$.*

Moreover, on $\mathrm{C}_\mathrm{c}^\infty(\mathbb{R} \setminus \{0\})$:

- *if $-3 < \alpha \leq -1$ for every $k \neq 0$ the operator $\hat{\Delta}_{\alpha,k}$ is essentially self-adjoint, while $\hat{\Delta}_{\alpha,0}$ is not;*
- *if $-1 < \alpha < 1$, every $\hat{\Delta}_{\alpha,k}$ is not essentially self-adjoint.*

As in the previous section, we turn on the Aharonov-Bohm perturbation considering the connection $\omega_b = -ibd\theta$. The corresponding Laplace-Beltrami operator is

$$\Delta_b = \partial_x^2 + |x|^{2\alpha}\partial_\theta^2 + |x|^{2\alpha}\left(\partial_\theta^2 - 2ib\partial_\theta - b^2\right).$$

Through the previous Fourier decomposition we get for each H_k the operator

$$\hat{\Delta}_{\alpha,k}^b = \partial_x^2 - \frac{\alpha}{x}\partial_x - |x|^{2\alpha}(b-k)^2. \tag{1.9}$$

The proof of Proposition 1.13 applied to (1.9) yields the following.

Theorem 1.14. *If $b \notin \mathbb{Z}$, the operator Δ_α^b with domain $\mathrm{C}_\mathrm{c}^\infty(M)$ is essentially self-adjoint in $L^2(M, d\omega)$ if $|\alpha| \geq 1$, and Proposition 1.13 still applies for $|\alpha| < 1$.*

On the other hand, if $b \in \mathbb{Z}$, Proposition 1.13 holds with the following change: if $-3 < \alpha \leq -1$ for every $k \neq b$ the operator $\hat{\Delta}_{\alpha,k}^b$ is essentially self-adjoint, while $\hat{\Delta}_{\alpha,b}^b$ is not.

The Aharonov-Bohm effect on the spectrum extends to this more general setting as follows.

Theorem 1.15. *For $\alpha > 0$, the operator $-\Delta_\alpha^b$ on $L^2(M, d\omega_\alpha)$ has a non-empty discrete spectral component $\sigma_\mathrm{d}(-\Delta_\alpha^b) \subset [0, +\infty)$.*

When $b \in \mathbb{Z}$ the operator has absolutely continuous spectrum $[0, +\infty)$ with embedded discrete spectrum. When $b \notin \mathbb{Z}$ the spectrum has no absolutely continuous part.

Proof. For $b \neq k$, the spectrum of the operators $\hat{\Delta}_{\alpha,k}^b$ (or of any of their self-adjoint extensions) is purely discrete (see e.g. [33, Chapter 5]). For $b = k$, on the other hand, the essential spectrum of $\hat{\Delta}_{\alpha,k}^b$ is non-empty and in particular it contains the half line $[0, +\infty)$ (see e.g. [34, Theorem 15.3]). □

The previous theorems suggest that, for $b = 0$ and $\alpha > 0$, the 0-th Fourier component, is the only responsible for the continuous spectrum. The Aharonov-Bohm perturbation, when $b \in \mathbb{Z}$, shift this role to the b-th Fourier component. When $b \notin \mathbb{Z}$ no Fourier component produces a continuous spectrum. This is a well-known phenomena in the case of asymptotically hyperbolic manifolds with finite volume [22], but completely new in this setting.

Further study of the cases $\alpha < 0$ is outside the scope of this paper. As a side remark, note that the case $\alpha = -1$ considered on $\mathbb{R}_+ \times \mathbb{S}^1$ coincides with the standard Aharonov-Bohm Laplacian in polar coordinates. Moreover, in the case $\alpha = -1/2$, $-\Delta_\alpha^b$ has discrete spectrum accumulating at 0 and absolutely continuous spectrum in $[0, +\infty)$. When $b \notin \mathbb{Z}$ an additional family of eigenvalues accumulating at 0 appears.

1.3. Structure of the paper

The paper is organized as follows. In Section 2 we study the Grushin metric on the cylinder. In Section 3 we study the Grushin metric on the sphere. Finally, in the Appendix we collect some technical details.

2. Spectral analysis and the Ahronov-Bohm effect for the Grushin cylinder

In this section we study the spectrum of the Grushin cylinder, with or without an Aharonov-Bohm magnetic field.

2.1. Grushin metric and associated Laplace-Beltrami operator

Recall from Section 1.1.1 that the Grushin almost-Riemannian structure on the cylinder defines on $M_+ \cup M_- = (\mathbb{R} \setminus \{0\}) \times \mathbb{S}^1$ the metric $g = dx^2 + x^{-2} d\theta^2$, the volume $\mathrm{d}\omega = \sqrt{|g|}\, dx\, d\theta = \frac{1}{|x|}\, dx\, d\theta$ and the Laplace-Beltrami operator

$$\Delta\, u \coloneqq \operatorname{div} \operatorname{grad} u = \partial_x^2 u - \frac{1}{x} \partial_x u + x^2 \partial_\theta^2 u.$$

As already mentioned, this operator with domain $\mathrm{C}_\mathrm{c}^\infty((\mathbb{R} \setminus \{0\}) \times \mathbb{S}^1)$ is essentially self-adjoint in $L^2(M, d\omega)$ and hence the evolutions on the two sides of the singularity are decoupled. Thus, we will henceforth consider the self-adjoint operator Δ acting only on M_+. Namely, the domain $\mathrm{D}(\Delta)$ will be the closure w.r.t. the graph norm of $\mathrm{C}_\mathrm{c}^\infty(M_+)$.

Recall the Fourier decomposition of $\mathrm{L}^2(M_+, \mathrm{d}\omega)$ w.r.t. the varaible θ introduced in Section 1.2.3,

$$\mathrm{L}^2(M_+, \mathrm{d}\omega) = \bigoplus_{k \in \mathbb{Z}} H_k, \qquad \text{where } H_k \simeq \mathrm{L}^2\left(\mathbb{R}_+, \frac{1}{x} dx\right).$$

The operator Δ decomposes as $\Delta = \bigoplus_{k \in \mathbb{Z}} \widehat{\Delta}_k$, where

$$\widehat{\Delta}_k = \partial_x^2 - \frac{1}{x} \partial_x - k^2 x^2.$$

Since Δ is essentially self-adjoint, each $\widehat{\Delta}_k$ is a self-adjoint operator on the closure w.r.t. the graph norm of $\mathrm{C}_c^\infty(\mathbb{R}_+)$.

Consider the unitary transformation $U : \mathrm{L}^2\left(\mathbb{R}_+, \frac{1}{x}dx\right) \to \mathrm{L}^2(\mathbb{R}_+, dx)$ defined by $Uv(x) := \sqrt{x}v(x)$. The operator $\widehat{\Delta}_k$ is then transformed to

$$L_k := U\widehat{\Delta}_k U^{-1} = \partial_x^2 - \frac{3}{4}\frac{1}{x^2} - k^2x^2, \quad \mathrm{D}(L_k) = U\,\mathrm{D}(\widehat{\Delta}_k). \tag{2.1}$$

2.2. Spectral properties of the Laplace-Beltrami operator

Since the spectrum is invariant under unitary transformations, it follows from (A.1) and (A.2) that we can reduce the study of the spectrum of Δ to that of the operators L_k. Exploiting this reduction we can easily prove Theorem 1.1.

Proof of Theorem 1.1. The operator $-L_0$ is the Schrödinger operator on the real line with a Calogero potential of strength $3/4$. It is well-known that this operator has continuous spectrum $[0, +\infty)$, see e.g., [31, Sec. VIII.10].

Let now $k \neq 0$ and let us compute the solutions of the eigenvalue problem

$$(L_k - \lambda)u = 0 \iff (\widehat{\Delta}_k - \lambda)U^{-1}u = 0. \tag{2.2}$$

Through the change of variables $|k|x^2 \mapsto z$ and multiplying by $4(k^2)z$, we obtain

$$\partial_z^2 v(z) + \left(-\frac{1}{4} + \frac{\lambda}{4z|k|}\right) = 0.$$

This is the well-known Whittaker equation, whose solutions are the Whittaker functions $M_{\frac{\lambda}{4|k|},\frac{1}{2}}(z)$ and $W_{\frac{\lambda}{4|k|},\frac{1}{2}}(z)$. The solutions of the eigenvalue problem (2.2) are then

$$u_1(x) = \frac{1}{\sqrt{x}} M_{\frac{\lambda}{4|k|},\frac{1}{2}}(|k|x^2), \qquad u_2(x) = \frac{1}{\sqrt{x}} W_{\frac{\lambda}{4|k|},\frac{1}{2}}(|k|x^2).$$

Through the asymptotic expansions of $M_{\nu,\mu}$ and $W_{\nu,\mu}$ (see e.g., [7]) one easily sees that u_1 is never square-integrable near infinity. On the other hand, $u_2 \in \mathrm{L}^2(\mathbb{R}_+)$ if and only if there exists a non-negative integer ℓ such that $-\ell = \frac{1}{2} - \nu + \mu = \frac{1}{2} - \frac{\lambda}{4|k|} + \frac{1}{2}$. Namely, for any $k \in \mathbb{N}$ there exists a sequence $\{\lambda_{n,k} = 4|k|n\}_{n\in\mathbb{N}}$ of eigenvalues with (non-normalized) eigenfunction $x \mapsto \psi_{n,k}(x) = W_{n,\frac{1}{2}}(|k|x^2)/\sqrt{x}$.

Let $k = 0$. Then, the operator L_0 given by (2.1) can be interpreted as a Laplace operator with a relatively infinitesimally-bounded perturbation. It is a well known result [31] that its spectrum is purely absolutely continuous and equal to $[0, \infty)$.

Finally, the statement follows from the definition of U^{-1} and the relations (A.1) and (A.2). $\square$

Remark 2.1. *Observe that* (2.1) *can be explicitly solved, it's solutions being of the form*

$$\begin{cases} c_1 x^{3/2} + \frac{c_2}{\sqrt{x}} & \text{for } \lambda = 0, \\ c_1\sqrt{x}\, J_1(\sqrt{\lambda}x) + c_2\sqrt{x}\, Y_1(\sqrt{\lambda}x) & \text{for } \lambda > 0, \end{cases} \tag{2.3}$$

where J_1 and Y_1 are the Bessel functions of order 1. In particular, for $\lambda \geq 0$ one has the explicit form of the generalized eigenfunctions of the absolutely continuous spectrum of L_0.

With the eigenvalue counting function $N(E)$ defined as in (1.1), we are able to prove Corollary 1.2.

Proof of Corollary 1.2. Obviously, by Theorem 1.1, the following holds,

$$\#\{\lambda \in \sigma_p(-\Delta) \mid \lambda \leq E\} = \#\{(n,k) \in \mathbb{N} \times \mathbb{Z} \setminus \{0\} \mid 4n|k| \leq E\}. \tag{2.4}$$

For fixed $k \in \mathbb{Z} \setminus \{0\}$, this implies that the couples (n,k) admissible in the above, are those such that $n \leq E/(4|k|)$. Moreover, it is clear that for any $|k| > E/4$ there exist no couple (n,k) is admissible. These facts and (2.4) yield the estimation

$$N(E) = \sum_{0<|k|\leq \frac{E}{4}} \frac{E}{4|k|} = \frac{E}{2} \sum_{\ell=1}^{\lfloor E/4 \rfloor} \frac{1}{\ell}.$$

The well-known asymptotic formula (see e.g., [19])

$$\sum_{m=1}^{n} \frac{1}{m} = \log(n) + \gamma + \frac{1}{2n} + O\left(\frac{1}{n^2}\right), \tag{2.5}$$

where γ is the Euler-Mascheroni constant, then implies the following asymptotic estimate as $E \to +\infty$,

$$\begin{aligned} N(E) &= \frac{E}{2}\left(\log\left(\frac{E}{4} + \gamma + O\left(\frac{1}{E}\right)\right)\right) \\ &= \frac{E}{2}\log(E) + (\gamma - 2\log(2))\,\frac{E}{2} + O(1). \end{aligned}$$

□

2.3. Aharonov-Bohm effect

Now we look at the Aharonov-Bohm effect on the Grushin cylinder. As already introduced in Section 1.2.2, the magnetic Laplace-Beltrami operator on M_+ with vector potential $\omega^b = -ib\,d\theta$, $b \in \mathbb{R}$, is

$$\Delta^b = \partial_x^2 - \frac{1}{x}\partial_x + x^2\partial_\theta^2 - 2ibx^2\partial_\theta - b^2x^2.$$

After the transformation U, introduced in Section 2.1, we obtain the following operator acting on $\mathrm{L}^2(M_+, dx\,d\theta)$,

$$L_b = U\,\Delta^b\,U^{-1} = \partial_x^2 - \frac{3}{4}\frac{1}{x^2} + x^2\left(\partial_\theta - ib\right)^2.$$

Through a straightforeward extension of the proof of Theorem 1.1, we immediately get Theorem 1.5. For $b \in \mathbb{Z}$ it is evident that the role of L_0 in the proof of Theorem 1.1 is now taken by L_b.

Proof of Corollary 1.6. W.l.o.g. we restrict ourselves to $b \in (-1/2, 1/2)$, therefore $\kappa = 0$. Clearly, if $b = 0$ the statement reduces to the one of Corollary 1.2. Thus we can assume $b \neq 0$.

Replacing k with $|k-b|$ in the proof of Corollary 1.2 we observe that for $k = 0$ the additional term $E/4|b|$ appears in the count. Thus, we can rewrite the counting function as

$$N(E) = \frac{E}{4} \sum_{k=1}^{\lfloor E/4 \rfloor} \frac{1}{k+b} + \frac{E}{4|b|} + \frac{E}{4} \sum_{k=1}^{\lfloor E/4 \rfloor} \frac{1}{k-b}$$

We now apply the following identity (see e.g., [29])

$$\sum_{k=1}^{n} \frac{1}{k+x} = \psi(n+x+1) - \psi(1+x),$$

and the asymptotic estimate as $x \to \infty$

$$\psi(x+1) = \log(x) + \gamma + \frac{1}{2x} + O\left(\frac{1}{x^2}\right),$$

where $\psi(x)$ is the digamma function and γ is the Euler-Mascheroni constant. By a straightforward computation we obtain

$$\begin{aligned} N(E) &= \frac{E}{4} \left(\psi\left(\lfloor E/4 \rfloor + b + 1\right) - \psi(1+b)\right) + \frac{E}{4|b|} + \frac{E}{4} \left(\psi\left(\lfloor E/4 \rfloor - b + 1\right) - \psi(1-b)\right) \\ &= \frac{E}{2} \log(E) + \frac{E}{2} \left(\frac{1}{2|b|} + \gamma - 2\log(2) - \frac{\psi(1-b) + \psi(1+b)}{2}\right) + O(1). \end{aligned}$$

The general result then follows by shifting the above computation with $b \mapsto |\kappa - b|$. □

We can now precisely determine the degeneracy of the eigenvalues, depending on the value of b.

Proof of Theorem 1.7. The proof is divided in three cases.

Case 1, $b \in \mathbb{R} \setminus \mathbb{Q}$: This immediately implies that $|k-b| \in \mathbb{R} \setminus \mathbb{Q}$. It is then straightforward to show that there exist no $(n', k') \neq (n, k)$ such that $\lambda^b_{n',k'} = \lambda^b_{n,k}$.

Case 2, $b \in \mathbb{Q}$: Let us write $b = p/q$ with $p, q \in \mathbb{Z}$ such that $(p, q) = 1$. Fix (n, k) and $(n', k') \neq (n, k)$ such that $\lambda^b_{n,k} = \lambda^b_{n',k'}$. Then,

$$4n'|qk' - p| = q\lambda^b_{n,k}. \tag{2.6}$$

W.l.o.g. assume that $qk' > p$. Then, since $4n'|qk' - p|$ cannot divide q because $(q, p) = 1$, we have that it must divide $\lambda_{n,k}$.

From $q \neq 1$, $\{|qk' - p| \mid k' \in \mathbb{Z}\} \subseteq (q\mathbb{Z} - p) \subsetneq \mathbb{Z}$, we obtain that the number of couples (n', k') such that $4n'|k' - b| = \lambda^b_{n,k}$ is bounded above by $2d(\lambda^b_{n,k}/4)$, where $d(n)$ denotes the number of divisors of n. In fact, if $|k' - b| = d_1$ for some $d_1 \in \mathbb{Q}$ divides $\lambda^b_{n,k}/4$, then $n' = \lambda^b_{n,k}/(4d_1)$. Observe that, due to the presence of a non integer b in the term $|k - b|$, not all the possible divisors can be considered. However, if a $k' > b$ can be taken, then there exists a $k'' < b$ that will give an additional couple (k'', n').

Case 3, $b \in \mathbb{Z}$: In this case, equation (2.6) reduces to $4n'|k' - b| = \lambda_{n,k}$. Then, for any (n, k) with $k \neq b$, a simple computation shows that

$$\lambda^b_{k,n+b} = \lambda^b_{n,k+b} = \lambda^b_{n,-k+b} = \lambda^b_{k,-n+b}.$$

However if $n|k|$ is even, the combination $n = k = \lambda^K_{n,k+K}/8$ is repeated twice. Therefore the degeneracy is given by

$$\begin{cases} 2d(\lambda/4), & \text{if } \lambda/4 \text{ is odd,} \\ 2d(\lambda/4) - 2, & \text{if } \lambda/4 \text{ is even.} \end{cases}$$

Finally, this degeneracy cannot be achieved for $b \in \mathbb{Q} \setminus \mathbb{Z}$. In fact, it would require $\mathbb{Z} \ni k' = (qn + p)/q$ which is impossible for $(q, p) = 1$. □

Corollary 1.6 suggests that in the limit $b \to k$, the number of eigenvalues in a finite interval explodes. Corollary 1.8 makes this statement more precise, namely

- for any fixed $k \in \mathbb{Z}$ and for any $n \in \mathbb{N}$, the spacing between the eigenvalues

$$|\lambda^b_{n,k} - \lambda^b_{n-1,k}| \to 0 \text{ as } b \to k;$$

- for any fixed interval $I = [x_1, x_2] \subset [0, \infty)$ and any $N \in \mathbb{N}$

$$\#\{n \in \mathbb{N} \mid \lambda^b_{n,k} \in I\} \geq N \text{ as } b \to k.$$

Proof of Corollary 1.8 (Corollary of Theorem 1.5). Observe that

$$|\lambda^b_{n,k} - \lambda^b_{n-1,k}| = 4|k - b|. \tag{2.7}$$

Taking the limit for $k \to b$ in the above yields immediately the first statement.

To prove the second statement, assume w.l.o.g. $k \geq 0$ and define

$$L(b) := \left\lceil \frac{x_1}{4|k-b|} \right\rceil, \quad R(b) := \left\lfloor \frac{x_2}{4|k-b|} \right\rfloor.$$

Then $\lambda^b_{L(b),k} \geq x_1$ and $\lambda^b_{R(b),k} \leq x_2$. If now

$$|k - b| \leq \frac{x_2 - x_1}{4(N+1)},$$

by (2.7) we obtain that

$$\#\{\lambda^b_{i,k} \mid L(b) \leq i \leq R(b)\} \geq N.$$

This completes the proof of the second statement and hence of the corollary. □

This limiting process affects also the eigenfunctions. Theorem 1.9 describes how the spectrum of the k-th Fourier components decompactifies in the limit $b \to k$ and produces the absolutely continuous part of the spectrum.

Proof of Theorem 1.9. Recall that $\psi^b_{n,k}(x,\theta) = e^{ik\theta} W_{n,\frac{1}{2}}(|k-b|x^2)/x$. Since w.l.o.g. we can assume $k = 0$, to complete the proof it suffices to show that

$$W_{n_j,\frac{1}{2}}(|b_j|x^2) \to \frac{\sqrt{\lambda}\,x}{2} J_1(\sqrt{\lambda}x). \tag{2.8}$$

Let us recall the following classical results (see resp. [26] and [7]).

$$W_{n,1/2}(z) = (-1)^{n-1} z e^{-\frac{1}{2}z} L^1_{n-1}(z),$$
$$\lim_{n\to\infty} n^{-\alpha} L^1_n(x/n) = x^{-\frac{1}{2}\alpha} J_\alpha(2\sqrt{x}).$$

Here L^α_n is the generalized Laguerre polynomial of degree n with parameter α and the limit is in the sense of uniform convergence on compact sets.

Define

$$n_j := 2j \quad \text{and} \quad b_j := \frac{\lambda}{4(n_j+1)},$$

so that $\lambda^{b_j}_{n_j+1,0} = \lambda$ for all $j > 0$. Then

$$\begin{aligned}\lim_{j\to\infty} W_{n_j,\frac{1}{2}}(|b_j|x^2) &= \lim_{j\to\infty} \frac{\lambda n_j x^2}{4(n_j+1)} \exp\left(-\frac{1}{2n_j}\frac{\lambda n_j x^2}{4(n_j+1)}\right) n_j^{-1} L^1_{n_j}\left(\frac{1}{n_j}\frac{\lambda n_j x^2}{4(n_j+1)}\right) \\ &= \frac{\sqrt{\lambda}\,x}{2} J_1(\sqrt{\lambda}x).\end{aligned}$$

This completes the proof of (2.8). □

Remark 2.2. *By changing the parity of the sequence n_j used in the previous proof, we could change the sign of the limit in* (1.5).

3. Spectral analysis and Ahronov-Bohm effect for the Grushin sphere

In this section we consider the Grushin almost-Riemannian metric on the sphere introduced in Section 1.1.2.

3.1. Grushin metric and associated Laplace-Beltrami operator

The Grushin almost-Riemannian structure introduced in Section 1.1.2 defines the metric $g = dx^2 + \tan(x)^{-2} d\theta^2$ on $S_+ \cup S_-$, the sphere $\mathbb{S}^2$ without the equatorial line. The natural volume form defined by this metric is $\mathrm{d}\omega = \sqrt{|g|}\, dx\, d\theta = |\tan(x)|^{-1}\, dx\, d\theta$ and the associated Laplace-Beltrami operator is

$$\Delta u = \partial_x^2 u - \frac{1}{\sin(x)\cos(x)}\partial_x u + \tan^2(x)\partial_\phi^2 u$$

As shown in [13], the operator Δ with domain $\mathrm{C}_\mathrm{c}^\infty(S_+ \cup S_-)$ is essentially self-adjoint in $L^2(\mathbb{S}^2, d\omega)$ and hence the evolutions on the two sides of the singularity are decoupled. Moreover it has purely discrete spectrum. In the following we will consider the self-adjoint operator Δ restricted to $\mathrm{L}^2(S_+, \mathrm{d}\omega)$. Namely, the domain $\mathrm{D}(\Delta)$ will be the closure w.r.t. the graph norm of $\mathrm{C}_\mathrm{c}^\infty(S_+)$.

As in the previous section we can separate the space using the orthoromal eigenbase of $\mathbb{S}^1$ getting

$$L^2(S_+, d\omega) = \bigoplus_{k=-\infty}^{\infty} H_k^{S_+}, \quad H_k^{S_+} \simeq L^2([0, \pi/2), \tan(x)dx). \tag{3.1}$$

On each $H_k^{S_+}$ the operator separates as

$$\tilde{\Delta}_k := \partial_x^2 - \frac{1}{\sin(x)\cos(x)}\partial_x - \tan^2(x)k^2.$$

3.2. Spectral properties of the Laplace-Beltrami operator

Exploiting decomposition (3.1), we can describe the spectrum of Δ.

Proof of Theorem 1.3. We look for solutions $\phi \in H_k^{S_+}$ of the eigenvalue equation

$$-\tilde{\Delta}_k \phi(x) = \lambda\phi(x).$$

Since k appears in $\tilde{\Delta}_k$ only squared, the eigenvalues are simmetric with respect to $k = 0$. To simplify the notation, in the following we will assume $k \geq 0$, but the same considerations hold for $k < 0$ substituting $|k|$ to k.

With the change of variables $z = \cos(x)^2$ and writing $\phi(x) = (-z)^{\frac{k}{2}}\varphi(z)$, the eigenvalue equation becomes

$$4(-z)^{\frac{k}{2}}\left(z(1-z)\partial_z^2\varphi(z) + (1+k)(1-z)\partial_z\varphi(z) + \frac{\lambda}{4}\varphi(z)\right) = 0.$$

The equation in bracket is a particular example of the well-known Euler's hypergeometric equations. Two linearly independent solutions can be found in terms of Gauss Hypergeometric Functions $F(a, b; c; z)$ (see [7, Vol. 1, Ch. 2]) as follows:

$$\phi_1(x) = i^{-k}\cos(x)^{-k} F\left(-\frac{k}{2} - \frac{\sqrt{\lambda + k^2}}{2}, -\frac{k}{2} + \frac{\sqrt{\lambda + k^2}}{2}; 1 - k; \cos(x)^2\right),$$

$$\phi_2(x) = i^{k}\cos(x)^{k} F\left(\frac{k}{2} - \frac{\sqrt{\lambda + k^2}}{2}, \frac{k}{2} + \frac{\sqrt{\lambda + k^2}}{2}; 1 + k; \cos(x)^2\right).$$

Notice here that in the case $\frac{k}{2} \pm \frac{\sqrt{\lambda+k^2}}{2}, k-1 \in \mathbb{N}_0$ the first solution is not defined, in fact we are in the so called *degenerate case* and the only regular solution is ϕ_2. Therefore we do not need to introduce the other corresponding linearly independent solution.

A solution is an eigenfunction of the Laplace-Beltrami operator if it is in $H_k^{S_+}$. For this to be true, the solutions has to be square-integrable with respect to the measure $d\omega := \tan(x)^{-1}dx$ near 0. This is equivalent to be $O(\sin(x))$ for $x \to 0$ and, in particular, it is requires that the solution be zero at $x = 0$.

Let us recall that

$$\phi_1(0) = i^{-k} \frac{\Gamma(1-k)}{\Gamma\left(-\frac{k}{2} - \frac{\sqrt{k^2+\lambda}}{2} + 1\right)\Gamma\left(-\frac{k}{2} + \frac{\sqrt{k^2+\lambda}}{2} + 1\right)}, \tag{3.2}$$

$$\phi_2(0) = i^{k} \frac{\Gamma(k+1)}{\Gamma\left(\frac{k}{2} - \frac{\sqrt{k^2+\lambda}}{2} + 1\right)\Gamma\left(\frac{k}{2} + \frac{\sqrt{k^2+\lambda}}{2} + 1\right)}. \tag{3.3}$$

Moreover, observing that

$$\pm\frac{k}{2} + \frac{\sqrt{k^2+\lambda}}{2} \geq 0 \text{ for all } k \in \mathbb{N}_0 \text{ and } \lambda \in \mathbb{R}_+^0,$$

it is immediate to obtain $\phi_1(0) \neq 0$ if $k \geq 1$. By the previous considerations, this implies that $\phi_1 \notin H_k^{S_+}$ for $k \geq 1$. Since $k = 0$ corresponds to the degenerate case, where the two solutions coincide, in the following we can restrict ourselves to consider only ϕ_2.

By (3.3), in order for $\phi_2(0) = 0$ to hold there has to exist $n \in \mathbb{N}_0$ such that λ satisfies

$$\frac{k}{2} + 1 - \frac{\sqrt{k^2+\lambda}}{2} = -n.$$

Solving the above for λ, yields the following expression for the candidate eigenvalue

$$\lambda = \lambda_{k,n}^+ := 4(1+n)(1+n+k).$$

In order to prove that the candidate eigenvalues $\lambda_{k,n}^+$ are indeed eigenvalues, we check the order of convergence of the solutions. For this purpose we use the well-known identity [29, 15.2(ii)] for $a = -m \in \mathbb{Z}_- \cup \{0\}$, $b > 0$ and $c \notin \mathbb{Z}_- \cup \{0\}$

$$F(-m, b; c; z) = \sum_{\ell=0}^{m} (-1)^m \binom{m}{\ell} \frac{(b)_\ell}{(c)_\ell} x^\ell. \tag{3.4}$$

Plugging the values of the parameters for ϕ_2 in the above, and setting $\lambda = \lambda_{k,n}^+$, we obtain

$$F\left(-(n+1), n+k+1; k+1; \cos(x)^2\right) = \sum_{\ell=0}^{n+1} (-1)^{n+1} \binom{n+1}{\ell} \frac{(n+k+1)_\ell}{(k+1)_\ell} \cos(x)^{2\ell}.$$

This shows that ϕ_2 and his derivative have the correct behaviour in 0 and are regular at $\pi/2$, completing the proof.

Finally, in order to obtain the expression of the eigenvalues and eigenfunctions given in the statement, it suffices to replace $n+1$ with n in the definition of $\lambda_{n,k}^b$. The theorem then follows by the symmetry w.r.t. $k = 0$ of the problem. $\square$

We are now in a position to derive the Weyl law for the Laplace-Beltrami operator of the Grushin sphere.

Proof of Corollary 1.4. By the simmetry of the eigenvalue problem w.r.t. $k = 0$, it follows that $N(E) = 2\#\{k \in \mathbb{N},\, n \in \mathbb{N} \mid \lambda_{n,k} \leq E\} + \#\{n \in \mathbb{N} \mid \lambda_{n,0} \leq E\}$. Let $N_0(E)$ be the counting function for this second sum. It is easy to see that $\lambda(0, n) \leq E$ for $n \in \left[0, \lfloor\sqrt{E}/2\rfloor\right]$. Therefore $N_0(E) = O(\sqrt{E})$.

Let $N_+(E)$ be the counting function for positive values of k. A simple computation shows that $\lambda(k, n) \leq E$ if and only if

$$0 < k \leq \left\lfloor \frac{E - 4n^2}{4n} \right\rfloor.$$

Additionally, notice that if $n > \lfloor\sqrt{E}/2\rfloor =: \eta_1(E)$, then $\frac{E-4n^2}{4n} < 0$.

Let

$$K(n) := \frac{E - 4n^2}{4n}. \tag{3.5}$$

Then we have

$$\lfloor K(n)\rfloor \leq \#\left\{k \in \left[0, \frac{E - 4n^2}{4n}\right] \cap \mathbb{N}\right\} \leq \lceil K(n)\rceil,$$

and consequently

$$\sum_{n=1}^{\eta_1(E)} \lfloor K(n)\rfloor \leq N_+(E) \leq \sum_{n=1}^{\eta_1(E)} \lceil K(n)\rceil.$$

Due to the asymptotic estimate (2.5), we immediately get the following asymptotic estimate as $E \to +\infty$

$$\begin{aligned}
N(E) = 2N_+(E) + N_0(E) &= 2\sum_{n=1}^{\eta_1(E)} K(n) + O(\sqrt{E}) \\
&= \frac{E}{2}\sum_{n=1}^{\eta_1(E)} \frac{1}{n} - 2\sum_{n=1}^{\eta_1(E)} n + O(\sqrt{E}) \\
&= \frac{E}{2}\left(\log(\sqrt{E}/2) + \gamma\right) - \frac{E}{4} + O(\sqrt{E}) \\
&= \frac{E}{4}\log(E) + \left(\gamma - \log(2) - \frac{1}{2}\right)\frac{E}{2} + O(\sqrt{E}).
\end{aligned}$$

This completes the proof. □

It follows from Theorem 1.3 that for $k \neq 0$ the operator $\tilde{\Delta}_k$ acting on H_k presents an infinite amount of eigenvalues accumulating at infinity that can be explicitly described by

$$\sigma_d(H_k) := \{\lambda_{n,|k|} = 4(1 + n)(1 + n + |k|) \mid n \in \mathbb{N}, k \in \mathbb{Z}\}.$$

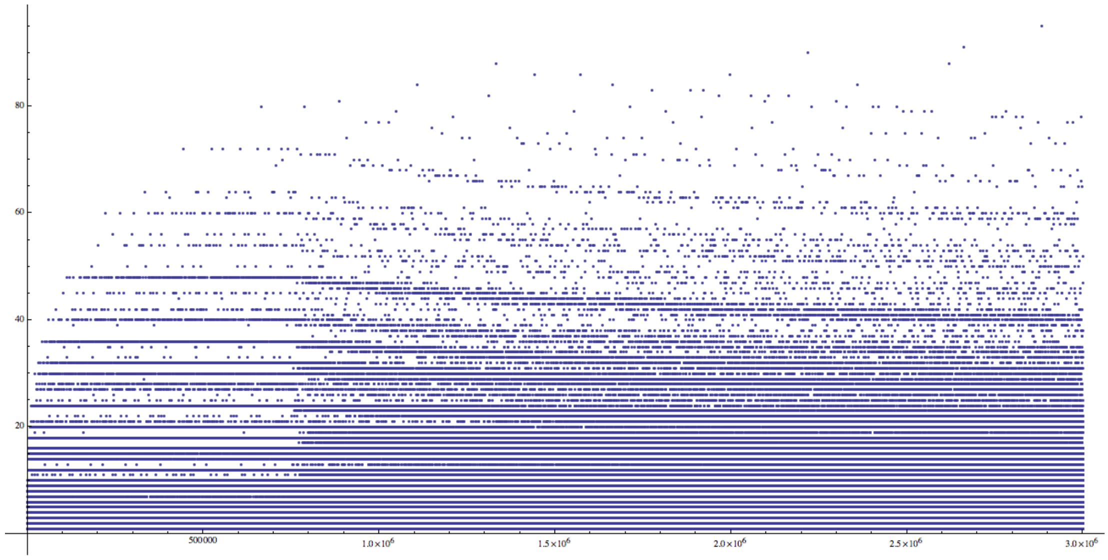

Figure 3.1: Degeneracy (halved) of the first $4.893.535$ eigenvalues (namely $\lambda_{n,k} < 3 \cdot 10^6$). Observe that only one of those eigenvalues attains the higher multiplicity of 110, far below than our upper bound.

Due to the symmetry w.r.t. k of $\lambda_{n,|k|}$, all the eigenvalues are at least double degenerate. Moreover, this degeneracy must be finite. In fact it is enough to observe that each operator has a ground state of energy greater than

$$\lambda_{1,|k|} = 4(|k| + 1),$$

and that the function $n \mapsto \lambda_{n,|k|}$ is increasing in $n \in \mathbb{N}$.

With little work it is possible to shows that the degeneracy of an eigenvalue λ can be easily bounded above by $(\lambda - 4)/2$ but this is far from being optimal. In fact, the computation of the first five million eigenvalues (see Figure 3.1) suggests the growth of the degeneracy to be irregular and slow as in the case of the Grushin cylinder (see Theorem 1.7). We remark that ploting more eigenvalues yielded qualitatively the same plot.

Unfortunately, it is not possible to obtain a more precise description of the degeneracy with the simple techniques employed in Theorem 1.7. Indeed, in this case the problem reduces to counting the number of solutions of a non-linear Diophantine equation, which is well-known to be an hard problem.

3.3. Aharonov-Bohm effect

We now consider the Aharonov-Bohm on the Grushin sphere. Since $\mathbb{S}^2$ is simply connected, any closed form is exact and hence we cannot hope to obtain an Aharonov-Bohm effect without artificially poking a hole in the manifold. This is the same phenomena as in the original Aharonov-Bohm effect [1, 20].

In this section we will thus consider the magnetic Laplace-Beltrami operator induced by the magnetic vector potential $\omega^b = -ib\, d\phi$, $b \in \mathbb{R}$, on the north hemisphere of $\mathbb{S}^2$ with the north pole removed, denoted by S_+°. Note that on S_+° the corresponding magnetic field is 0. The resulting operator is

$$\Delta^b = \partial_x^2 - \frac{1}{\sin(x)\cos(x)}\partial_x + \tan(x)^2\left(\partial_\phi^2 - 2ib\partial_\phi - b^2\right). \tag{3.6}$$

That decomposes in the Fourier components

$$\tilde{\Delta}_k^b = \partial_x^2 - \frac{1}{\sin(x)\cos(x)}\partial_x - \tan(x)^2(k-b)^2$$

It is important to remark that, due to the forceful removal of the origin (the north pole), this magnetic Laplace-Beltrami operator is not essentially self-adjoint on $\mathrm{C}_\mathrm{c}^\infty(S_+^\circ)$. As is customary for the standard Aharonov-Bohm effect on $\mathbb{R}^2$, we will consider the self-adjoint extension obtained by posing Dirichlet boundary conditions on the origin. Namely, we will take as domain the closure of $\mathrm{C}_\mathrm{c}^\infty(S_+^\circ)$ w.r.t. the Sobolev norm $W_0^{1,2}(S_+^\circ)$.

We immediately get Theorem 1.10 by the same arguments of Theorem 1.3, replacing k with $k-b$. We then easily obtain Corollary 1.11.

Proof of Corollary 1.11. Without loss of generality, we can assume $b \in (-1/2, 1/2)$, i.e., $\kappa = 0$. If $b = 0$ then the statement reduces to the one of Corollary 1.4, so let us assume, by the simmetry of the eigenvalue expression, that $b < 0$.

We then proceed similarly to the proof of Corollary 1.4, splitting the counting function in two components $N_-(E)$ and $N_+(E)$, depending on wether k is smaller or bigger than b. The estimates on $N_+(E)$ and $N_-(E)$ are then obtained as in Corollary 1.4, paying attention to the presence of b. Indeed, in the notation of the proof of that corollary, in both cases we obtain

$$K(n) = \frac{E - 4n^2}{4n} + |b|.$$

Since the sum has to be computed for $n \leq \eta_1(E)$, given by

$$\eta_1(E) = \left\lfloor \frac{|b| + \sqrt{E + |b|^2}}{2} \right\rfloor,$$

it is easy to see that b only appears (linearly) in the $O(\sqrt{E})$ term. Since $|b| \leq 1/2$ this completes the proof. □

As already anticipated in the previous section, the degeneracy of the spectrum for the Grushin sphere seems to be of similar nature as for the Grushin cylinder, at least from a numerical point of view, but having a precise control on it is much more involved and probably not possible at present. However, one can still prove that the degeneracy is very unstable with respect to the parameter b and, in particular, that the spectrum is simple for $b \in \mathbb{R} \setminus \mathbb{Q}$ and finitely degenerate for $b \in \mathbb{Q}$. This is summarised in Corollary 1.11 and it follows from an argument very close to the one in the proof of Corollary 1.7.

A. Appendix

A.1. The Covariant Laplacian

For more details we refer the reader to [25] or any other standard differential geometry book.

Consider a Riemannian Manifold M whose metric is denoted g (e.g. $\mathbb{R}^3$ with a non-flat metric). Let E a complex vector bundle with connection ω (e.g. a one-form $\omega = -i\left(\frac{e}{h}\right)A$ for a magnetic field with vector potential A).

Then the covariant derivative is a cross section of the bundle $E \otimes T^*M$ defined

$$\psi_{/j} = \partial_j \psi + \omega_j \psi.$$

In presence of a nonzero connection ω, the Laplace-Beltrami operator is rewritten using Einstein notation as

$$\nabla^2 \psi := g^{jk} \psi_{/jk}$$

where

$$\psi_{/jk} = \partial_k \psi_{/j} + \omega_k \psi_{/j} + \Gamma^r_{kj} \psi_{/r}$$

is defined in terms of the Christoffel symbols

$$\Gamma^r_{kj} = \frac{1}{2} g^{ir} \left(\frac{\partial g_{ji}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{kj}}{\partial x^i} \right).$$

A slightly more explicit computation gives

$$\nabla^2 \psi = g^{jk} \left(\partial_k \partial_j \psi - \Gamma^r_{kj} \partial_r \psi \right) + g^{jk} \left(\partial_k (\omega_j \psi) + \omega_k (\partial_j \psi + \omega_j \psi) - \Gamma^r_{kj} \omega_r \psi \right)$$

Note that the first block of the RHS can be rewritten as

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left(\sqrt{g} g^{jk} \frac{\partial \psi}{\partial x^k} \right),$$

i.e. it is just the usual Laplace-Beltrami operator on M.

A.2. Spectrum of direct sums

The following relations for the spectrum of direct sums $A = \oplus_{k \in \mathbb{Z}} A_k$ are well known (see e.g., [31]):

$$\sigma_p(A) = \bigcup_{k \in \mathbb{Z}} \sigma_p(A_k) \tag{A.1}$$

$$\begin{aligned} \sigma_c(A) = &\left(\left(\bigcup_{k \in \mathbb{Z}} \sigma_p(A_k) \right)^c \cap \left(\bigcup_{k \in \mathbb{Z}} \sigma_r(A_k) \right)^c \cap \left(\bigcup_{k \in \mathbb{Z}} \sigma_p(A_k) \right) \right) \\ &\cup \left\{ \lambda \in \bigcap_{k \in \mathbb{Z}} \rho(A_k) \mid \sup_k \| R_\lambda(A_k) \| = +\infty \right\}. \end{aligned} \tag{A.2}$$

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References

[1] R. Adami and A. Teta, *On the Aharonov-Bohm Hamiltonian*, Lett. Math. Phys. **43** (1998), no. 1, 43–53. MR 1607550 (99a:81030)

[2] A. Agrachev, D. Barilari, and U. Boscain, *Introduction to Riemannian and sub-Riemannian geometry (Lecture Notes)*, http://people.sissa.it/agrachev/agrachev_files/notes.htm, 2012.

[3] A. Agrachev, U. Boscain, and M. Sigalotti, *A Gauss-Bonnet-like formula on two-dimensional almost-Riemannian manifolds*, Discrete Contin. Dyn. Syst. **20** (2008), no. 4, 801–822. MR 2379474 (2009i:53023)

[4] A. A. Agrachev, U. Boscain, G. Charlot, R. Ghezzi, and M. Sigalotti, *Two-dimensional almost-Riemannian structures with tangency points*, Ann. Inst. H. Poincaré Anal. Non Linéaire **27** (2010), no. 3, 793–807. MR 2629880 (2011e:53032)

[5] Tom M. Apostol, *Introduction to analytic number theory*, Springer-Verlag, New York-Heidelberg, 1976, Undergraduate Texts in Mathematics. MR 0434929 (55 #7892)

[6] Mohamed Salah Baouendi, *Sur une classe d'opérateurs elliptiques dégénérés*, Bull. Soc. Math. France **95** (1967), 45–87. MR 0228819 (37 #4398)

[7] H. Bateman and A. Erdelyi, *Higher transcendental functions. vol. 1–3*, McGraw-Hill, 1953–1956.

[8] A. Bellaïche, *The tangent space in sub-Riemannian geometry*, Sub-Riemannian geometry, Progr. Math., vol. 144, Birkhäuser, Basel, 1996, pp. 1–78. MR 1421822 (98a:53108)

[9] Bernard Bonnard, Jean-Baptiste Caillau, Robert Sinclair, and Minoru Tanaka, *Conjugate and cut loci of a two-sphere of revolution with application to optimal control*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), no. 4, 1081–1098. MR 2542715 (2010h:49030)

[10] David Borthwick, *Spectral theory of infinite-area hyperbolic surfaces*, Progress in Mathematics, vol. 256, Birkhäuser Boston, Inc., Boston, MA, 2007. MR 2344504 (2008h:58056)

[11] U. Boscain, G. Charlot, and R. Ghezzi, *Normal forms and invariants for 2-dimensional almost-Riemannian structures*, Differential Geom. Appl. **31** (2013), no. 1, 41–62. MR 3010077

[12] U. Boscain, G. Charlot, R. Ghezzi, and M. Sigalotti, *Lipschitz classification of almost-Riemannian distances on compact oriented surfaces*, J. Geom. Anal. **23** (2013), no. 1, 438–455. MR 3010287

[13] U. Boscain and C. Laurent, *The Laplace-Beltrami operator in almost-Riemannian geometry*, Annales de l'institut Fourier **63** (2013), no. 5, 1739–1770.

[14] U. Boscain and D. Prandi, *The heat and schrödinger equations on conic and anticonic-type surfaces*, Preprint arXiv:1305.5271v2 (2013).

[15] Ugo Boscain, Thomas Chambrion, and Grégoire Charlot, *Nonisotropic 3-level quantum systems: complete solutions for minimum time and minimum energy*, Discrete Contin. Dyn. Syst. Ser. B **5** (2005), no. 4, 957–990 (electronic). MR 2170218 (2006h:81337)

[16] Ugo Boscain and Grégoire Charlot, *Resonance of minimizers for n-level quantum systems with an arbitrary cost*, ESAIM Control Optim. Calc. Var. **10** (2004), no. 4, 593–614 (electronic). MR 2111082 (2005g:49008)

[17] Ugo Boscain, Grégoire Charlot, Jean-Paul Gauthier, Stéphane Guérin, and Hans-Rudolf Jauslin, *Optimal control in laser-induced population transfer for two- and three-level quantum systems*, J. Math. Phys. **43** (2002), no. 5, 2107–2132. MR 1893663 (2003f:81294)

[18] W.-L. Chow, *Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung*, Math. Ann. **117** (1939), 98–105. MR 0001880 (1,313d)

[19] John H. Conway and Richard K. Guy, *The book of numbers*, Copernicus, New York, 1996. MR 1411676 (98g:00004)

[20] César R. de Oliveira and Marciano Pereira, *Mathematical justification of the Aharonov-Bohm Hamiltonian*, J. Stat. Phys. **133** (2008), no. 6, 1175–1184. MR 2462016 (2010f:81113)

[21] B. Franchi and E. Lanconelli, *Une métrique associée à une classe d'opérateurs elliptiques dégńérés*, Conference on linear partial and pseudodifferential operators (Torino, 1982).

[22] Sylvain Golénia and Sergiu Moroianu, *Spectral analysis of magnetic Laplacians on conformally cusp manifolds*, Ann. Henri Poincaré **9** (2008), no. 1, 131–179. MR 2389893 (2009b:58069)

[23] V. V. Grušin, *A certain class of hypoelliptic operators*, Mat. Sb. (N.S.) **83 (125)** (1970), 456–473. MR 0279436 (43 #5158)

[24] V. Ja. Ivriĭ, *The second term of the spectral asymptotics for a Laplace-Beltrami operator on manifolds with boundary*, Funktsional. Anal. i Prilozhen. **14** (1980), no. 2, 25–34. MR 575202 (82m:58057)

[25] Shoshichi Kobayashi and Katsumi Nomizu, *Foundations of differential geometry. Vol. I*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1996, Reprint of the 1963 original, A Wiley-Interscience Publication. MR 1393940 (97c:53001a)

[26] Wilhelm Magnus, Fritz Oberhettinger, and Raj Pal Soni, *Formulas and theorems for the special functions of mathematical physics*, Third enlarged edition. Die Grundlehren der mathematischen Wissenschaften, Band 52, Springer-Verlag New York, Inc., New York, 1966. MR 0232968 (38 #1291)

[27] R. Montgomery, *A tour of subriemannian geometries, their geodesics and applications*, Mathematical Surveys and Monographs, vol. 91, American Mathematical Society, Providence, RI, 2002. MR 1867362 (2002m:53045)

[28] Abderemane Morame and Françoise Truc, *Magnetic bottles on the Poincaré half-plane: spectral asymptotics*, J. Math. Kyoto Univ. **48** (2008), no. 3, 597–616. MR 2511052 (2010b:58033)

[29] Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert, and Charles W. Clark (eds.), *NIST handbook of mathematical functions*, U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010, With 1 CD-ROM (Windows, Macintosh and UNIX). MR 2723248 (2012a:33001)

[30] L. B. Parnovski, *Asymptotics of Dirichlet spectrum on some class of noncompact domains*, Math. Nachr. **174** (1995), 253–263. MR 1349049 (96g:58190)

[31] Michael Reed and Barry Simon, *Methods of modern mathematical physics. I*, second ed., Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1980, Functional analysis. MR 751959 (85e:46002)

[32] E. Shargorodsky, *Semi-elliptic operators generated by vector fields*, Dissertationes Math. (Rozprawy Mat.) **409** (2002), 96. MR 1903026 (2004i:58045)

[33] E.C. Titchmarsch, *Eigenfunction expansions. part 1. second edition*, Oxford University Press, 1962.

[34] Joachim Weidmann, *Spectral theory of ordinary differential operators*, Lecture Notes in Mathematics, vol. 1258, Springer-Verlag, Berlin, 1987. MR 923320 (89b:47070)