Fractional calculus and generalized Mittag-Leffler type functions

Christian Lavault[*]

Abstract

In this paper, the generalized fractional integral operators of two generalized Mittag-Leffler type functions are investigated. The special cases of interest involve the generalized Fox–Wright function and the generalized M–series and K-function.

In the next Section 2 we first recall some generalized fractional integral operators among the most widely used in fractional calculus. Section 3 is devoted to the definitions of M-series and K-function and their relations to special functions. In Sections 4 and 5, effective fractional calculus of the generalized M-series and the K-function is carried out. The last section briefly concludes and opens up new perspectives.

The results established herein generalize recent properties of generalized Mittag-Leffler type functions using left- and right-sided generalized fractional differintegral operators. The note results also in important applications in physics and mathematical engineering.

Keywords: Fox–Wright psi function; Generalized hypergeometric function; M-series and K-function; Mittag-Leffler type functions; Riemann–Liouville's, Saigo's and Saigo–Maeda's generalized fractional calculus operators.

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Contents

[*]LIPN, CNRS UMR 7030. *E-mail:* lavault@lipn.univ-paris13.fr

1 Introduction and motivations

During the last two decades, the interest in Mittag-Leffler type functions has considerably developed. This is due to their vast potential of applications in applied sciences and engineering, and their steadily increasing importance in physics researches. More precisely, deviations of physical phenomena having an exponential behavior may be governed by physical laws (exponential and power laws) with the help of *generalized Mittag-Leffler type functions*. For example, they appear especially important in research domains such as stochastic systems theory, dynamical systems theory, statistical distribution theory, disordered and chaotic systems, etc., with special emphasis placed on applications to fractional differential equations—although this topic is not addressed herein. Furthermore, geometric properties including starlikeness, convexity and close-to-convexity for the Mittag-Leffler type functions were also recently investigated, e.g. by Bansal and Prajapat in [4], Kilbas *et al.* [21] and Kiryakova [23, 24, 26]. This makes these functions directly and naturally amenable to fractional calculus techniques as studied by Gorenflo *et al.* [13], Kilbas *et al.* [17, 19, 21], Kiryakova [22], Kumar–Saxena [28], Saigo [37, 38, 39], Samko *et al.* [43], Saxena–Saigo [44], Sharma [46, 47], Srivastava *et al.* [49, 50], etc.

1.1 The Mittag-Leffler and generalized Mittag-Leffler type functions

The one-parametric *Mittag-Leffler function* (M-L for short) $E_\alpha(z)$ was first introduced by the swedish mathematician G. M. Mittag-Leffler in five notes [31, 1903] [32, 1905] and also studied by Wiman [53, 1905]. It is a special function of $z \in \mathbb{C}$ which depends on the complex parameter α and is defined by the power series

$$(1.1) \qquad E_\alpha(z) := \sum_{n \geq 0} \frac{z^n}{\Gamma(\alpha n + 1)} \qquad (\alpha \in \mathbb{C}).$$

One can see that the series (1.1) converges in the whole complex plane for all $\Re(\alpha) > 0$. For all $\Re(\alpha) < 0$, it diverges everywhere on $\mathbb{C} \setminus \{0\}$ and, for $\Re(\alpha) = 0$, its radius of convergence is $R = e^{\pi/2 |\Im(z)|}$.

A first generalization of $E_\alpha(z)$ introduced by Wiman [53, 1905], and later studied by Agarwal *et al.* [1, 16, 1953], is the two-parametric M-L function of $z \in \mathbb{C}$, defined by the series

$$(1.2) \qquad E_{\alpha,\beta}(z) := \sum_{n \geq 0} \frac{z^n}{\Gamma(\alpha n + \beta)} \qquad (\alpha, \beta \in \mathbb{C};\ \Re(\alpha) > 0\ \Re(\beta) > 0).$$

In the case when α and β are real positive, the series converges for all values of $z \in \mathbb{C}$, while when $\alpha, \beta \in \mathbb{C}$, the conditions of convergence closely follow the ones for $E_{\alpha,1}(z) := E_\alpha(z)$.

$E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ are entire functions of $z \in \mathbb{C}$ of order $\rho = 1/\Re(\alpha)$ and type $\sigma = 1$; in a sense, they are the simplest two entire functions of this order (see, e.g., [13, §3.1]).[1] The one- and two-parametric M-L function are fractional extensions of the basic functions $E_1(\pm z) := E_{1,1}(\pm z) = \mathrm{e}^{\pm z}$, $E_{1,2}(z) := (\mathrm{e}^z - 1)/z$, $E_2(z) := E_{2,1}(z) = \cosh(\sqrt{z})$, $E_{2,2}(z) = \sinh(\sqrt{z})/\sqrt{z}$, etc. (see, e.g., [2, §7.1], [10], [13, Sec.1–4], [14], [17, 19, 22, 23, 53], and references therein). They also appear as solutions of fractional integro-differential equations, e.g. in [9, 21, 22, 24, 26].

Among the numerous generalizations of the M-L function let us point out the *standard Wright function*, defined by Wright in a bunch of papers (from 1933 to 1940) [54, 1934] by the series

$$(1.3) \qquad \phi(\alpha, \beta; z) := \sum_{n \geq 0} \frac{1}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!} = {}_0\psi_1\left({}^{--}_{(\beta,\alpha)}; z\right) \qquad (z, \alpha, \beta \in \mathbb{C}),$$

where ${}_0\psi_1(-; (\beta, \alpha); z)$ is a special case of the *generalized Fox–Wright psi function*, which is given in Definition 1.2, Eq. (1.24) of §1.3. The standard Wright function, which is very close to the two-parametric M-L function, may be rewritten also in terms of the *Fox H-function* introduced by Fox [11] (see Definition 1.4 in §1.3 and, e.g., [3, 17]). If $\alpha > -1$, this series is absolutely convergent for all $z \in \mathbb{C}$, while for $\alpha = -1$, it is absolutely convergent for $|z| < 1$, and for $|z| = 1$ and $\Re(\beta) > -1$ [21, Sec. 1.11].

For $\alpha > -1$, $\phi(\alpha, \beta; z)$ is an entire function of z, wherefrom one can deduce that for $\alpha > -1$, the function has order $\rho = 1/(\alpha + 1)$ and type $\sigma = (\alpha + 1)\alpha^{\frac{1}{\alpha+1}} = \alpha^\rho/\rho$.

Wright investigated the function $\phi(\delta, \nu + 1; -z) := J_\nu^\delta(z)$, known as the *Bessel-Wright function* (or the Wright generalized Bessel function) and derived also the asymptotic behaviour of $\phi(\alpha, \beta; z)$ at infinity by means of its integral representation in terms of a *Mellin–Barnes contour integral* (see, e.g., [22, App. E.ii],[21, 1.11]). Additionally, for $z \in \mathbb{C} \setminus (-\infty, 0]$ and $\nu \in \mathbb{C}$, the functions $\phi(1, \nu + 1; \pm z^2/4)$ can be expressed also in terms of the *Bessel function of the first kind*,

$$J_\nu(z) := (z/2)^\nu \sum_{k=0}^\infty (-1)^k \frac{(z^2/4)^k}{k!\Gamma(\nu + k + 1)} = \frac{z/2}{\Gamma(\nu + 1)} \, {}_0F_1\left(\nu + 1; z^2/4\right)$$

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[1]Following Levin [29, Lect. 1], every entire function $f(z)$ is represented by a power series $f(z) = \sum_{n \geq 0} c_n z^n$ which converges everywhere in the complex plane; the radius of convergence is infinite, which implies that $\lim_{n \to \infty} \sqrt[n]{|c_n|} = 0$. Any power series satisfying this criterion will represent an entire function. The global behaviour of an entire function of finite order is characterized by its *order* ρ and its *type* σ represented by the formulas

$$\rho = \limsup_{n \to \infty} \frac{n \log n}{\log (1/|c_n|)} \qquad \text{and} \qquad \sigma = \frac{1}{\rho e} \limsup_{n \to \infty} \left(n \sqrt[n]{|c_n|^\rho}\right).$$

Moreover, the asymptotic behaviour ($|z| \to \infty$) of an entire function is usually studied via its restriction to rays in an angle $\theta_1 \leq |\arg z| \leq \theta_2$ (cf. Phargmén–Lindelöf). The so-called *indicator function* of an entire function of order ρ is introduced as

$$h(\theta) = \limsup_{r \to \infty} \frac{\log\left|f(re^{\mathrm{i}\theta})\right|}{r^\rho} \qquad (\theta_1 \leq \theta \leq \theta_2).$$

Thus, the entire function $f(z) = \sum \left(\frac{\sigma e \rho}{n}\right)^{n/\rho} z^n$ has order ρ and type σ. For instance, the M-L function $\sum_{n=0}^\infty \frac{(A^\alpha z)^n}{\Gamma(\alpha n+1)}$, where $A > 0$ and $\alpha > 0$, is an entire function of order $\rho = 1/\alpha$ and type $\sigma = A$ (by Stirling formula).

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and of the *modified Bessel function*

$$I_\nu(z) := (z/2)^\nu \sum_{k=0}^{\infty} \frac{(z^2/4)^k}{k!\,\Gamma(\nu+k+1)} = \frac{z/2}{\Gamma(\nu+1)} \, {}_0F_1\big(\nu+1; -z^2/4\big),$$

as follows,

$$\phi(1, \nu+1; -z^2/4) = 2z^{-\nu} J_\nu(z) \qquad \text{and} \qquad \phi(1, \nu+1; z^2/4) = 2z^{-\nu} I_\nu(z).$$

Both Bessel functions are analytic functions of $z \in \mathbb{C}$, except for a branch point at $z = 0$ when ν is not an integer. The principal branches of $J_\nu(z)$ and $I_\nu(z)$ correspond to the principal value of $(z/2)^\nu$ and is analytic in the z-plane cut along the interval $(-\infty, 0]$. When $\nu \in \mathbb{Z}$, they are entire in z. For fixed $z \neq 0$ each branch of the functions $J_\nu(z)$ and $I_\nu(z)$ is entire in ν.[2]

Prabhakar [35, 1971] introduced a three-parametric generalization of $E_{\alpha,\beta}(z)$ defined in (1.2) as a kernel of certain fractional differential equations in terms of the series

$$(1.4) \qquad E_{\alpha,\beta}^\gamma(z) := \sum_{n \geq 0} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!} \qquad (\alpha, \beta, ; \, \Re(\alpha) > 0, \, \Re(\beta) > 0),$$

where $(\lambda)_n$ denotes the usual Pochhammmer symbol defined by the identity

$$(\lambda)_n := \lambda(\lambda+1)\cdots(\lambda+n-1) \text{ if } n = 1, 2, 3, \ldots, \qquad (\lambda)_0 = 1 \qquad (\lambda \neq 0).$$

In (1.4), no condition is imposed on λ, provided it is not zero; for example, λ can be a negative integer, in which case the series is terminating into a polynomial. By contrast, whenever $(\lambda)_n$ is to be written in terms of a gamma function as

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \qquad (\Re(\lambda) > 0),$$

then the condition $\Re(\lambda) > 0$ must be fulfilled, or at least the constraint $\lambda \neq 0, -1, -2, \ldots$ is required. Note that if $E_{\alpha,\beta}^\gamma(z)$ is represented as a Fox H-function, α is real and positive: in that case, such a condition becomes a requirement.

Prabhakar's three-parametric M-L function is an entire function of $z \in \mathbb{C}$ of order $\rho = 1/\Re(\alpha)$ and type $\sigma = 1$. If $\gamma = 1$, then $E_{\alpha,\beta}^1(z) = E_{\alpha,\beta}(z)$ and, if $\gamma = \beta = 1$, then $E_{\alpha,1}^1(z) := E_\alpha(z)$.

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[2]The one- and two-parametric M-L functions are generalized to the multiple M-L function $F_{\alpha,\beta}^{(\mu)}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)^\mu}$ for real values α, β and $\mu > 0$ (see Appendix A.3). They are also tightly related to the M-L type function $E_{\alpha,1}^{(a)}(s; z)$ due to Barnes: $E_\alpha(z) = \lim_{s\to 0} E_{\alpha,1}^{(a)}(s; z)$ and $E_{\alpha,\beta}(z) = \lim_{s\to 0} E_{\alpha,\beta}^{(a)}(s; z)$ (with complex parameters). Moreover,

$$\lim_{\alpha\to 0} E_{\alpha,1}^{(a)}(s; z) = \frac{1}{\Gamma(\beta)} \Phi(z, s, a),$$

where the series $\Phi(z, s, a) := \sum_{n \geq 0} \frac{z^n}{(n+a)^s}$ defines the *Lerch's transcendent zeta* function, which is analytic when $|z| < 1$ for $a \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ and $s \in \mathbb{C}$; the series also converges when $|z| = 1$, provided that $\Re(s) > 1$.

As special cases, $\Phi(z, s, a)$ contains the *Riemann zeta function* $\zeta(s) = \Phi(1, s, 1)$, the *Hurwitz (or generalized) zeta* function $\zeta(s, a) = \Phi(1, s, a)$ and the *Lerch zeta* function $\ell(\xi, s, a) = e^{2\pi i \xi} \Phi\big(e^{2\pi i \xi}, s, a\big)$ ($\xi \in \mathbb{R}$, $\Re(s) > 1$), but also the *Polylogarithmic function* $\mathrm{Li}_s(z) = z\Phi(z, s, 1)$ (for $s \in \mathbb{C}$, it is analytic when $|z| < 1$ and the series converges when $|z| = 1$ provided that $\Re(s) > 1$) and the *Lipschitz–Lerch* zeta function $L(\xi, s, a) := \Phi\big(e^{2\pi i \xi}, s, a\big)$ (for $a \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, it is analytic when $\Re(s) > 0$ for $\xi \in \mathbb{R} \setminus \mathbb{Z}$ and when $\Re(s) > 1$ for $\xi \in \mathbb{Z}$). (For other values of z, these functions are defined by analytic continuation.)

Due to its integral representation (see, e.g., [13, Chap. 5, §5.1.2]), $E_{\alpha,\beta}^{\gamma}(z)$ is considered as a special case of Fox's H-function as well as of *Wright's generalized hypergeometric* $_p\psi_q$, so-called *Fox–Wright psi function* of $z \in \mathbb{C}$. (see, e.g., [3, 17] and the definitions of these higher transcendental functions, including the *Meijer G-function*, in §1.3, Definitions 1.2, 1.4 and 1.5).

It is straightforward to verify that

$$(1.5) \qquad E_{\alpha,\beta}(z) = {}_1\psi_1\left(\begin{matrix}(1,1)\\(\beta,\alpha)\end{matrix}; z\right) = H_{1,2}^{1,1}\left[-z\,\middle|\,\begin{matrix}(0,1)\\(0,1),(1-\beta,\alpha)\end{matrix}\right] \qquad \text{and}$$

$$(1.6) \qquad E_{\alpha,\beta}^{\gamma}(z) = \frac{1}{\Gamma(\gamma)}\,{}_1\psi_1\left(\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix}; z\right) = \frac{1}{\Gamma(\gamma)}\,H_{1,2}^{1,1}\left[-z\,\middle|\,\begin{matrix}(1-\gamma,1)\\(0,1),(1-\beta,\alpha)\end{matrix}\right]$$

$$(1.7) \qquad = \frac{1}{\Gamma(\gamma)}\,{}_1F_m\left(\gamma; \frac{\beta}{m}, \frac{\beta+1}{m}, \ldots, \frac{\beta+m-1}{m}; \frac{z}{m^m}\right) \qquad \text{for} \quad \alpha = m \in \mathbb{N}.$$

In particular, when $\alpha = 1$ the H-function coincides with the Meijer G-function (see §1.3, Eq. (1.32))

$$E_{1,\beta}^{\gamma}(z) = \frac{1}{\Gamma(\gamma)}\,{}_1\psi_1\left(\begin{matrix}(\gamma,1)\\(\beta,1)\end{matrix}; z\right) = \frac{1}{\Gamma(\gamma)}\,H_{1,2}^{1,1}\left[-z\,\middle|\,\begin{matrix}(1-\gamma,1)\\(0,1),(1-\beta,1)\end{matrix}\right] = \frac{1}{\Gamma(\gamma)}\,G_{1,2}^{1,1}\left(\begin{matrix}1-\gamma\\0,1-\beta\end{matrix}\,\middle|\,z\right)$$

$$(1.8) \qquad = \frac{1}{\Gamma(\beta)}\,{}_1F_1(\gamma;\beta;z) = \frac{1}{\Gamma(\beta)}\,M(\gamma;\beta;z),$$

where $M(\gamma;\beta;z)$ denotes Kummer's confluent hypergeometric function (sometimes denoted by $\Phi(\gamma;\beta;z)$). Similarly, if we set $\beta = 1$ in Eq. (1.5), we find that

$$(1.9) \qquad E_{\alpha,1}(z) = \frac{1}{\Gamma(\gamma)}\,{}_1\psi_1\left(\begin{matrix}(1,1)\\(1,\alpha)\end{matrix}; z\right) = H_{1,2}^{1,1}\left[-z\,\middle|\,\begin{matrix}(0,1)\\(0,1),(0,\alpha)\end{matrix}\right] \qquad (\alpha \in \mathbb{C},\ \Re(\alpha) > 0)$$

Further extensions of the M-L function to four and six parameters were defined by Salim [41] and, associated with Weyl fractional integral and differential operators, by Salim and Faraj [42], respectively, by the power series

$$(1.10) \qquad E_{\alpha,\beta}^{\gamma,\delta}(z) := \sum_{n\geq 0} \frac{(\gamma)_n\, z^n}{\Gamma(\alpha n + \beta)\,(\delta)_n} \qquad \text{and} \qquad E_{\alpha,\beta,r}^{\gamma,\delta,s}(z) := \sum_{n\geq 0} \frac{(\gamma)_{sn}\, z^n}{\Gamma(\alpha n + \beta)\,(\delta)_{rn}}\,.$$

In the first case (with four parameters), $\alpha,\beta,\gamma,\delta \in \mathbb{C}$, $\min(\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\delta) > 0)$. While in the second case (with six parameters), $\alpha,\beta,\gamma,\delta \in \mathbb{C}$, $\min(\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\delta) > 0)$, with $r, s \in \mathbb{R}_+$ and $s \leq \Re(\alpha) + r$. In the latter case, $(\gamma)_{ns}$ denotes an extended variant of the Pochhammer symbol, defined by $(\gamma)_{sn} := \Gamma(\gamma + sn)/\Gamma(\gamma)$, which reduces to $s^{sn} \prod_{j=1}^{s}\left(\frac{\gamma+j-1}{s}\right)_n$ when s is a non-negative integer.[3] Both functions are entire in the complex z-plane of order $\rho = 1/\Re(\alpha - \delta + 1)$ and type $\sigma = \frac{1}{\rho}\left(\Re(\delta)^{\Re(\delta)}/\Re(\alpha)^{\Re(\alpha)}\right)^{\rho}$.

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[3] The Pochhammer symbol $(x)_{nk}$ must not be confused with the *generalized Pochhammer k-symbol* $(x)_{n,k}$ itself, which was introduced by Diaz and Pariguan in [6] and is defined in terms of $\Gamma_k(x) := \lim_{n\to\infty} \dfrac{n!(nk)^{x/k-1}}{(x)_{n,k}}$ $(k > 0)$ for $x \in \mathbb{C} \setminus k\mathbb{Z}_{<0}$ and $k \in \mathbb{R}$ by the relation

$$(x)_{n,k} := x(x+k)(x+2k)\cdots(x+(n-1)k) = \Gamma_k(x+nk)/\Gamma_k(x).$$

When $k = 1$, this reduces to the standard Pochhammer symbol and gamma function.

Remarks 1. Among the M-L functions with three parameters are the *generalized (Kilbas–Saigo) M-L type* special functions. These functions were introduced in 1995 by Kilbas and Saigo [18] in the form

$$(1.11) \qquad E_{(\alpha,m,\ell)}(z) = 1 + \sum_{k=1}^{\infty} \prod_{j=0}^{n-1} \frac{\Gamma(\alpha\lfloor jm + \ell \rfloor + 1)}{\Gamma(\alpha\lfloor jm + \ell + 1 \rfloor + 1)} \, z^k \qquad (z \in \mathbb{C}),$$

where an empty product is assumed to be equal to 1 ('empty product convention'). The generalized M-L type function (1.11) is defined for real α, $m \in \mathbb{R}$ and $\ell \in \mathbb{C}$ meeting the conditions $\alpha > 0$, $m > 0$ and $\alpha(jm + \ell) + 1 \neq -1, -2 \ldots (j = 1, 2, \ldots)$.

When α, m and ℓ are real numbers which fulfill the above conditions, then $E_{(\alpha,m,\ell)}(z)$ is an entire function of z of order $\rho = 1/\alpha$ and type $\sigma = 1/m$. In particular, if $m = 1$, the function reduces to $E_{(\alpha,1,\ell)}(z) := \Gamma(\alpha\ell + 1)E_{(\alpha,\alpha\ell+1)}(z)$ (see, e.g., [13, §5.2] for a detailed study of these generalized (Kilbas–Saigo) M-L type special functions).

A wider extended class of special functions of M-L type was further introduced and studied, e.g., by Kiryakova [22, 23, 25]. This class, based on Luchko–Kilbas–Kiryakova's approach, consists of multi-parameters analogues of $E_{\alpha,\beta}(z)$ called *multi-indices M-L functions*, such that the parameters $\alpha := 1/\rho$ and $\beta := \mu$ are replaced by two sets of multi-parameters.

The M-L functions with $2n$ parameters are defined for $\alpha_j \in \mathbb{R}$ $(\alpha_1^2 + \cdots + \alpha_n^2 \neq 0)$ and $\beta_j \in \mathbb{C}$ $(j = 1, \ldots, n \in \mathbb{N})$ by the series

$$(1.12) \qquad E_{(\alpha,\beta)_n}(z) := \sum_{k \geq 0} \frac{z^k}{\prod\limits_{j=1}^{n} \Gamma(\alpha_j k + \beta_j)} \qquad (z \in \mathbb{C}).$$

Under the additional condition that $\Sigma_n = \alpha_1 + \cdots + \alpha_n > 0$, the generalized $2n$-parametric function $E_{(\alpha,\beta)_n}(z)$ is an entire function of $z \in \mathbb{C}$ of order $\rho = 1/\Sigma_n$ and type $\sigma = \prod\limits_{i=1}^{n} (\Sigma_i/|\alpha_i|)^{\alpha_i/\Sigma_i}$. When $n = 1$, the definition in (1.12) coincides with the definition of the two-parametric M-L function

$$(1.13a) \qquad E_{(\alpha,\beta)_1}(z) := E_{\alpha,\beta}(z) = \sum_{k \geq 0} \frac{z^k}{\Gamma(\alpha k + \beta)} \qquad (z \in \mathbb{C}),$$

and similarly for $n = 2$, where $E_{(\alpha,\beta)_n}(z)$ coincides with the four-parametric M-L function

$$(1.13b) \qquad E_{(\alpha,\beta)_2}(z) = E_{(\alpha_1,\beta_1;\alpha_2,\beta_2)}(z) = \sum_{k \geq 0} \frac{z^k}{\Gamma(\alpha_1 k + \beta_1)\Gamma(\alpha_2 k + \beta_2)} \qquad (z \in \mathbb{C}).$$

The *generalized $2n$-parametric M-L function* $E_{(\alpha,\beta)_n}(z)$ can be represented in terms of the generalized Fox–Wright hypergeometric function ${}_p\psi_q(z)$ by

$$(1.14a) \qquad E_{(\alpha,\beta)_n}(z) = {}_1\psi_n \left(\begin{matrix} (1,1) \\ (\beta_1,\alpha_1),\ldots,(\beta_n,\alpha_n) \end{matrix} ; z \right) \qquad (z \in \mathbb{C})$$

and, via a Mellin–Barnes integral (see §1.2 below), by

$$(1.14b) \qquad E_{(\alpha,\beta)_n}(z) = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(s)\Gamma(1-s)}{\prod\limits_{j=1}^{n} \Gamma(\beta_j - \alpha_j s)} (-z)^{-s} \mathrm{d}s \qquad (z \neq 0).$$

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For $\Re(\Sigma_n) > 0$, one can choose the left loop $\mathcal{L}_{-\infty}$ as a contour of integration in (1.14b). Carrying out this integral by use of the theory of residues immediately yields the series representation given in (1.12). The extension of $E_{(\alpha,\beta)_n}(z)$ with the right loop $\mathcal{L}_{+\infty}$ chosen as the contour of integration is entailed through the Mellin–Barnes integral representation in (1.14b). This yields the convergence of the contour integral for all values of parameters $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, $\beta_1, \ldots, \beta_n \in \mathbb{C}$ such that $\Re(\Sigma_n) < 0$.

1.2 Integral representations of Mittag-Leffler type functions

The integral representations of M-L type functions play a prominent role in the analysis of entire functions. As for any function of the M-L type, the M-L functions with one to three parameters can be represented via *Mellin–Barnes integrals* obtained, for example, by a calculus of residues.

Lemma 1.1. *Let $\alpha \in \mathbb{R}_+$, β, $\gamma \in \mathbb{C}$ and $\Re(\gamma) > 0$. Then the Mellin–Barnes integral representation of Prabhakar's three-parametric function is*

$$(1.15) \qquad E_{\alpha,\beta}^{\gamma}(z) = \frac{1}{\Gamma(\gamma)} \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\Gamma(s)\Gamma(\gamma - s)}{\Gamma(\beta - \alpha s)} (-z)^{-s} \mathrm{d}s \qquad (z \in \mathbb{C}, \ |\arg z| < \pi)$$

The contour of integration $\mathcal{C} = (c - \mathrm{i}\infty, c + \mathrm{i}\infty)$, $0 < c < \Re(\gamma)$, separates all the poles of $\Gamma(s)$ at the points $s = -\nu$ ($\nu \in \mathbb{N}$) to the left from those of $\Gamma(\gamma - s)$ at the points $s = \nu + \gamma$ ($\nu \in \mathbb{N}$) to the right.[4]

Proof. The contour integral in (1.15) is obtained by the sum of residues technique evaluated at the poles $s = 0, -1, -2, \ldots$. Hence,

$$\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\Gamma(s)\Gamma(\gamma - s)}{\Gamma(\beta - \alpha s)} (-z)^{-s} \mathrm{d}s = \sum_{k=0}^{\infty} \lim_{s \to -k} \left(\frac{(s + k)\Gamma(s)\Gamma(\gamma - s)(-z)^{-s}}{\Gamma(\beta - \alpha s)} \right)$$

$$(1.16) \qquad = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(\gamma + k)}{\Gamma(\beta + \alpha k)} (-z)^k = \Gamma(\gamma) \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\beta + \alpha k)} \frac{z^k}{k!} = \Gamma(\gamma) E_{\alpha,\beta}^{\gamma}(z).$$

Eq. (1.16) is the expression of $E_{\alpha,\beta}^{\gamma}(z)$ as a Mellin–Barnes integral, and the lemma follows. $\qquad\square$

Mellin–Barnes integral representations of the one- and two parametric M-L functions are immediately deduced from the one given in Lemma 1.1 by setting respectively $\gamma = \beta = 1$ and $\gamma = 1$. The following related representations of $E_{\alpha,1}^{1}(z) := E_{\alpha}(z)$ and $E_{\alpha,\beta}^{1}(z) := E_{\alpha,\beta}(z)$ write in terms of Mellin–Barnes integrals

$$E_{\alpha}(z) = \frac{1}{2\pi i} \int_{\mathcal{C}_1} \frac{\Gamma(s)\Gamma(1 - s)}{\Gamma(1 - \alpha s)} (-z)^{-s} \mathrm{d}s \qquad (z \in \mathbb{C}, \ |\arg z| < \pi) \qquad \text{and}$$

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i} \int_{\mathcal{C}_2} \frac{\Gamma(s)\Gamma(1 - s)}{\Gamma(\beta - \alpha s)} (-z)^{-s} \mathrm{d}s \qquad (z \in \mathbb{C}, \ |\arg z| < \pi),$$

where each contour of integration $\mathcal{C}_1$ and $\mathcal{C}_2$ is again a straight line which starts at $c - \mathrm{i}\infty$ and ends at $c + \mathrm{i}\infty$ ($0 < c < 1$), leaving all poles of each integrand respectively separated to the left and to the right of each line contour.

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[4]The present path of integration $\mathcal{C}$ is simpler than the ones considered, for example, in Eq. (1.14b) and in the case of a Hankel contour, in Eq. (1.19) below (see also Appendix C and Appendix D), but however runs exactly along the same lines.

Remark 1. M-L type functions play a basic role in the solution of fractional differential equations and integral equations of Abel type. Therefore, studying and developing their theory and stable methods is a first important step for their numerical computation. In this respect, integral representations of M-L type functions in terms of Mellin–Barnes integrals are sometimes not the easiest nor the more useful ones to handle. This is indeed the case of the two-parametric M-L function, for example, which enjoys various integral representations considered by Džr-bašjan [8, 9], Erdélyi *et al.* [10, Vol 3, §8.1] and Wright [54] that are far more fruitful to the design of performing numerical algorithms (see Appendix A for more detail on such integral representations).

Also, on evaluating the residues at the poles of $\Gamma(1 - s)$, there results the following analytic continuations of the one- and two-parametric M-L functions,

$$(1.18) \qquad E_\alpha(z) = -\sum_{n=1}^\infty \frac{z^{-n}}{\Gamma(1 - \alpha n)} \quad \text{and} \quad E_{\alpha,\beta}(z) = -\sum_{n=1}^\infty \frac{z^{-n}}{\Gamma(\alpha n + \beta)}.$$

Many important propertie of $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ follow from the integral representation of the two-parametric M-L function given by Džrbašjan [7, 8, 9, 1952, 1960, 1966] and Erd'elyi *et al.* [10, Vol. I-III, 1953] (which reduces to $E_\alpha(z) := E_{\alpha,1}(z)$ when $\beta = 1$)

$$(1.19) \qquad E_{\alpha,\beta}(z) = \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{H}} \frac{t^{\alpha-\beta}\, \mathrm{e}^t}{t^\alpha - z}\, \mathrm{d}t \qquad (z,\, \alpha,\, \beta \in \mathbb{C},\, \Re(\alpha) > 0,\, \Re(\beta) > 0),$$

where the contour of integration (or *Hankel path*) $\mathcal{H}$ is the loop starting and ending at $-\infty$, and encircling the disk $|t| \le |z|^{1/\alpha}$ in the positive sense (counterclockwise): $|\arg(t)| < \pi$ on $\mathcal{H}$. The integrand in (1.19) has a branch point at $t = 0$, and the complex t-plane is cut along the negative real axis. The integrand is single-valued in the cut plane, where the principal branch of t^α is taken.[5] (Detailed studies of M-L and M-L type functions may be found, e.g., in Džrbašjan [9], Erdélyi *et al.* [10], Gorenflo *et al.* [13, Chap. 4, §4.8], Kilbas *et al.* [21, Chap. 1, §1.8], etc.)

The integral representation (1.19) of the two-parametric M-L function is used for instance to obtain the asymptotic expansion of $E_{\alpha,\beta}(z)$ ($|z| \to \infty$) which is different in the case $0 < \alpha < 2$ from the case when $\alpha \ge 2$. According to the values of $|\arg z|$, there result the corresponding asymptotic expansions of $E_{\alpha,\beta}(z)$ and $E_\alpha := E_{\alpha,1}$ when $|z|$ is large (see Appendix A).

The two-parametric M-L function defined in the form of Eq. (1.2) exists only for the values on parameters $\Re(\alpha) > 0$ and $\beta \in \mathbb{C}$. However, an analytic continuation of $E_{\alpha,\beta}(z)$ depending on real parameters α, β may be performed by extending its domain to negative values of α. The integral representation (1.20) is similar to (1.19) along the Hankel path $\mathcal{H}$ (see, e.g., [8, 13, 21]),

$$(1.20) \qquad E_{\alpha,\beta}(z) = \frac{1}{2\pi} \int_{\mathcal{H}} \frac{t^{\alpha-\beta}\, \mathrm{e}^t}{t^\alpha - z}\, \mathrm{d}t \qquad (z \in \mathbb{C},\, \alpha,\, \beta \in \mathbb{R},\, \alpha < 0).$$

The integral representation (1.20) makes it possible to define the function $E_{-\alpha,\beta}(z)$ for real negative values of the first parameter (see Appendix B for a proof).

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[5]The representation (1.19) can be proved by expanding the integrand in powers of t and integrating term by term by making use of Hankel's integral path $\mathcal{H}$ for the reciprocal of the gamma function, namely

$$\frac{1}{\Gamma(\beta)} = \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{H}} \mathrm{e}^t\, t^{-\beta} \mathrm{d}t \qquad (\beta \in \mathbb{C}).$$

Hankel's integral representation of $1/\Gamma(z)$ ($z \in \mathbb{C}$) is shown in Appendix C (see also, e.g., Temme [52, Chap. 3, §3.2]).

As another example, consider the Mellin–Barnes contour integral represention of Wright's function $\phi(\alpha, \beta; z)$ defined in (1.3). It may be given by the use of the sum of residues technique, as in Lemma 1.1,

$$(1.21) \qquad \phi(\alpha, \beta; z) = \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{C}} \frac{\Gamma(s)}{\Gamma(\beta - \alpha s)} (-z)^{-s} \mathrm{d}s,$$

where the path of integration $\mathcal{C}$ separates all the poles at $s = -\nu$ ($\nu \in \mathbb{N}$) to the left. If $\mathcal{C} = (c - \mathrm{i}\infty, c + \mathrm{i}\infty)$ ($c \in \mathbb{R}$), then the representation (1.21) is valid provided either of the following two conditions holds:

$$(i) \ 0 < \alpha < 1, \ |\arg(-z)| < (1-\alpha)\pi/2 \quad \text{and} \quad z \neq 0 \qquad \text{or}$$
$$(ii) \ \alpha = 1, \ \Re(\beta) > 1 + 2c, \ \arg(-z) = 0 \quad \text{and} \quad z \neq 0.$$

(See Kilbas *et al.* in [19, Cor. 4.1] for more detailed conditions on $\mathcal{C}$.)

Remark 2. The Laplace transforms $\mathcal{L}[\phi]$ and $\mathcal{L}[E_{\alpha,\beta}]$ can be expressed in terms of each other:

$$\mathcal{L}\big(\phi(\alpha, \beta; t)\big)(s) = \int_{0}^{\infty} \mathrm{e}^{-st} \phi(\alpha, \beta; t) \mathrm{d}t = \frac{1}{s} E_{\alpha,\beta}\left(\frac{1}{s}\right) \qquad \text{and}$$

$$\mathcal{L}\big(E_{\alpha,\beta}(t)\big)(s) = \frac{1}{s} \phi\left(\alpha, \beta; \frac{1}{s}\right) = \frac{1}{s} \, {}_2\psi_1\left({}^{(1,1),(1,1)}_{(\beta,\alpha)}; \frac{1}{s}\right) \quad (\alpha > -1, \ \alpha, \beta \in \mathbb{C}, \ \Re(s) > 0).$$

Further, under the constraints $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $s > r^{1/\Re(\alpha)}$, the Laplace transform of Prabhakar's three-parametric M-L function is equal to

$$(1.23\mathrm{a}) \qquad \mathcal{L}\big(E_{\alpha,\beta}^{\gamma}(t)\big)(s) := E_{\alpha,\beta}^{\gamma}\left(t^{\alpha}\right) = \frac{1}{s^{\beta}} \left(1 - \frac{1}{s^{\alpha}}\right)^{-\gamma} = \frac{1}{s} \, {}_2\psi_1\left({}^{(\gamma,1),(1,1)}_{(\beta,\alpha)}; \frac{1}{s}\right)$$

and, by applying the Mellin inversion formula to the Mellin–Barnes representation of $E_{\alpha,\beta}(z)$ (with $\alpha \in \mathbb{R}_+$, $\beta, \gamma \in \mathbb{C}$, $\beta \neq 0$, $|\arg z| < \pi$. See Appendix A), its Mellin transform is equal to

$$(1.23\mathrm{b}) \ \ \mathcal{M}\big(E_{\alpha,\beta}^{\gamma}(-wt)\big)(s) := \int_{0}^{\infty} t^{-s} E_{\alpha,\beta}^{\gamma}\left(t^{\alpha}\right) \mathrm{d}t = \frac{\Gamma(s)\Gamma(\gamma - s)}{\Gamma(\gamma)\Gamma(\beta - \alpha s)} \, w^{-s} \qquad (0 < \Re(s) < \Re(\gamma)).$$

1.3 Higher transcendental functions

For any positive integers p, q, the generalized complex function ${}_p\psi_q(z)$ has been defined in the seminal papers of Fox [11] and Wright's papers of 1934–1940 [54] (see also, e.g., [3]). Later on, the so-called Fox–Wright (or Fox–Wright psi) function has been extensively studied by Gorenflo *et al.* [13, Chaps. 3–6], Kilbas *et al.* [17, 19, 21], Kiryakova [22, 23, 24, 25, 26],

Here and in the sequel, the following synthetic vector notations are used for convenience whenever no ambigity may arise:

$$\mathbf{a_p} := (a_1, \ldots, a_p), \qquad \Gamma(\mathbf{a_p}) := \prod_{i=1}^{p} \Gamma(a_i) \qquad \text{and} \qquad (a_i, \alpha_i)_1^p := (a_1, \alpha_1), \ldots, (a_p, \alpha_p).$$

Definition 1.2. The *generalized Fox–Wright psi function* of $z \in \mathbb{C}$ is defined formally by the series

$$(1.24) \qquad {}_p\psi_q(z) = {}_p\psi_q \begin{pmatrix} (a_i, \alpha_i)_1^p \\ (b_j, \beta_j)_1^q \end{pmatrix}; z \end{pmatrix} := \sum_{n \geq 0} \frac{\prod\limits_{i=1}^p \Gamma(a_i + \alpha_i n)}{\prod\limits_{j=1}^q \Gamma(b_j + \beta_j n)} \frac{z^n}{n!},$$

where $a_i, b_j \in \mathbb{C}$, $\alpha_i, \beta_j \in \mathbb{R}$ $(i = 1, \ldots, p$ and $j = 1, \ldots, q)$.

Wright proved several results on the asymptotic expansion of ${}_p\psi_q(z)$ for all values of the argument z which fullfill the property $\sum_{j=1}^q \beta_j - \sum_{i=1}^p \alpha_i > -1$. Under this constraint, it was proved in [20] that the series is an entire function of $z \in \mathbb{C}$. Conditions of convergence are provided by the following lemma.

Lemma 1.3 (Kilbas–Srivastava–Trujillo [21, Thm. 1]). *Under the assumptions stated in Definition 1.2, let*

$$(1.25) \qquad \Delta := \sum_{j=1}^q \beta_j - \sum_{i=1}^p \alpha_i, \quad \delta := \prod_{i=1}^p |\alpha_i|^{-\alpha_i} \prod_{j=1}^q |\beta_j|^{\beta_j} \quad and \quad \mu := \sum_{j=1}^q b_j - \sum_{i=1}^p a_i + \frac{p - q}{2}.$$

Then, the conditions of convergence are as follows

  (i) *If $\Delta > -1$, then the series in (1.24) is absolutely convergent for all $z \in \mathbb{C}$.*
  (ii) *If $\Delta = -1$, then the series in (1.24) is absolutely convergent for $z \leq \delta$, and for $z = \delta$ and $\Re(\mu) > 1/2$.*

When $a_i, b_j \in \mathbb{R}$ $(i = 1, \ldots, p$ and $j = 1, \ldots, q)$, the Fox–Wright function ${}_p\psi_q(z)$ has the following integral representation as a Mellin–Barnes contour integral

$$(1.26) \qquad {}_p\psi_q \begin{pmatrix} (a_i, \alpha_i)_1^p \\ (b_j, \beta_j)_1^q \end{pmatrix}; z \end{pmatrix} = \frac{1}{2\pi i} \int_{\mathcal{H}} \frac{\prod\limits_{i=1}^p \Gamma(a_i + \alpha_i s)}{\prod\limits_{j=1}^q \Gamma(b_j + \beta_j s)} \Gamma(s) \, (-z)^{-s} \mathrm{d}s,$$

where the contour integration $\mathcal{H}$ separates all the poles of $\Gamma(s)$ at $s = -\nu$ $(\nu \in \mathbb{N})$ to the left from all the poles of $\Gamma(a_i + \alpha_i s)$ at $s = (a_i + \nu_i)/\alpha_i$ $(i = 1, \ldots, p$ and $\nu_i \in \mathbb{N}$ to the right. If $\mathcal{H} = (c - i\infty, c + i\infty)$ $(c \in \mathbb{R})$, then representation (1.26) is valid provided either of the following two conditions hold:

$$(i) \ \Delta < 1, \ |\arg(-z)| < (1 - \Delta)\pi/2, \ z \neq 0 \qquad \text{or}$$
$$(ii) \ \Delta = 1, \ \Re(\mu) > (\Delta + 1)c + 1/2 \quad \text{and} \quad \arg(-z) = 0, \ z \neq 0.$$

Conditions for the representation (1.26) are also given for the cases when $\mathcal{H} = \mathcal{L}_{-\infty}$ ($\mathcal{L}_{+\infty}$, resp.) is a loop located in a horizontal strip starting at the point $-\infty + i\tau_1$ ($+\infty + i\tau_1$, resp.) and terminating at the point $-\infty + i\tau_2$ ($+\infty + i\tau_2$, resp.) with $-\infty < \tau_1 < \tau_2 < +\infty$.

In turns, the normalized Fox–Wright psi function is defined by

$$(1.27) \qquad {}_p\psi_q^* \begin{pmatrix} (a_i, \alpha_i)_1^p \\ (b_j, \beta_j)_1^q \end{pmatrix}; z \end{pmatrix} := \frac{\Gamma(\mathbf{b_q})}{\Gamma(\mathbf{a_p})} \, {}_p\psi_q \begin{pmatrix} (a_i, \alpha_i)_1^p \\ (b_j, \beta_j)_1^q \end{pmatrix}; z \end{pmatrix},$$

and trivially occurs as a generalization of the generalized hypergeometric function $_pF_q(z)$ (see also, e.g., Eq. (1.38)). Whenever $p = q = 1$, the normalization reduces to

$$_1\psi_1^* \begin{pmatrix} (a,\alpha) \\ (b,\beta) \end{pmatrix}; z \end{pmatrix} := \frac{\Gamma(b)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+\alpha n)}{\Gamma(b+\beta n)} \frac{z^n}{n!}.$$

The importance of $_1\psi_1^*$ appears in Fermat's last theorem and in applied problems, such as the solution of trinomial equations.

Now, if we put $\alpha_i = \beta_j = 1$ for all $1 \le i \le p$ and $1 \le j \le q$ in Definition 1.2, then we obtain the following relation between the generalized Fox–Wright function $_p\psi_q$ and the generalized hypergeometric function $_pF_q$,[6]

$$(1.28) \qquad\qquad _p\psi_q \begin{pmatrix} (a_i,1)_1^p \\ (b_j,1)_1^q \end{pmatrix}; z \end{pmatrix} = \frac{\Gamma(\mathbf{a_P})}{\Gamma(\mathbf{b_q})} \, _pF_q \begin{pmatrix} \mathbf{a_P} \\ \mathbf{b_q} \end{pmatrix}; z \end{pmatrix}.$$

The $_p\psi_q$ and $_pF_q$ functions are special cases of the more general special functions Fox H- and Meijer G-functions.

General conditions of existence of M-L type series defined in (1.1) to (1.4) and their relations with generalized Fox–Wright-, Fox H- and Meijer G-functions (in Definitions 1.2, 1.4 and 1.5, resp.) are fully studied, for example, in [13, 15, 21, 30], along with their representations with suitable Mellin–Barnes contour integrals.

Remark 3. The generalized *Lommel–Wright function* denoted by $J_{\rho,\lambda}^{\mu,\nu}(z)$ is defined by

$$(1.29) \qquad \begin{aligned} J_{\rho,\lambda}^{\mu,\nu}(z) &:= \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\lambda+n+1)^\nu \Gamma(\rho+\lambda+\mu n+1)} \left(\frac{z}{2}\right)^{\rho+2\lambda+2n} \\ &= \left(\frac{z}{2}\right)^{\rho+2\lambda} {}_1\psi_{\nu+1} \begin{pmatrix} (1,1) \\ (\lambda+1,1)^\nu,(\rho+\lambda+1,\mu) \end{pmatrix}; -z^2/4 \end{pmatrix}, \end{aligned}$$

where $\mu > 0$, $\rho, \lambda \in \mathbb{C}$ and $\nu \in \mathbb{N}$. Turning to Wright's generalization of the classical Bessel function $J_\rho(z)$, that is the *Bessel–Wright function* $J_\rho^\mu(z)$, it is defined by

$$J_\rho^\mu(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(\rho+\mu n+1)} = {}_0\psi_1(-;(\rho+1,\mu);-z).$$

Eq. (1.29) readily yields the following relationships with the classical Bessel function, its generalization and a specialized Lommel–Wright function: $J_\rho(z)(z) = J_{\rho,0}^{1,1}(z)$, $J_{\rho,\lambda}^\mu(z) = J_{\rho,\lambda}^{\mu,1}(z)$, as well as $J_\rho^\mu(z) = z^{-\rho/2} J_{\rho,\lambda}^{\mu,1}(2\sqrt{2})$.

Definition 1.4. The *H-function* was introduced by Fox in [11] as a generalized hypergeometric function defined by an integral representation in terms of the Mellin–Barnes contour integral

$$(1.30) \qquad H_{p,q}^{m,n} \left[z \,\middle|\, \begin{matrix} (a_i,\alpha_i)_1^p \\ (b_j,\beta_j)_1^q \end{matrix} \right] := \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\prod\limits_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod\limits_{i=1}^{n} \Gamma(1 - a_i + \alpha_i s)}{\prod\limits_{j=m+1}^{q} \Gamma(1 - b_j + \beta_j s) \prod\limits_{i=n+1}^{p} \Gamma(a_i - \alpha_i s)} \, z^s \mathrm{d}s.$$

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[6]From Eq. (1.28), the conditions of convergence of $_pF_q(z)$ are easily recovered. When $p \le q$ and $\Delta \ge 0$, $_pF_q(z)$ is an entire function of $z \in \mathbb{C}$. But if $p = q + 1$, it is absolutely convergent in the unit disk $U = \{z \,|\, |z| < 1\}$ ($\delta = 1$) and diverges for all $z \ne 0$ if $p > q+1$. In the case when the variable z in $_{q+1}F_q(z)$ is not equal to unity, it is assumed tacitly that $|z| < 1$, while if $z = 1$, the condition $\Re \left(\sum\limits_{j=1}^{q} \beta_j - \sum\limits_{i=1}^{q+1} \alpha_i \right) > 0$ is required (see [10, §4.1]).

Here $\mathcal{L}$ is a suitable contour in $\mathbb{C}$ and $z^s = \exp\left(s \ln|z| + \mathrm{i}\arg z\right)$. Due to the occurrence of the factor z^s in the integrand, the H-function is, in general, multi-valued, but it can be made one-valued on the Riemann surface of $\log z$ by choosing a proper branch. In Eq. (1.30), an empty product, when it occurs, is taken to be equal to 1. The order (m, n, p, q) consists in integers with $0 \leq m \leq q$ and $0 \leq n \leq p$. The parameters fulfills the conditions a_i, $b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}_+$ $(i = 1, \ldots, p)$, $(j = 1, \ldots, q)$. As for the conditions on the types of contours ensuring the existence and analyticity of the Fox H-function in disks $D \subset \mathbb{C}$, one can see [3, 11],[21],[22, App.],[51, Chap. 1],[54], etc.

Definition 1.4 is still valid when the α_i's and the β_j's are positive rational numbers. Therefore, the H-function contains, as special cases, all of the functions which are expressible in terms of the G-function. More importantly, it contains the generalized hypergeometric Fox–Wright function defined in Definition 1.2, the generalized Bessel function J_ν^μ, the generalizations of the M-L function $E_{\alpha,\beta}$, etc. For example, ${}_p\psi_q(z)$ is one of these (additional) special cases of the H-function, which is obviously not contained in the class of G-function. By Definition 1.2, it is also easily extended to the complex plane as follows,

$$(1.31) \qquad {}_p\psi_q\left(\begin{matrix}(a_i,\alpha_i)_1^p\\(b_j,\beta_j)_1^q\end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i n)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j n)} \frac{z^n}{n!} = H_{p,q+1}^{1,p}\left[-z \,\middle|\, \begin{matrix}(1-a_i,\alpha_i)_1^p\\(0,1),(1-b_j,\alpha_j)_1^q\end{matrix}\right].$$

Proper conditions of convergence are shown from Definition 1.2 in [3, §2.1].

The special case for which the Fox H-function reduces to the Meijer G-function is when $\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = \kappa$, $\kappa > 0$ (see same references as above). In that case,

$$(1.32) \qquad H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix}(a_i,\kappa)_1^p\\(b_j,\kappa)_1^q\end{matrix}\right] := \frac{1}{\kappa}\, G_{p,q}^{m,n}\left(\begin{matrix}\mathbf{a_P}\\\mathbf{b_q}\end{matrix}\,\middle|\, z^{1/\kappa}\right),$$

Additionally, when setting $\alpha_i = \beta_j = 1$ in Eq.(1.31) (i.e. $\kappa = 1$ in Eq. (1.32)), the Fox H- and the Fox–Wright Psi functions turn readily into the Meijer G-function. A general definition of the complex G-function is given by the following line integral of Mellin–Barnes type and may be viewed as an inverse Mellin transform.

Meijer's G-function $G_{p,q}^{m,n}\left(\begin{matrix}\mathbf{a_P}\\\mathbf{b_q}\end{matrix}\,\middle|\, z\right)$ was actually first introduced as a generalization of the Gauß hypergeometric function presented in the form of the series

$$(1.33) \qquad G_{p,q}^{m,n}\left(\begin{matrix}1-\mathbf{a_P}\\0,1-\mathbf{b_q}\end{matrix}\,\middle|\, -z\right) = \frac{\Gamma(\mathbf{a_P})}{\Gamma(\mathbf{b_q})}\, {}_pF_q\left(\mathbf{a_P}; \mathbf{b_q}; z\right).$$

Later this definition was replaced by the Mellin–Barnes representation of the G-function.

Definition 1.5. The *Meijer G-function* of the order (m, n, p, q) is defined by an integral representation of Mellin–Barnes contour integral given by

$$(1.34) \qquad G_{p,q}^{m,n}\left(\begin{matrix}\mathbf{a_P}\\\mathbf{b_q}\end{matrix}\,\middle|\, z\right) := \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{L}} \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{i=1}^{n} \Gamma(1 - a_i + s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + s) \prod_{i=n+1}^{p} \Gamma(a_i - s)}\, z^s \mathrm{d}s,$$

where $\mathcal{L}$ is a suitably chosen contour such that $z \neq 0$, $z^s = \exp\left(s \ln|z| + \mathrm{i}\arg z\right)$ with a single valued branch of $\arg z$. Eq. (1.34) is valid under the following assumptions:

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(a) The empty product is assumed to be equal to 1.

(b) Parameters (m, n, p, q) satisfy the relation $0 \leq m \leq q$ and $0 \leq n \leq p$, where m, n, p and q are non-negative integers.

(c) The complex numbers $a_i - b_j \neq 1, 2, 3, \ldots$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, which implies that no pole of any $\Gamma(b_j - s)$, $j = 1, 2, \ldots, m$, coincides with any pole of $\Gamma(1 - a_i + s)$, $i = 1, 2, \ldots, n$.

1.3.1 Elementary identities of higher transcendental functions

Among the most useful elementary properties of the Meijer G-function are two straightforward consequences of Definition 1.5, from which the first identity is readily derived,

$$(1.35) \qquad z^\mu G_{p,q}^{m,n}\left(\begin{matrix} \mathbf{a_p} \\ \mathbf{b_q} \end{matrix} \middle| z\right) = G_{p,q}^{m,n}\left(\begin{matrix} \mathbf{a_p} + \mu \\ \mathbf{b_q} + \mu \end{matrix} \middle| z\right),$$

while the second one exhibits the fact that we can, w.l.o.g., suppose that $p \leq q$ in the discussion of the G-function,

$$(1.36) \qquad G_{p,q}^{m,n}\left(\begin{matrix} \mathbf{a_p} \\ \mathbf{b_q} \end{matrix} \middle| 1/z\right) = G_{q,p}^{n,m}\left(\begin{matrix} 1 - \mathbf{b_q} \\ 1 - \mathbf{a_p} \end{matrix} \middle| z\right).$$

According to the general translation formulas of the Fox H- and Meijer G-functions (see [3, 21]), we also have

$$(1.37\text{a}) \qquad H_{p,q}^{m,n}\left[z \, \middle| \, \begin{matrix} (a_i, \alpha_i)_1^p \\ (b_j, \beta_j)_1^q \end{matrix} \right] = H_{q,p}^{n,m}\left[1/z \, \middle| \, \begin{matrix} (1 - b_j, \beta_j)_1^q \\ (1 - a_i, \alpha_i)_1^p \end{matrix} \right] \qquad \text{and}$$

$$(1.37\text{b}) \qquad {}_pF_q\left(\begin{matrix} \mathbf{a_p} \\ \mathbf{b_q} \end{matrix}; z\right) = \frac{\Gamma(\mathbf{b_q})}{\Gamma(\mathbf{a_p})} G_{p,q+1}^{1,p}\left(\begin{matrix} 1 - \mathbf{a_p} \\ 0, 1 - \mathbf{b_q} \end{matrix} \middle| -z\right) = \frac{\Gamma(\mathbf{b_q})}{\Gamma(\mathbf{a_p})} G_{q+1,p}^{p,1}\left(\begin{matrix} 1, \mathbf{b_q} \\ \mathbf{a_p} \end{matrix} \middle| -1/z\right).$$

Unless there occurs a non-positive integer value for at least one of its upper parameters $\mathbf{a_p}$, the latter formula (1.37b) holds, in which case the hypergeometric function terminates in a finite polynomial. Then, the gamma prefactor of either G-function vanishes and the parameter sets of the G-functions violate the requirement $a_i - b_j \neq 1, 2, 3, \ldots$ for $i = 1, 2 \ldots, n$ and $j = 1, 2, \ldots m$ from Definition 1.5 of $G_{p,q}^{m,n}$ (The conditions of convergence are exposed in details in [3].) Therefore, calculations on L-S and R-S fractional integration operators are amenable to Laplace and Mellin integral transforms for example, as shown in [9],[23, App.],[43, Chap. 2, §7].

The G- and H-functions encompass almost all the elementary and special functions. Among them, the M-L function and the Fox–Wright psi function. From Eq. (1.35), ${}_p\psi_q(z)$ can easily simplify to the generalized hypergeometric function ${}_pF_q(z)$ with parameters a_i's and b_j's and a gamma prefactor which consists of these parameters. According to the relationship between the H- and G-functions in Eq. (1.32) and to the expression of ${}_p\psi_q(z)$ in terms of the H-function in Eq. (1.31), the following useful identities hold

$$\begin{aligned} {}_p\psi_q\left(\begin{matrix} (a_i, 1)_1^p \\ (b_j, 1)_1^q \end{matrix}; z\right) &= H_{p,q+1}^{1,p}\left[-z \, \middle| \, \begin{matrix} (1 - a_i, 1)_1^p \\ (0,1), (1 - b_j, 1)_1^q \end{matrix} \right] \\ (1.38) \qquad\qquad &= G_{p,q+1}^{1,p}\left(\begin{matrix} 1 - \mathbf{a_p} \\ 0, 1 - \mathbf{b_q} \end{matrix} \middle| -z\right) = \frac{\Gamma(\mathbf{a_p})}{\Gamma(\mathbf{b_q})} \, {}_pF_q\left(\begin{matrix} \mathbf{a_p} \\ \mathbf{b_q} \end{matrix}; z\right), \end{aligned}$$

where $z, a_i, b_j \in \mathbb{C}$ ($i = 1, \ldots, p$; $j = 1, \ldots, q$) and $\Re\left(\sum_{j=1}^q b_j - \sum_{i=1}^p a_i\right) > 0$. Conversely, setting $a_i = b_j = 1$ in Eq. (1.31) yields the generalized hypergeometric function ${}_pF_q\left(\begin{matrix} \alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q \end{matrix}; z\right)$.

Finally, notice that the *H*- and *G*-functions are analytic functions of z with a branch point at the origin. Especially, the kernel functions $H_{m,m}^{m,0}$ and $G_{m,m}^{m,0}$ ($n = 0$, $m = p = q$) of the operators of generalized fractional calculus that we consider, are analytic functions in the unit disc and vanish identically outside it, i.e. for $|z| > 1$.

2   Fractional calculus of generalized M-L type functions

Here and throughout the paper, the class of functions we deal with is a space of analytic functions in some domain $D \subseteq \mathbb{C}$. More precisely such functions are currently taken in

$$\mathfrak{H}_\mu(D) := \big\{ f(z) = z^\mu \tilde{f}(z) \,|\, \tilde{f} \in \mathfrak{H}(D) \big\},$$

where $\mu \geq 0$ is a real number and $\mathfrak{H}(D)$ is the space of analytic (single-valued) functions in a complex domain D, starlike with respect to the origin $z = 0$. For $\alpha \in \mathbb{C}$ and $\Re(\alpha) > 0$, composition relations between a very large number of generalized M-L type functions and the operators of fractional integration $\mathcal{I}^\alpha$ and differentiation $\mathcal{D}^\alpha$ are well established.

A fairly natural functional space where the *generalized fractional differintegration operators* may be considered is the space of *weakly differentiable functions* (i.e. Fréchet a.e.) in $L^1_{\text{loc}}(\Omega)$, where $L^1_{\text{loc}}(\Omega)$ denotes the set of locally integrable functions on a compact subset Ω of $\mathbb{R}$ (see, e.g., Bergounioux *et al.* [5], Kiryakova [22, Chap. 1], [24, 25], Samko *et al.* [43, §1.1]) and Srivastava [49].

2.1   Riemann–Liouville fractional calculus

For a suitable class of weakly differentiable functions $\varphi \in L^1_{\text{loc}}\big((a, b]\big)$ ($a < b \in \mathbb{R}$), it is convenient to set $a = 0$, without loss of generality (see definition below). Then, the related fractional integration operators are referred to by the notation $\mathcal{I}^\alpha_{0+}$ and $\mathcal{I}^\alpha_{x-}$, respectively and the *left-sided* and *right-sided Riemann–Liouville fractional integrals* of order $\alpha \in \mathbb{C}$ are defined by

$$(2.1a) \qquad \mathcal{I}^\alpha_{0+} f(x) := \frac{1}{\Gamma(\alpha)} \int_0^x \frac{1}{(x-t)^{1-\alpha}}\, f(t)\, \mathrm{d}t \qquad (\alpha \in \mathbb{C},\ \Re(\alpha) > 0,\ x > 0),$$

$$(2.1b) \qquad \mathcal{I}^\alpha_{x-} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^\infty \frac{1}{(t-x)^{1-\alpha}}\, f(t)\, \mathrm{d}t \qquad (\alpha \in \mathbb{C},\ \Re(\alpha) > 0,\ x < 0).$$

This notion is a natural consequence of the well known formula (Cauchy–Dirichlet?), that reduces the calculation of the *n*-fold primitive ($n \in \mathbb{N}$) of a real function $f(x)$ to a single integral of convolution type

$$I_{a+} f(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)\, dt,$$

which equals 0 at $t = a$ with its derivatives of order $1, 2, \ldots, n-1$. The definition also requires $f(x)$ and $I_{a+} f(x)$ to be *causal functions*, that is, vanishing for $x < 0$.

As for the corresponding *Riemann–Liouville fractional differentiation operators*, $\mathcal{D}^\alpha_x$, they are

defined by

$$
\text{(2.2a)} \quad \left(\mathcal{D}_{0+}^{\alpha} f \right)(x) := \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^n \left(\mathcal{I}_{0+}^{n-\alpha} f \right)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^n \int_0^x \frac{f(t)}{(x-t)^{1-n+\alpha}} \, \mathrm{d}t \quad (x > 0),
$$

$$
\text{(2.2b)} \quad \left(\mathcal{D}_{x-}^{\alpha} f \right)(x) := \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^n \left(\mathcal{I}_{x-}^{n-\alpha} f \right)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^n \int_x^{\infty} \frac{f(t)}{(t-x)^{1-n+\alpha}} \, \mathrm{d}t \quad (x < 0),
$$

where $\lfloor \Re(\alpha) \rfloor$ denotes the integral part of $\Re(\alpha)$, $\{\alpha\} := \alpha - \lfloor \Re(\alpha) \rfloor$ denotes its fractional part and $n := \lceil \Re(\alpha) \rceil = \lfloor \Re(\alpha) \rfloor + 1$ (see, e.g., [19] and [43, §§2.1–2.4]). All such well-defined Riemann–Liouville (R–L for short) fractional operators are called respectively the left-and right-sided fractional integrals and fractional derivatives of order α. In a synthetic form, the R-L operators are denoted by $\mathcal{R}_z^{\alpha}$ ($\alpha \in \mathbb{C}$) and written as

$$
\mathcal{R}_z^{\alpha} f(z) := \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1} f(t) \mathrm{d}t \qquad \text{and} \qquad \frac{1}{\Gamma(\alpha)} \int_z^{\infty} (t-z)^{\alpha-1} f(t) \mathrm{d}t \qquad (\Re(\alpha) > 0), \\ \frac{\mathrm{d}^n}{\mathrm{d}z^n} \left(\mathcal{R}_z^{\alpha+n} f \right)(z) \qquad (-n < \Re(\alpha) \leq 0, \quad n = \lfloor \Re(\alpha) \rfloor + 1 \in \mathbb{N}), \end{cases}
$$

provided that the defining integrals exists.

For example, it is easily seen that, for $Re(\mu) > -1$, $z > 0$,[7]

$$
\text{(2.3)} \qquad \mathcal{D}_{0+}^{\alpha}(z^{\mu}) = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\alpha)} z^{\mu-\alpha}, \qquad \text{and} \qquad \mathcal{D}_{0+}^{\alpha}(z^{-\mu}) = \frac{\Gamma(\mu+\alpha)}{\Gamma(\mu)} z^{-\mu-\alpha}
$$

and, more importantly, that

$$
\text{(2.4)} \qquad\qquad E_{\alpha,\beta}^{\gamma}(z) = \frac{1}{\Gamma(\gamma)} \mathcal{D}_{0+}^{\alpha} \left(z^{\gamma-1} E_{\alpha,\beta}(z) \right) \qquad (\Re(\gamma) > 0).
$$

2.2 Saigo's fractional differintegration operators

Samko *et al.* [43, Sec. 2, §2] and Erdélyi–Kober extended the R–L fractional operators (see, e.g., Kilbas *et al.* in [21, §2.6]) and references therein). A useful generalization of fractional integral operators was later introduced by Saigo [37, 39] by adding Gauß's hypergeometric function in the kernel. Saigo's operators include Weyl's type and Erdélyi–Kober's ones.

Definition 2.1 (Saigo [40, p. 393]). Let $\alpha, \beta, \gamma \in \mathbb{C}$ and $\Re(\alpha) > 0$. For a suitable class $\mathcal{C}$ of complex functions specified below, Saigo's left- and right-sided operators of fractional integration

---

[7]From Eq. (2.3) we can check the intuitive idea that $\mathcal{D}^{-1}$ is the integration operator. Examine indeed the operator $\mathcal{D}^{-1}$ applied to a monomial.

$$
\mathcal{D}_{0+}^{-1}(z^{\mu}) = \frac{\Gamma(\mu+1)}{\Gamma(\mu+2)} z^{\mu+1} = \frac{z^{\mu+1}}{\mu+1},
$$

from which we see that effectively this is the integral operator. Now, by linearity of the operator we obtain the standard definition of the integral and in general we write

$$
\mathcal{D}_{0+}^{-1}\left(f(z) \right) = \int_0^z f(t) \mathrm{d}t.
$$

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are defined for $x \in \mathbb{R}^+$ by

$$(2.5\text{a}) \qquad \left(\mathcal{I}_{0+}^{\alpha,\beta,\gamma} f\right)(x) := \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int\limits_0^x (x-t)^{\alpha-1} \,_2F_1\left(\begin{matrix} \alpha+\beta,-\gamma \\ \alpha \end{matrix}; 1-t/x\right) f(t)\,\mathrm{d}t \qquad \text{and}$$

$$(2.5\text{b}) \qquad \left(\mathcal{I}_{x-}^{\alpha,\beta,\gamma} f\right)(x) := \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int\limits_x^\infty (t-x)^{\alpha-1}\, t^{-\alpha-\beta} \,_2F_1\left(\begin{matrix} \alpha+\beta,-\gamma \\ \alpha \end{matrix}; 1-x/t\right) f(t)\,\mathrm{d}t.$$

Similarly, Saigo's left- and right-sided fractional differentiation operators $\mathcal{D}_{0+}^{\alpha,\beta,\gamma}$ and $\mathcal{D}_{x-}^{\alpha,\beta,\gamma}$ corresponding respectively to (2.5a) and (2.5b), write as follows with $\Re(\alpha) > 0$ and $n := \lfloor\Re(\alpha)\rfloor + 1$,

$$(2.6\text{a}) \qquad \left(\mathcal{D}_{0+}^{\alpha,\beta,\gamma} f\right)(x) := \left(\mathcal{I}_{0+}^{-\alpha,-\beta,-\gamma} f\right)(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\mathcal{I}_{0+}^{-\alpha,-\beta,-\gamma+n} f\right)(x) \qquad \text{and}$$

$$(2.6\text{b}) \qquad \left(\mathcal{D}_{x-}^{\alpha,\beta,\gamma} f\right)(x) := \left(\mathcal{I}_{x-}^{-\alpha,-\beta,-\gamma} f\right)(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\mathcal{I}_{x-}^{-\alpha,-\beta,-\gamma+n} f\right)(x).$$

When $\beta = -\alpha$, the above fractional operators (2.5a) to (2.6b) include the classical Riemann–Liouville's in (2.1a)–(2.2b):

$$\mathcal{I}_{0+}^{\alpha,-\alpha,\gamma} = \mathcal{I}_{0+}^{\alpha}, \quad \mathcal{I}_{x-}^{\alpha,-\alpha,\gamma} = \mathcal{I}_{x-}^{\alpha} \qquad \text{and} \qquad \mathcal{D}_{0+}^{\alpha,-\alpha,\gamma} = \mathcal{D}_{0+}^{\alpha}, \quad \mathcal{D}_{x-}^{\alpha,-\alpha,\gamma} = \mathcal{D}_{x-}^{\alpha}.$$

Here, $\mathcal{C}$ is the class of analytic functions $f(z)$ in a simply-connected region of the z-plane containing the origin (for example the unit disk), of the order $f(z) = \mathcal{O}\left(|z|^\varepsilon\right)$ $(|z| \to 0)$, where $\varepsilon > \max\{0, \beta - \eta\}$ and the multiplicity of $(x-t)^{\alpha-1}$ $((t-x)^{\alpha-1}$, resp.) is removed by requiring $\ln(x-t)$ $(\ln(t-x)$, resp.) to be real when $x > t$ $(x < t$, resp.). (Exhaustive conditions of existence and convergence of the r.h.s. of Eqs. (2.5a) and (2.5b) may be found, e.g., in [3, 22, 23].) Saigo's left- and right-sided fractional integrals are consistently employed to prove the results in Sections 4.1 and 4.2.[8]

Remark 4. For suitable functions f of a real variable and $\alpha \in \mathbb{R}_+$, there holds the fundamental property of operators of fractional integration, namely the additive index law (or semigroup property), according to which

$$\mathcal{J}_{a+}^\alpha \, \mathcal{J}_{a+}^\beta = \mathcal{J}_{a+}^{\alpha+\beta} \qquad \text{and} \qquad \mathcal{J}_{b-}^\beta \, \mathcal{J}_{b-}^\alpha = \mathcal{J}_{b-}^{\alpha+\beta} \qquad (\alpha, \beta \geq 0),$$

where, for complementation we define $\mathcal{J}_{a+}^0 = \mathcal{J}_{b-}^0 := \mathbb{I}$ (Identity operator), i.e., $\mathcal{J}_{a+}^\alpha f(z) = \mathcal{J}_{b-}^\alpha f(z) = f(z)$. It can be shown however that the fractional differentiation operator $\mathcal{D}^\alpha$ is neither commutative nor additive in general. Indeed, denoting by $\mathcal{D}^n$ the operator of ordinary differentiation of order $n \in \mathbb{N}$, then $\mathcal{D}^n \mathcal{J}^n = \mathbb{I}$, whereas $\mathcal{J}^n \mathcal{D}^n \neq \mathbb{I}$; i.e., $\mathcal{D}^n$ is left-inverse (but not right-inverse) to the corresponding integral operator $\mathcal{J}^n$. Furthermore, by $\mathcal{I}_t^\alpha f(0+)$ we mean $\lim\limits_{t\to 0+} \mathcal{I}_t^\alpha f(t)$ (if the limit exists) and this limit may be infinite.

---

[8]As shown e.g. by Samko *et al.* in [43, Chap. 4, §22], the fractional integration process can also be carried out by use of the formula for differentiating the Cauchy type integral to the (real or complex) order α:

$$\mathcal{D}_z^\alpha(z) = \frac{\Gamma(1+\alpha)}{2\pi\mathrm{i}} \int_\mathcal{L} \frac{f(t)}{(t-z)^{1+\alpha}}\,\mathrm{d}t.$$

2.3 Saigo–Maeda's fractional differintegration operators

A natural extension of Saigo's generalized fractional differintegral operators stated in Definition 2.1 (Eqs. (2.5a)–(2.2b)) was introduced further. This extension involves Appell's two variable hypergeometric series F_3 in the kernel (a generalization of Gauß's hypergeometric function of one variable). Appell's two variable F_3 is defined by the double series

$$(2.7) \qquad F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y) := \sum_{m,n \geq 0} \frac{(\alpha)_m (\alpha')_n (\beta)_m (\beta)_n}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \qquad (\max(|x|, |y|) < 1).$$

Several generalized fractional differintegral operators involving Appell's two variable hypergeometric function F_3 in the kernel have been addressed in [13, 43]. For a suitable class of function such as $\mathcal{C}$, *Saigo–Maeda's (or Marichev–Saigo–Maeda's) fractional differintegral operators* are introduced therefrom.

Definition 2.2 (Saigo–Maeda [40, p. 393, Eqs. (4.12)–(4.13)]). Let α, α', β, β', $\gamma \in \mathbb{C}$ and $\Re(\gamma) > 0$. Then, for $x \in \mathbb{R}_+$,

$$(2.8a) \quad \left(\mathcal{I}_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f \right)(x) := \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3\left(\alpha, \alpha', \beta, \beta', \gamma; 1 - t/x, 1 - x/t\right) f(t) \mathrm{d}t$$

and

$$(2.8b) \quad \left(\mathcal{I}_{x-}^{\alpha,\alpha',\beta,\beta',\gamma} f \right)(x) := \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_x^\infty (t-x)^{\gamma-1} t^{-\alpha} F_3\left(\alpha, \alpha', \beta, \beta', \gamma; 1 - x/t, 1 - t/x\right) f(t) \mathrm{d}t.$$

Similarly, set $n = \lfloor \Re(\gamma) \rfloor + 1$, then

$$\left(\mathcal{D}_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f \right)(x) := \left(\mathcal{I}_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f \right)(x)$$

$$(2.9a) \qquad\qquad = \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^n \left(\mathcal{I}_{0+}^{-\alpha',-\alpha,-\beta'+n,-\beta,-\gamma+n} f \right)(x) \qquad \text{and}$$

$$\left(\mathcal{D}_{x-}^{\alpha,\alpha',\beta,\beta',\gamma} f \right)(x) := \left(\mathcal{I}_{x-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f \right)(x)$$

$$(2.9b) \qquad\qquad = (-1)^n \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^n \left(\mathcal{I}_{x-}^{-\alpha',-\alpha,-\beta',-\beta+n,-\gamma+n} f \right)(x).$$

As a direct consequence of the above definitions, the following lemma holds.

Lemma 2.3 (Saigo–Maeda [40, p. 394, Eqs. (4.18)–(4.19)]). *Let α, α', β, β', γ, $\rho \in \mathbb{C}$. For $f(x) := x^{\rho-1}$, we get the formulas*
(i) If $\Re(\gamma) > 0$ and $\Re(\rho) > \max\left(0, \Re(\alpha + \alpha' + \beta - \gamma), \Re(\alpha' - \beta')\right)$, then

(2.10a)
$$\left(\mathcal{I}_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1} \right)(x) = \frac{\Gamma(\rho)\Gamma(\rho + \gamma - \alpha - \alpha' - \beta)\Gamma(\rho - \alpha' + \beta')}{\Gamma(\rho + \beta')\Gamma(\rho + \gamma - \alpha - \alpha')\Gamma(\rho + \gamma - \alpha' - \beta)} x^{\rho - \alpha - \alpha' + \gamma - 1}.$$

(ii) If $\Re(\gamma) > 0$ and $\Re(\rho) < 1 + \min\left(0, \Re(-\beta), \Re(\alpha + \alpha' - \gamma), \Re(\alpha + \beta' - \gamma)\right)$, then

(2.10b)
$$\left(\mathcal{I}_{x-}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1} \right)(x) = \frac{\Gamma(1 + \alpha + \alpha' - \gamma - \rho)\Gamma(1 + \alpha + \beta' - \gamma - \rho)\Gamma(1 - \beta - \rho)}{\Gamma(1 - \rho)\Gamma(1 + \alpha + \alpha' + \beta' - \gamma - \rho)\Gamma(1 + \alpha - \beta - \rho)} x^{\rho - \alpha - \alpha' + \gamma - 1}.$$

Remarks 2. Reducing Saigo–Maeda's fractional differintegral operators (2.8a)–(2.9b) to Saigo's (2.5a)–(2.5b) is commonly done by means of the following relations,

(2.11a) $\qquad \mathcal{I}_{0+}^{\alpha,0,\beta,\beta',\gamma} = \mathcal{I}_{0+}^{\gamma,\alpha-\gamma,-\beta} \qquad$ and $\qquad \mathcal{I}_{x-}^{\alpha,0,\beta,\beta',\gamma} = \mathcal{I}_{x-}^{\gamma,\alpha-\gamma,-\beta} \qquad (\gamma \in \mathbb{C}).$

and, similarly,

(2.11b) $\quad \mathcal{D}_{0+}^{0,\alpha',\beta,\beta',\gamma} = \mathcal{D}_{0+}^{\gamma,\alpha'-\gamma,\beta'-\gamma} \qquad$ and $\qquad \mathcal{D}_{x-}^{0,\alpha',\beta,\beta',\gamma} = \mathcal{D}_{x-}^{\gamma,\alpha'-\gamma,\beta'-\gamma} \qquad (\Re(\gamma) > 0).$

Lemma 2.3 and Eqs. (2.11a)–(2.11b) are useful and often required in fractional calculus.

The Appell function F_3 in two variables which appears in the definitions (2.8a) and (2.8b) satisfies a system of two linear partial differential equations of the second order and reduces to the Gauss hypergeometric function $_2F_1$ in the form

$$F_3(\alpha, \gamma - \alpha, \beta, \gamma - \beta; \gamma; x, y) = {}_2F_1\left({\alpha,\beta \atop \gamma}; x + y - xy\right).$$

Moreover, it is easily observed that

$$F_3(\alpha, 0, \beta, \beta'; \gamma; x, y) = F_3(\alpha, \alpha', \beta, 0; \gamma; x, y) = {}_2F_1\left({\alpha,\beta \atop \gamma}; x\right) \qquad \text{and}$$

$$F_3(0, \alpha', \beta, \beta'; \gamma; x, y) = F_3(\alpha, \alpha', 0, \beta'; \gamma; x, y) = {}_2F_1\left({\alpha',\beta' \atop \gamma}; x\right).$$

In view of the above reduction formula, the general operators reduce to the aforementioned Saigo's operators $\mathcal{I}_{0+}^{\alpha,\beta,\gamma}$ and $\mathcal{I}_{x-}^{\alpha,\beta,\gamma}$ defined in Definition 2.1, Eqs. (2.5a) and (2.5b).

3   The generalized M-series and K-function

The *generalized M-series and the K-function* introduced in [46, 47] both extend the Fox–Wright generalized hypergeometric function $_p\psi_q(z)$ stated in Definition 1.2 and of Prabakhar's three-parametric generalized M-L function defined in Eq. (1.4). Namely,

Definition 3.1. Let $z, \alpha, \beta \in \mathbb{C}$ and $\Re(\alpha) > 0$, then

(3.1) $\qquad\qquad {}_pM_q^{\alpha,\beta}(z) = {}_pM_q^{\alpha,\beta}\left({\mathbf{a_P} \atop \mathbf{b_q}}; z\right) := \sum_{n \geq 0} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{\Gamma(\alpha n + \beta)}.$

Definition 3.2. Let $z, \alpha, \beta, \gamma \in \mathbb{C}$ and $\Re(\alpha) > 0$, then

(3.2) $\qquad\qquad {}_pK_q^{\alpha,\beta;\gamma}(z) = {}_pK_q^{\alpha,\beta;\gamma}\left({\mathbf{a_P} \atop \mathbf{b_q}}; z\right) := \sum_{n \geq 0} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}.$

As a matter of fact, the K-function stands for a three-parametric variant of the (two-parametric) M-series. Of course, if $\gamma = 1$, $_pK_q^{\alpha,\beta;1}(z)$ coincides with $_pM_q^{\alpha,\beta}(z)$.

Both series are defined provided that none of the bottom parameters b_j's ($j = 1, 2, \ldots, q$) is a non-positive integer. If any of the top parameters a_j is in $\mathbb{Z}_{\leq 0}$, then the series terminate. From the ratio test, the series are convergent for all z if $p \leq q$, and they are divergent if $p > q + 1$. On the circle $|z| = 1$ and when $p = q + 1$, the conditions of convergence of the series depend on the parameters. More precisely, they are absolutely convergent when $\Re\left(\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i\right) > 0$, they are conditionally convergent when $-1 < \Re\left(\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i\right) \leq 0$ at $z = -1$ and they

are divergent if $\Re\left(\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i\right) \leq -1$. The exhaustive proofs of convergence of both series in (3.1)–(3.2) are deduced from the general theory of the Fox–Wright function (see, e.g., [3, §16.2] and [21, Thm. 1.5]).

3.1 Relations to the Fox–Wright and M-L type functions

According to some well-chosen specific values of the parameters, the M-series and the K-function can be easily be reduced to classical generalized special functions. As a straighforward consequence of Definitions 1.2, 3.1 and 3.2, the next two expressions involve the generalized Fox–Wright psi function, its normalized variant and the H-function.

Proposition 3.3. *Let* $z, \alpha, \beta \in \mathbb{C}$ *and* $\Re(\alpha) > 0$, *then*

$$
{}_pM_q^{\alpha,\beta}(z) = \frac{\Gamma(\mathbf{b_q})}{\Gamma(\mathbf{a_p})} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i+n)}{\prod_{j=1}^{q} \Gamma(b_j+n)} \frac{z^n}{\Gamma(\alpha n + \beta)} = \frac{\Gamma(\mathbf{b_q})}{\Gamma(\mathbf{a_p})} \; {}_{p+1}\psi_{q+1}\left(\begin{matrix}(a_i,1)_1^p,(1,1)\\(b_j,1)_1^q,(\beta,\alpha)\end{matrix}; z\right)
$$

(3.3a) $\qquad = {}_{p+1}\psi_{q+1}^*\left(\begin{matrix}(a_i,1)_1^p,(1,1)\\(b_j,1)_1^q,(\beta,\alpha)\end{matrix}; z\right) = \frac{\Gamma(\mathbf{b_q})}{\Gamma(\mathbf{a_p})} H_{p+1,q+2}^{1,p+1}\left[-z \,\middle|\, \begin{matrix}(1-\alpha_i,1)_1^p,(0,1)\\(0,1),(1-b_j,1)_1^q,(1-\beta,\alpha)\end{matrix}\right]$

and, similarly,

$$
{}_pK_q^{\alpha,\beta;\gamma}(z) = \frac{1}{\Gamma(\gamma)} \frac{\Gamma(\mathbf{b_q})}{\Gamma(\mathbf{a_p})} \; {}_{p+2}\psi_{q+2}\left(\begin{matrix}(a_i,1)_1^p,(\gamma,1),(1,1)\\(b_j,1)_1^q,(1,1),(\beta,\alpha)\end{matrix}; z\right) = \frac{1}{\Gamma(\gamma)} \; {}_{p+2}\psi_{q+2}^*\left(\begin{matrix}(a_i,1)_1^p,(\gamma,1),(1,1)\\(b_j,1)_1^q,(1,1),(\beta,\alpha)\end{matrix}; z\right)
$$

(3.3b) $\qquad = \frac{1}{\Gamma(\gamma)} \frac{\Gamma(\mathbf{b_q})}{\Gamma(\mathbf{a_p})} H_{p+2,q+3}^{1,p+2}\left[-z \,\middle|\, \begin{matrix}(1-\alpha_i,1)_1^p,(\gamma,1),(1,1))\\(0,1),(1-\beta_j,1)_1^q,(1,1),(1-\beta,\alpha)\end{matrix}\right],$

provided that each member of the equations exists.

The above Eqs. (3.3a) and (3.3b) exhibit the fact that the generalized M-series and K-function are in fact two obvious (rather trivial) variants of the generalized Fox-Wright psi function ${}_p\psi_q$ and its normalized form ${}_p\psi_q^*$ defined in Eq. (1.27).

A natural unification and generalization of the Fox-Wright function ${}_p\psi_q^*$ defined by Eq. (1.27) anf of Lerch's transcendent zeta function $\Phi(z,s,a)$ defined in Note 2 is indeed accomplished by introducing essentially arbitrary numbers of numerator and denominator parameters in the definition of $\Phi(z,s,\tau)$.

When there exists neither top nor bottom parameters in the definitions of the M-series and the K-function in Eqs. (3.1) and (3.2), one obtains, respectively,

(3.4) $\qquad {}_0M_0^{\alpha,\beta}(-;-;z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} =: E_{\alpha,\beta}(z) \qquad$ and

(3.5) $\qquad {}_0K_0^{\alpha,\beta;\gamma}(-;-;z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!} = E_{\alpha,\beta}^{\gamma}(z) = {}_1M_1^{\alpha,\beta}(z).$

The formulas coincide with the two-parametric M-L function in (3.4) and with Prabhakar's generalized form in (3.5), respectively. Now, if we let $\gamma = 1$ in formula (3.5), the two-parametric M-L function may be rewritten in the form

(3.6) $\qquad {}_0K_0^{\alpha,\beta;1}(-;-;z) = E_{\alpha,\beta}^1(z) := E_{\alpha,\beta}(z) = {}_0M_0^{\alpha,\beta}(\gamma;1;z),$

and setting $\beta = 1$ in (3.6)entails $E_{\alpha,1}(z) := E_\alpha(z)$. Finally, putting $\alpha = 1$ in the above relation (3.6) yields $E_{1,1}(z) := E_1(z)$, which is also the exponential function e^z [36].

4 Fractional calculus of the M-series and the K-function

With the help of the left-sided (L-S for short) Riemann–Liouville fractional integral and derivative of order $\nu \in \mathbb{C}$, it can be shown first in the following Theorem 4.1, that as naturally expected for fractional calculus operators of generalized hypergeometric special functions, the M-series $_pM_q^{\alpha,\beta}$ and the K-function $_pK_q^{\alpha,\beta;\gamma}$ write respectively in terms of $_{p+1}M_{q+1}^{\alpha,\beta}$ and $_{p+1}K_{q+1}^{\alpha,\beta;\gamma}$ up to prefactors.

Theorem 4.1. *Let z, α, β, γ, $\nu \in \mathbb{C}$ and $\Re(\alpha)$, $\Re(\gamma)$, $\Re(\nu) > 0$ and a_i, $b_j \in \mathbb{C}$ ($i = 1, \ldots, p$, $j = 1, \ldots, q$), then*

$$\mathcal{I}_{0+}^{\nu}\left(_pM_q^{\alpha,\beta}\right)(z) = \frac{z^{\nu}}{\Gamma(\nu+1)}\ _{p+1}M_{q+1}^{\alpha,\beta}\begin{pmatrix} \mathbf{a_p},1 \\ \mathbf{b_q},\nu+1 \end{pmatrix};z \qquad and$$

$$(4.1) \qquad \mathcal{I}_{0+}^{\nu}\left(_pK_q^{\alpha,\beta;\gamma}\right)(z) = \frac{z^{\nu}}{\Gamma(\nu+1)}\ _{p+1}K_{q+1}^{\alpha,\beta;\gamma}\begin{pmatrix} \mathbf{a_p},1 \\ \mathbf{b_q},\nu+1 \end{pmatrix};z.$$

Similarly,

$$\mathcal{D}_{0+}^{\nu}\left(_pM_q^{\alpha,\beta}\right)(z) = \frac{z^{-\nu}}{\Gamma(1-\nu)}\ _{p+1}M_{q+1}^{\alpha,\beta}\begin{pmatrix} \mathbf{a_p},1 \\ \mathbf{b_q},1-\nu \end{pmatrix};z \qquad and$$

$$(4.2) \qquad \mathcal{D}_{0+}^{\nu}\left(_pK_q^{\alpha,\beta;\gamma}\right)(z) = \frac{z^{-\nu}}{\Gamma(1-\nu)}\ _{p+1}K_{q+1}^{\alpha,\beta;\gamma}\begin{pmatrix} \mathbf{a_p},1 \\ \mathbf{b_q},1-\nu \end{pmatrix};z.$$

Proof. Consider the L-S fractional R–L operator $\mathcal{I}_{0+}^{\nu}$ of the generalized M-series, defined in (2.1a).

$$\mathcal{I}_{0+}^{\nu}\left(_pM_q^{\alpha,\beta}\right)(z) = \frac{1}{\Gamma(\nu)}\int_0^z (z-t)^{\nu-1}\sum_{k=0}^{\infty}\frac{(\mathbf{a_p})_k}{(\mathbf{b_q})_k}\frac{t^k}{\Gamma(\alpha k+\beta)}\,\mathrm{d}t.$$

The uniform convergence of the M-series follows, for example, from the convergence of the integral involved. Thus, term by term integration yields

$$\mathcal{I}_{0+}^{\nu}\left(_pM_q^{\alpha,\beta}\right)(z) = \frac{1}{\Gamma(\nu)}\sum_{k=0}^{\infty}\frac{(\mathbf{a_p})_k}{(\mathbf{b_q})_k}\frac{z^{\nu-1}}{\Gamma(\alpha k+\beta)}\int_0^z (1-t/z)^{\nu-1}t^k\mathrm{d}t.$$

Substituting the variable u for t/z, by definition of $(x)_k = \Gamma(x+k)/\Gamma(x)$, we have

$$\mathcal{I}_{0+}^{\nu}\left(_pM_q^{\alpha,\beta}\right)(z) = \frac{1}{\Gamma(\nu)}\sum_{k=0}^{\infty}\frac{(\mathbf{a_p})_k}{(\mathbf{b_q})_k}\frac{z^{\nu+k}}{\Gamma(\alpha k+\beta)}\int_0^1 (1-u)^{\nu-1}u^k\mathrm{d}u$$

$$= \frac{z^{\nu}}{\Gamma(\nu)}\sum_{k=0}^{\infty}\frac{(\mathbf{a_p})_k}{(\mathbf{b_q})_k}\frac{(1)_k}{(\nu+1)_k}\frac{z^k}{\Gamma(\alpha k+\beta)} \qquad and$$

$$(4.3) \qquad \mathcal{I}_{0+}^{\nu}\left(_pM_q^{\alpha,\beta}\right)(z) = \frac{z^{\nu}}{\Gamma(\nu+1)}\ _{p+1}M_{q+1}^{\alpha,\beta}\begin{pmatrix} \mathbf{a_p},1 \\ \mathbf{b_q},\nu+1 \end{pmatrix};z.$$

The proof for $_pK_q^{\alpha,\beta;\gamma}(z)$ proceeds exactly along the same lines. In other words, a L-S fractional integral of an M-series (a K-function, resp.) can be rewritten as an M-series (a K-function, resp.), up to a prefactor, with the values of the indices p and q each incremented by one. This proves the first (integration) part of the theorem.

As for the derivation part of the theorem, consider likewise the L-S fractional R–L derivative operator $\mathcal{D}_{0+}^{\nu}$ of the generalized M-series, defined in (2.2a).

$$\mathcal{D}_{0+}^{\nu}\left({}_{p}M_{q}^{\alpha,\beta}\right)(z) = \frac{1}{\Gamma(n-\nu)}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n}\int\limits_{0}^{z}(z-t)^{n-\nu-1}\sum_{k=0}^{\infty}\frac{(\mathbf{a_p})_k}{(\mathbf{b_q})_k}\frac{t^k}{\Gamma(\alpha k+\beta)}\,\mathrm{d}t,$$

where $n = \lfloor\Re(\nu)\rfloor + 1$. As above, term by term integration leads to

$$\mathcal{D}_{0+}^{\nu}\left({}_{p}M_{q}^{\alpha,\beta}\right)(z) = \frac{1}{\Gamma(n-\nu)}\left(\frac{\mathrm{d}}{\mathrm{d}z}r\right)^{n}\sum_{k=0}^{\infty}\frac{(\mathbf{a_p})_k}{(\mathbf{b_q})_k}\frac{1}{\Gamma(\alpha k+\beta)}\int\limits_{0}^{z}(z-t)^{n-\nu-1}t^k\mathrm{d}t$$

$$= \frac{1}{\Gamma(n-\nu)}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n}\sum_{k=0}^{\infty}\frac{(\mathbf{a_p})_k}{(\mathbf{b_q})_k}\frac{1}{\Gamma(\alpha k+\beta)}\,B(n-k,k-1)$$

$$= \frac{1}{\Gamma(n-\nu)}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n}\sum_{k=0}^{\infty}\frac{(\mathbf{a_p})_k}{(\mathbf{b_q})_k}\frac{z^{n-\nu+k}}{\Gamma(\alpha k+\beta)}\frac{\Gamma(n-\nu)\Gamma(k+1)}{\Gamma(n-\nu+k+1)},$$

where $k+1 > 0$ and $n-\nu > 0$. Now, by the modified beta function formula

$$\int\limits_{a}^{b}(t-a)^{\alpha-1}(b-t)^{\beta-1} = (b-a)^{\alpha+\beta-1}\,B(\alpha,\beta) \qquad (\Re(\alpha),\ \Re(\beta) > 0),$$

differentiating n times the term $z^{n-\nu+k}$ yields, provided that $k+1 > 0$ and $(k-\nu+1)_n \neq 0$,

$$\mathcal{D}_{0+}^{\nu}\left({}_{p}M_{q}^{\alpha,\beta}\right)(z) = z^{-\nu}\sum_{k=0}^{\infty}\frac{(\mathbf{a_p})_k}{(\mathbf{b_q})_k}\frac{z^k}{\Gamma(\alpha k+\beta)}\frac{\Gamma(k+1)}{\Gamma(k-\nu+1)} \qquad \text{and}$$

$$(4.4) \quad \mathcal{D}_{0+}^{\nu}\left({}_{p}M_{q}^{\alpha,\beta}\right)(z) = \frac{z^{-\nu}}{\Gamma(1-\nu)}\sum_{k=0}^{\infty}\frac{(\mathbf{a_p})_k}{(\mathbf{b_q})_k}\frac{z^k}{\Gamma(\alpha k+\beta)} = \frac{z^{-\nu}}{\Gamma(1-\nu)}\,{}_{p+1}M_{q+1}^{\alpha,\beta}\left(\begin{matrix}\mathbf{a_p},1\\\mathbf{b_q},1-\nu\end{matrix};z\right).$$

The L-S fractional integral and derivative of the K-function are achieved exactly along the same lines, since the pochhammer symbol $(\gamma)_k$ in the numerator of the summand $\frac{(\mathbf{a_p})_k}{(\mathbf{b_q})_k}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!}$ is transported all along the proof steps.

Therefore, the L-S fractional derivative of an M-series (a K-function, resp.) is still an M-series (a K-function, resp.), up to a prefactor, with the value of the indices p and q each incremented by one. This proves the second (derivation) part of the theorem. $\square$

More generally, the classes of Wright, Fox–Wright, Fox H- functions, etc. are all closed under Riemann–Liouville fractional integrals and derivatives, but involve a greater order of functions in each class (see Section 2).

All proofs in the following Subsections 4.1 and 4.2 are also very similar since, being merely parametric variants, the generalized M-series and the K-function share themselves quite similar features. Saigo's generalized fractional integral operators recalled in Section 2 are basic means and tools for the purpose of integrating M-L type functions such as the M-series and the K-function in the complex plane. It must be pointed out also that the prefactors $t^{\sigma-1}$ and $t^{-\alpha-\sigma}$ which take place respectively in their left- and right-sided fractional integrands are required for proving the results.

4.1 Left-sided generalized fractional integrations

In the present cases, a theorem and a corollary are deduced for the M-series in §4.1.1 and the K-function in §4.1.2. By contrast with Theorem 4.1, both proofs use Saigo's left-sided fractional integration formula (2.5a) applied to the M-series and the K-function as functions in $\mathcal{C}$ (see [23, 19, 39, 44]). The results are due in part to Sharma [46][9] but was recently revisited for the M-series by Kumar and Saxena [28] by using Samko–Maeda's fractional operators.

4.1.1 Left-sided fractional integration of the M-series

The generalized Fox–Wright psi functions which are obtained in this case naturally expresses as $_{p+3}\psi_{q+3}$ functions, up to prefactors.

Theorem 4.2. *Let $z > 0$ and suppose also that the parameters $\alpha, \beta, \gamma,\ \eta,\ \xi,\ \sigma \in \mathbb{C}$ are constrained by $\Re(\alpha) > 0$, $\Re(\xi) > 0$, $\mu > 0$, $c \in \mathbb{R}$ and $a_i,\ b_j \in \mathbb{C}$ $(i = 1, \ldots, p,\ j = 1, \ldots, q)$. Let $\mathcal{I}_{0+}^{\alpha,\beta,\gamma}$ be Saigo's L-S generalized fractional integral, then the following formula holds true.*

$$
\mathcal{I}_{0+}^{\alpha,\beta,\gamma}\left(t^{\sigma-1}\, _pM_q^{\xi,\eta}\big(ct^\mu\big)\right)(z) = \frac{\Gamma(\mathbf{b_q})}{\Gamma(\mathbf{a_p})}\, z^{\sigma-\beta-1}
$$

$$
\text{(4.5)} \qquad\qquad \times\ _{p+3}\psi_{q+3}\left(\begin{matrix}(a_i,1)_1^p,(\sigma,\mu),(-\beta+\gamma+\sigma,\mu),(1,1)\\(b_j,1)_1^q,(-\beta+\sigma,\mu),(\alpha+\gamma+\sigma,\mu),(\eta,\xi)\end{matrix}; cz^\mu\right),
$$

provided that each member of the equation exists.

Proof. By Definition 2.1 of Saigo's L-S fractional integration operator $\mathcal{I}_{0+}^{\alpha,\beta,\gamma}$ applied to the M-function in the class $\mathcal{C}$, Eq. (2.5a) rewrites as follows,

$$
\mathcal{I}_{0+}^{\alpha,\beta,\gamma}\left(t^{\sigma-1}\, _pM_q^{\xi,\eta}\big(ct^\mu\big)\right)(z) =
$$

$$
\text{(4.6)} \qquad\qquad \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1}\, _2F_1\left(\begin{matrix}\alpha+\beta,-\gamma\\\alpha\end{matrix};1-t/z\right)\,t^{\sigma-1}\, _pM_q^{\xi,\eta}\big(ct^\mu\big)\,\mathrm{d}t.
$$

Now, since

$$
_2F_1\left(\begin{matrix}\alpha+\beta,-\gamma\\\alpha\end{matrix};1-t/z\right) := \sum_{n=0}^\infty \frac{(\alpha+\beta)_n(-\gamma)_n}{(\alpha)_n}\frac{(1-t/z)^n}{n!}
$$

$$
\text{(4.7)} \qquad\qquad = \sum_{n=0}^\infty \frac{\Gamma(\alpha+\beta+n)\Gamma(-\gamma+n)\Gamma(\alpha)}{\Gamma(\alpha+\beta)\Gamma(-\gamma)\Gamma(\alpha+n)}\frac{(1-t/z)^n}{n!},
$$

we may plug Eq. (4.7) into Eq. (4.6). Under the constaints of the theorem, interchanging the order of integration and summation within the integrand can be justified by the absolute convergence of the integral and the uniform convergence of the series involved. After substituting the variable

---

[9]Kishan Sharma actually considers in [46] the L-S and the R-S fractional integrals of a simpler M-series: more precisely, $I_{0+}^{\alpha,\beta,\gamma}\left(t^{\eta-1}\, _pM_q^{\xi,\eta}\left(ct^\xi\right)\right)(x)$ and $I_-^{\alpha,\beta,\gamma}\left(t^{-\alpha-\eta}\, _pM_q^{\xi,\eta}\left(ct^\xi\right)\right)(x)$. This leads to formulas which write in terms of a Fox–Wright function $_{p+2}\psi_{q+2}$ in the L-S case and of a Fox–Wright function $_{p+3}\psi_{q+3}$ in the R-S case, respectively . Same remarks about the K-function $_pK_q^{\xi,\eta}\left(ct^\xi\right)$ considered in [47].

u for t/z in the integral, a few simplifications lead to

$$(4.8) \quad \mathcal{I}_{0+}^{\alpha,\beta,\gamma}\left(t^{\sigma-1} {}_pM_q^{\xi,\eta}(ct^\mu)\right)(z) = z^{\sigma-\beta-1}$$

$$\times \sum_{n=0}^\infty \frac{\Gamma(\alpha+\beta+n)\Gamma(-\gamma+n)}{\Gamma(\alpha+n)\Gamma(n+1)} \int_0^1 (1-u)^{\alpha-1+n}\, u^{\sigma-1}\, {}_pM_q^{\xi,\eta}(c(zu)^\mu)\, \mathrm{d}t,$$

where, by Definition 3.1,

$${}_pM_q^{\xi,\eta}(c(zu)^\mu) := \frac{\Gamma(\mathbf{b_q})}{\Gamma(\mathbf{a_p})} \sum_{n=0}^\infty \frac{\Gamma(\mathbf{a_p}+n)}{\Gamma(\mathbf{b_q}+n)} \frac{u^{\mu n}\,(cz^\mu)^n}{\Gamma(\xi n+\eta)}\,.$$

Next, using the beta function and Gauß's summation theorem, Eq. (4.8) simplifies and may be reexpanded into the final expression

$$\mathcal{I}_{0+}^{\alpha,\beta,\gamma}\left(t^{\sigma-1} {}_pM_q^{\xi,\eta}(ct^\mu)\right)(z) = \frac{\Gamma(\mathbf{b_q})}{\Gamma(\mathbf{a_p})}\, z^{-\sigma-\beta-1}$$

$$(4.9) \qquad \times \sum_{n=0}^\infty \frac{\Gamma(\mathbf{a_p}+n)}{\Gamma(\mathbf{b_q}+n)} \frac{\Gamma(\sigma+\mu n)\Gamma(-\beta+\gamma+\sigma+\mu n)\Gamma(n+1)}{\Gamma(\alpha+\gamma+\sigma+\mu n)\Gamma(-\beta+\sigma+\mu n)\Gamma(\eta+\xi n)} \frac{(cz^\mu)^n}{n!},$$

Finally, by interpreting the r.h.s. in (4.9) by means of Eq. (1.24) in Definition 1.2, we can write this last summation in terms of the Fox–Wright function ${}_{p+3}\psi_{q+3}$ in Eq. (4.5), and the theorem follows. $\qquad\square$

4.1.2 Left-sided fractional integration of the K-function

The following result makes also use of Saigo's L-S fractional integration operator.

Corollary 4.3. *Let $z > 0$ and suppose also that the parameters $\alpha, \beta, \gamma,\ \eta,\ \nu, \xi,\ \sigma \in \mathbb{C}$ are constrained by $\Re(\alpha) > 0$, $\Re(\nu) > 0$, $\Re(\xi) > 0$, $\mu > 0$, $c \in \mathbb{R}$ and $a_i, b_j \in \mathbb{C}$ ($i = 1, \ldots, p$, $j = 1, \ldots, q$). By Definition 2.1, Saigo's L-S fractional integration operator $\mathcal{I}_{0+}^{\alpha,\beta,\gamma}$ yields*

$$\mathcal{I}_{0+}^{\alpha,\beta,\gamma}\left(t^{\sigma-1} {}_pK_q^{\xi,\eta;\nu}(ct^\mu)\right)(z) = \frac{\Gamma(\mathbf{b_q})}{\Gamma(\mathbf{a_p})} \frac{z^{\sigma-\beta-1}}{\Gamma(\nu)}$$

$$(4.10) \qquad \times {}_{p+3}\psi_{q+3}\left({\substack{(a_i,1)_1^p,(\sigma,\mu),(-\beta+\gamma+\sigma,\mu),(\nu,1) \\ (b_j,1)_1^q,(-\beta+\sigma,\mu),(\alpha+\gamma+\sigma,\mu),(\eta,\xi)}}; cz^\mu\right),$$

provided each member of the equation exists.

Proof. Along the same lines as in Theorem 4.2 and in view of Definition 2.1, Eq. (2.5a) applied to the K-function in the class $\mathcal{C}$ writes now

$$\mathcal{I}_{0+}^{\alpha,\beta,\gamma}\left(t^{\sigma-1} {}_pK_q^{\xi,\eta;\nu}(ct^\mu)\right)(z) = \frac{z^{\sigma-\beta-1}}{\Gamma(\alpha)}$$

$$(4.11) \qquad \int_0^z (z-t)^{\alpha-1}\, {}_2F_1\left({\substack{\alpha+\beta,-\gamma \\ \alpha}}; 1-t/z\right)\, t^{\sigma-1}\, {}_pK_q^{\xi,\eta;\nu}(ct^\mu)\, \mathrm{d}t.$$

Upon interchanging the order of integration and summation (which is guaranteed by the absolute convergence of the integral and the uniform convergence of the series), the integrand can be

evaluated after the substitution of variables $u = t/z$. By means of the beta function and Gauß's summation theorem, formula (4.11) then coincides with an infinite summation similar to Eq. (4.9) (up to the parameter ν):

$$
\mathcal{I}_{0+}^{\alpha,\beta,\gamma}\left(t^{\sigma-1}\,{}_pK_q^{\xi,\eta;\nu}\big(ct^\mu\big)\right)(z) = \frac{\Gamma(\mathbf{a_p})}{\Gamma(\mathbf{b_q})}\,\frac{z^{\eta-\beta-1}}{\Gamma(\nu)}
$$

(4.12)
$$
\times \sum_{n=0}^\infty \frac{\Gamma(\mathbf{a_p}+n)}{\Gamma(\mathbf{b_q}+n)}\,\frac{\Gamma(\sigma+\mu n)\Gamma(-\beta+\gamma+\sigma+\mu n)\Gamma(\nu+n)}{\Gamma(-\beta+\sigma+\mu n)\Gamma(\alpha+\gamma+\sigma+\mu n)\Gamma(\eta+\xi n)}\,\frac{\big(cz^\mu\big)^n}{n!}\,.
$$

By Eq. (1.24) in Definition 1.2, Corollary 4.3 runs parallel to Theorem 4.2 and Eq. (4.12) may be expressed in terms of the Fox–Wright function ${}_{p+3}\psi_{q+3}$ in Eq. (4.5). Therefore, the corollary is completed. □

Note that if we set $\nu = \eta = 1$, in Eq. (4.10), then we get the L-S fractional integral of the M-series ${}_pM_q^{\xi,1}\big(ct^\mu\big)$.

4.2 Right-sided generalized fractional integrations

The proofs are along the same lines as in the L-S case in Section 4.1. As basic tool, the Definition 2.1 of Saigo's right-sided (R-S for short) fractional integration formula (2.5b) is applied to the M-series and the K-function as functions in $\mathcal{C}$ (see also [23, 19, 39, 44]). This provides the following two results in §§4.2.1–4.2.2.

4.2.1 Right-sided fractional integration of the M series

This theorem is the R-S counterpart of the L-S operator of fractional integration carried out in Theorem 4.2.

Theorem 4.4. *Let $z > 0$ and suppose also that the parameters $\alpha, \beta, \gamma,\ \eta, \xi,\ \sigma \in \mathbb{C}$ are constrained by $\Re(\alpha) > 0$, $\Re(\xi) > 0$, $\mu > 0$, $c \in \mathbb{R}$ and $a_i, b_j \in \mathbb{C}$ $(i = 1,\ldots,p,\ j = 1,\ldots,q)$. Let $\mathcal{I}_{z-}^{\alpha,\beta,\gamma}$ be the R-S operator of generalized fractional integration. Then, the following formula holds true.*

$$
\mathcal{I}_{z-}^{\alpha,\beta,\gamma}\left(t^{-\alpha-\sigma}\,{}_pM_q^{\xi,\eta}\big(ct^{-\mu}\big)\right)(z) = \frac{\Gamma(\mathbf{b_q})}{\Gamma(\mathbf{a_p})}\,z^{-2\alpha-2\beta-\sigma}
$$

(4.13)
$$
\times\ {}_{p+3}\psi_{q+3}\left(\begin{smallmatrix}(a_i,1)_1^p,(\alpha+\beta+\sigma,\mu),(\alpha+\gamma+\sigma,\mu),(1,1)\\(b_j,1)_1^q,(\eta,\xi),(2\alpha+\beta+\gamma+\sigma,\mu),(\alpha+\sigma,\mu)\end{smallmatrix};cz^{-\mu}\right),
$$

provided that each member of the equation exists.

Proof. By Definition 2.1 of Saigo's R-S fractional integration operator $\mathcal{I}_{z-}^{\alpha,\beta,\gamma}$, Eq. (2.5a) applied to the M-function in the class $\mathcal{C}$ satisfies

(4.14)$\quad \mathcal{I}_{z-}^{\alpha,\beta,\gamma}\left(t^{-\alpha-\sigma}\,{}_pM_q^{\xi,\eta}\big(ct^{-\mu}\big)\right)(z) =$

$$
\frac{1}{\Gamma(\alpha)}\int_z^\infty (t-z)^{\alpha-1}\,t^{-\alpha-\beta}\,{}_2F_1\big(\begin{smallmatrix}\alpha+\beta,-\gamma\\\alpha\end{smallmatrix};1-z/t\big)\,t^{-\alpha-\sigma}\,{}_pM_q^{\xi,\eta}\big(ct^{-\mu}\big)\ \mathrm{d}t.
$$

The proof now follows the same lines as the proofs in Theorem 4.2 and Corollary 4.3. Interchanging the order of integration and summation within the integrand in (4.14) is justified by

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absolute and uniform convergence of the integral and the series, respectively. After substituting the variable u for z/t and simplification, the R-S generalized fractional integral may be rewritten

$$(4.15) \quad \mathcal{I}_{z-}^{\alpha,\beta,\gamma}\left(t^{-\alpha-\sigma}\,{}_pM_q^{\xi,\eta}\big(ct^{-\mu}\big)\right)(z) = \frac{z^{-\alpha-\beta-\sigma}}{\Gamma(\alpha+\beta)\Gamma(-\gamma)}$$

$$\times \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+\beta+n)\Gamma(-\gamma+n)}{\Gamma(\alpha+n)\Gamma(n+1)} \int_{0}^{1} u^{\alpha-\beta-\sigma-1}(1-u)^{\alpha+n-1}\,{}_pM_q^{\xi,\eta}\big(cz^{-\mu}u^{\mu}\big)\,\mathrm{d}u.$$

Next, the inner integral in the above Eq. (4.15) may be evaluated by means the beta function and Gauß's summation theorem. The formula develops into an infinite summation, in the same vein as the sum in Eq. (4.5), which yields

$$\mathcal{I}_{z-}^{\alpha,\beta,\gamma}\left(t^{-\alpha-\sigma}\,{}_pM_q^{\xi,\eta}\big(ct^{-\mu}\big)\right)(z) = \frac{\Gamma(\mathbf{b_q})}{\Gamma(\mathbf{a_p})}\,z^{-2\alpha-2\beta-\sigma}$$

$$(4.16) \qquad \times \sum_{n=0}^{\infty} \frac{\Gamma(\mathbf{a_p}+n)}{\Gamma(\mathbf{b_q}+n)}\,\frac{\Gamma(\alpha+\beta+\sigma+\mu n)\Gamma(\alpha+\gamma+\sigma+\mu n)\Gamma(n+1)}{\Gamma(2\alpha+\gamma+\beta+\sigma+\mu n)\Gamma(\alpha+\sigma+\mu n)\Gamma(\eta+\xi n)}\,\frac{\big(cz^{-\mu}\big)^n}{n!}.$$

Last, by interpreting the r.h.s. in Eq. (4.16) by means of Eq. (1.24) in Definition 1.2, the Fox–Wright function ${}_{p+3}\psi_{q+3}$ in (4.13) is readily derived and the theorem follows. $\qquad\square$

4.2.2 Right-sided fractional integration of the K function

This last result is again the R-S counterpart of Theorem 4.4.

Corollary 4.5. *Let $z > 0$ and suppose also that the parameters $\alpha, \beta, \gamma,\ \eta,\ \nu, \xi,\ \sigma\ \in \mathbb{C}$ are constrained by $\Re(\alpha) > 0$, $\Re(\nu) > 0$, $\Re(\xi) > 0$, $\mu > 0$, $c \in \mathbb{R}$ and $a_i, b_j \in \mathbb{C}$ $(i = 1,\dots,p,$ $j = 1,\dots,q)$. Let $\mathcal{I}_{z-}^{\alpha,\beta,\gamma}$ be the R-S operator of the generalized fractional integration, then there holds*

$$\mathcal{I}_{z-}^{\alpha,\beta,\gamma}\left(t^{-\alpha-\sigma}\,{}_pK_q^{\xi,\eta;\nu}\big(ct^{-\mu}\big)\right)(z) = \frac{z^{-2\alpha-2\beta-\sigma}}{\Gamma(\nu)}\,\frac{\Gamma(\mathbf{b_q})}{\Gamma(\mathbf{a_p})}$$

$$(4.17) \qquad\qquad \times {}_{p+3}\psi_{q+3}\left(\begin{matrix}(a_i,1)_1^p,(\alpha+\beta+\sigma,\mu),(\alpha+\gamma+\sigma,\mu),(\nu,1)\\(b_j,1)_1^q,(\alpha+\sigma,\mu),(2\alpha+\beta+\gamma+\sigma,\mu),(\eta,\xi)\end{matrix};cz^{-\mu}\right),$$

provided that each member of the equation exists.

Proof. By Definition 2.1 applied to the K-function in $\mathcal{C}$, Saigo's R-S fractional integration operator is deduced from Eq. (2.5b) by the equation

$$(4.18) \quad \mathcal{I}_{z-}^{\alpha,\beta,\gamma}\left(t^{-\alpha-\sigma}\,{}_pK_q^{\xi,\eta;\nu}\big(ct^{-\mu}\big)\right)(z) = \frac{1}{\Gamma(\alpha)}$$

$$\int_{z}^{\infty}(t-z)^{\alpha-1}\,t^{-\alpha-\beta}\,{}_2F_1\left(\begin{matrix}\alpha+\beta,-\gamma\\\alpha\end{matrix};1-z/t\right)t^{-\alpha-\sigma}\,{}_pK_q^{\mu,\eta;\nu}\big(ct^{-\mu}\big)\,\mathrm{d}t.$$

As in the former proofs, the generalized hypergeometric series in the kernel is defined as an infinite sum in the same form as Eq. (4.7). Upon interchanging the order of integration and summation is again guaranteed under the constarints of the theorem (by absolute convergence and uniform convergence of the integral and the series, respectively). The evaluation of the

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inner integral is thus carried out by making the substitution $u = z/t$, with the help of the beta function and by Gauß's summation theorem. Eq. (4.18) rewrites again in the form of an infinite summation similar to the one in Eq. (4.9). As in the previous corollary, we obtain

$$(4.19) \quad \mathcal{I}_{z-}^{\alpha,\beta,\gamma} \left(t^{-\alpha-\sigma} \, _pK_q^{\xi,\eta;\nu}\left(ct^{-\mu}\right) \right)(z) = \frac{z^{-2\alpha-2\beta-\sigma}}{\Gamma(\nu)} \, \frac{\Gamma(\mathbf{b_q})}{\Gamma(\mathbf{a_p})}$$

$$\times \sum_{n=0}^{\infty} \frac{\Gamma(\mathbf{a_p}+n)}{\Gamma(\mathbf{b_q}+n)} \, \frac{\Gamma(\alpha+\beta+\sigma+\mu n)\Gamma(\alpha+\gamma+\sigma+\mu n)\Gamma(\nu+n)}{\Gamma(\alpha+\sigma+\mu n)\Gamma(2\alpha+\beta+\gamma+\sigma+\mu n)\Gamma(\eta+\xi n)} \, \frac{\left(cz^{-\mu}\right)^n}{n!} \, .$$

There remains to derive the Fox–Wright function $_{p+3}\psi_{q+3}$ in Eq. (4.17) from the infinite summation in Eq. (4.19). This simply proceeds from Eq. (1.24) in Definition 1.2, and the result follows. □

Note that if we set $\nu = \eta = 1$, in Eq. (4.17), then we get the R-S fractional integral of the M-series $_pM_q^{\mu,1}\left(ct^{-\mu}\right)$.

Remarks 3. In conclusion to this section it is interesting to point out that Saigo's (L-S and R-S) fractional integration of the M-series and the K-function bring along Fox–Wright functions of the $_{p+3}\psi_{q+3}$ type in every proof. This is partly due to the fact that all integrands contain L-S prefactors $t^{\sigma-1}$ and R-S prefactors $t^{\alpha-\sigma}$.

Following Kiryakova e.g in [22, 23, 24], Saigo's fractional integrals can be considered as examples of operators for generalized fractional integration (of R–L type) in the complex plane, with a suitable analytic kernel-function (in the class $\mathcal{C}$, for example). As such, all proofs involving the fractional calculus of the Fox–Wright function can be shown by taking a Mellin–Barnes type contour integral $\mathcal{L}$ in $\mathbb{C}$, for which the conditions ensuring the existence and analyticity of all function involved can be seen in [3, 21],[22, App.],[24, 43], etc. (see also §1.2).

5 Fractional calculus involving F_3

The present section, makes full use of the extended L-S and R-S fractional calculus involving the Appell's two variable hypergeometric function F_3 defined in Eq. (2.7), Definition 2.2 (see, e.g., [28]). It must be pointed out that, for suited results, the prefactor $t^{\sigma-1}$ is required in Saigo–Maeda's L-S and R-S fractional integrands and derivatives, as in Subsections 4.1 and 4.1 (see Remarks 3).

5.1 Left- and right-sided fractional integration of the M-series and the K-function

In this part, the L-S and R-S generalized fractional integration formulas of the M-series and the K-function are derived.

5.1.1 L-S fractional integral formulas of the M-series and the K-function

The theorem makes use of Saigo–Maeda's L-S fractional integral defined by 2.2.

Theorem 5.1. *Let α, α', β, β', γ, η, ν, ξ, $\sigma \in \mathbb{C}$, $\Re(\nu) > 0$, $\Re(\xi) > 0$, $\Re(\gamma) > 0$, $\mu > 0$, $c \in \mathbb{R}$ and a_i, $b_j \in \mathbb{C}$ $(i = 1, \ldots, p, j = 1, \ldots, q)$, then, for $z > 0$, the following relations hold true*

$$\mathcal{I}_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\sigma-1}\,_pM_q^{\xi,\eta}\left(ct^\mu\right)\right)(z) = \frac{\Gamma(\mathbf{b_q})}{\Gamma(\mathbf{a_p})}\, z^{\sigma-\alpha-\alpha'+\gamma-1}$$

(5.1a)
$$\times\,_{p+4}\psi_{q+4}\left({{(a_i,1)_1^p,(\sigma,\mu),(\sigma+\gamma-\alpha,\mu),(\sigma+\beta'-\alpha',\mu),(1,1)}\atop{(b_j,1)_1^q,(1-\sigma,\mu),(\sigma+\gamma-\alpha'-\beta,\mu),(\sigma+\beta',\mu),(\eta,\xi)}}\,;cz^\mu\right),$$

provided that each member of the equation exists.

$$\mathcal{I}_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\sigma-1}\,_pK_q^{\xi,\eta;\nu}\left(ct^\mu\right)\right)(z) = \frac{\Gamma(\mathbf{b_q})}{\Gamma(\mathbf{a_p})}\,\frac{z^{\sigma-\alpha-\alpha'+\gamma-1}}{\Gamma(\nu)}$$

(5.1b)
$$\times\,_{p+4}\psi_{q+4}\left({{(a_i,1)_1^p,(\sigma,\mu),(\sigma+\gamma-\alpha,\mu),(\sigma+\beta'-\alpha',\mu),(\nu,1)}\atop{(b_j,1)_1^q,(1-\sigma,\mu),(\sigma+\gamma-\alpha'-\beta,\mu)(\sigma+\beta',\mu),(\eta,\xi)}}\,;cz^\mu\right),$$

provided that each member of the equation exists.

5.1.2 Right-sided fractional integral formulas of the M-series and the K-function

Following the definition of Saigo–Maeda's R-S fractional integral defined by 2.2, we get the following theorem.

Theorem 5.2. *Let α, α', β, β', γ, η, ν, ξ, $\sigma \in \mathbb{C}$, $\Re(\nu) > 0$, $\Re(\xi) > 0$, $\Re(\gamma) > 0$, $\mu > 0$, $c \in \mathbb{R}$ and a_i, $b_j \in C$ $(i = 1, \ldots, p, j = 1, \ldots, q)$, then, for $z > 0$, the following two relations holds true*

$$\mathcal{I}_{z-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\sigma-1}\,_pM_q^{\xi,\eta}\left(ct^{-\mu}\right)\right)(z) = \frac{\Gamma(\mathbf{b_q})}{\Gamma(\mathbf{a_p})}\, z^{\sigma-\alpha-\alpha'+\gamma-1}$$

(5.2a)
$$\times\,_{p+4}\psi_{q+4}\left({{(a_i,1)_1^p,(1+\alpha+\alpha'-\gamma-\sigma,\mu),(1+\alpha+\beta'-\gamma-\sigma,\mu),(1-\beta-\sigma,\mu),(1,1)}\atop{(b_j,1)_1^q,(1-\sigma,\mu),(1+\alpha-\alpha'-\gamma-\sigma,\mu),(1-\alpha-\beta-\sigma,\mu),(\eta,\xi)}}\,;cz^{-\mu}\right),$$

provided that each member of the equation exists.

$$\mathcal{I}_{z-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\sigma-1}\,_pK_q^{\xi,\eta;\nu}\left(ct^{-\mu}\right)\right)(z) = \frac{\Gamma(\mathbf{b_q})}{\Gamma(\mathbf{a_p})}\,\frac{z^{\sigma-\alpha-\alpha'+\gamma-1}}{\Gamma(\nu)}$$

(5.2b)
$$\times\,_{p+4}\psi_{q+4}\left({{(a_i,1)_1^p,(1+\alpha+\alpha'-\gamma-\sigma,\mu),(1+\alpha+\beta'-\gamma-\sigma,\mu),(1-\beta-\sigma,\mu),(\nu,1)}\atop{(b_j,1)_1^q,(1-\sigma,\mu),(1+\alpha-\alpha'-\gamma-\sigma,\mu),(1-\alpha-\beta-\sigma,\mu),(\eta,\xi)}}\,;cz^{-\mu}\right),$$

provided that each member of the equation exists.

Short proofs. Both proofs of Theorems 5.1 and 5.2 run along the very same lines. Under the assumptions of the theorems, let the function $\varphi(z)$ denote $_pM_q^{\xi,\eta}(z)$ and $_pK_q^{\xi,\eta;\nu}$ (resp.). By Eqs. (2.8a) and (2.8b) in Definition 2.2, Saigo–Maeda's L-S and R-S fractional integrals of φ write respectively,

$$\mathcal{I}_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\sigma-1}\varphi\left(ct^\mu\right)\right)(z) := \frac{z^{-\alpha}}{\Gamma(\gamma)}$$

(5.3a)
$$\int_0^\infty (z-t)^{\gamma-1}t^{\alpha'+\sigma-1}\,F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-t/z;1-z/t\right)\varphi\left(ct^\mu\right)\,\mathrm{d}t$$

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and

$$\mathcal{I}_{z-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\sigma-1}\varphi\left(ct^{-\mu}\right)\right)(z) := \frac{z^{-\alpha'}}{\Gamma(\gamma)}$$

(5.3b)
$$\int\limits_z^\infty (t-z)^{\gamma-1}t^{-\alpha+\sigma-1}\,F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-z/t;1-t/z\right)\,\varphi\left(ct^{-\mu}\right)\,\mathrm{d}t.$$

By definition of Appell's two variable series F_3 and in view of Eq. (2.10a) in Lemma 2.3, expanding F_3 and interchanging the summations and the integrals is justified, under the constraints of the theorems, by the absolute convergence of integrals and the uniform convergence of series. Next, the evaluation of the inner integrals by using the beta function and Gauß's summation theorem, yields Eqs. (5.1a)–(5.1b) and Eqs. (5.2a)–(5.2b). This provides the sketchproofs of Theorems 5.1 and 5.2. □

5.2 Left- and right-sided fractional differentiation of the M-series and the K-function

In the subsection, the L-S and R-S generalized fractional derivative formulas of the M-series and the K-function are deduced from Saigo–Maeda's L-S fractional derivative introduced in Definition 2.2.

5.2.1 L-S fractional derivative formulas of the M-series and the K-function

Theorem 5.3. *Let α, α', β, β', γ, η, ν, ξ, $\sigma \in \mathbb{C}$, $\Re(\nu) > 0$, $\Re(\xi) > 0$, $\Re(\gamma) > 0$, $\mu > 0$, $c \in \mathbb{R}$ and and a_i, $b_j \in \mathbb{C}$ $(i = 1, \ldots, p, \ j = 1, \ldots, q)$, then, for $z > 0$, the following formula hold true*

$$\mathcal{D}_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\sigma-1}{}_pM_q^{\xi,\eta}\left(ct^\mu\right)\right)(z) = \frac{\Gamma(\mathbf{b_q})}{\Gamma(\mathbf{a_p})}\,z^{\sigma-\alpha-\alpha'+\gamma-1}$$

(5.4a)
$$\times {}_{p+4}\psi_{q+4}\left(\begin{matrix}(a_i,1)_1^p,(\sigma,\mu),(\sigma+\alpha+\alpha'+\beta'-\gamma,\mu),(\sigma+\alpha-\beta,\mu),(1,1)\\(b_j,1)_1^q,(\sigma+\alpha+\alpha'-\gamma,\mu),(\sigma+\alpha+\beta'-\gamma,\mu),(\sigma-\gamma,\mu),(\eta,\xi)\end{matrix}; cz^\mu\right),$$

provided that each member of the equation exists.

$$\mathcal{D}_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\sigma-1}{}_pK_q^{\xi,\eta;\nu}\left(ct^\mu\right)\right)(z) = \frac{\Gamma(\mathbf{b_q})}{\Gamma(\mathbf{a_p})}\frac{z^{\sigma-\alpha-\alpha'+\gamma-1}}{\Gamma(\nu)}$$

(5.4b)
$$\times {}_{p+4}\psi_{q+4}\left(\begin{matrix}(a_i,1)_1^p,(\sigma,\mu),(\sigma+\gamma-\alpha,\mu),(\sigma+\beta'-\alpha',\mu),(\nu,1)\\(b_j,1)_1^q,(1-\sigma,\mu),(\sigma+\gamma-\alpha'-\beta,\mu)(\sigma+\beta',\mu),(\eta,\xi)\end{matrix}; cz^\mu\right),$$

provided that each member of the equation exists.

5.2.2 R-S generalized fractional derivative formulas of the M-series and the K-function

Theorem 5.4. *Let α, α', β, β', γ, η, ν, ξ, $\sigma \in \mathbb{C}$, $\Re(\nu) > 0$, $\Re(\xi) > 0$, $\Re(\gamma) > 0$, $\mu > 0$, $c \in \mathbb{R}$ and a_i, $b_j \in \mathbb{C}$ $(i = 1, \ldots, p, \ j = 1, \ldots, q)$, then, for $z > 0$, the following two relations holds true*

$$\mathcal{D}_{z-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\sigma-1}{}_pM_q^{\xi,\eta}\left(ct^{-\mu}\right)\right)(z) = \frac{\Gamma(\mathbf{b_q})}{\Gamma(\mathbf{a_p})}\,z^{\sigma-\alpha-\alpha'+\gamma-1}$$

(5.5a)
$$\times {}_{p+4}\psi_{q+4}\left(\begin{matrix}(a_i,1)_1^p,(1-\alpha-\alpha'+\gamma-\sigma,\mu),(1-\alpha'-\beta+\gamma-\sigma,\mu),(1+\beta'-\sigma,\mu),(1,1)\\(b_j,1)_1^q,(1-\sigma,\mu),(1-\alpha-\alpha'-\beta+\gamma-\sigma,\mu),(1-\alpha'+\beta'-\sigma,\mu),(\eta,\xi)\end{matrix}; cz^{-\mu}\right),$$

provided that each member of the equation exists.

$$\mathcal{D}_{z-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\sigma-1}\,{}_pK_q^{\xi,\eta;\nu}\left(ct^{-\mu}\right)\right)(z) = \frac{\Gamma(\mathbf{b_q})}{\Gamma(\mathbf{a_p})}\,\frac{z^{\sigma-\alpha-\alpha'+\gamma-1}}{\Gamma(\nu)}$$

$$(5.5b) \qquad \times\ {}_{p+4}\psi_{q+4}\left(\begin{matrix}(a_i,1)_1^p,(1+\alpha+\alpha'-\gamma-\sigma,\mu),(1+\alpha+\beta'-\gamma-\sigma,\mu),(1-\beta-\sigma,\mu),(\nu,1)\\(b_j,1)_1^q,(1-\sigma,\mu),(1+\alpha-\alpha'-\gamma-\sigma,\mu),(1-\alpha-\beta-\sigma,\mu),(\eta,\xi)\end{matrix};cz^{-\mu}\right),$$

provided that each member of the equation exists.

Short proofs. Both proofs of Theorems 5.3 and 5.4 are quite similar to those of Theorems 5.1 and 5.2. Under the above assumptions, let the function $\varphi(z)$ denote ${}_pM_q^{\xi,\eta}(z)$ and ${}_pK_q^{\xi,\eta;\nu}$, according to the case. From Saigo–Maeda's L-S and R-S Eqs. (2.9a) and (2.9b) in Definition 2.2, fractional derivatives are written respectively, with $n = \lfloor\Re(\gamma)\rfloor + 1$,

$$\mathcal{D}_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\sigma-1}\varphi\left(ct^\mu\right)\right)(z) := \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \mathcal{I}_{0+}^{-\alpha',-\alpha,-\beta'+n,-\beta,-\gamma+n}\left(t^{\sigma-1}\varphi\left(ct^\mu\right)\right)(z) \quad \text{and}$$

$$\mathcal{D}_{z-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\sigma-1}\varphi\left(ct^{-\mu}\right)\right)(z) := (-1)^n\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \mathcal{I}_{z-}^{-\alpha',-\alpha,-\beta',-\beta+n,-\gamma+n}\left(t^{\sigma-1}\varphi\left(ct^{-\mu}\right)\right)(z).$$

By definition of F_3 and in view of Eq. (2.10a) in Lemma 2.3, expanding the series F_3 and interchanging the order of integration and summation is again guaranteed by the conditions of the theorems: absolute of integrals and uniform convergence of series (resp.). Next, from Eqs. (5.4a)–(5.4b) and Eqs. (5.5a)–(5.5b), the identity $\frac{\mathrm{d}^n}{\mathrm{d}z^n}z^m = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}z^{m-n}$, where $m \geq n$, involves a few simplifications, which yield the generalized Fox–Wright functions ${}_{p+4}\psi_{q+4}$ in Theorems 5.1 and 5.2. This completes the sketchproofs of the theorems. $\qquad\square$

Remarks 4. By setting parameters (p, q, α, α', β, β', γ, η, μ, ξ, σ, etc.) to various specific values, a number of special cases of the above theorems may be obtained by virtue of relations (2.11a) and (2.11b) in Remarks 2.

For example, if $\alpha' = 0$, $\sigma = \eta$, $\mu = \xi$ and in view of Eq. (2.11a), Theorems 5.1 and 5.2 coincide with Saigo's L-S and R-S fractional integrals of the M-series and the K-function, as carried out in §§4.1–4.2 (Section 4). Along the same lines, Riemann–Liouville and Erdélyi–Kober fractional differintegral operators can be also obtained by putting $\beta = -\alpha$ and $\beta = 0$, respectively in Theorems 5.3–5.4 and in Theorems 5.1–5.2.

Notice that Riemann–Liouville's, Saigo's and Saigo–Maeda's operators of fractional calculus operate as well for a very great number of generalized hypergeometric special functions, such as the generalized hypergeometric function ${}_pF_q$, the generalized Mittag-Leffler and Wright type functions, Fox H- and Meijer G-functions, Fox–Wright ${}_p\psi_q$ and Lommel–Wright,functions, Appell's one- and multi-variable functions, generalized incomplete gamma type functions ${}_p\gamma_q$ and ${}_p\Gamma_q$, incomplete Pochhammer symbols $(\lambda;x)_\nu$ and $[\lambda;x]_\nu$ ($\lambda,\nu \in \mathbb{C}$, $x \geq 0$), etc. (see a few recent papers such as, e.g., [26, 27, 28, 34, 50]).

6 Conclusion & perspectives

The results obtained so far are extensions of the works carried out by many authors, actually in increasing number and ability within the past years. On account of the general nature of the differintegral operators of Mittag-Leffler type functions, highly transcendental functions, etc. a number of known results can be easily found as special cases of the present results and conversely. Fractional calculus also offers wide perspectives in various domains of mathematics spreading out from pure and applied analysis, such as q-analogues for instance, to algebra of operators.

Appendices

A Asymptotic expansion of M-L type functions ($|z| \to \infty$)

A.1 Asymptotic expansion of $E_{\alpha,\beta}(z)$ [13, 15]

The asymptotic expansion of the two-parametric M-L function is based on the integral representation of $E_{\alpha,\beta}(z)$ in the form (1.19) given in §1.2 with the same Hankel path $\mathcal{H}$. Following Džrbašjan [7, 8, 9, 1952, 1960, 1966], Erdélyi *et al.* [10, Vol. III, 1953], the case when $\beta = 1$ and the general case with arbitrary $\beta \in \mathbb{C}$ are treated in the survey of Gorenflo *et al.* [13, Sect. 4]. The following two representations are considered.

$$(A.1a) \qquad E_{\alpha,\beta}(z) = \frac{1}{2\pi i \alpha} \int_{\gamma(\epsilon;\delta)} \frac{e^{s^{1/\alpha}} s^{(1-\beta)/\alpha}}{s - z} \, ds \qquad \text{for } z \in \Omega^{(-)}(\epsilon;\delta)$$

(A.1b)

and, for $z \in \Omega^{(+)}(\epsilon;\delta)$,

$$(A.1c) \qquad E_{\alpha,\beta}(z) = \frac{1}{\alpha} e^{s^{1/\alpha}} z^{(1-\beta)/\alpha} + \frac{1}{2\pi i \alpha} \int_{\gamma(\epsilon;\delta)} \frac{e^{s^{1/\alpha}} s^{(1-\beta)/\alpha}}{s - z} \, ds,$$

under the constraints

$$(A.2) \qquad 0 < \alpha < 2 \qquad \text{and} \qquad \pi\alpha/2 < \delta < \min(\pi, \pi\alpha).$$

The above contour denoted by $\gamma(\epsilon;\delta) = \{\epsilon > 0, \ 0 < \delta \leq \pi\}$ is oriented by non-decreasing $\arg s$. It consists in the two rays $S_{-\delta} = \{\arg s = -\delta, \ |s| \geq \epsilon\}$ and $S_{\delta} = \{\arg s = \delta, \ |s| \geq \epsilon\}$, and the circular arc $C_{\delta}(0;\epsilon) = \{|s| = \epsilon, \ -\delta \leq \arg s \leq \delta\}$.

If $0 < \delta < \pi$, then the Hankel contour $\gamma(\epsilon;\delta)$ divides the complex s-plane into two unbounded regions, namely $\Omega^{(-)}(\epsilon;\delta)$ to the left of $\gamma(\epsilon;\delta)$ by orientation and $\Omega^{(+)}(\epsilon;\delta)$ to the right of it. If $\delta = \pi$, then the contour consists of the circle $|s| = \epsilon$ and the twice passable ray $-\infty < s \leq -\epsilon$.

Using the integral representations in (A.1b) and (A.1c) entails asymptotic expansions for the M-L function in the complex plane (see, e.g., [13, Thm. 4.3]). Let $0 < \alpha < 2$, $\beta \in \mathbb{C}$ be an arbitrary number, and $\delta \in \mathbb{R}$ be chosen to fullfill the condition (A.2). Thus, for any $m \in \mathbb{N}$ (and for $m = 0$ if the 'empty sum convention' is adopted), for all z such that $|\arg z| \leq \delta$,

$$(A.3a) \qquad E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} e^{z^{1/\alpha}} - \sum_{n=1}^{m} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + \mathcal{O}\left(\frac{1}{z^{m+1}}\right) \qquad (|z| \to \infty)$$

and analogously, for all z such that $\delta < |\arg z| < \pi$,

$$(A.3b) \qquad E_{\alpha,\beta}(z) = -\sum_{n=1}^{m} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + \mathcal{O}\left(\frac{1}{z^{m+1}}\right) \qquad (|z| \to \infty).$$

Similarly, in the case when $\alpha \geq 2$, the following asymptotic formula holds for $\beta \in \mathbb{C}$ and $m \in \mathbb{N}$ (see, e.g., [13, Thm. 4.4]).

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} \sum_{|\arg z + 2\pi n| < \frac{3\pi\alpha}{4}} \left(z^{1/\alpha} e^{2\pi i n/\alpha}\right)^{1-\beta} \exp\left(z^{1/\alpha} e^{2\pi i n/\alpha}\right)$$

$$(A.4) \qquad\qquad\qquad - \sum_{n=1}^{m} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + \mathcal{O}\left(\frac{1}{z^{m+1}}\right) \qquad (|z| \to \infty).$$

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In the case of $\beta = 1$, $E_\alpha(z) := E_{\alpha,1}(z)$ reduces to related asymptotic expansion stated, e.g., in [13, §. 3.4].

A.2   Asymptotic expansion of Prabhakar's M-L function [13, 33]

By Lemma 1.1, the function $E_{\alpha,\beta}^\gamma(z)$ can be represented via the Mellin–Barnes integral given in Eq. (1.15) (cf. [13, § 5.1.2]), under the constraints $z \in \mathbb{C}$ and $|\arg z| < \pi$ for $\alpha \in \mathbb{R}_+$, β, $\gamma \in \mathbb{C}$ and $\Re(\gamma) > 0$. Now, when β is a sufficiently large real number, one can use Stirling's formula, valid for any fixed a,

$$\Gamma(z + a) \approx \sqrt{2\pi}\, z^{z+a-1/2}\, \mathrm{e}^{-z} \quad \text{as}\ \ |z| \to \infty,$$

in order to get the following asymptotic formula as $x \to \infty$ ($a > 0$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$).

$$\Gamma(\alpha)E_{\alpha,\beta}^\gamma\big(a(\alpha x)^\gamma\big) \approx \sum_{n=0}^\infty \frac{(\beta)_n\, a^n\, x^{\gamma n}}{n!}\, \frac{\sqrt{2\pi}\, \alpha^{\alpha-1/2}\, \mathrm{e}^{-\alpha}}{\sqrt{2\pi}\, \alpha^{\alpha-1/2+\gamma n}\, \mathrm{e}^{-\alpha}}$$

$$\text{(A.5)} \qquad = \sum_{n=0}^\infty \frac{(\beta)_n}{n!}\, \big(a(x/\alpha)^\gamma\big)^n = \frac{1}{\big(1 + a(x/\alpha)^\gamma\big)^\beta}\,.$$

As in the case of the M-L function with two parameters, the asymptotic behaviour of the three parametric function $E_{\alpha,\beta}^\gamma(z)$ critically depends on the values of the parameters α, β, γ and cannot easily be described. In principle, an asymptotic expansion of Prabhakar's function can be found from its representation via a generalized Wright function or H-function (cf. [13, § 5.1.5] or [21]). To the best of our knowledge, the asymptotic behaviour of $E_{\alpha,\beta}^\gamma(z)$ in different domains of the complex plane (similar, for example, to that of Prop. 3.6, and of Theorems 4.3 and 4.4 in [13]) has not yet been described in an explicit form.

Another step towards an asymptotic expansion was obtained however in [33] for non-negative integer values of $\beta = n$, when n gets large. Prabhakar's M-L function is then naturally defined by

$$\text{(A.6)} \qquad E_{\alpha,n}^\gamma(z) := \sum_{k=0}^\infty \frac{(\gamma)_k}{\Gamma(\alpha k + n)} \frac{z^k}{k!} \qquad (\alpha, \gamma \in \mathbb{C},\ \Re(\alpha) > 0,\ n \in \mathbb{N}).$$

Next, given a number γ, suppose that some coefficients in Def. (A.6) equal zero; that is, there exists an integer $p \in \mathbb{N}$, such that (A.6) can be written as follows,

$$\text{(A.7)} \qquad E_{\alpha,n}^\gamma(z) = z^p \sum_{k=p}^\infty \frac{(\gamma)_k}{\Gamma(\alpha k + n)} \frac{z^{k-p}}{k!}.$$

Now, three main cases may be considered.

(i) If $\gamma \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, then $p = 0$ for $n \in \mathbb{N}$ and $p = 1$ for $n = 0$. The functions $E_{\alpha,n}^\gamma(z)$ are entire of order $\rho = 1/\Re(\alpha)$ and type $\sigma = 1$.

(ii) Otherwise, the functions $E_{\alpha,n}^\gamma(z)$ actually reduce to polynomials of power $-\gamma$ and, denoting $m = -\gamma$, the representation (A.7) can be rewriten in the form

$$E_{\alpha,n}^\gamma(z) = z^p \sum_{k=p}^m \frac{(\gamma)_k}{\Gamma(\alpha k + n)} \frac{z^{k-p}}{k!}$$

and, since $(-m)_k = \binom{m}{k}$ when $k \leq m$,

$$\text{(A.8)} \qquad E_{\alpha,n}^{-m}(z) = z^p \sum_{k=p}^m (-1)^k \binom{m}{k} \frac{z^{k-p}}{\Gamma(\alpha k + n)}.$$

(iii) If $\gamma = 0$, then $E^0_{\alpha,n}(z) = 1/\Gamma(n)$ for $n \in \mathbb{N}$ and $E^0_{\alpha,n}(z) = 0$ for $n = 0$.

Summarizing, if γ is a non-positive integer, as seen above, the three-parametric M-L functions reduce to polynomials, while when $\gamma \notin \mathbb{Z}_{\leq 0}$, they are entire functions of order $\rho = 1/\Re(\alpha)$ and type $\sigma = 1$. Finally, the fonctions $E^\gamma_{\alpha,n}(z)$ admit asymptotic behaviours written in the following form as $n \to \infty$.

$$(A.9) \qquad E^\gamma_{\alpha,n}(z) = \frac{(\gamma)_p}{\Gamma(\alpha p + n)} \, z^p \left(1 + \theta^\gamma_{\alpha,n}(z)\right) \qquad \text{with}$$

$$(A.10a) \qquad \theta^\gamma_{\alpha,n}(z) = \sum_{k=p+1}^{\infty} \frac{\Gamma(\alpha p + n)}{\Gamma(\alpha k + n)} \frac{z^{k-p}}{k!} \qquad \text{for } \gamma \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$$

and respectively, for $\gamma = -m$, $m \in \mathbb{N}$,

$$(A.10b) \qquad \theta^{-m}_{\alpha,n}(z) = \sum_{k=p+1}^{\infty} \frac{(-m)_k}{(-m)_p} \frac{\Gamma(\alpha p + n)}{\Gamma(\alpha k + n)} \frac{z^{k-p}}{k!} = \sum_{k=p+1}^{\infty} \frac{(-1)^{k-p}\binom{m}{k}}{\binom{m}{p}} \frac{\Gamma(\alpha p + n)}{\Gamma(\alpha k + n)} z^{k-p}.$$

In representations (A.10a)–(A.10b), $\gamma \neq 0$ and the parameter p is determined as follows: $p = 0$ for all $n \in \mathbb{Z}_{>0}$ and $p = 1$ for $n = 0$.

A.3 Asymptotics for the multiple M-L function $F^{(\mu)}_{\alpha,\beta}(z)$ [12]

The two-parametric M-L function $E_{\alpha,\beta}(z)$ can be generalized by attaching an exponent to its Taylor coefficients. For real values α, β and $\mu > 0$, the series

$$(A.11) \qquad F^{(\mu)}_{\alpha,\beta}(z) := \sum_{n \geq 0} \frac{z^n}{\Gamma(\alpha n + \beta)^\mu}$$

defines an entire function of $z \in \mathbb{C}$ of order $\rho = 1/(\alpha\mu)$. Special cases are the two-parametric M-L function and the Bessel function $I_0(2\sqrt{z}) = \sum_{n=0}^{\infty} z^n/n!$.

If μ is a natural number, then the function in (A.11) is an instance of the multiple M-L function investigated, e.g., by Kiryakova [25], in connection with numerous applications to fractional calculus; but it seems that the asymptotic behaviour of the multiple M-L function had not been studied earlier than Gerhold's paper [12]. The asymptotic behaviour of $F^{(\mu)}_{\alpha,\beta}(z)$ as $z \to \infty$ is considered in a sector of the complex plane, containing the positive real line. For the two-parametric M-L function ($\mu = 1$), this is usually analysed by an integral representation (see previous §A.1), which appears to have no immediate extension to $\mu = 1$. Whatsoever, the asymptotic result can be established by approximating the sum by an integral and then use the Laplace method.

Theorem A.1. *Let μ, $\alpha, \beta > 0$ and $\epsilon > 0$ be arbitrary. Then, as $z \to \infty$ in the sector*

$$|\arg z| \leq \begin{cases} \alpha\mu\pi/2 - \epsilon, & 0 < \alpha\mu < 2 \\ (2 - \alpha\mu/2)\pi & 2 \leq \alpha\mu < 4 \\ 0 & \alpha\mu \geq 4, \end{cases}$$

we have the asymptotics

$$(A.12) \qquad F^{(\mu)}_{\alpha,\beta}(z) \sim \frac{1}{\alpha\sqrt{\mu}} \, (2\pi)^{(1-\mu)/2} \, z^{(\mu-2\beta\mu+1)/2\alpha\mu} \, e^{\mu z^{1/\alpha\mu}}.$$

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Applying the Laplace method directly does not work for non-real z; the absolute values of the summands in (A.11) are peaked near $n \approx \alpha^{-1}|z|^{1/\alpha\mu}$, but it seems that one cannot balance the local expansion and the tails. This is caused by oscillations in the summands, which can be dealt with by shifting the problem to the asymptotic evaluation of an integral. The Laplace method then succeeds, after moving the integration contour through a saddle point located approximately at $\alpha^{-1}z^{1/\alpha\mu}$. For that purpose, an integral representation of (A.11) is needed as $z \to \infty$ in the sector $|\arg z| \le \max\big(0, (2 - \alpha\mu/2)\pi\big)$, which leads to

$$F_{\alpha,\beta}^{(\mu)}(z) = \int_0^\infty \frac{z^t}{\Gamma(\alpha t + \beta)^\mu}\, dt + \mathcal{O}(z).$$

The saddle point method may then be applied to the above integral, first by locating the saddle point, next by changing the integration contour to a line $\mathcal{L}$ which begins at 0 and passes through t_0 (valid for large $|z|$) and finally by picking up the dominant contribution of the integral from the range $|t - t_0| \le |t_0|^\tau$ ($\tau \in]1/2, 2/3[$) around the saddle point. Substituting $t = t_0(y + 1)$ with $y \ge -1$ in the line $\mathcal{L}$ divides the r.h.s. in three integrals in the form

(A.13)
$$\int_{\mathcal{L}} \frac{z^t}{\Gamma(\alpha t + \beta)^\mu}\, dt = t_0 \int_{-1}^\infty \frac{z^{t_0(y+1)}}{\Gamma\big(\alpha t_0(y+1) + \beta\big)^\mu}\, dy$$
$$= t_0 \left(\int_{-1}^{-|t_0|^{\tau-1}} + \int_{-|t_0|^{\tau-1}}^{|t_0|^{\tau-1}} + \int_{|t_0|^{\tau-1}}^\infty \right) \frac{z^{t_0(y+1)}}{\Gamma\big(\alpha t_0(y+1) + \beta\big)^\mu}\, dy.$$

For large $|z|$, the central integral provides the r.h.s. of Eq. (A.12) in Theorem A.1 and the two side integrals are negligible. A full asymptotic expansion can be obtained easily by pushing the local expansion around the saddle point further.

A possible question for future research remains the finding of an analytic continuation for the multiple M-L function. For $\mu \in \mathbb{C}$ and fixed z, one might ask whether the function defined by (A.11) has an analytic continuation for $\Re(\mu) \le 0$.

Remark 5. Whenever $\mu > 0$ and $\alpha = \beta = 1$, the asymptotics as $n \to \infty$ along the real line of

$$F_{1,1}^{(\mu)}(z) := \sum_{n=0}^\infty \frac{z^n}{n!^\mu}$$

is a special case of the above asymptotic behaviour of A.11. Namely, in the sector defined in Theorem A.1 (with $\alpha = \beta = 1$), we get the asymptotics

$$F_{1,1}^{(\mu)}(z) \sim \frac{1}{\sqrt{\mu}} (2\pi)^{(1-\mu)/2}\, z^{(1-\mu)/2\mu}\, e^{\mu z^{1/\mu}} \qquad (z \to \infty).$$

Notice that the (possibly formal) series $F_{1,1}^{(\mu)}(z)$ is not *D-finite* (or *non-holonomic*) for any $\mu \in \mathbb{R} \setminus \mathbb{Q}$.

Besides, this function satisfies the integral relation

$$\int_0^\infty e^{-t/z} F_{1,1}^{(\mu+1)}(t)\, dt = z F_{1,1}^{(\mu)}(z) \qquad (z \ne 0).$$

The proof is immediate by simple expansion and substitution $s = t/z$ in the above integral.

$$\int\limits_0^\infty e^{-t/z} F_{1,1}^{(\mu+1)}(t)\, dt = \int\limits_0^\infty e^{-t/z} \sum_{n=0}^\infty \frac{t^n}{n!^{(\mu+1)}}\, dt = \sum_{n=0}^\infty \frac{1}{n!^{(\mu+1)}} \int\limits_0^\infty t^n e^{-t/z}\, dt$$

$$= \sum_{n=0}^\infty \frac{z^{n+1}}{n!^{(\mu+1)}} \int\limits_0^\infty s^n e^{-s}\, ds = \sum_{n=0}^\infty \frac{z^{n+1}}{n!^{(\mu)}}.$$

B   Determination of $E_{-\alpha,\beta}(z)$ with negative value of the first parameter

To find a defining equation of the function $E_{-\alpha,\beta}(z)$ rewrite the integral representation (1.20) of the M-L function with parameters $\alpha,\ \beta \in \mathbb{R},\ \alpha < 0$ as

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi} \int\limits_{\mathcal{H}} \frac{e^\zeta}{\zeta^\beta - z\zeta^{-\alpha+\beta}}\, d\zeta \qquad (z \in \mathbb{C}).$$

After expanding part of the above integrand and then substituting it into Eq. (1.20) we obtain the following definition of $E_{-\alpha,\beta}(z)$ with negative value of the first parameter,

(B.1a) $$\qquad E_{-\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} - E_{\alpha,\beta}(\tfrac{1}{z}) \qquad (\alpha > 0,\ z \in \mathbb{C} \setminus \{0\}).$$

In particular, for $\beta = 1$,

$$E_{-\alpha}(z) := E_{-\alpha,1}(z) = 1 - E_\alpha(\tfrac{1}{z}) \qquad (\alpha > 0\ z \in \mathbb{C} \setminus \{0\}).$$

Now, from the known recurrence formula

$$E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + z E_{\alpha,\alpha+\beta}(z),$$

another variant of the definition in Eq. (B.1a) is derived,

(B.2a) $$\qquad E_{-\alpha,\beta}(z) = -\frac{1}{z} E_{\alpha,\alpha+\beta}(\tfrac{1}{z}) \qquad (\alpha > 0,\ \beta \in \mathbb{R},\ z \in \mathbb{C} \setminus \{0\}).$$

The definitions (B.1a)–(B.2a) actually determine the same function, analytic in $\mathbb{C} \setminus \{0\}$.

From the definition of $E_{\alpha,\beta}(z)$ in (1.2), also results the following series representation of the extended M-L function (i.e. the function corresponding to real negative values of the first parameter),

$$E_{-\alpha,\beta}(z) = -\sum_{n=1}^\infty \frac{z^{-1}}{\Gamma(\alpha n + \beta)} \qquad (z \in \mathbb{C} \setminus \{0\}).$$

By using this series representation and the definitions ((B.1a) or (B.2a)), the extended two-parametric M-L function with real negative first parameter provides several classical functional, differential and recurrence relations, which are analogous to corresponding relations for $E_{\alpha,\beta}(z)$ with positive valued first parameter.

C   Complex contour for the reciprocal gamma and the beta functions

Recall the classical definitions of Euler's gamma and beta functions (1729 and 1772, resp.) and *Euler's reflection formula for the gamma function* (1771); all three are required in this appendix. The latter reflection formula (C.2c) connects the gamma function with the sine function.[10]

$$(C.2a) \qquad \Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt \qquad (\Re(z) > 0)$$

$$(C.2b) \qquad B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1} dt \qquad (\Re(x) > 0, \Re(y) > 0)$$

$$(C.2c) \qquad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \qquad (z \notin \mathbb{Z}).$$

Hankel's contour integral provides one of the most beautiful and useful integral representations of the reciprocal gamma function (see, e.g., Temme [52, Chap. 3, §3.2.6]). It has the form

$$(C.3) \qquad \frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{\mathcal{H}} e^s s^{-z} ds \qquad (z \in \mathbb{C}).$$

The contour of integration $\mathcal{H}$ is the Hankel contour that runs from $-\infty$, $\arg s = -\pi$, encircles the origin in the positive direction (that is counterclockwise) ends at $-\infty$, now with $\arg s = +\pi$ (this is the reason why the notation $\int_{-\infty}^{(0+)}$ is sometimes used instead of notation $\int_{\mathcal{H}}$). The multi-valued function s^{-z} is assumed to be real for real values of z and s, $s > 0$.

A proof of representation (C.3) follows immediately from the theory of Laplace transforms: from the well-known integral

$$\frac{\Gamma(z)}{s^z} = \int_0^\infty t^{z-1} e^{-st} dt,$$

---

[10]To prove the reflection formula (C.2c), first set $t = s/(s-1)$ in Definition (C.2b), which gives rise to the *second beta integral* with integration along the real half line:

$$(C.1) \qquad B(x,y) = \int_0^\infty \frac{s^{x-1}}{(1+s)^{x+y}} ds = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Next, set $y = 1 - x$, $0 < x < 1$ in Eq. (C.1) to obtain $\Gamma(x)\Gamma(1-x) = \int_0^\infty \frac{t^{x-1}}{(1+t)} dt$.

This integral is computed by considering the contour integral $I_x := \int_{\mathcal{C}} \frac{z^{x-1}}{(1-z)} dz$, where $\mathcal{C}$ consists of two circles about the origin of radii r and ε, respectively, which are joined along the negative real axis from $-r$ to $-\varepsilon$. Move along the outer circle in the counterclockwise direction, and along the inner circle in the clockwise direction. By the residue theorem, $I_x = -2\pi i$ when z^{x-1} has its principal value. Thus,

$$-2\pi i = \int_{-\pi}^{\pi} \frac{i r^x e^{ix\theta}}{1 - r e^{i\theta}} d\theta + \int_r^\varepsilon \frac{t^{x-1} e^{ix\pi}}{1+t} dt + \int_\pi^{-\pi} \frac{i\varepsilon^x e^{ix\theta}}{1 - \varepsilon e^{i\theta}} d\theta + \int_\varepsilon^r \frac{t^{x-1} e^{-ix\pi}}{1+t} dt.$$

Let $r \to \infty$ and $\varepsilon \to 0$, so that the first and third integrals tend to zero and the second and fourth combine to provide Euler's reflection formula (C.2c) for $0 < x < 1$. The full result follows next by analytic continuation. This, in a sense, shows that $1/\Gamma(z)$ is 'half of the sine function'. (See, e.g., Temme [52, Chap. 3, §3.2.5].)

Eq. (C.3) is obtained as a special case of the inversion formula. A direct proof follows from a special choice of $\mathcal{H}$, that is the negative real axis. This is only possible when $\Re(z) < 1$. Under this condition, the contribution from a small circle around the origin, with radius tending to zero, can be neglected. Thus, the r.h.s. of (C.3) yields,

$$\frac{1}{2\pi i}\left(-\int\limits_{\infty}^{0}\left(se^{-i\pi}\right)^{-z}e^{-s}ds - \int\limits_{0}^{\infty}\left(se^{i\pi}\right)^{-z}e^{-s}ds\right) = \frac{\sin \pi z}{\pi}\Gamma(1-z).$$

With the help of Euler's reflection formula (C.2c), it is easily shown that the above relation equals indeed the l.h.s. of (C.3). In a final step, we can deduce from the principle of analytic continuation that Eq. (C.3) holds true for all finite complex values of z. Namely, both the l.h.s. and the r.h.s. of Eq. (C.3) are entire functions of z. As pointed out in §1.2, the integral representation of the one- and two-parametric M-L functions is given by Eq. (1.19) by means of the Hankel contour $\mathcal{H}$ for the $1/\Gamma(z)$ integral in (C.3) (see Note 5).

Another form of (C.3) gives rise to the following integral representation of $\Gamma(z)$

(C.4)
$$\Gamma(z) = \frac{1}{2i\sin(\pi z)}\int\limits_{\mathcal{H}} s^{z-1}e^{s}ds.$$

The substitution $s = -t$ yields an integrand as in the starting point (C.2a) (that is, the definition of Γ). The main idea which stands behind (C.2a) is that the many-valued function t^{z-1} arising in definition (C.2a) can be used to open up the original contour along $[0, \infty)$, and obtain a representation that is valid in a larger domain of the parameter z. This approach can be useful with other special functions, e.g. the complex contour for the beta integral, which works as follows [52, Chap. 3, §3.2].

A complex contour for the beta function

Consider the integral

$$I_{x,y} = \frac{1}{2\pi i}\int\limits_{0}^{(1+)} \omega^{x-1}(1-\omega)^{y-1}d\omega,$$

where $\Re(x) > 0$ and $y \in \mathbb{C}$. The contour starts and ends at the origin, and encircles the point 1 in the positive (counterclockwise) direction. The argument (or *phase*) of $\omega - 1$ is zero at positive points larger than 1. When $\Re(y) > 0$ we can deform the contour along $(0, 1)$. Then we get $I_{x,y} = B(x, y)\sin(\pi y)/y$ and it follows that

(C.5)
$$B(x, y) = \frac{y}{\sin(\pi y)}\frac{1}{2i\pi}\int\limits_{0}^{(1+)} \omega^{x-1}(1-\omega)^{y-1}d\omega.$$

The integral is defined for any complex value of y. For $y = 1, 2, \ldots$, the integral vanishes; this absorbs the infinite values of the term in front of the integral.

Double contour integral

It is possible to replace the integral for $\Gamma(z)$ along a half line by a contour integral which converges for all values of z. A similar process can be carried out for the beta integral.

Let P be any point between 0 and 1. The following Pochhammer's extension of the beta integral holds:

$$\int\limits_{P}^{(1+,0+,1-,0-)} t^{x-1}(1-t)^{y-1}\mathrm{d}t = \frac{4\pi\mathrm{e}^{\pi\mathrm{i}(x+y)}}{\Gamma(1-x)\Gamma(1-y)\Gamma(x+y)}\,.$$

Here the contour starts at P, encircles the point 1 in the positive (counterclockwise) direction, returns to P, then encircles the origin in the positive direction, and returns to P. The $1-$, $0-$ indicates that now the path of integration is in the clockwise direction, first around 1 and then 0. The formula is proved by the same method as Hankel's formula. Notice that it is true for any complex x and y: both sides are entire functions of x and y.

D   Integral representation of the Gauß hypergeometric function

The Mellin–Barnes integral is a contour integral representation involving a product of gamma functions. The *confluent* hypergeometric function $_1F_1(z)$, for example, can be represented by such a Mellin–Barnes type contour integral, in which case the integral is valid in the sector $|\arg(-z)| < \pi/2$. More generally, all confluent hypergeometric functions, such as $_1F_1(z)$, Kummer's hypergeometric function $M(a;c;z) = \lim\limits_{b\to\infty} {}_2F_1(a,b;c;z/b)$, etc., have integral representations in the form of a Mellin–Barnes type contour integral, each one giving rise to an analytic continuation in a complex domain included in (all or part of) the complex plane (see, e.g., [3, §12.5(iii)]),[13, App. D and F],[21, §1.6], etc.).

Now, the following theorem deals with the hypergeometric function, in which case the line integral is usually taken along a contour $\mathcal{C}$ which is a deformation of the imaginary axis such that the integration contour separates all the poles of $\Gamma(a+s)\Gamma(b+s)$ from those of $\Gamma(-s)$, and $(-z)^s$ has its principal value.

Theorem D.1. *Provided that a is a positive integer, the Mellin–Barnes contour integral representation of Gauß's hypergeometric function $_2F_1(a,b;c;z)$ is given by*

(D.1) $$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\,{}_2F_1(a,b;c;z) = \frac{1}{2\pi\mathrm{i}}\int\limits_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)}\,\Gamma(-s)\,(-z)^s\mathrm{d}s.$$

Here, $|\arg(-z)| < \pi$ and the path of integration can pass to infinity parallel to the imaginary axis with any finite value of $c = \Re(s)$, on condition that the contour $\mathcal{C}$ can be indented, if necessary, to separate all the poles of $\Gamma(-s)$ at the points $s = \nu$ ($\nu \in \mathbb{N}$) to the left from all the poles of $\Gamma(a+s)$ and $\Gamma(b+s)$ to the right, at the points $s = -a-\nu$ and $s = -b-\nu$ ($\nu \in \mathbb{N}$), respectively. (Such a contour may always be drawn if a and b are non-negative integers or zero. Moreover, Eq. (D.1) remains valid also for negative or zero values of a.)

Proof. This result can be obtained by moving the contour to the right while picking up the residues at $s = 0,\ 1,\ 2,\ldots$. In establishing the result, it was necessary to suppose that $|z| < 1$. From the principle of analytic continuation, the integral (D.1) converges and defines an analytic function of z, which is holomorphic in the sector $|\arg(-z)| < \pi$. Hence, $_2F_1(a,b;c;z)$ is defined not only inside the unit circle but also in the complex s-plane cut along the real z-axis from 0 to ∞, provided that the parameters are such that the integration path can be drawn to separate the poles of $\Gamma(-s)$ from the poles of $\Gamma(a+s)$ and $\Gamma(b+s)$. ☐

If the path of integration is displaced to the left we can proceed to evaluate the integral in (D.1) in a similar manner. In this case, such poles are encountered which correspond to two sequences of simple poles: the poles of $\Gamma(a+s)$ with residues

$$(D.2) \quad (-1)^n \frac{\Gamma(b-a-n)\Gamma(a+n)}{n!\Gamma(c-a-n)}(-z)^{-a-n} = \frac{\Gamma(a)\Gamma(b-a)}{\Gamma(c-a)}(-z)^{-a}\frac{(1+a-c)_n(a)_n}{(1+a-b)_n n!}(-z)^{-n}$$

at the points $s = -a - \nu$, and the poles of $\Gamma(b+s)$ at $s = -b - \nu$, respectively; the related residues of which are obtained by interchanging a and b in Eq. (D.2). (This is straightforward by symmetry in the series, $F(a,b;c;z) = F(b,a;c;z)$.) It is first assumed also that $a - b$ is not an integer, so that these poles are indeed simple poles [10, Vol. 1, §2.1.4].

The use of analytic continuation enables to remove the restriction $|z| > 1$ and establish that the following Eq. (D.3) holds for $|\arg(-z)| < \pi$ and $a - b \notin \mathbb{Z}$.

$$\begin{aligned}
(D.3) \quad {}_2F_1(a,b;c;z) =& \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}\,{}_2F_1\left({a,1+a-c \atop 1+a-b};\tfrac{1}{z}\right) \\
&+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}\,{}_2F_1\left({b,1+b-c \atop 1+b-a};\tfrac{1}{z}\right).
\end{aligned}$$

Now, when $|z| > 1$, the integral round the translated path vanishes as the path moves to infinity. In that case, the asymptotic expansion of Gauß's hypergeometric function for large $|z|$ results easily from formula (D.3) under the above conditions. Whence the asymptotic behaviour of ${}_2F_1(a,b;c;z)$ as $|z| \to \infty$ in the sector $|\arg(-z)| < \pi$ given by

$$_2F_1(a,b;c;z) = A(-z)^{-a}\left(1 + \mathcal{O}\left(z^{-1}\right)\right) + B(-z)^{-b}\left(1 + \mathcal{O}\left(z^{-1}\right)\right) \qquad \text{if } a - b \notin \mathbb{Z}$$

and by

$$(D.4) \quad {}_2F_1(a,b;c;z) = C(-z)^{-a}\log(-z)\left(1 + \mathcal{O}\left(z^{-1}\right)\right) \qquad \text{if } a - b \in \mathbb{Z},$$

where $A, B, C \in \mathbb{C}$ are constants. More precisely, $A = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}$ and $B = \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}$ by Eq. (D.3), with commuting parameters a and b [10, §§2.1.4 and 2.3.1].

When $a - b$ is an integer, some of the poles become double poles and then, the residues involve terms in $\ln(-z)$ as may be checked in Eq. (D.4). For instance, if $b - a$ is a non-negative integer,

$$\begin{aligned}
{}_2F_1\left({a,a+m \atop c};z\right) =& (-z)^{-a}\frac{\Gamma(c)}{\Gamma(a+m)}\sum_{n=0}^{m-1}\frac{(a)_n(m-n-1)!}{k!\Gamma(c-a-n)}z^{-n} \\
&+ (-z)^{-a}\frac{\Gamma(c)}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{(a+m)_n}{n!(n+m)!\Gamma(c-a-n-m)}(-1)^n z^{-n-m} \\
&\times \Big(\ln(-z) + \psi(n+1) + \psi(n+m+1) - \psi(a+n+m) - \psi(c-a-n-m)\Big),
\end{aligned}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the usual psi (or digamma) function, $|z| > 1$ and $|\arg(-z)| < \pi$. (see Askey *et al.* [3, Chap. 15, §15.8(ii)] and Erdélyi *et al.* [10, Vol. 1, §§2.1.4 and 2.3.1] for exhaustive proofs).

Given proper conditions of convergence, one can relate more general Mellin–Barnes type contour integrals to the generalized hypergeometric function ${}_pF_q(z)$ in a similar way. Some special function related to ${}_pF_q(z)$ include the dilogarithm $\text{Li}_2(z) = \sum_{n \geq 0} n^{-2}z^n = z\,{}_3F_2(1,1,1;2,2;z)$,

Hahn and Wilson polynomials, etc.

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Christian LAVAULT

E-mail: lavault@lipn.univ-paris13.fr

LIPN, UMR CNRS 7030 – Laboratoire d'Informatique de Paris-Nord

Université Paris 13, Sorbonne Paris Cité, F-93430 Villetaneuse.

URL: http://lipn.univ-paris13.fr/~lavault