Appendix B

Special Functions

Key properties of special functions as they are often used in plasma physics are summarized in this appendix.

B.1 Heaviside Step Function

The Heaviside step function is usually just called the step function. It was introduced by Oliver Heaviside in the late 1800s to represent the idealized switching on (in time) of a voltage or other source in electrical engineering problems. It is defined by

$$H(x - x_0) = \left\{ \begin{array}{ll} 1, & x > x_0, \\ 0, & x < x_0, \end{array} \right. \qquad \text{Heaviside step function.} \qquad \text{(B.1)}$$

The step function H is in general undefined at $x = x_0$ (because it is discontinuous there); it will be taken to be $1/2$ there so that it is equal to its average value at this jump discontinuity. The derivative of the step function is usually taken to be the Dirac delta function, which is discussed in the next Section:

$$H'(x - x_0) \equiv \frac{d}{dx} H(x - x_0) = \delta(x - x_0). \qquad \text{(B.2)}$$

The x dependence of the step and delta functions are shown schematically in Fig. B.1.

B.2 Dirac Delta Function

The Dirac delta function, which is usually just called the delta function, is a concentrated "spike" or impulse of unit area. It was introduced by P.A.M. Dirac in the 1920s in the context of developing a physical interpretation of quantum mechanics. The delta function is often used in plasma physics to represent the spatial distribution of "point" charged particles. It also often arises in functions that represent singular responses to resonant perturbations.

Figure B.1: Schematic of x dependence of the Heaviside step function and Dirac delta function.

The one-dimensional delta function is defined by the following properties:

$$\delta(x - x_0) = 0 \quad \text{for} \quad x \neq x_0,$$

$$\int_a^b dx\, f(x)\, \delta(x - x_0) = \left\{ \begin{array}{ll} f(x_0), & a < x_0 < b, \\ 0, & \text{otherwise,} \end{array} \right. \tag{B.3}$$

for any function $f(x)$ that is continuous at $x = x_0$. Thus, the delta function is zero except at the point where its argument vanishes; there, it is so large (i.e., singular) that the integral of it over that point (its area) is unity [i.e., for $f(x) = 1$ we have $\int_a^b dx\, \delta(x - x_0) = 1$]. Note that hence the delta function has units of one over the units of its argument.

The delta function is a mathematically improper function — because it is unbounded where its argument vanishes. However, it is a generalized function whose integral can be defined through a limiting process in distribution theory. Specifically, for a unity area distribution function $w(x)$, one defines a delta sequence $w(x; \Delta)$ which becomes progressively more peaked (height $\sim 1/\Delta$) and narrower (width $\sim \Delta$) in the limit that $\Delta \to 0$ such that it becomes a unit area "spike." In terms of such a delta sequence, one defines

$$\int_a^b dx\, \delta(x)\, f(x) \equiv \lim_{\Delta \to 0} \int_a^b dx\, w(x; \Delta)\, f(x) = f(0). \tag{B.4}$$

Symbolically, we can write

$$\delta(x) \doteq \lim_{\Delta \to 0} w(x; \Delta), \quad \text{Dirac delta function.} \tag{B.5}$$

which is only valid in evaluating integrals in the form given in (B.4).

A function $w(x)$ is a suitable basis for a delta sequence if it is nonnegative and has unity area [i.e., $\int_{-\infty}^{\infty} dx\, w(x) = 1$]. Oscillatory, unity area functions that decay as their argument increases [i.e., $\lim_{|x| \to \infty} |w(x)| \to 0$] can also be suitable basis functions for delta sequences. For a suitable distribution function $w(x)$, a delta sequence is defined by $w(x; \Delta) \equiv w(x/\Delta)/\Delta$. Examples of delta sequences based on Gaussian [$w_G = e^{-x^2}/\sqrt{\pi}$], Lorentzian [$w_L = 1/\pi(x^2 + 1)$],

and correlation function $[w_C = \sin x/\pi x]$ distributions are, respectively,

$$w_G(x; \Delta) = \frac{1}{\sqrt{\pi}\,\Delta} e^{-(x/\Delta)^2}, \quad \text{Gaussian,}$$

$$w_L(x; \Delta) = \frac{\Delta}{\pi\,(x^2 + \Delta^2)}, \qquad \text{Lorentzian,} \tag{B.6}$$

$$w_C(x; \Delta) = \frac{\sin\,(x/\Delta)}{\pi\,x}, \qquad \text{correlation function.}$$

Basic properties of delta functions include

$$\delta(x_0 - x) = \delta(x - x_0), \tag{B.7}$$

$$f(x)\,\delta(x - x_0) = f(x_0)\,\delta(x - x_0), \quad x\,\delta(x) = 0, \tag{B.8}$$

$$\int_a^b dx\,\delta(x - x_0)\,\delta(x - x_1) = \delta(x_0 - x_1) \quad \text{for } a < x_0,\,x_1\,< b, \tag{B.9}$$

$$\delta(x^2 - x_0^2) = \frac{1}{2\,|x_0|}\,[\delta(x - x_0) + \delta(x + x_0)], \quad \text{and} \tag{B.10}$$

$$\delta[f(x)] = \sum_i \frac{\delta(x - x_i)}{|df/dx|_{x_i}}, \quad \delta(ax) = \frac{1}{|a|}\delta(a), \tag{B.11}$$

in which x_i are the (assumed) simple zeros of f [i.e., $f(x_i) = 0$]. The derivative of a delta function is a "couple," which is a positive spike followed by a negative spike. It can be integrated by parts to yield

$$\int_a^b dx\,f(x)\,\delta'(x - x_0) = -f'(x_0), \quad \text{effect of derivative of delta function,} \tag{B.12}$$

where the prime denotes differentiation with respect to the argument. The effect of the jth derivative of a delta function can be calculated by integrating by parts j times:

$$\int_a^b dx\,f(x)\,\delta^{(j)}(x - x_0) = (-1)^j f^{(j)}(x_0). \tag{B.13}$$

Differentiation properties of delta functions are

$$\frac{\partial}{\partial x}\,\delta(x - x_0) = -\frac{\partial}{\partial x_0}\,\delta(x - x_0), \tag{B.14}$$

$$\frac{d}{dt}\,\delta[x(t)] = \frac{d\delta[x]}{dx}\,\frac{dx(t)}{dt} \equiv \delta'\,\frac{dx(t)}{dt}. \tag{B.15}$$

For more than one dimension one simply takes products of delta functions in the various directions. Thus, a delta function at the point $\mathbf{x}_0 \equiv (x_0, y_0, z_0)$ in three-dimensional Cartesian coordinate space is written as

$$\delta(\mathbf{x} - \mathbf{x}_0) \equiv \delta(x - x_0)\,\delta(y - y_0)\,\delta(z - z_0), \quad \text{Cartesian.} \tag{B.16}$$

In other coordinate systems the three-dimensional delta function is just the product of the delta functions in the new coordinates divided by the Jacobian

of the coordinate transformation. Thus, three-dimensional delta functions in cylindrical and spherical coordinates are given, respectively, by

$$\delta(\mathbf{x} - \mathbf{x}_0) = \frac{\delta(r - r_0)\, \delta(\theta - \theta_0)\, \delta(z - z_0)}{r}, \quad \text{cylindrical,} \quad \text{(B.17)}$$

$$\delta(\mathbf{x} - \mathbf{x}_0) = \frac{\delta(r - r_0)\, \delta(\vartheta - \vartheta_0)\, \delta(\varphi - \varphi_0)}{r^2 |\sin\vartheta|}, \quad \text{spherical.} \quad \text{(B.18)}$$

An integral of a three-dimensional delta function over a volume ΔV vanishes unless ΔV includes the point $\mathbf{x}_0$:

$$\int_V^{V+\Delta V} d^3x\, f(\mathbf{x})\, \delta(\mathbf{x} - \mathbf{x}_0) = \begin{cases} f(\mathbf{x}_0) & \text{if } \Delta V \text{ contains } \mathbf{x} = \mathbf{x}_0, \\ 0 & \text{otherwise.} \end{cases} \quad \text{(B.19)}$$

Some key summations, integrals, limits, and differentials that result in delta functions are

$$\sum_{n=-\infty}^{\infty} e^{inx} = 2\pi\, \delta(x), \quad \text{(B.20)}$$

$$\int_{-\infty}^{\infty} dk\, e^{ikx} = 2\pi\, \delta(x), \quad \text{(B.21)}$$

$$\int d^3k\, e^{i\mathbf{k}\cdot\mathbf{x}} = (2\pi)^3\, \delta(\mathbf{x}), \quad \text{(B.22)}$$

$$\lim_{x\to\infty} \frac{1 - e^{-ikx}}{ikx} = \pi\, \delta(k) \quad \text{(B.23)}$$

$$\nabla^2 \frac{1}{|\mathbf{x} - \mathbf{x}_0|} = -4\pi\, \delta(\mathbf{x} - \mathbf{x}_0). \quad \text{(B.24)}$$

Delta functions are treated simply but rigorously in

Lighthill, *Intoduction to Fourier Analysis and Generalized Functions* (1958) [**?**]

Dennery and Krzywicki, *Mathematics for Physicists*, Section III.13 (1967) [**?**].

A comprehensive treatment of generalized functions is given in

Gel'fand and Shilov, *Generalized Functions*, Vol. I (1964) [**?**].

B.3   Plasma Dispersion Function

A singular integral that often arises in calculating linear plasma responses for a Maxwellian equilibrium distribution is

$$Z(w) \equiv \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} du\, \frac{e^{-u^2}}{u - w}, \quad \mathcal{I}m\{w\} > 0, \quad \text{plasma dispersion function.} \quad \text{(B.25)}$$

Figure B.2: Behavior of the plasma dispersion function $Z(w)$ and its derivative $Z'(w)$ as a function of $w_R \equiv \mathcal{R}e\{w\}$. In the figures on the left the w_R dependences of the real (subscript R) and imaginary (subscript I) parts of Z and Z' are shown for $\mathcal{I}m\{w\} = 0$ by solid and dashed lines, respectively. Corresponding polar plots are shown on the right, which also indicate the behavior for selected values of $\mathcal{I}m\{w\} \equiv w_I$.

Analytic continuation to $\mathcal{I}m\{w\} \leq 0$, which is obtained by deforming the integration contour to always pass beneath the pole at $u = w$ (see Fig. **??**), yields the complete specification

$$
\sqrt{\pi}\, Z(w) \equiv
\begin{cases}
\displaystyle \int_{-\infty}^{\infty} du\, \frac{e^{-u^2}}{u - w}, & \mathcal{I}m\{w\} > 0, \\[2ex]
\displaystyle \mathcal{P} \int_{-\infty}^{\infty} du\, \frac{e^{-u^2}}{u - w} + \pi i\, e^{-w^2}, & \mathcal{I}m\{w\} = 0, \\[2ex]
\displaystyle \int_{-\infty}^{\infty} du\, \frac{e^{-u^2}}{u - w} + 2\pi i\, e^{-w^2}, & \mathcal{I}m\{w\} < 0.
\end{cases}
\tag{B.26}
$$

Here, $\mathcal{P}$ is the Cauchy principal value operator (**??**) that defines (i.e., makes convergent) the integration over the singularity at $u = w$ when w is real.

While the definition of $Z(w)$ might appear to be discontinuous at $\mathcal{I}m\{w\} = 0$, it is in fact continuous there. Its continuity there can be verified by taking the $\mathcal{I}m\{w\} \to 0$ limit [for finite $\mathcal{R}e\{w\} \neq 0$] of the forms given above for $\mathcal{I}m\{w\} > 0$ and $\mathcal{I}m\{w\} < 0$, and showing that they are identical to the $\mathcal{I}m\{w\} = 0$ definition — see (**??**). The behavior of the real and imaginary parts of $Z(w)$ and its derivative $Z'(w)$ are shown in Fig. B.3. As indicated in Fig. B.3, the plasma dispersion function has the following symmetry properties:

$$
Z(-w) = 2i\sqrt{\pi}\, e^{-w^2} - Z(w), \tag{B.27}
$$

$$
Z(w^*) = -[Z(-w)]^* = Z^*(w) + 2i\sqrt{\pi}\, e^{-w^2}, \tag{B.28}
$$

where the * superscript indicates the complex conjugate.

A complementary function $\bar{Z}$, which is defined by the integral in (B.25) but for $\mathcal{I}m\{w\} < 0$, is related to $Z(w)$ by

$$
\bar{Z}(w) = -Z(-w) = Z(w) - 2i\sqrt{\pi}\, e^{-w^2}. \tag{B.29}
$$

The $w_I \equiv \mathcal{I}m\{w\} \to 0$ representations of the plasma dispersion function and its complement can be obtained directly using the Plemelj formulas (**??**).

An alternative definition of $Z(w)$ that is valid for all finite $\mathcal{I}m\{w\}$ is

$$W(w) = \frac{Z(w)}{i\sqrt{\pi}} = \frac{2\,e^{-w^2}}{\sqrt{\pi}} \int_{-iw}^{\infty} dt\, e^{-t^2} = e^{-w^2}[1 - \mathrm{erf}(-iw)] \equiv e^{-w^2}\mathrm{erfc}(-iw),$$

(B.30)

which indicates the close relationship to the error function. This is the form of the plasma dispersion function most commonly used in the Russian plasma physics literature, and in error function reference books.

The plasma dispersion function satisfies the differential equation

$$Z'(w) \equiv \frac{dZ}{dw} = -2\,[\,1 + w\,Z(w)\,].$$

(B.31)

This differential equation can be used to write higher order derivatives in terms of lower order derivatives:

$$Z^{(n)} \equiv \frac{d^n Z}{dw^n} = -2\frac{d^{n-1}(wZ)}{dw^{n-1}} = -2[(n-1)Z^{(n-2)} + wZ^{(n-1)}] \quad \text{for } n \geq 2.$$

(B.32)

The plasma dispersion function has a power series expansion about $w = 0$ given by

$$\begin{aligned}
Z(w) &= i\sqrt{\pi}\,e^{-w^2} - w\sqrt{\pi}\sum_{n=0}^{\infty} \frac{(-w^2)^n}{\Gamma(n+1/2)} \\
&= i\sqrt{\pi}\,e^{-w^2} - 2w\left(1 - \frac{2w^2}{3} + \frac{4w^4}{15} - \frac{8w^6}{105} + \cdots\right).
\end{aligned}$$

(B.33)

Its asymptotic expansion for $|w| >> 1$ is

$$\begin{aligned}
Z(w) &\sim i\sigma\sqrt{\pi}\,e^{-w^2} - \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty} w^{-(2n+1)}\,\Gamma(n-1/2) \\
&= i\sigma\sqrt{\pi}\,e^{-w^2} - \frac{1}{w}\left(1 + \frac{1}{2w^2} + \frac{3}{4w^4} + \frac{15}{8w^6} + \cdots\right),
\end{aligned}$$

(B.34)

where

$$\sigma \equiv \begin{cases} 0, & \mathcal{I}m\{w\} > 0, \\ 1, & \mathcal{I}m\{w\} = 0, \\ 2, & \mathcal{I}m\{w\} < 0. \end{cases}$$

(B.35)

Corresponding power series and asymptotic expansions for the derivative of the plasma dispersion function are given, respectively, by

$$Z' = -2iw\sqrt{\pi}\,e^{-w^2} - 2\left(1 - 2w^2 + \cdots\right) \quad \text{for } |w| << 1 \text{ and} \quad \text{(B.36)}$$

$$Z' \sim -2i\sigma w\sqrt{\pi}\,e^{-w^2} + \frac{1}{w^2}\left(1 + \frac{3}{2w^2} + \cdots\right) \quad \text{for } |w| >> 1. \quad \text{(B.37)}$$

Related, but more complicated integrals of the form

$$Z_n(w) \equiv \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} du \, \frac{u^n \, e^{-u^2}}{u - w}, \quad n \geq 0, \quad \mathcal{I}m\{w\} > 0, \qquad \text{(B.38)}$$

can be calculated in terms of derivatives of $Z(w)$ as follows. First, taking n successive derivatives of $Z(w)$ and integrating by parts n times one obtains

$$Z^{(n)}(w) \equiv \frac{d^n Z(w)}{dw^n} = \frac{n!}{\sqrt{\pi}} \int_{-\infty}^{\infty} du \, \frac{e^{-u^2}}{(u - w)^{n+1}} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{du}{u - w} \, \frac{d^n(e^{-u^2})}{du^n}.$$

From the Rodrigues formula for Hermite polynomials $H_n(u)$,

$$\frac{d^n}{du^n}(e^{-u^2}) = (-1)^n \, e^{-u^2} H_n(u),$$

it is clear that the nth derivative of $Z(w)$ can be written as

$$\frac{d^n Z(w)}{dw^n} = \frac{(-1)^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} du \, \frac{H_n(u) \, e^{-u^2}}{u - w}. \qquad \text{(B.39)}$$

Now, any power of the variable u can be expressed in terms of a series of Hermite polynomials with orders up to and including the power of u through the relation

$$u^n = \frac{1}{2^n} \sum_{m=0}^{M} d_m(n) \, H_{n-2m}(u), \qquad \text{(B.40)}$$

in which $d_m(n)$ are the coefficients given in Table 22.12 of Abromowitz and Stegun [**?**] and the upper limit of the sum is $M \equiv [n/2]$, the largest integer less than or equal to $n/2$. Substituting (B.40) into (B.38) and utilizing (B.39) yields

$$Z_n(w) = \frac{1}{2^n} \sum_{m=0}^{[n/2]} (-1)^{n-2m} d_m(n) \frac{d^{n-2m} Z(w)}{dw^{n-2m}}.$$

The first four of these functions are

$$Z_0(w) = Z \qquad \qquad \text{(B.41)}$$
$$Z_1(w) = -(1/2)Z' = 1 + wZ, \qquad \qquad \text{(B.42)}$$
$$Z_2(w) = (1/4)[2Z + Z''] = -(w/2)Z' = w + w^2 Z, \qquad \text{(B.43)}$$
$$Z_3(w) = -(1/8)[6Z' + Z'''] = (1/2)[1 + 2w^2(1 + wZ)], \qquad \text{(B.44)}$$

in which the primes denote differentiation with respect to the argument and in the last equalities we have made use of the definitions of differentials of Z given in (B.31) and (B.32).

The plasma dispersion function is tabulated in

Fried and Conte, *The Plasma Dispersion Function* (1961) [**?**].

Figure B.3: Variation of the basic and modified Bessel functions with their arguments.

B.4   Bessel Functions

Bessel functions arise naturally from second order differential equations in the radial coordinate for a cylindrical geometry. Their governing differential equation is

$$z^2 \frac{d^2y}{dz^2} + z \frac{dy}{dz} + (z^2 - n^2) y = 0, \quad \text{Bessel's equation,} \qquad \text{(B.45)}$$

in which z is the independent (usually radial coordinate) variable and n is an integer. The basic (fundamental) solution of this differential equation is $y = J_n(z)$, which is the Bessel function of the first kind of order n. The linearly independent solution $Y_n(z)$ of (B.45) is called the Bessel function of the second kind of order n. It is singular at $z = 0$. Hence it is not a valid solution for most cylindrical geometry problems where the response must be finite at the origin of the cylindrical geometry (where $z \propto r = 0$). The z dependence of J_n for $n = 0, 1, 2$ is shown in Fig. B.4. The first few zeros $[J_n(j_{n,i}) \equiv 0]$ of the fundamental Bessel functions are (for $i = 1, 2, 3, \cdots$)

$$j_{0,i} = 2.405, 5.520, 8.654, \cdots, \quad j_{1,i} = 3.832, 7.016, 10.173, \cdots. \qquad \text{(B.46)}$$

Changing z from a real to an imaginary argument ($z \rightarrow iz$) in (B.45) changes the sign of the $z^2 y$ term in the defining differential equation. The corresponding solutions of this modified differential equation are the modified Bessel functions of the first and second kind of order n, respectively: $I_n(z)$ and $K_n(z)$. The variation of $e^{-z} I_n(z)$ with the argument z is shown for $n = 0, 1, 2$ in Fig. B.4.

Useful recursion relations for the basic Bessel functions include

$$J_{n-1}(z) + J_{n+1}(z) = \frac{2n}{z} J_n(z), \qquad \text{(B.47)}$$

$$J_{n-1}(z) - J_{n+1}(z) = 2 J_n' \equiv 2 \frac{dJ_n(z)}{dz}, \qquad \text{(B.48)}$$

$$J_{-n}(z) = (-1)^n J_n(z). \qquad \text{(B.49)}$$

Analogous recursion relations for the modified Bessel functions are:

$$I_{n-1}(z) - I_{n+1}(z) = \frac{2n}{z} I_n(z), \qquad \text{(B.50)}$$

$$I_{n-1}(z) + I_{n+1}(z) = 2\,I'_n \equiv 2\,\frac{dI_n(z)}{dz}, \tag{B.51}$$

$$I_{-n}(z) = I_n(z). \tag{B.52}$$

A fundamental (generating function) identity that is useful for calculating the effects on plane waves of the gyromotion of charged particles about a magnetic field is

$$e^{\pm iz\sin\varphi} = \sum_{n=-\infty}^{\infty} J_n(z)\,e^{\pm in\varphi}. \tag{B.53}$$

The product of this equation with its complex conjugate yields

$$1 = \sum_{m,n} J_m(z)\,J_n(z)\,e^{\pm i(m-n)\varphi}. \tag{B.54}$$

Integrating this equation over φ from 0 to 2π yields the summation relation

$$1 = \sum_{n=-\infty}^{\infty} J_n^2(z). \tag{B.55}$$

Multiplying (B.54) by $\cos\varphi$ and then integrating over φ and using (B.47) gives

$$0 = \sum_{n=-\infty}^{\infty} n\,J_n^2(z). \tag{B.56}$$

By similar means it can be shown that

$$0 = \sum_n (-1)^n n\,J_n^2 = \sum_n J_n J'_n = \sum_n n\,J_n J'_n = \sum_n (-1)^n n\,J_n J'_n, \tag{B.57}$$

$$\frac{1}{2} = \sum_n \left(J'_n\right)^2, \tag{B.58}$$

$$\frac{z^2}{2} = \sum_n n^2 J_n^2. \tag{B.59}$$

The fundamental integration identity that is useful in calculating velocity-space integrals over the product of two Bessel functions times a Maxwellian speed distribution is

$$\int_0^\infty dx\,x\,e^{-p^2 x^2} J_n(ax) J_n(bx) = \frac{1}{2p^2}\exp\left(-\frac{a^2+b^2}{4p^2}\right) I_n\left(\frac{ab}{2p^2}\right) \tag{B.60}$$

which, for $a = b = s$ and $p = 1$, becomes simply

$$\int_0^\infty dx\,x\,e^{-x^2} J_n^2(sx) = \frac{1}{2}e^{-s^2/2} I_n\left(\frac{s^2}{2}\right). \tag{B.61}$$

Summing this equation, and n times it, over n from $-\infty$ to ∞ utilizing (B.55) and (B.56) yields

$$1 = \sum_{n=-\infty}^{\infty} I_n\left(\frac{s^2}{2}\right), \tag{B.62}$$

$$0 = \sum_{n=-\infty}^{\infty} n\, I_n\left(\frac{s^2}{2}\right). \tag{B.63}$$

Integrals of Bessel functions with higher powers of the integration variable in the integrand can be calculated by differentiating (B.60) with respect to p^2. For example, with $a = b = s$ we obtain

$$\int_0^{\infty} dx\, x^3 e^{-x^2} J_n^2(sx) = -\lim_{p^2 \to 1} \frac{\partial}{\partial p^2} \int_0^{\infty} dx\, x\, e^{-p^2 x^2} J_n^2(sx)$$

$$= \frac{1}{2} e^{-s^2/2}\left[\left(1 - \frac{s^2}{2}\right) I_n\left(\frac{s^2}{2}\right) + \left(\frac{s^2}{2}\right) I_n'\left(\frac{s^2}{2}\right)\right]. \tag{B.64}$$

Power series representations (rapidly convergent for $z << n$) of the Bessel functions are

$$J_n(z) = \left(\frac{z}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m}}{m!\,(m+n)!} = \frac{1}{n!}\left(\frac{z}{2}\right)^n - \frac{1}{1!\,(n+1)!}\left(\frac{z}{2}\right)^{n+2} + \cdots, \tag{B.65}$$

$$I_n(z) = \left(\frac{z}{2}\right)^n \sum_{m=0}^{\infty} \frac{(z/2)^{2m}}{m!\,(m+n)!} = \frac{1}{n!}\left(\frac{z}{2}\right)^n + \frac{1}{1!\,(n+1)!}\left(\frac{z}{2}\right)^{n+2} + \cdots. \tag{B.66}$$

Specific power series expansions of particular interest in plasma physics include:

$$J_0(z) = 1 - \frac{z^2}{4} + \frac{z^4}{64} - \cdots, \tag{B.67}$$

$$J_1(z) = -J_0'(z) = \frac{z}{2} - \frac{z^3}{16} + \cdots, \tag{B.68}$$

$$e^{-z} I_0(z) = 1 - z + \frac{3z^2}{4} - \cdots, \tag{B.69}$$

$$e^{-z} I_1(z) = \frac{z}{2} - \frac{z^2}{2} + \frac{5z^3}{16} - \cdots. \tag{B.70}$$

Asymptotic expansions for large arguments compared to the order (i.e., for $|z| >> n$) are

$$J_n(z) \sim \sqrt{\frac{2}{\pi z}}\left[\cos\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right) + \mathcal{O}\left(\frac{1}{|z|}\right)\right], \tag{B.71}$$

$$e^{-z} I_n(z) \sim \frac{1}{\sqrt{2\pi z}}\left[1 - \frac{4n^2 - 1}{8z} + \mathcal{O}\left(\frac{1}{|z|^2}\right)\right]. \tag{B.72}$$

The classic, comprehensive reference for Bessel functions is:

Watson, *A Treatise on the Theory of Bessel Functions* (1962), 2nd Edition [**?**]

B.5   Legendre Polynomials

Legendre polynomials are the natural (orthogonal basis) polynomials in which to expand the latitude angle (ϑ) part of a distribution function in spherical velocity space — see (**??**) in Appendix A.4. Legendre polynomials are governed by the differential equation ($\zeta \equiv \cos\vartheta$)

$$\frac{d}{d\zeta}\left[(1-\zeta^2)\frac{dP_l(\zeta)}{d\zeta}\right] + l(l+1)\,P_l(\zeta) = 0, \quad \text{Legendre's equation}, \qquad \text{(B.73)}$$

and satisfy the symmetry and boundary conditions

$$P_l(-\zeta) = (-1)^l P_l(\zeta), \quad P_l(1) = 1. \qquad \text{(B.74)}$$

Legendre polynomials are given in general by

$$P_l(\zeta) = \frac{1}{2^l}\sum_{m=0}^{M}\frac{(-1)^m\,(2l-2m)!}{m!\,(l-m)!\,(l-2m)!}\,\zeta^{l-2m} = \frac{1}{2^l\,l!}\frac{d^l}{d\zeta^l}\,(\zeta^2-1)^l \qquad \text{(B.75)}$$

in which the upper limit of the sum is $M \equiv [l/2]$, the largest integer less than or equal to $l/2$. The lowest order Legendre polynomials are

$$P_0 = 1, \quad P_1 = \zeta, \quad P_2 = (3\zeta^2-1)/2, \quad P_3 = (5\zeta^3-3\zeta)/2. \qquad \text{(B.76)}$$

Useful recurrence relations are

$$(l+1)P_{l+1}(\zeta) + l\,P_{l-1}(\zeta) = (2l+1)\,\zeta\,P_l(\zeta), \qquad \text{(B.77)}$$

$$l\,\zeta\,P_l(\zeta) - l\,P_{l-1}(\zeta) = (\zeta^2-1)\frac{dP_l(\zeta)}{d\zeta}. \qquad \text{(B.78)}$$

The orthgonality and values of relevant angular integrals of products of Legendre polynomials are given in (**??**) in Appendix C.3. A useful expansion of a delta function in terms of Legendre polynomials is

$$\delta(\zeta-\zeta_0) = \sum_{l=0}^{\infty}P_l(\zeta)\,P_l(\zeta_0). \qquad \text{(B.79)}$$

B.6   Laguerre Polynomials

Laguerre polynomials are the natural (orthogonal basis) energy weighting functions in which to expand a distribution function in spherical velocity space — see (**??**) in Appendix A.4. The relevant forms for kinetic theory and plasma physics are defined in general by

$$L_n^{(l+1/2)}(x) = \sum_{m=0}^{n}\frac{\Gamma(n+l+3/2)\,(-x)^m}{m!\,(n-m)!\,\Gamma(m+l+3/2)} = \frac{e^x}{n!\,x^{l+1/2}}\frac{d^n}{dx^n}\left(e^{-x}x^{n+l+1/2}\right),$$
$$\text{(B.80)}$$

in which ($x \equiv mv^2/2T = v^2/v_T^2$) is the normalized kinetic energy variable and l is the integer subscript of the (Legendre) polynomial expansion in spherical velocity space. These "generalized" Laguerre polynomials satisfy the differential equation

$$x \, \frac{d^2 L_n^{(l+1/2)}}{dx^2} + (l - x + 3/2) \, \frac{dL_n^{(l+1/2)}}{dx} + n \, L_n^{(l+1/2)} = 0. \qquad \text{(B.81)}$$

They have a generating function given by

$$\frac{1}{(1-z)^{l+3/2}} \, \exp\left(\frac{xz}{z-1}\right) = \sum_{n=0}^{\infty} L_n^{(l+1/2)}(x) \, z^n, \quad \text{for } |z| < 1. \qquad \text{(B.82)}$$

Laguerre polynomials are closely related to Hermite polynomials (their Cartesian velocity space equivalents) and Sonine polynomials (reversed indices, different normalization). The lowest order ($n = 0, 1, 2$ and $l = 0, 1, 2$) Laguerre polynomials are

$$L_0^{(1/2)} = 1, \quad L_1^{(1/2)} = \frac{3}{2} - x, \quad L_2^{(1/2)} = \frac{15}{8} - \frac{5x}{2} + \frac{x^2}{2}, \quad \cdots,$$

$$L_0^{(3/2)} = 1, \quad L_1^{(3/2)} = \frac{5}{2} - x, \quad L_2^{(3/2)} = \frac{35}{8} - \frac{7x}{2} + \frac{x^2}{2}, \quad \cdots, \qquad \text{(B.83)}$$

$$L_0^{(5/2)} = 1, \quad L_1^{(5/2)} = \frac{7}{2} - x, \quad L_2^{(5/2)} = \frac{63}{8} - \frac{9x}{2} + \frac{x^2}{2}, \quad \cdots.$$

The orthogonality and values of relevant energy integrals of products of these Laguerre polynomials are given in (**??**) and (**??**) in Appendix C.2.

REFERENCES

Some general compendia of properties of special functions are:

Abromowitz and Stegun, *Handbook of Mathematical Functions* (1965) [**?**]

Magnus, Oberhettinger and Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (1966) [**?**]

Jahnke and Emde, *Table of Functions* (1945) [**?**].