Floyd's Algorithm

Introduction

- Used to find shortest paths in a weighted graph
- Travel maps containing driving distance from one point to another
  - Represented by tables
  - Shortest distance from point A to point B given by intersection of row and column
  - Route may pass through other cities represented in the table
- Navigation systems

All-pairs shortest-path problem

- Graph $G = (V, E)$
- Weighted and directed graph
- Problem: Find the length of the shortest path between every pair of vertices
  - Length of the path is strictly determined by the weight of its edges
  - It is not based on the number of edges traversed
- Representation of weighted directed graph by adjacency matrix
  - $n \times n$ matrix for a graph with n vertices – adjacency matrix
    - Chosen for constant time access to every edge
  - Nonexistent edges may be assigned a value ∞

	0	1	2	3	4	5
0	0	2	5	∞	∞	∞
1	∞	0	7	1	∞	8
2	∞	∞	0	4	∞	∞
3	∞	∞	∞	0	3	∞
4	∞	∞	2	∞	0	3
5	∞	5	∞	2	4	0

- Algorithm

```
Input n: Number of vertices
      a[0..n-1][0..n-1] -- Adjacency matrix

Output: Transformed a that contains shortest path lengths

for ( k = 0; k < n; k++ )
    for ( i = 0; i < n; i++ )
        for ( j = 0; j < n; j++ )
            a[i][j] = min ( a[i][j], a[i][k] + a[k][j] );
```

  - Easy to see that the algorithm is $\Theta(n^3)$

- Solution

	0	1	2	3	4	5
0	0	2	5	3	6	9
1	∞	0	6	1	4	7
2	∞	15	0	4	7	10
3	∞	11	5	0	3	6
4	∞	8	2	5	0	3
5	∞	5	6	2	4	0

Creating arrays at run time

- Allocate arrays on heap at run-time to make the program more useful, such as

```
int * arr;
arr = ( int * ) malloc ( n * sizeof ( int ) );
```

- 2D arrays in C are represented by a 1D array of pointers, using `malloc`
- Problem in transmission as memory should be contiguous
- Allocate them by first allocating space, and then, putting pointers in place

```
int ** arr;                    // The main array
int  * storage;                // Contiguous storage for array
storage = ( int * ) malloc ( m * n * sizeof ( int ) );
arr = ( int ** ) malloc ( m * sizeof ( int * ) );
for ( i = 0; i < m; i++ )
    arr[i] = storage + ( i * n );
```

- Initialize the array elements either by using `a[i][j]` notation, or by using `storage` if initialized *en masse*

Designing parallel algorithm

- Partitioning
  - Choose either domain decomposition or functional decomposition
  - Same assignment statement executed n^3 times
    - No functional parallelism
  - Easy to do domain decomposition
    - Divide matrix A into n^2 elements
    - Associate a primitive task with each element
- Communication
  - Each update of element `a[i][j]` requires access to elements `a[i][k]` and `a[k][j]`
  - For a given value of `k`
    - Element `a[k,m]` is needed by every task associated with elements in column `m`
    - Element `a[m,k]` is needed by every task associated with elements in row `m`
  - In iteration k, each element in row k gets broadcast to the tasks in the same column
  - Each element in column k gets broadcast to tasks in the same row
  - Do we need to update every element of matrix concurrently?
    - Values of `a[i][k]` and `a[k][j]` do not change during iteration k

  - Update to `a[i][k]` is
    ```
    a[i][k] = min ( a[i,k], a[i,k] + a[k,k] );
    ```
  - Update to `a[k][j]` is
    ```
    a[k][j] = min ( a[k,j], a[k,k] + a[k,j] );
    ```
  - In both the above updates, `a[i][k]` and `a[k][j]` cannot decrease (all numbers are positive)
  - Hence, no dependence between updates of `a[i][j]` and the updates of `a[i][k]` and `a[k][j]`
  - For each iteration k of the outer loop, perform broadcasts and update every element of matrix in parallel

Agglomeration and mapping

- Use the decision tree to determine agglomeration and mapping strategy
  - Number of tasks: static
  - Communication among tasks: structured
  - Computation time per task: constant
- Agglomerate tasks to minimize communication
  - One task per MPI process
- Agglomerate n^2 primitive tasks into p tasks using either of the following two methods
  - Row-wise block striped
    - Agglomerate tasks in the same row
    - Broadcast within rows eliminated; data values local to task
    - During every iteration of outer loop, one task broadcasts n elements to all other tasks
    - Time for each broadcast: $\lceil \log p \rceil (\lambda + n/\beta)$
  - Column-wise block striped
    - Agglomerate tasks within same column
    - Broadcast within columns eliminated
    - Time for each broadcast: $\lceil \log p \rceil (\lambda + n/\beta)$
  - Both the above methods result in same performance, look outside the computational kernel
  - Final choice: Simpler to read matrix from file with row-wise block striped if the file stores data as row-major order
    - Internally, C stores the matrices as row major as well

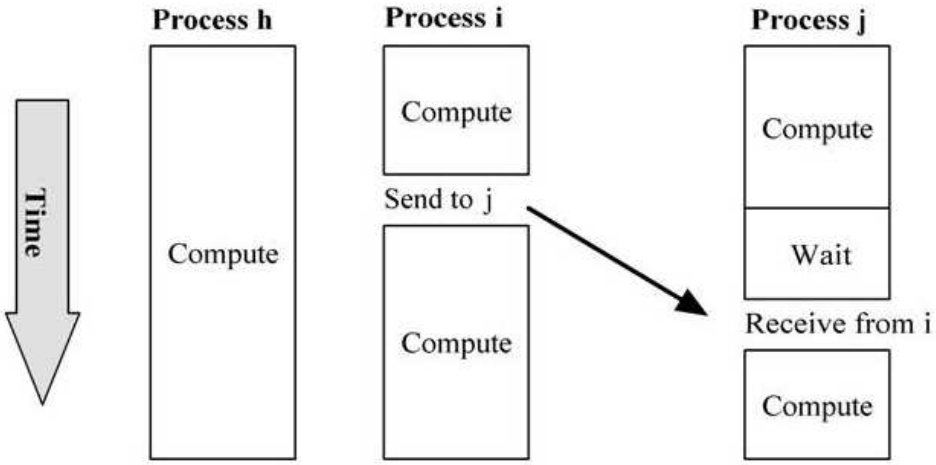
Matrix I/O

- Choice to let each process open the file, access the appropriate row (seek to it), and read the row
- Another method to read the matrix rows in last process and send them to appropriate process
  - Process i responsible for rows $\lfloor in/p \rfloor$ through $\lfloor (i+1)n/p \rfloor - 1$
  - Last process responsible for $\lceil n/p \rceil$ rows (largest buffer)
    - No extra space needed for file input buffering
  - Last process reads the rows for each process in a loop and sends the data to appropriate task
- All the printing done by process 0, by getting data from other processes
  - Print all the output in correct order
  - Processes $1, 2, \ldots, p-1$ simply wait for a message from process 0, then send process 0 their portion of the matrix

    - Process 0 never receives more than one submatrix at a time

Point-to-point communication

- Function to read matrix from file
  - Executed by process $p-1$
  - Reads a contiguous group of matrix rows
  - Sends a message containing these rows directly to the process responsible to manage them
- Function to print the matrix
  - Each process sends the group of matrix rows to process 0
  - Process 0 receives the message and prints the rows to standard output
- Communication involves a pair of processes
  - One process sends a message
  - Other process receives the message



  - Process h is not involved in communication
  - Both *Send* and *receive* are blocking
  - Both send and receive need to be conditionally executed by process rank

```
...
if ( id == i )
{
    ...
    /* Send message to process j */
    ...
}
else if ( id == j )
{
    ...
    /* Receive message from process i */
    ...
}
```

  - Communication calls must be in conditionally executed code

- Function `MPI_Send`
  - Perform a blocking send

    ```
    int MPI_Send ( void * buffer, int count, MPI_Datatype datatype, int dest,
        int tag, MPI_Comm comm );
    ```

    `buffer` Starting address of the array of data items to send
    `count` Number of data items in array (nonnegative integer)
    `datatype` Data type of each item (uniform since it is an array); defined by an MPI constant
    `dest` Rank of destination (integer)
    `tag` Message tag, or integer label; allows identification of message purpose
    `comm` Communicator; group of processes participating in this communication function
  - Function blocks until the message buffer is again available
  - Message buffer is free when
    - Message copied to system buffer, or
    - Message transmitted (may overlap computation)
- Function `MPI_Recv`
  - Blocking receive for a message

    ```
    int MPI_Recv ( void * buffer, int count, MPI_Datatype datatype, int src,
        int tag, MPI_Comm comm, MPI_Status * status );
    ```

    `buffer` Starting address of receive buffer
    `count` Maximum number of data items in receive buffer
    `datatype` Data type of each item (uniform since it is an array); defined by an MPI constant
    `src` Rank of source (integer)
      - Can be specified as `MPI_ANY_SOURCE` to receive the message from any source in the communicator
      - Process rank in this case can be determined through `status`

    `tag` Desired tag value (integer)
      - Can be specified as `MPI_ANY_TAG`
      - Received tag can be determined through `status`

    `comm` Communicator; group of processes participating in this communication function
    `status` Status objects; must be allocated before call to `MPI_Recv`
  - Blocks until the message has been received, or until an error condition causes the function to return
  - `count` contains the maximum length of message
    - Actual length of received message can be determined with `MPI_Get_count`
  - `status` contains information about the just-completed function

    `status->MPI_source` Rank of the process sending message
    `status->MPI_tag` Message's tag value
    `statis->MPI_ERROR` Error condition
  - Function blocks until message arrives in buffer
  - If message never arrives, function never returns
- Deadlock
  - Process blocked on a condition that will never become true
  - Easy to write send/receive code that deadlocks
    - Two processes with rank 0 and 1, each wanting to compute the average of two numbers in an array `a[2]`

    - Process i has the updated value of $a[i]$
  - Both receive before send

```
float      a[2], avg;
int        me;             // Process rank
MPI_Status status;
...
int other = 1 - me;
MPI_Recv ( a + other, 1, MPI_FLOAT, other, 0, MPI_COMM_WORLD, &status );
MPI_Send ( a + me, 1, MPI_FLOAT, other, 0, MPI_COMM_WORLD );
avg = ( a[0] + a[1] ) / 2.0;
```

    - Process 0 blocks inside `MPI_Recv` waiting for message from process 1 to arrive
    - Process 1 blocks inside `MPI_Recv` waiting for message from process 0 to arrive
    - Both are deadlocked
  - Send tag does not match receive tag

```
float      a[2], avg;
int        me;             // Process rank
MPI_Status status;
...
int other = 1 - me;
MPI_Isend ( a + me, 1, MPI_FLOAT, other, me, MPI_COMM_WORLD, &request );
MPI_Recv ( a + other, 1, MPI_FLOAT, other, me, MPI_COMM_WORLD, &status );
avg = ( a[0] + a[1] ) / 2.0;
```

    - Processes block in `MPI_Recv` because the tag is not correct
  - Other problems could be wrong destination in send or wrong source in receive

Documenting the parallel program

- Housekeeping, starting in `floyd.c`
  - Use a `typedef` and a `#define` macro to indicate matrix data types
- `floyd.c:main()` responsible to read and print original matrix
  - It verifies that the matrix is square
  - All work done by `compute_shortest_paths` with four parameters

Analysis and benchmarking

- Sequential version performance: $\Theta(n^3)$
- Analysis of parallel algorithm
  - Innermost loop has complexity $\Theta(n)$
  - Middle loop executed at most $\lceil n/p \rceil$ times due to rowwise block-striped decomposition of matrix
  - Overall computational complexity: $\Theta(n^2/p)$
- Communication complexity
  - No communication in inner loop
  - No communication in middle loop
  - Broadcast in outer loop

  - Passing a single message of length n from one PE to another has time complexity $\Theta(n)$
  - Broadcasting to p PEs requires $\lceil \log p \rceil$ message-passing steps
  - Complexity of broadcasting: $\Theta(n \log p)$

- Outermost loop
  - For every iteration of outermost loop, parallel algorithm must compute the root PE taking constant time
  - Root PE copies the correct row of A to array `tmp`, taking $\Theta(n)$ time
  - The loop itself executes n times
- Overall time complexity
$$\Theta(n(1 + n + n \log p + n^2/p)) = \Theta(n^3/p + n^2 \log p)$$
- Prediction for the execution time on commodity cluster
  - n broadcasts, with $\lceil \log p \rceil$ steps each
  - Each step passes messages of $4n$ bytes
  - Expected communication time of parallel program: $n\lceil \log p \rceil(\lambda + 4n/\beta)$
  - Average time to update a single cell: χ
  - Expected computation time for parallel program: $n^2\lceil n/p \rceil \chi$
  - Execution time
$$n^2\lceil n/p \rceil \chi + n\lceil \log p \rceil(\lambda + 4n/\beta)$$
  - This expression overestimates the parallel execution time because it ignores the fact that there can be considerable overlap between computation and communication