PROBLEM SET 4

16822 GEOMETRY-BASED METHODS IN VISION (FALL 2022)
https://piazza.com/cmu/fall2022/16822

OUT: Oct. 11, 2022
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Instructor: Shubham Tulsiani
TAs: Mosam Dabhi, Kangle Deng, Jenny Nan

START HERE: Instructions

- **Collaboration policy:** All are encouraged to work together BUT you must do your own work (code and write up). If you work with someone, please include their name in your write up and cite any code that has been discussed. If we find highly identical write-ups or code without proper accreditation of collaborators, we will take action according to university policies, i.e. you will likely fail the course. See the Academic Integrity Section detailed in the initial lecture for more information.
- **Late Submission Policy:** There are **no** late days for Problem Set submissions.
- **Submitting your work:**
  - We will be using Gradescope (https://gradescope.com/) to submit the Problem Sets. Please use the provided template. Submissions can be written in LaTeX. Regrade requests can be made, however this gives the TA the opportunity to regrade your entire paper, meaning if additional mistakes are found then points will be deducted. Each derivation/proof should be completed on a separate page. For short answer questions you **should** include your work in your solution.
- **Materials:** The data that you will need in order to complete this assignment is posted along with the writeup and template on Piazza.

For multiple choice or select all that apply questions, replace `\choice` with `\CorrectChoice` to obtain a shaded box/circle, and don't change anything else.

Instructions for Specific Problem Types

For "Select One" questions, please fill in the appropriate bubble completely:

Select One: Who taught this course?

- ● Shubham Tulsiani
- ○ Deepak Pathak
- ○ Fernando De la Torre
- ○ Deva Ramanan

For "Select all that apply" questions, please fill in all appropriate squares completely:

Select all that apply: Which are scientists?

- ■ Stephen Hawking
- ■ Albert Einstein
- ■ Isaac Newton
- □ None of the above

For questions where you must fill in a blank, please make sure your final answer is fully included in the given space. You may cross out answers or parts of answers, but the final answer must still be within the given space.

Fill in the blank: What is the course number?

16-822

1 Two-view Geometry [11 pts]

1. **[2 pts]** Given $\mathbf{F} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 4 & 6 & \mathbf{x} \end{bmatrix}$:

   (a) Find $\mathbf{x}$ if $\mathbf{F}$ is a valid fundamental matrix.

   (b) Compute epipoles $\mathbf{e}$ and $\mathbf{e}'$ for the computed value of $\mathbf{x}$.

(a) For $\mathbf{F}$ to be a valid fundamental matrix, its rank should be 2 $\Rightarrow \mathbf{F} = \begin{bmatrix} f_1 & f_2 & \alpha f_1 + \beta f_2 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 0 \\ 1 \\ \mathbf{x} \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} \Rightarrow 2\alpha = 0; \quad 3\beta = 1; \quad \mathbf{x} = 4\alpha + 6\beta \Rightarrow \mathbf{x} = 2$$

(b) For the epipoles, we know $\mathbf{Fe} = 0; \quad \mathbf{e}'^T\mathbf{F} = 0$

Let $\mathbf{e} = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 4 & 6 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = 0 \Rightarrow \mathbf{e} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$

Similarly, Let $\mathbf{e}' = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} \Rightarrow \begin{bmatrix} a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 4 & 6 & 2 \end{bmatrix} = 0 \Rightarrow \mathbf{e}' = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$

2. **[2 pts]** Given SVD decomposition of an essential matrix $\mathbf{E} = \mathbf{UDV}^T$, what is the relative translation $\mathbf{t}$ between the two cameras (expressed in terms of elements of $\mathbf{U}$ and $\mathbf{V}$)?

Given the decomposition, $\mathbf{E} = \mathbf{UDV}^T$, we know $\mathbf{t} = \pm\mathbf{u}_3$
where, $\mathbf{u}_3$ is the third column vector of the unitary matrix $\mathbf{U}$

3. **[3 pts]** Given two affine cameras $\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{P}' = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ 0 & 0 & 0 & 1 \end{bmatrix}$, show that any two distinct epipolar lines in the second image are parallel.

Assuming the camera center for the first camera to be, $\mathbf{C} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

$\mathbf{C}$ can be resolved from $\mathbf{PC} = 0 \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} a \\ b \\ d \end{bmatrix} = 0 \Rightarrow \mathbf{C} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

Now, we know the projection of $\mathbf{C}$ in the second image plane, the epipole $\mathbf{e}' = \mathbf{P}'\mathbf{C}$

$$\Rightarrow \mathbf{e}' = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \mathbf{e}' = \begin{bmatrix} p_{13} \\ p_{23} \\ 0 \end{bmatrix}$$

So, as we can see that the epipole $\mathbf{e}'$ lies at infinity in the second image plane and as any 2 distinct epipolar lines have to pass through it $\Rightarrow$ these 2 distinct lines are parallel to each other (because they 'meet' at infinity)

4. **[4 pts]** Are these statements true or false?

   (a) Given a camera with zero-skew undergoing a translation-only motion, the epipolar lines are parallel if and only if the translation compoenent along the view direction is 0.

   (b) $\mathbf{F} = \mathbf{E}$ if and only if $\mathbf{K} = \mathbf{K}' = \mathbf{I}$.

   (c) Assuming $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{e}$ are distinct, $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{e}$ are collinear if and only if $\mathbf{F}\mathbf{x}_1 = \mathbf{F}\mathbf{x}_2$.

   (d) If the vanishing line of a plane contains the epipole, then the plane is parallel to the baseline.

(a) True
(b) False
(c) True
(d) True

2 Two-view Calibration [19 pts]

5. In class, we used the 8-pt algorithm to compute $\mathbf{F}$ given correspondences of form $(\mathbf{x}, \mathbf{x}')$. In this question, you need to instead compute $\mathbf{F}$ using points in the first image with known corresponding epipolar lines in the second image $(\mathbf{x}, \mathbf{l}')$.

   (a) **[3 pts]** Design an algorithm to compute $\mathbf{F}$ in such case.

   (b) **[1 pt]** What is the minimum number of such correspondences needed?

(a) Given one point-line correspondence, $(\mathbf{x}, \mathbf{l}')$, we know that $\mathbf{l}' = \mathbf{F}\mathbf{x} \Rightarrow \mathbf{l}' \times \mathbf{F}\mathbf{x} = 0 \Rightarrow [\mathbf{l}']_\times \mathbf{F}\mathbf{x} = 0$ where, $[\mathbf{l}']_\times$ is the skew-symmetric format of $\mathbf{l}'$

Assuming, $\mathbf{l}' = \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix}^T$; $,\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ and $\mathbf{F} = \begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0 & -l_3 & l_2 \\ l_3 & 0 & -l_1 \\ -l_2 & l_1 & 0 \end{bmatrix} \begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow$ we can build a linear system, similar to calculating homographies, $\mathbf{A}\mathbf{f} = 0$, as $\begin{bmatrix} 0 & -l_3\mathbf{x}^T & l_2\mathbf{x}^T \\ l_3\mathbf{x}^T & 0 & -l_1\mathbf{x}^T \end{bmatrix} \mathbf{f} = 0$, where $\mathbf{f}$ is a flattened $\mathbf{F}$ matrix.

(b) Using a minimum of 4 such correspondences, a system $\mathbf{A}\mathbf{f} = 0$ can be built, where $\mathbf{A}$ is a 8×9 matrix and $\mathbf{f}$ is a 9×1 vector and $\mathbf{f}$ can be obtained by an **SVD** of $\mathbf{A}$

6. **[3 pts]** In the translation only case with intrinsics being unchanged, design an algorithm to compute $\mathbf{F}$ using just 2 point correspondences.

In the translation only case the fundamental matrix, $\mathbf{F} = [\mathbf{e}']_\times$. Let, $\mathbf{e}' = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}^T$

$\Rightarrow \mathbf{F} = \begin{bmatrix} 0 & -e_1 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix}$. We know, $\mathbf{x}'^T \mathbf{F} \mathbf{x} = 0 \Rightarrow \begin{bmatrix} x'_1 & x'_2 & x'_3 \end{bmatrix} \begin{bmatrix} 0 & -e_1 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

Solving this, boils down to, $\begin{bmatrix} x'_3 x_2 - x'_2 x_3 & x'_1 x_3 - x'_3 x_1 & x'_2 x_1 - x'_1 x_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = 0$

We need just 2 points to make a linear system of equations, $\mathbf{A}\mathbf{f} = 0$ can be built, where $\mathbf{A}$ is a 2×3 matrix and $\mathbf{f}$ is a 3×1 vector and $\mathbf{f}$ can be obtained by an **SVD** of $\mathbf{A}$
Just 2 points are needed although we have 3 elements in $\mathbf{f}$, because of scale ambiguity.

7. In 8-pt algorithm, we computed $\mathbf{f}$ such that $\mathbf{A}\mathbf{f} = 0$, $||\mathbf{f}|| = 1$, where $\mathbf{A}$ is a $N \times 9$ matrix. Assuming the matrix $\mathbf{A}$ is computed from perfect correspondences (i.e., without any noise), what is the rank of $\mathbf{A}$ if:

   (a) **[1 pt]** Assuming $N >> 9$, and several points are chosen in a non-degenerate way.

   (b) **[2 pts]** Assuming $N >> 9$, but all points lie on a common plane in 3D space.

(a) 8, this is a general case of 8-point algorithm. As we need to resolve 8 dofs in the linear equation, so rank(A) =8
(b) 6. Here, $\mathbf{F} = \mathbf{S}\mathbf{H}_\pi$. $\mathbf{H}_\pi$ is the planar homography So, $\mathbf{F}$ forms a 2-parameter family of 3-dof matrix, due to the skew-symmetric matrix $\mathbf{S}$

8. Consider the problem of estimating the fundamental matrix from a set of 8 correspondences $(\mathbf{p}, \mathbf{p}')_i \ \forall\, i \in [1, 8]$. We denote the image coordinates as $\mathbf{p} = (u, v, 1)$, $\mathbf{p}' = (u', v', 1)$, the fundamental matrix as $\mathbf{F}$ with each entry as $\mathbf{F}_{ij}$. Assuming $\mathbf{F}_{33} \neq 0$, we can set $\mathbf{F}_{33} = 1$, and obtain a set of 8 linear equations of the form $\mathbf{A}\mathbf{f} = -\mathbf{1}_8$, where $\mathbf{1}_8$ is the 8 vector of ones, $\mathbf{f} = \begin{bmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} & \dots & \mathbf{F}_{32} \end{bmatrix}^\top$ (note that this $\mathbf{f}$ is different from the vector used in lectures for the 8-pt algorithm – it only has 8 elements.)

   (a) **[1 pts]** Express $\mathbf{A}$ in terms of (u, v) (expressing one row suffices).

   (b) **[3 pts]** If $\mathbf{A}$ is singular, show that there exists a 3×3 matrix $\mathbf{Q}$ that is different from $\mathbf{F}$, such that for all 8 correspondence, we have $\mathbf{p}_i'^\top \mathbf{Q} \mathbf{p}_i = 0$.

   (c) **[3 pts]** Show that the eight points in $\mathbb{P}^3$ corresponding to $(\mathbf{p}, \mathbf{p}')_i \ \forall\, i \in [1, 8]$ must lie on a quadric surface. (Hint, a quadric $\mathbf{S}$ is defined by the equation $\mathbf{X}^T \mathbf{S} \mathbf{X} = 0$, where $\mathbf{S}$ is a symmetric 4×4 matrix).

   (d) **[2 pts]** Show that the optical centers $\mathbf{C}$ and $\mathbf{C}'$ of the two cameras lie on this quadric.

3 Two-view Reconstruction [5 pts]

9. Suppose $\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and $\mathbf{P} = [\mathbf{I}|0]$:

   (a) **[2 pts]** Find a feasible $\mathbf{P}' = [\mathbf{M}|\mathbf{m}]$.

   (b) **[1 pt]** Find another distinct solution for $\mathbf{P}'$.

(a)A feasible solution for this case, is $\mathbf{P}' = \begin{bmatrix}[\mathbf{e}']_\times \mathbf{F} & \mathbf{e}'\end{bmatrix}$. $\mathbf{e}'$ can be found as left null space of $\mathbf{F}$

$$\Rightarrow \mathbf{e}'^T\mathbf{F} = 0 \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \Rightarrow \mathbf{e}' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow [\mathbf{e}']_\times = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \mathbf{M} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \mathbf{m} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(b) Another solution can be found out be using a homography of the form, $\mathbf{H} = \begin{bmatrix} k^{-1}\mathbf{I} & 0 \\ k^{-1}v^T & k \end{bmatrix}$

this translates to the $\mathbf{P}'$ matrix as $\mathbf{P}' = \begin{bmatrix}[\mathbf{e}']_\times \mathbf{F} + \mathbf{e}'\mathbf{v}^T & k\mathbf{e}'\end{bmatrix}$. Let $\mathbf{v} = 0, k = 2$

$$\Rightarrow \text{Another feasible solution, } \mathbf{M} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \mathbf{m} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

10. **[2 pts]** Given two cameras, $\mathbf{KR}[I|\mathbf{C}]$ and $\mathbf{K}'\mathbf{R}'[I|\mathbf{C}']$, which of these leave $\mathbf{F}$ unchanged:

(a) Change $(\mathbf{C}, \mathbf{C}')$ to $(\mathbf{C}/2, \mathbf{C}'/2)$.

(b) Change $\left(f_x, f_y, f_x', f_y'\right)$ to $\left(f_x/2, f_y/2, f_x'/2, f_y'/2\right)$.

Both the cases can be achieved using a homography of 4×4. So, both (a) and (b) will leave $\mathbf{F}$ unchanged

Collaboration Questions Please answer the following:

1. Did you receive any help whatsoever from anyone in solving this assignment?
   - ○ Yes
   - ● No
   - If you answered 'Yes', give full details:
   - (e.g. "Jane Doe explained to me what is asked in Question 3.4")

2. Did you give any help whatsoever to anyone in solving this assignment?
   - ○ Yes
   - ● No
   - If you answered 'Yes', give full details:
   - (e.g. "I pointed Joe Smith to section 2.3 since he didn't know how to proceed with Question 2")

3. Did you find or come across code that implements any part of this assignment ?
   - ○ Yes
   - ● No
   - If you answered 'Yes', give full details: No
   - (book & page, URL & location within the page, etc.).