Problem Set 2

16822 Geometry-based Methods in Vision (Fall 2022)
https://piazza.com/cmu/fall2022/16822

OUT: Sep. 20, 2022
DUE: Sep. 27, 2022 11:59 PM
Instructor: Shubham Tulsiani
TAs: Mosam Dabhi, Kangle Deng, Jenny Nan

START HERE: Instructions

- **Collaboration policy:** All are encouraged to work together BUT you must do your own work (code and write up). If you work with someone, please include their name in your write up and cite any code that has been discussed. If we find highly identical write-ups or code without proper accreditation of collaborators, we will take action according to university policies, i.e. you will likely fail the course. See the Academic Integrity Section detailed in the initial lecture for more information.
- **Late Submission Policy:** There are **no** late days for Problem Set submissions.
- **Submitting your work:**
  - We will be using Gradescope (https://gradescope.com/) to submit the Problem Sets. Please use the provided template. Submissions can be written in LaTeX. Regrade requests can be made, however this gives the TA the opportunity to regrade your entire paper, meaning if additional mistakes are found then points will be deducted. Each derivation/proof should be completed on a separate page. For short answer questions you **should** include your work in your solution.
- **Materials:** The data that you will need in order to complete this assignment is posted along with the writeup and template on Piazza.

For multiple choice or select all that apply questions, replace `\choice` with `\CorrectChoice` to obtain a shaded box/circle, and don't change anything else.

Instructions for Specific Problem Types

For "Select One" questions, please fill in the appropriate bubble completely:

Select One: Who taught this course?

● Shubham Tulsiani

○ Deepak Pathak

○ Fernando De la Torre

○ Deva Ramanan

For "Select all that apply" questions, please fill in all appropriate squares completely:

Select all that apply: Which are scientists?

■ Stephen Hawking

■ Albert Einstein

■ Isaac Newton

□ None of the above

For questions where you must fill in a blank, please make sure your final answer is fully included in the given space. You may cross out answers or parts of answers, but the final answer must still be within the given space.

Fill in the blank: What is the course number?

$\boxed{\text{16-822}}$

1 2D Projective Geometry [18 pts]

1. (a) **[2 pts]** Calculate the line passing through 2 given points: (1) $\mathbf{p}_1 = [3, 4, 1]^T, \mathbf{p}_2 = [4, 3, 0]^T$, (2) $\mathbf{p}_1 = [3, 4, 2022]^T, \mathbf{p}_2 = [3, 4, -1967]^T$.

   (b) **[2 pts]** Calculate the intersection point between 2 given lines: (1) $\mathbf{l}_1 = [3, 4, 1]^T, \mathbf{l}_2 = [0, 0, 1]^T$, (2) $\mathbf{l}_1 = [3, 4, 1]^T, \mathbf{l}_2 = [3, 4, 2]^T$.

(a) For any two points, p_1 and p_2, the line passing through them can be given as the cross product, $L = p_1 \times p_2$ This can be written as a matrix multiplication, $L = p_{1\times}\, p_2$, where $p_{1\times}$ is the matrix equivalent of p_1 in a cross product, where if $p_1 = \begin{bmatrix} a & b & c \end{bmatrix}^T \Rightarrow p_{1\times} = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$

(1) For the given points, p_1 and p_2, $L = \begin{bmatrix} 0 & -1 & 4 \\ 1 & 0 & -3 \\ -4 & 3 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ -7 \end{bmatrix}$

(2) For the given points, $L = \begin{bmatrix} 0 & -2022 & 4 \\ 2022 & 0 & -3 \\ -4 & 3 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ -1967 \end{bmatrix} = 3989 \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} \equiv \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$

(b) As lines and points have a duality in the projective space, so, the point of intersection of any two lines can also be written as, $p = L_1 \times L_2$

(1) For the given lines, $p = \begin{bmatrix} 0 & -1 & 4 \\ 1 & 0 & -3 \\ -4 & 3 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}$

(2) For the given lines, $p = \begin{bmatrix} 0 & -1 & 4 \\ 1 & 0 & -3 \\ -4 & 3 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}$

2. **[3 pts]** Suppose a conic in 2D projective space is given by $\mathbf{C} = \mathbf{l}\mathbf{m}^T + \mathbf{m}\mathbf{l}^T$, where $\mathbf{l}$ and $\mathbf{m}$ are 2 lines. Show that a point belongs to $\mathbf{C}$ if and only if it is on $\mathbf{m}$ or $\mathbf{l}$.

For any point x in projective space, to lie on a conic C in projective space, $x^TCx = 0$
Given, $C = lm^T + ml^T \Rightarrow \quad x^TCx = x^T(lm^T + ml^T)x = 0 \Rightarrow \quad x^Tlm^Tx + x^Tml^Tx = 0$
$\Rightarrow (l^Tx)^T(m^Tx) + (m^Tx)^T(l^Tx) = 0 \Rightarrow$ To satisfy this equation, either of l^Tx or m^Tx should be 0
For any line k, if $k^Tx = 0$, this means point x lies on it.
This means, point x should lie on either line l or m, for it to lie on the conic $C = lm^T + ml^T$

3. **[4 pts]** Given a transformation $\mathbf{H} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

   (a) transform a point $\mathbf{p} = [3, 4, 1]^T$,

   (b) transform a line $\mathbf{l} = [-4, 3, 0]$

   (c) transform a conic $\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

   (d) does this transformation leaves the circular points at infinity unchanged? Explain the reason without calculation.

(a) For the given point, $\mathbf{p}$, the transformed point, $p' = \mathbf{Hp} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 1 \end{bmatrix}$

(b) For the line $\mathbf{l}$, the transformed line, $l' = \mathbf{H}^{-T}\mathbf{l}$. As $\mathbf{H}$ is a diagonal matrix, $\mathbf{H}^T = \mathbf{H} \Rightarrow \mathbf{H}^{-T} = \mathbf{H}^{-1}$
As $\mathbf{H}$ is a diagonal matrix, the inverse can be attained by just inverted the individual diagonal elements
$\Rightarrow \mathbf{H}^{-1} = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow l' = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -4/3 \\ 3/2 \\ 0 \end{bmatrix}$

(c) For a conic, the transformation is given as $C' = \mathbf{H}^{-T}\mathbf{C}\mathbf{H}^{-1}$. From the last part, we know that, $\mathbf{H}^{-T} = \mathbf{H}^{-1}$
$\Rightarrow C' = \mathbf{H}^{-1}\mathbf{C}\mathbf{H}^{-1} = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/9 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(d) No, this transformation does not leave the circular points at infinity unchanged. The reason is that, this transformation is an affine transformation and not a similarity transformation. Had this been a similarity transformation, the circular points at infinity would have been preserved

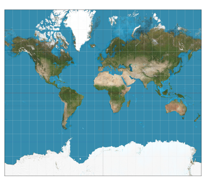
4. **[4 pts]** Are these 2D projective transformations?

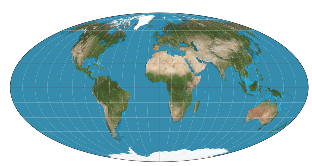
   (a) Reflection along a line,

   (b) Doubling spherical coordinates: $(r, \theta) \to (2r, 2\theta)$,k

   (c) A picture hanging on a wall and its image taken by a camera,

   (d) Transformation between these 2 world maps.





(a) True
(b) True
(c) True
(d) False

5. **[3 pts]** Are these statements true or false?

   (a) Given a line $\mathbf{l}$, if both $\mathbf{H}_A$ and $\mathbf{H}_B$ map $\mathbf{l}$ to $[0, 0, 1]^T$, then $\mathbf{H}_A\mathbf{H}_B^{-1}$ is an affine transformation.

   (b) Instead of annotating orthogonal lines, if we annotate multiple pairs of lines that form 45 degree angles in the metric space, we can still calculate $\mathbf{C}^{\star}_{\infty}{}'$.

   (c) If we are allowed to annotate pairs of parallel and orthogonal lines, we need at least 5 pairs of them to calculate $\mathbf{C}^{\star}_{\infty}{}'$.

(a) False
(b) True
(c) False

2 3D Projective Geometry [12 pts]

6. **[3 pts]** Show that the Plucker Representation of a 3D line $\mathbf{L} = \mathbf{x}_1\mathbf{x}_2^T - \mathbf{x}_2\mathbf{x}_1^T$ is equivalent to representing the line as $(\tilde{\mathbf{d}}, \tilde{\mathbf{x}} \times \tilde{\mathbf{d}})$, i.e. show they have the same elements up to scale.

   Notations: $\tilde{\mathbf{d}}$ is the unit direction vector along the line, and $\tilde{\mathbf{x}}$ is any point on the line. Note that $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{d}}$ are 3-dim Euclidean coordinates while $\mathbf{x}_1$ and $\mathbf{x}_2$ are 4-dim homogeneous coordinates.

Assuming the two points as, $\mathbf{x}_1 = \begin{bmatrix} a_1 & b_1 & c_1 & 1 \end{bmatrix}^T$ and $\mathbf{x}_2 = \begin{bmatrix} a_2 & b_2 & c_2 & 1 \end{bmatrix}^T$
As the scale is 1, their euclidean representations would be, $\tilde{\mathbf{x}}_1 = \begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix}^T$, $\tilde{\mathbf{x}}_2 = \begin{bmatrix} a_2 & b_2 & c_2 \end{bmatrix}^T$
PS: Assuming these euclidean values to be normalized

Computing the Plucker representation,

$$\mathbf{L} = \mathbf{x}_1\mathbf{x}_2^T - \mathbf{x}_2\mathbf{x}_1^T = \begin{bmatrix} 0 & a_1b_2 - a_2b_1 & a_1c_2 - a_2c_1 & a_1 - a_2 \\ -(a_1b_2 - a_2b_1) & 0 & b_1c_2 - b_2c_1 & c_1 - c_2 \\ -(a_1c_2 - a_2c_1) & -(b_1c_2 - b_2c_1) & 0 & c_1 - c_2 \\ -(a_1 - a_2) & -(b_1 - b_2) & -(c_1 - c_2) & 0 \end{bmatrix}$$

Using euclidean co-ordinates, the plucker matrix can be re-written as, $\mathbf{L} = \begin{bmatrix} \tilde{\mathbf{x}}_1 \times \tilde{\mathbf{x}}_2 & \tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2 \\ -(\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2) & 0 \end{bmatrix}$

If we look at the line representation as $(\tilde{\mathbf{d}}, \tilde{\mathbf{x}} \times \tilde{\mathbf{d}})$ and substitute the values of the euclidean points $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2$
$\tilde{\mathbf{d}} = \tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2$
$\tilde{\mathbf{x}} \times \tilde{\mathbf{d}} \equiv \tilde{\mathbf{x}}_1 \times \tilde{\mathbf{x}}_2$, the normal vector to the 3D line
So, clearly we can see that up to a scale, both the plucker coordinate and plucker matrix representation of a line have the same elements.

7. **[2 pts]** Suppose $\mathbf{U}$ is a 4×4 matrix. $\mathbf{U}_{4\times 4} = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4]$ and $\mathbf{U}^T\mathbf{U} = \mathbf{I}$.

   (a) Suppose $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ represent 3 points in the 3D space. What is the plane passing through these 3 points?

   (b) Suppose $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ represent 4 points in the 3D space. Let $\mathbf{l}_1$ be the line passing through $\mathbf{u}_1, \mathbf{u}_2$, and $\mathbf{l}_2$ be the line passing through $\mathbf{u}_3, \mathbf{u}_4$. Do $\mathbf{l}_1$ and $\mathbf{l}_2$ intersect or not? (Only consider real-number points.)

(a) As $\mathbf{U}$ is orthogonal, this means, all the column vectors, $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4$ are also mutually orthogonal.
$\Rightarrow \vec{u}_4 \cdot \vec{u}_1 = 0, \vec{u}_4 \cdot \vec{u}_2 = 0, \vec{u}_4 \cdot \vec{u}_3 = 0$
If we consider u_1, u_2, u_3 to be points, the above equations can be written as $u_4^T u_1 = 0, u_4^T u_2 = 0, u_4^T u_3 = 0$
Plane going these points is $p = u_4$

(b) The lines $l_1 = u_1u_2^T - u_2u_1^T$, $l_2 = u_3u_4^T - u_4u_3^T$, in the plucker matrix form
The point of intersection can be given as $l_1l_2 = (u_1u_2^T - u_2u_1^T)(u_3u_4^T - u_4u_3^T)$
$\Rightarrow l_1l_2 = u_1u_2^Tu_3u_4^T - u_1u_2^Tu_4u_3^T - u_2u_1^Tu_3u_4^T + u_2u_1^Tu_4u_3^T = 0$, as the column vectors are orthogonal
As 0 isn't defined in projective space, the lines do not intersect

8. **[4 pts]** (a) Calculate the 3D transformation $\mathbf{H}$ that represents the projection onto a plane $\pi = [\mathbf{n}^T, 0]^T$, where $\mathbf{n} = [a, b, c]^T$ is a unit vector.

   (b) Calculate the 3D transformation $\mathbf{H}$ that represents the reflection along a plane $\pi = [\mathbf{n}^T, 0]^T$, where $\mathbf{n} = [a, b, c]^T$ is a unit vector.

9. **[3 pts]** In the lecture, we introduced an algorithm to compute homography between images from 4 pairs of point correspondences. Design an algorithm that instead uses pairs of line correspondences. Write the constraints provided by each correspondence, and how to compute the **H** that satisfies these.

As line transformations are defined as $l' = H^{-T}l$, we can use algorithm similar to the point correspondences and just substitute the points by lines and the H matrix by H^{-T}.

Collaboration Questions Please answer the following:

1. Did you receive any help whatsoever from anyone in solving this assignment?

   ○ Yes

   ● No

   - If you answered ‘Yes’, give full details:
   - (e.g. “Jane Doe explained to me what is asked in Question 3.4”)

2. Did you give any help whatsoever to anyone in solving this assignment?

   ○ Yes

   ● No

   - If you answered ‘Yes’, give full details:
   - (e.g. “I pointed Joe Smith to section 2.3 since he didn’t know how to proceed with Question 2”)

3. Did you find or come across code that implements any part of this assignment ?

   ○ Yes

   ● No

   - If you answered ‘Yes’, give full details: No
   - (book & page, URL & location within the page, etc.).