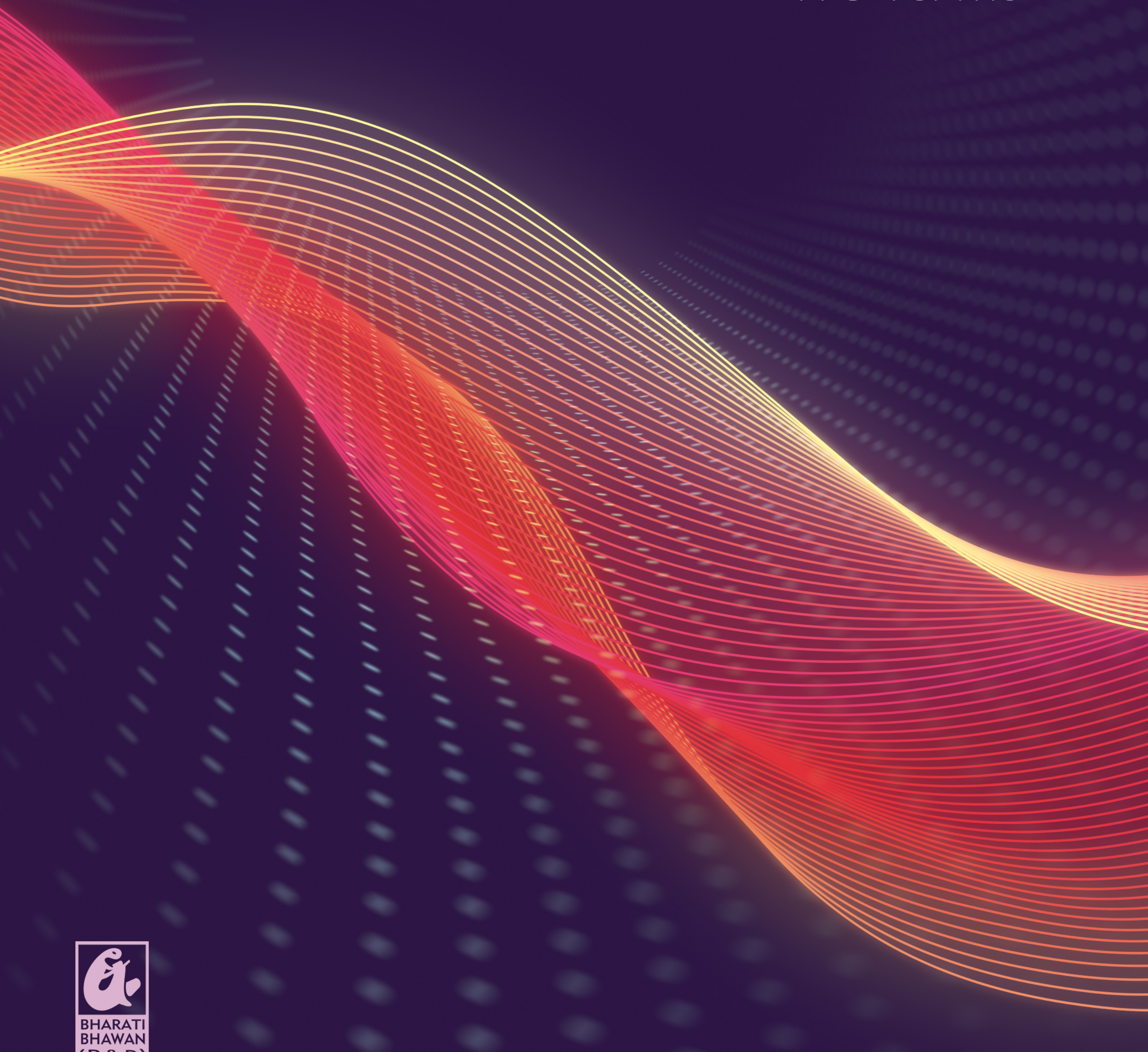
Classical ELECTROMAGNETISM

H C Verma



BHARATI BHAWAN (P & D)

CLASSICAL ELECTROMAGNETISM

H C VERMA, PhD
Retired Professor
Department of Physics
IIT, Kanpur

Bharati Bhawan
PUBLISHERS & DISTRIBUTORS

PREFACE

Electricity and magnetism are an integral part of our lives. How uncomfortable we feel when there is a power cut and no electricity in our house! Most of the functions of our body, including those of the brain, are governed by tiny electric currents flowing in millions of circuits, which are crucial for our survival. And do we not live all the time in the magnetic field created by the earth, which protects us from the harmful cosmic rays? Light from any source is nothing but a combination of electric and magnetic fields. Mobile phones, which have become commonplace, transfer our conversations and pictures from one place to another through electric and magnetic fields. Banking, ticket reservations, online classes, entertainment and many other operations are done through the internet using electric and magnetic fields.

As a student of physics, you must be curious to learn how electric and magnetic fields are produced and how they interact with materials. This book will help you understand the very basics of these aspects. Engineers, however, look at the subject from a different perspective—they are concerned with making products utilising electricity and magnetism. This book will not focus much on this aspect and will discuss the basic laws governing the behaviour of electric and magnetic fields. Though these fields behave differently, they are facets of the same physical quantity. You may realise this fact if you change your frame of observation. It is possible that there is no electric field in a frame, but as you change your frame, such a field appears. The situation is similar with magnetic fields. Electricity and magnetism are not different at a higher level. That is why the term "electromagnetism". There is also the quantum character of electromagnetic fields. We will not address it here and will confine ourselves to what is called the classical theory of electromagnetism.

The level of mathematics used in the book is intermediate, vector calculus being our main tool. This and more such mathematical tools are discussed briefly in the appendices. If you have not practised these in your mathematics courses, the appendices will help you. The flow of the main text will not be interrupted by these discussions.

Some numerical problems have been solved during the development of the subject in the chapters. These have been included to help you understand the concepts more clearly. The main concepts developed in the chapter are summarized at the end before offering you the exercises. These exercises have been divided into two groups—"Questions Based on Concepts" and "Problems". Do spend time on the first group before solving the problems. Remember, the problems are designed to improve your understanding. You may need to make several attempts to finally get the correct answer in some cases. But it will be a good investment. The more you struggle to work out a problem, the more you will learn.

There are many well-written books on this subject. I used *Introduction to Electrodynamics* by D J Griffiths and *Electricity and Magnetism* by A S Mahajan and A A Rangwala extensively while I taught at IIT Kanpur and at Patna Science College. Each author and teacher has his/her characteristic style. You get to see things from different perspectives when you read different books on the same topic. You will not only enjoy reading them but will also expand your horizon.

If you find any mistake in the book, do write to me or the publisher so that necessary action may be taken.

Finally, join me in thanking the whole team of Bharati Bhawan for the attractive design and layout, and also for improving the presentation by editing the language and putting forth many queries to me regarding clarity and correctness.

Wish you all the best and enjoy your journey through the world of electromagnetism.

H C Verma

CONTENTS

❑

1 Charges and Electric Fields

All matter is largely made up of electrons, protons and neutrons. Electrons and protons have electric charges whereas neutrons are neutral. The nature of the charges on protons is different from that on electrons. By convention, the charge on a proton is considered positive and that on an electron, negative. In most cases, positive and negative charges on an object are exactly equal and the object does not bear a net charge. However, there are forces between these charges (electrons and protons), which are responsible for the way an object behaves. Thus you can hold a pen between your fingers because of the force between the electrons and protons in your finger's skin and the electrons and protons in the body of the pen. The pen itself is intact and maintains its shape because of molecular attraction, which again results from the forces between the charges within the molecules.

Apart from material objects, the most common and important thing we encounter is light. As you know, light is electromagnetic radiation, which is a combination of an electric field and a magnetic field. Both types of field are created from charges. There are other forms of electromagnetic radiation too. Radio and television channels continuously beam electromagnetic radiation to our homes, which our radio and TV sets receive when we switch them on. Our mobile phones also send and receive electromagnetic radiation.

Thus, all of us are surrounded by lots of charges, and electric and magnetic fields. In this book, we will describe some relations between the charges and the fields, and some of their behaviour. Most of the discussion will be centred on the questions of how charges at rest and charges in motion set up electric and magnetic fields, and how electric and magnetic fields affect charges. Apart from this, we will discuss the propagation of electromagnetic waves. The entire subject is called classical electromagnetism.

We will use spherical and cylindrical coordinate systems extensively. Also, there will be some references to the Dirac delta function. In addition, we will use vector calculus in almost all the chapters. These topics are briefly discussed in Appendices 1–5.

1.1 Unit of Charge

The SI unit of charge is the coulomb. How much charge is one coulomb? To answer this question, we must examine the concept of electric current and know how much current one ampere (A) is. If two long wires carrying equal currents placed at a separation of 1 m attract each other with a force of $2\pi \times 10^{-7}$ newtons on each metre, each of the currents will be 1 A. And the charge flowing in one second through any cross section of a wire carrying a current of 1 A is called one coulomb (C). If you find this definition complicated, don't worry. A rigorously correct description of a physical quantity is often complicated. Fortunately, we can often intuitively understand the quantity even without considering a rigorous description. You can take it that a certain charge is defined as 1 C in SI units. The charge on a proton is approximately 1.6×10^{-19} C and that on an electron is of exactly the same magnitude but has a negative sign. The charge on a proton is expressed as $+e$ and that on an electron, as $-e$.

1.2 Charge is Quantized and Conserved

When you rub a plastic scale with a cloth, the scale and the cloth get charged. How much charge appears on the scale depends on how strongly and how long you rub it. You can vary the charge by rubbing it in different fashions but all the time the charge will be an integral multiple of e. You can increase or decrease the charge only in steps of e. You may wonder that if the charge q on an object can be varied only in steps of e, how can one use the methods of integral and differential calculus, which are based on limits $dq \to 0$, and so on. Strictly speaking, this objection is valid. But physicists do not always insist on exactness. They deal with realistic situations where approximations are essential. For many applications, the step size e is so small that one can treat the possible variation in charge as continuous. By rubbing the scale, we put a charge of the order of a nanocoulomb on it. A nanocoulomb means more than 6,000,000,000 units of electronic charge. If we are only trying to put charges of the order of a nanocoulomb, a millicoulomb or more, how does it matter if we are not allowed to increase or decrease charge by $e/2$ or $e/3$? In this book, we will mostly be concerned with situations where charges will either be zero or thousands of times e.

It is a wonder that all charges in the universe are multiples of e. Not only electrons and protons but all other charged particles such as positrons, mesons, etc., obey this rule. We do talk about quarks, which have charges of $\pm e/3$ or $\pm 2e/3$, but so far no one has been able to isolate them. They always occur in combinations so that an observable particle gets a charge which is an integral multiple of e. Thus a proton has three quarks, of charges $+2e/3$, $+2e/3$ and $-e/3$, meaning that the total charge on a proton is $+e$. While free protons are easily found, a free quark has not been observed so far.

Another amazing property of nature is that the net charge on an isolated system is never changed and the total charge of the universe is constant. It could well be zero, meaning that there is as much positive charge as negative charge in the universe. However, the positive charge itself or the negative charge itself is not conserved. We can produce a new positive charge which never existed anywhere and also destroy a positive charge. Similarly, we can produce a negative charge and also destroy one. But we have to create or destroy charges in pairs (each charge of a pair bearing an opposite sign to the other) so that we do not produce or destroy the net charge. The positron is a particle with charge $+e$ (and mass equal to that of an electron). When it meets an electron, often the two destroy each other and gamma ray photons, having no charge, are created. Thus the total positive charge of the universe is reduced and so is the total negative charge, but the net charge of the universe remains unchanged.

1.3 Force between Two Point Charges: Coulomb's Law

What is a point charge? Charge is always carried by a particle or a body and any particle occupies some volume. "Point" here does not mean the point one encounters in geometry, that is, something which has only location, and no length, no breadth and no height. In the situations we will discuss, a point charge will mean a small particle or object carrying a charge. How small? That depends on the other distances involved in the problem. If we are talking about the force between two charged particles, the size of the particles should be small compared to the separation between them. Only then can we call the charges point charges.

The force exerted by one point charge q_1 on another point charge q_2 placed at a separation r from it is

$$F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}.$$

It is attractive if the two charges are of opposite signs and repulsive if they are of the same sign. This is how we generally express Coulomb's law in more elementary texts. This law is complete in itself, giving the magnitude as well as the direction of the force. Let us express Coulomb's law in a little more complex fashion, using position vectors.

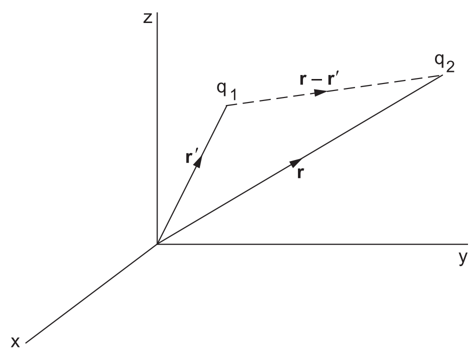


Fig. 1.1

Choose a coordinate system to describe the positions of points in terms of position vectors. Suppose the position vector of the charge q_1 is $\boldsymbol{r}'$ and that of q_2 is $\boldsymbol{r}$ (Figure 1.1). The force on charge q_2 due to charge q_1 is

$$\boldsymbol{F} = \frac{q_1 q_2 (\boldsymbol{r} - \boldsymbol{r}')}{4\pi\varepsilon_0 |\boldsymbol{r} - \boldsymbol{r}'|^3}. \tag{1.1}$$

The vector $\boldsymbol{r}-\boldsymbol{r}'$ is the one that starts from the position of q_1 and ends at q_2. Its magnitude $|\boldsymbol{r}-\boldsymbol{r}'|$ is equal to the separation between the two charges. Check that Equation 1.1 gives the correct magnitude as well as direction of the force. If q_1 and q_2 have the same sign, q_1q_2 is positive and the direction of $\boldsymbol{F}$ is the same as that of $\boldsymbol{r}-\boldsymbol{r}'$. This means that the force is repulsive. (Remember that $\boldsymbol{F}$ is the force on q_2 due to q_1.) If the signs of q_1 and q_2 are opposite, q_1q_2 is negative and the direction of $\boldsymbol{F}$ is opposite to that of $\boldsymbol{r}-\boldsymbol{r}'$, that is, the force is attractive.

What is ε_0 in Equation 1.1? It is a constant with a value very close to $8.85\times10^{-12}\ \mathrm{C^2/N\cdot m^2}$. The unit $\mathrm{C^2/N\cdot m^2}$ is the same as F/m. Here F is written for farad and is the SI unit of capacitance. The constant ε_0 is called *permittivity of vacuum*, or *permittivity of free space*.

Is Coulomb's law valid in all conditions? Not exactly. It is valid if the charges q_1 and q_2 are at rest for a sufficiently long time in the coordinate system we are using. Let us elaborate on this statement by describing an imaginary but, in principle, valid situation, described in Figure 1.2.

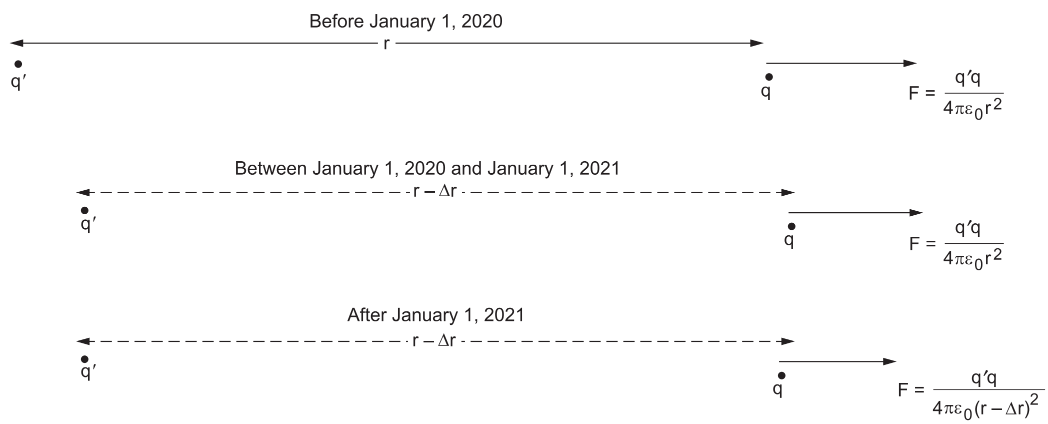


Fig. 1.2

Suppose two charges q' and q are placed (and held) at a separation $r = 1$ light year. As you know, a light year is the distance light moves in one year in vacuum. The charges are at rest for quite some time and the force on q has a magnitude of $q'q/4\pi\varepsilon_0r^2$. Suppose someone shifts q' closer to q on January 1, 2020 by a few metres, say Δr. This shifting may hardly take a few seconds. According to Coulomb's law, the force on q should increase to $q'q/4\pi\varepsilon_0(r-\Delta r)^2$. As soon as the force on q is increased, an observer at q should know that q' has been shifted. But *no information can be passed on with a speed greater than that of light in vacuum—this is a very basic law of nature*, and one of the main results of the theory of relativity. As the separation between q' and q is 1 light year, information about any event at q' cannot reach q before January 1, 2021. This means that for the whole of 2020, the force on q remains $q'q/4\pi\varepsilon_0r^2$ while the separation between the charges is $r-\Delta r$. Coulomb's law does not work too well during this period.

There is another significant aspect. Though the charge q registers the decrease in separation only after one year of shifting, the charge q' registers it immediately. So, right from January 1, 2020, when the charge q' is shifted, q' starts experiencing the larger force due to q. So for the whole of 2020, the force on q' due to q is larger in magnitude than the force on q due to q'. Newton's third law also does not work too well if you consider the force due to q on q' and due to q' on q.

Now, Newton's third law leads to the conservation of linear momentum, a principle so well respected in physics. If Newton's third law is violated, the principle of conservation of linear momentum will be in danger.

This anomaly occurs because we have assumed that a charge located at one place can directly exert force on another charge placed at a distance. This "action at a distance" viewpoint is not consistent with the principles of relativity, which have proved themselves to be unquestionable so far. We therefore introduce the field viewpoint, according to which one charge interacts with another through a physical entity, called *electric field*.

1.4 Electric Field

The force on charge q_2 placed at $\boldsymbol{r}$, due to charge q_1 placed at $\boldsymbol{r}'$, is given by the equation

$$\boldsymbol{F} = \frac{q_1 q_2 (\boldsymbol{r} - \boldsymbol{r}')}{4\pi\varepsilon_0 |\boldsymbol{r} - \boldsymbol{r}'|^3}.$$

This can be written as

$$\boldsymbol{F} = q_2 \boldsymbol{E}, \tag{1.2}$$

where

$$\boldsymbol{E} = \frac{q_1 (\boldsymbol{r} - \boldsymbol{r}')}{4\pi\varepsilon_0 |\boldsymbol{r} - \boldsymbol{r}'|^3}. \tag{1.3}$$

The quantity $\boldsymbol{E}$ is called the electric field at point $\boldsymbol{r}$ due to q_1 at $\boldsymbol{r}'$. Note that $\boldsymbol{E}$ is a function of space points. At different points $\boldsymbol{r}$, the electric field may have different values, meaning that it may have different magnitudes and different directions. Such a function of space points is, in general, called a *field*. As $\boldsymbol{E}$ itself is a vector quantity, it is called a *vector field*.

Electric field is not just a mathematical quantity. It describes an altogether different viewpoint. The force is seen here as a two-step process. The charge q_1 produces an electric field $\boldsymbol{E}$ according to Equation 1.3 and this electric field exerts a force on q_2, which is immersed in the field at $\boldsymbol{r}$. Even if the charge q_2 is not there, the field due to q_1 exists there. Think of the electric field as a physical entity which can exist in space and exert force on any charge placed in it. Having a magnitude and a direction, it is a vector quantity. So a charge q' placed at point $\boldsymbol{r}'$

establishes its electric field everywhere in space, independent of whether or not there are other charges. If the charge is stationary and has been there for a long time, the field at $\boldsymbol{r}$ due to it is given by

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{q'(\boldsymbol{r}-\boldsymbol{r}')}{4\pi\varepsilon_0|\boldsymbol{r}-\boldsymbol{r}'|^3}. \tag{1.4}$$

The meaning of "long time" should be clearly understood. If the distance involved in a given situation (here $|\boldsymbol{r}-\boldsymbol{r}'|$) is of the order of r_0, any effect of a change in the source position will be felt at the other position in time r_0/c. So if the charge is stationary for a time longer than r_0/c, we can assume that it is stationary for a long time. The situation in which all the charges producing an electric field are stationary for a long time is called *electrostatics*, and the electric field produced thus is called an *electrostatic field*. When a charge q is placed at a point where the electric field due to all other charges is $\boldsymbol{E}$, the force on q is given by $\boldsymbol{F} = q\boldsymbol{E}$. Note that the field created by a charge does not exert force on itself.

1.5 Electric Field due to a Charge Distribution

Discrete charge distribution

Equation 1.4 gives the value of the electric field at a point $\boldsymbol{r}$ due to a single point charge q placed at $\boldsymbol{r}'$. If several point charges $q'_1, q'_2, q'_3, \ldots$ are placed at positions $\boldsymbol{r}'_1, \boldsymbol{r}'_2, \boldsymbol{r}'_3, \ldots$ respectively, we say that this group of charges has a *discrete charge distribution* (Figure 1.3). Why do we say this charge distribution is discrete? Because we can label the charges as the first charge, the second charge, and so on. We can count them. The charges are discretely placed. Each of the charges in this distribution creates its own electric field according to Equation 1.4 and at any point $\boldsymbol{P}$, all these fields add vectorially to give the net electric field. This is called the principle of superposition. Superposition is not observed in case of all physical quantities. In Young's double-slit experiment, for example, the intensity at a given point on the screen due to the two slits is not equal to the sum of the intensities at that point due to the individual slits. The electric field due to charges, however, follows the principle of superposition. Thus the field at point $\boldsymbol{P}$, having position vector $\boldsymbol{r}$, due to this charge distribution is

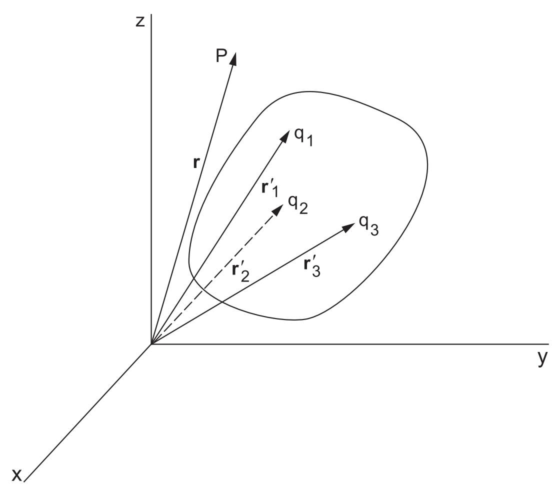


Fig. 1.3

$$\boldsymbol{E}(\boldsymbol{r}) = \sum_i \frac{q_i'(\boldsymbol{r} - \boldsymbol{r}_i')}{4\pi\varepsilon_0 |\boldsymbol{r} - \boldsymbol{r}_i'|^3}. \tag{1.5}$$

You can see that we need two kinds of position vectors—one for the given charge distribution and the other for the position where the field is needed. We have used dashed arrows for the positions of charges and solid arrows for the field points. Whenever we need to use these two kinds of coordinates, we will use this convention.

EXAMPLE 1.1 **Three charges, q each, are placed at (0, 0, 0), (a, 0, 0) and (0, a, 0). Find the electric field at the point (0, 0, a) due to this charge distribution.**

Solution The situation is described in Figure 1.4. We have to find the field at P. The position vector of P is $\boldsymbol{r} = 0\hat{\boldsymbol{i}} + 0\hat{\boldsymbol{j}} + a\hat{\boldsymbol{k}}$.

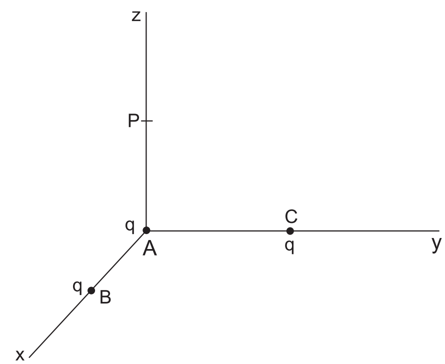


Fig. 1.4

Also, $\boldsymbol{r}_1' = 0\hat{\boldsymbol{i}} + 0\hat{\boldsymbol{j}} + 0\hat{\boldsymbol{k}}$, $\boldsymbol{r}_2' = a\hat{\boldsymbol{i}} + 0\hat{\boldsymbol{j}} + 0\hat{\boldsymbol{k}}$

and $\boldsymbol{r}_3' = 0\hat{\boldsymbol{i}} + a\hat{\boldsymbol{j}} + 0\hat{\boldsymbol{k}}$.

Thus $\boldsymbol{r} - \boldsymbol{r}_1' = a\hat{\boldsymbol{k}}$, $\boldsymbol{r} - \boldsymbol{r}_2' = -a\hat{\boldsymbol{i}} + a\hat{\boldsymbol{k}}$,

$\boldsymbol{r} - \boldsymbol{r}_3' = -a\hat{\boldsymbol{j}} + a\hat{\boldsymbol{k}}$

and $|\boldsymbol{r} - \boldsymbol{r}_1'| = a$, $|\boldsymbol{r} - \boldsymbol{r}_2'| = \sqrt{2}\,a$

and $|\boldsymbol{r} - \boldsymbol{r}_3'| = \sqrt{2}\,a$.

The electric field at P due to this charge distribution is

$$\begin{aligned}
\boldsymbol{E} &= \sum_i \frac{q_i'(\boldsymbol{r} - \boldsymbol{r}_i')}{4\pi\varepsilon_0 |\boldsymbol{r} - \boldsymbol{r}_i'|^3} \\
&= \frac{q}{4\pi\varepsilon_0}\left[\frac{a\hat{\boldsymbol{k}}}{a^3} + \frac{-a\hat{\boldsymbol{i}} + a\hat{\boldsymbol{k}}}{(\sqrt{2}\,a)^3} + \frac{-a\hat{\boldsymbol{j}} + a\hat{\boldsymbol{k}}}{(\sqrt{2}\,a)^3}\right] \\
&= \frac{q}{4\pi\varepsilon_0 a^2}\left[-\frac{1}{2\sqrt{2}}\hat{\boldsymbol{i}} - \frac{1}{2\sqrt{2}}\hat{\boldsymbol{j}} + \left(1 + \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\right)\hat{\boldsymbol{k}}\right] \\
&= \frac{q}{8\sqrt{2}\,\pi\varepsilon_0 a^2}[-\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2(1 + \sqrt{2})\hat{\boldsymbol{k}}].
\end{aligned}$$

Continuous charge distributions

Say a metallic ball has some charge on it. The charge spreads over the entire surface of the ball. Take any small area on the surface and there will be some charge on this area. You cannot label the charges as the first charge, the second charge, and so on. You cannot express them as $q_1, q_2, q_3, \ldots$. You cannot count them. This is an example of a continuous charge distribution.

(Remember, by a small area, we mean an area small compared to other lengths relevant to the situation, but large enough to have thousands of atoms.) We have three basic varieties of continuous charge distribution—linear, surface and volume.

Linear charge distribution

Suppose some charge is distributed along the length of a thin wire or thread (*ABC* in Figure 1.5). We have to find the electric field due to this charge distribution at a point P. The thickness of the wire is small compared to the distance of any portion of the wire from P. Thus the wire may be treated as a line and the charge may be assumed to be distributed along this line.

Such a charge distribution is called a *linear charge distribution*. Consider a small portion of the wire of length $\boldsymbol{dl'}$ situated at the position vector $\boldsymbol{r'}$. Let the charge on this portion of the wire be dq'. Then

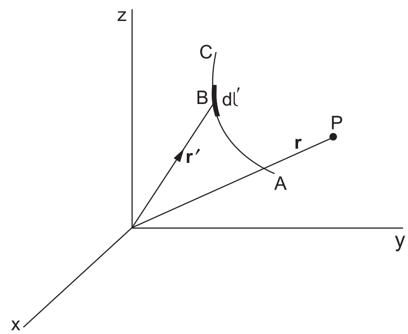


Fig. 1.5

$$\lambda(\boldsymbol{r'}) = \frac{dq'}{dl'} \tag{1.6}$$

is called the *linear charge density* at $\boldsymbol{r'}$. Obviously this is defined only for points $\boldsymbol{r'}$ of the wire and dl' must be taken along the wire. You can express the charge dq on $\boldsymbol{dl}$ as $dq' = \lambda(\boldsymbol{r'})\,dl'$.

How do we express the electric field **E** at P due to this linear charge distribution? We can use Equation 1.5, which is relevant to a discrete charge distribution. In place of the point charge q_i', write the charge dq' or $\lambda(\boldsymbol{r'})dl'$ and in place of summation over $\boldsymbol{i}$, use integration over the whole length of the linear charge distribution. The position $\boldsymbol{r}_i'$ is replaced by $\boldsymbol{r'}$.

Thus the field at P is

$$\boldsymbol{E}(\boldsymbol{r}) = \int_{\text{line}} \frac{\lambda(\boldsymbol{r'})(\boldsymbol{r}-\boldsymbol{r'})\,dl'}{4\pi\varepsilon_0|\boldsymbol{r}-\boldsymbol{r'}|^3}\,. \tag{1.7}$$

See Appendix 1 if you have problems using line integration. You will also find descriptions of surface and volume integrals, which we will use shortly.

Surface charge distribution

Next, consider charge spread over a surface. When we say that the charge is distributed on a surface, we do not mean the surface of zero thickness that one comes across in mathematics. Indeed it is a material that gets charged and there will be some thickness of the charged layer. But that thickness is so small that you need not look into what happens inside it. Thus we assume that the charge exists only on the surface.

How do we describe such a surface charge distribution? Look at Figure 1.6. Take a small area da' on the surface at the position vector $\boldsymbol{r}'$. Let the charge on this small area be dq'. Then

$$\sigma(\boldsymbol{r}') = \frac{dq'}{da'} \tag{1.8}$$

is called the *surface charge density* at $\boldsymbol{r}'$. Obviously it is defined only for points $\boldsymbol{r}'$ on the given surface. The charge dq' on the area da' is $dq' = \sigma(\boldsymbol{r}')da'$.

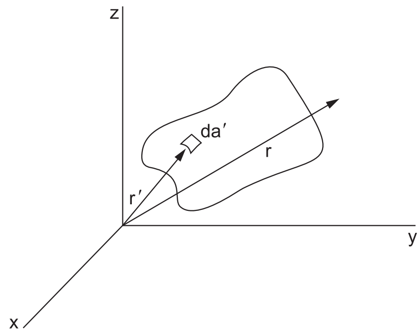


Fig. 1.6

We can express the electric field due to this surface charge distribution by taking a clue from Equation 1.5. In place of q_i', we will have $dq' = \sigma(\boldsymbol{r}')da'$, in place of summation over i, we will have integration over the entire surface containing the charge, and in place of $\boldsymbol{r}_i'$, we will have just $\boldsymbol{r}'$. So, the field at position $\boldsymbol{r}'$ due to this distribution will be

$$\boldsymbol{E}(\boldsymbol{r}) = \int_{\text{surface}} \frac{\sigma(\boldsymbol{r}')(\boldsymbol{r}-\boldsymbol{r}')da'}{4\pi\varepsilon_0|\boldsymbol{r}-\boldsymbol{r}'|^3}. \tag{1.9}$$

Volume charge distribution

Finally, suppose charge is distributed in a given volume (Figure 1.7). To describe such a distribution, take a small volume element $d\tau'$ at the position vector $\boldsymbol{r}'$. If the charge contained in this small volume is dq', the quantity

$$\rho(\boldsymbol{r}') = \frac{dq'}{d\tau} \tag{1.10}$$

is called the volume *charge density* at $\boldsymbol{r}'$. The charge in this element is $dq' = \rho(\boldsymbol{r}')d\tau'$. The electric field at the position $\boldsymbol{r}$ due to this volume charge distribution may be expressed in the same manner as we did for linear and surface charge distributions. This time, the expression will be

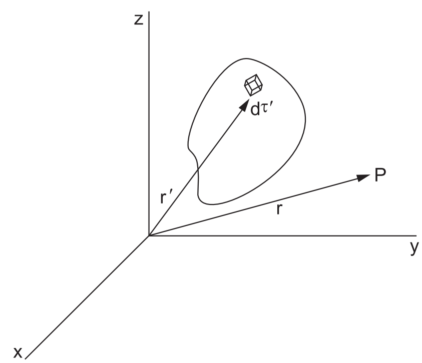


Fig. 1.7

$$\boldsymbol{E}(\boldsymbol{r}) = \int_{\text{volume}} \frac{\rho(\boldsymbol{r}')(\boldsymbol{r}-\boldsymbol{r}')d\tau'}{4\pi\varepsilon_0|\boldsymbol{r}-\boldsymbol{r}'|^3}. \tag{1.11}$$

In fact, all charge distributions are volume charge distributions. Be it a discrete, line or surface charge distribution, the charge will occupy some volume. But, depending on the geometry, sometimes we approximate the charges to point charges, line charges or surface charges. If you are familiar with the Dirac delta function, you can express any charge distribution as a volume charge distribution. Thus a point charge q' at a point $\boldsymbol{r}'$ can be described by

$\rho(\boldsymbol{r}') = q'\delta^3(\boldsymbol{r}-\boldsymbol{r}')$ and a collection of point charges q_i' at positions $\boldsymbol{r}_i'$ as $\rho(\boldsymbol{r}') = \Sigma q_i' \delta^3(\boldsymbol{r}-\boldsymbol{r}_i')$. A line charge density $\lambda(\boldsymbol{r}')$, for example, along the z-axis may be described as a volume charge density $\rho(\boldsymbol{r}') = \lambda(z')\delta(x')\delta(y')$. A surface charge density $\sigma(\boldsymbol{r}')$ on the x–y plane can be written as a volume charge density $\rho(\boldsymbol{r}') = \sigma(x', y')\delta(z')$.

A surface charge density $\sigma(\theta, \phi)$ on a spherical surface of radius R, centred on the origin, can be written as a volume charge density $\rho(\boldsymbol{r}') = \sigma(\theta', \phi')\delta(r'-R)$ in spherical polar coordinates. If you are not familiar with spherical coordinates and cylindrical coordinates, go through Appendix 2.

If you are not familiar with the Dirac delta function, don't worry. Simply understand that all charge distributions are volume charge distributions. Thus, most of the time we will derive general results in terms of volume charge distributions. These can be reformed for surface, linear or discrete distributions by simple substitutions such as $\rho d\tau' \to \sigma da' \to \lambda dl' \to q_i'$. We have briefly discussed the Dirac delta function in Appendix 3.

EXAMPLE 1.2 **Charge is distributed on a circular ring of radius *R*, made of plastic, placed on the *x*–*y* plane. The charge density at a point where the radius makes an angle ϕ' with the *x*-axis is $\lambda(\phi') = \lambda_0 \sin\phi'$, where λ_0 is a positive constant. Find the electric field due to this charge distribution at the point $(0, 0, z_0)$.**

Solution The charge distribution is qualitatively shown in Figure 1.8. At $\phi' = 0, \sin\phi' = 0$ and $\lambda = 0$. For $0 < \phi' < \pi, \sin\phi'$ is positive and so the charge is also positive. For $\pi < \phi' < 2\pi$, the charge is negative. The linear charge density has maximum magnitude at $\phi' = \pi/2$ and $3\pi/2$. This is a linear charge distribution. To take the element dl', locate the point A on the ring where the radius makes an angle ϕ' with the x-axis. Now vary ϕ' to $\phi' + d\phi'$ to get an element AB of length $dl' = Rd\phi'$. The linear charge density at this element is $\lambda(\phi') = \lambda_0 \sin\phi'$. The coordinates of the point A are $(R\cos\phi', R\sin\phi', 0)$ and so its position vector is

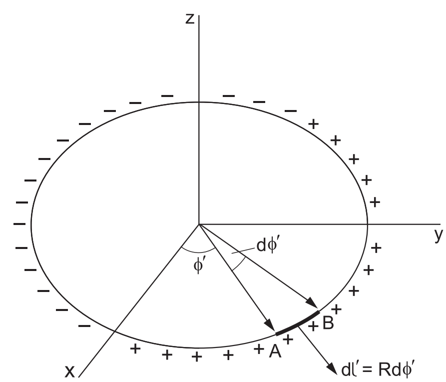


Fig. 1.8

$$\boldsymbol{r}' = R\cos\phi'\hat{\boldsymbol{i}} + R\sin\phi'\hat{\boldsymbol{j}}.$$

We have to find the field at the point $P(0, 0, z_0)$. The position vector of this point is

$$\boldsymbol{r} = z_0\hat{\boldsymbol{k}}.$$

Thus $\boldsymbol{r}-\boldsymbol{r}' = -R\cos\phi'\hat{\boldsymbol{i}} - R\sin\phi'\hat{\boldsymbol{j}} + z_0\hat{\boldsymbol{k}}$ and $|\boldsymbol{r}-\boldsymbol{r}'| = (R^2 + z_0^2)^{1/2}$. The electric field at P is, therefore,

$$\boldsymbol{E} = \int_{\text{line}} \frac{\lambda(\boldsymbol{r}')(\boldsymbol{r}-\boldsymbol{r}')\,dl'}{4\pi\varepsilon_0|\boldsymbol{r}-\boldsymbol{r}'|^3}$$

$$= \int_0^{2\pi} \frac{\lambda_0 \sin\phi'\,(-R\cos\phi'\,\hat{\boldsymbol{i}} - R\sin\phi'\,\hat{\boldsymbol{j}} + z_0\hat{\boldsymbol{k}})\,R\,d\phi'}{4\pi\varepsilon_0(R^2+z_0^2)^{3/2}}$$

$$= \frac{\lambda_0 R}{4\pi\varepsilon_0(R^2+z_0^2)^{3/2}}\left[\left(-R\int_0^{2\pi}\sin\phi'\cos\phi'\,d\phi'\right)\hat{\boldsymbol{i}} + \left(-R\int_0^{2\pi}\sin^2\phi'\,d\phi'\right)\hat{\boldsymbol{j}} + \left(z_0\int_0^{2\pi}\sin\phi'\,d\phi'\right)\hat{\boldsymbol{k}}\right]$$

$$= \frac{\lambda_0 R}{4\pi\varepsilon_0(R^2+z_0^2)^{3/2}}(-\pi R\hat{\boldsymbol{j}}) = -\frac{\lambda_0 R^2}{4\varepsilon_0(R^2+z_0^2)^{3/2}}\hat{\boldsymbol{j}}.$$

EXAMPLE 1.3 **A hemisphere of radius R is placed with its centre at the origin and the rim on the x–y plane. The hemisphere is on the positive z side as suggested in Figure 1.9. Charge is distributed on its surface with surface charge density $\sigma(\theta) = \sigma_0\cos\theta$, where θ is the angle made by the radius with the z-axis at the given position of the surface. Find the electric field at the origin due to this charge distribution.**

Solution The surface can be best described in terms of spherical polar coordinates. Any point on the surface will have spherical coordinates (r, θ, ϕ), where $r = R$, θ can take values from 0 to $\pi/2$, and ϕ can vary from 0 to 2π. A small surface element at (r, θ, ϕ) can be constructed by varying θ from θ to $\theta + d\theta$, and ϕ from ϕ to $\phi + d\phi$. This gives an elementary surface area $da = (R\,d\theta)(R\sin\theta\,d\phi)$. The position vector of the point (r, θ, ϕ) where this elementary surface area is constructed is given by

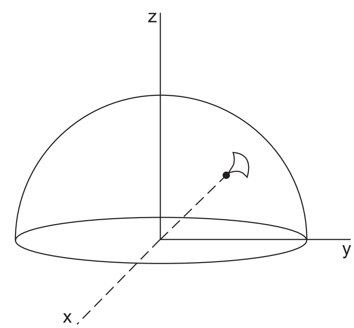


Fig. 1.9

$$\boldsymbol{r}' = R\sin\theta\cos\phi\,\hat{\boldsymbol{i}} + R\sin\theta\sin\phi\,\hat{\boldsymbol{j}} + R\cos\theta\,\hat{\boldsymbol{k}}.$$

The field is needed at the origin, which has the position vector $\boldsymbol{r} = 0$. Thus,

$$\boldsymbol{r}-\boldsymbol{r}' = -R\sin\theta\cos\phi\,\hat{\boldsymbol{i}} - R\sin\theta\sin\phi\,\hat{\boldsymbol{j}} - R\cos\theta\,\hat{\boldsymbol{k}}, \text{ and } |\boldsymbol{r}-\boldsymbol{r}'| = R.$$

The field at the origin is

$$\boldsymbol{E} = \int_{\text{surface}} \frac{\sigma(\boldsymbol{r}')(\boldsymbol{r}-\boldsymbol{r}')\,da}{4\pi\varepsilon_0\,|\boldsymbol{r}-\boldsymbol{r}'|^3}$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} \frac{(\sigma_0\cos\theta)[-R\sin\theta\cos\phi\,\hat{\boldsymbol{i}} - R\sin\theta\sin\phi\,\hat{\boldsymbol{j}} - R\cos\theta\,\hat{\boldsymbol{k}}]R^2\sin\theta\,d\theta\,d\phi}{4\pi\varepsilon_0 R^3}$$

$$= \frac{-\sigma_0}{4\pi\varepsilon_0}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}[\sin^2\theta\cos\theta\cos\phi\,\hat{\boldsymbol{i}} + \sin^2\theta\cos\theta\sin\phi\,\hat{\boldsymbol{j}} + \cos^2\theta\sin\theta\,\hat{\boldsymbol{k}}]\,d\theta\,d\phi.$$

Integrating the first term, we get

$$\int_{\theta=0}^{\pi/2}\sin^2\theta\cos\theta\,d\theta\int_0^{2\pi}\cos\phi\,d\phi = 0.$$

And the second term may be integrated as follows.

$$\int_{\theta=0}^{\pi/2} \sin^2\theta \cos\theta \, d\theta \int_0^{2\pi} \cos\phi \, d\phi = 0.$$

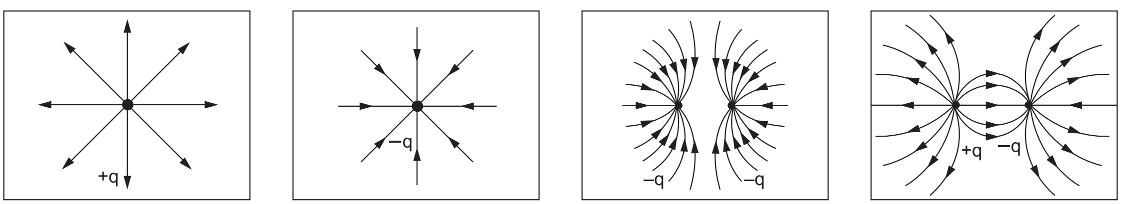
Thus,

$$\boldsymbol{E} = \frac{-\sigma}{4\pi\varepsilon_0}\left[\int_{\theta=0}^{\pi/2} \cos^2\theta \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi\right]\hat{\boldsymbol{k}} = \frac{-\sigma}{4\pi\varepsilon_0}\left|-\frac{\cos^3\theta}{3}\right|_{\theta=0}^{\pi/2} \cdot 2\pi\hat{\boldsymbol{k}} = \frac{-\sigma}{6\varepsilon_0}\hat{\boldsymbol{k}}.$$

You could have argued by looking at the charge distribution itself that the electric field at the origin should be in the $-\hat{\boldsymbol{k}}$ direction. The charge density $\sigma = \sigma_0 \cos\theta$ is independent of ϕ. Think of the hemispherical surface as made of rings parallel to the *x–y* plane. Each ring will have a uniform charge density. The field at the origin from each such ring will be in the $-\hat{\boldsymbol{k}}$ direction, and hence the net field will also be in this direction.

1.6 Electric Field Lines

Electric field $\boldsymbol{E}(\boldsymbol{r})$ is a vector field. At any given point it has a magnitude and a direction. At different points, $\boldsymbol{E}$ can have different magnitudes and different directions. The variation of electric field can be qualitatively shown by a geometrical construction in terms of what we call *electric field lines*. An electric field line is drawn in such a way that the tangent drawn to this line from any point on it gives the direction of the electric field at that point. We have shown electric field lines for some charge configurations in Figure 1.10.



(a) A positive point charge

(b) A negative point charge

(c) Two point charges, both negative

(d) Two point charges, one positive and one negative

Fig. 1.10

1.7 Electric Field without Any Charge Distribution

Electric charges produce an electric field in space and in this chapter, we have considered many charge distributions and calculated the electric field due to each of them. However, electric charges are not the only source of an electric field. If you shake a bar magnet, an electric field is produced although it is electrically neutral everywhere. If kept stationary, the same magnet does not produce any electric field. Indeed, the main job of a magnet is to produce another kind of field called magnetic field ***B***, which we will discuss in detail in other chapters. When we move a magnet, the magnetic field ***B*** around it changes. The changing magnetic field produces an electric field.

So, an electric field can be produced either by electric charges or by a changing magnetic field. Note that a magnetic field is created by charges in motion. Thus the ultimate cause of a changing magnetic field and the corresponding electric field is a charge in motion.

Concepts at a Glance

1. Electric charge is quantized. Quarks have charges of plus or minus $e/3$, $2e/3$, but they combine to form observable particles having charge 0, or in units of e. The electron has a charge of $-e$.
2. Charge cannot be created or destroyed, but charged particles can be created or destroyed.
3. Charges produce an electric field. A charge placed in an electric field due to other charges experiences a force due to this electric field.
4. Coulomb's law and the corresponding equation for the electric field are valid for electrostatic conditions, in which the charges are assumed to be at rest for a long time.
5. The most general charge distribution is volume charge distribution. Point, line and surface charge distributions can be expressed as volume charge distributions in terms of the Dirac delta function.

EXERCISES

Based on Concepts

1. Give an example in which an electron is created, that is, manufactured afresh. Give an example in which an electron is destroyed.
2. A charge A is placed at the origin and another charge B, at a distance of 60 cm from it. If A is shifted quickly to a nearby point, how much time will pass before B starts experiencing the changed force?
3. A charge is brought from somewhere and kept at the origin at $t = 0$.
   (a) What will be the electric field at the point (3 cm, 4 cm, 0) a long time after the charge is kept at the origin?
   (b) At what time after putting the charge at the origin will the electric field at (3 cm, 4 cm, 0) be that given by the expression you have obtained in part (a)?
4. It is known that any charge given to a metallic sphere spreads uniformly on its outer surface. Suppose a charge of 1 femtocoulomb is given to a metallic sphere of radius 20 cm. If we look at a part of the sphere having area 1 mm^2, what will be the charge on it? Can we still say that the charge is uniformly spread over the whole surface?

5. A positive charge q is placed at the origin. Give the electric field due to this charge at a general point $\boldsymbol{r}$ in Cartesian, cylindrical and spherical polar coordinates.

Problems

1. Find the total positive charge on all the protons contained in 2 g of hydrogen gas. [**Ans.** 1.92×10^5 C]
2. Two charges, of 5 nC and 10 nC, are placed at (10 cm, 10 cm, 10 cm) and (20 cm, 20 cm, 20 cm) respectively. Find the electric field at the origin due to this charge pair. [**Ans.** $-1.1 \times 10^{-5} (\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$ N/C]
3. Charges of 1, 2, 3 and 4 nanocoulombs are placed respectively at the four corners A, B, C and D of a square of side 20 cm. Take AB as the x-axis and AD as the y-axis. Find the electric field at the centre of the square. [**Ans.** $-900\sqrt{2}\,\hat{\boldsymbol{j}}$ N/C]
4. Two particles, each having charge q, are placed at a separation. What charge Q should be placed at the midpoint between them, so that all the three may remain in equilibrium? Is this a stable equilibrium? [**Ans.** $-q/4$]
5. A charge of 2.0 nC is spread uniformly on a rod of length 1 cm. A very small particle having a charge of 4.0 nC is kept on the perpendicular bisector of the rod, at a distance of 5 cm from its centre. (a) Find the force on this particle. (b) If we assume all the charge on the rod to be concentrated at its centre to calculate the force on the particle, what percentage error will we make? (c) Repeat parts (a) and (b) if the particle is kept at a distance of 20 cm from the centre of the rod. [**Ans.** (b) 0.47%]
6. A thin rod of length L contains a uniform linear charge density λ. Find the electric field at a point on the perpendicular bisector at a distance (a) $z = L/4$, (b) $z = L$, and (c) $z = 4L$. In each case, what will be the percentage error if we use $E = \lambda/(2\pi\epsilon_0 z)$? [**Ans.** (a) 11% (b) 30% (c) 76%]
7. A rod of length L, kept along the x-axis, has a charge Q uniformly distributed on it. Let P be a point in the x–y plane at a distance L from the midpoint O of the rod, OP making an angle $\theta = 60^\circ$ with the rod. Find the expression for the electric field at P. [**Ans.** $\dfrac{2Q}{4\sqrt{21}\,\pi\varepsilon_0 L^2}[\sqrt{7} - \sqrt{3}\,\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{i}}]$]
8. Consider the situation given in the previous problem. (a) It is given that the magnitude of the electric field is minimum for only one value of θ in the range $0 < \theta < \pi$. Find the value of this minimum and the corresponding θ. (b) It is given that the magnitude of the electric field is maximum for only one value of θ in the range $0 < \theta < \pi$. Find the value of this maximum and the corresponding θ. [**Ans.** (a) $\dfrac{2Q}{4\pi\varepsilon_0\sqrt{5}\,L^2}$ (b) $\dfrac{Q}{3\pi\varepsilon_0 L^2}$]
9. A charge q is uniformly distributed over the length $-a < z < a$ along the z-axis. Find the electric field at the point $(0, a, a)$ due to this charge distribution. [**Ans.** $\dfrac{Q}{\pi\varepsilon_0 a^2}\left\{\dfrac{2}{\sqrt{5}}\hat{\boldsymbol{j}} + \left(1 - \dfrac{1}{\sqrt{5}}\right)\hat{\boldsymbol{k}}\right\}$]
10. A rod of length L is given a charge with uniform linear charge density λ. Consider a small portion of the rod, of length δ, situated at a distance $L/4$ from the centre. Find the electrostatic force on this portion due to the rest of the rod. What balances this force? [**Ans.** $\dfrac{2\lambda}{3\pi\varepsilon_0 L}$]

11. A charge is distributed along the z-axis with a linear charge density

$$\lambda = \lambda_0\left(1 - \frac{|z|}{z_0}\right) \text{ for } |z| < z_0, \text{ and } \lambda = 0 \text{ for } |z| > z_0.$$

Find the electric field at $(a, 0, 0)$. [**Ans.** $\dfrac{\lambda_0\sqrt{z_0^2 + a^2}}{2\pi\varepsilon_0 z_0 a}$]

12. A wire bent in the form of a semicircle of radius R contains a uniform linear charge density λ. Find the magnitude of the electric field at the centre of the circle. [**Ans.** $\dfrac{\lambda}{2\pi\varepsilon_0 R}$]

13. A plastic ring of radius R has a charge Q, uniformly spread over one semicircular portion, and $-Q$, uniformly spread over the other semicircular portion. Find the magnitude of the electric field at a point on the axis of the ring at a distance z from the centre. [**Ans.** $\dfrac{QR}{\pi^2\varepsilon_0(R^2 + z^2)^{3/2}}$]

14. A square plate of edge a is placed with its centre at the origin and the edges parallel to the x- and y-axes. The plate carries a charge with surface charge density $\sigma = k|y|$.
 (a) What is the SI unit of k?
 (b) Find the electric field at the origin.
 (c) Find the electric field at $(0, 0, \sqrt{2}a)$. [**Ans.** (c) $\dfrac{\sqrt{2}ak}{\pi\varepsilon_0}\ln\dfrac{\sqrt{20} - \sqrt{2}}{3}\hat{k}$]

15. A semicircular annular disk of inner and outer radii R_1 and R_2 carries a uniform surface charge density σ on one surface (Figure 1E.1). Find the electric field at the centre O. [**Ans.** magnitude $= \dfrac{\sigma}{2\pi\varepsilon_0}\ln\dfrac{R_2}{R_1}$]

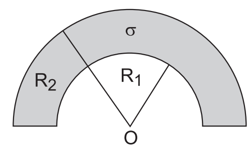


Fig. 1E.1

16. A disk of radius R carries a surface charge density that varies as $\sigma(r) = \sigma_0 \cos\phi$ as shown in Figure 1E.2. Find the electric field at a distance z from the centre on the axis of the disk where $z >> R$. [**Ans.** $-\dfrac{\sigma_0 R^3}{12\varepsilon_0 z^3}\hat{i}$]

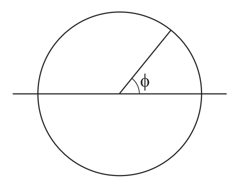


Fig. 1E.2

17. A charge is distributed on one surface of a disk of radius R with surface charge density $\sigma(s, \varphi) = -\sigma_0 s \cos\phi$. Here s and ϕ are cylindrical coordinates with the origin at the centre of the disk and the z-axis along the axis of the disk. Find (a) the total charge on the disk, and (b) the electric field at the centre. [**Ans.** (a) zero (b) $\dfrac{R\sigma_0}{4\varepsilon_0}\hat{i}$]

18. The charge density on a spherical surface is given by $\sigma = \sigma_0 \cos\theta$ in spherical polar coordinates. Find the charge on the hemisphere given by (a) $\theta < \pi/2$, and (b) $\theta > \pi/2$. [**Ans.** (a) $\pi\sigma_0 R^2$ (b) $-\pi\sigma_0 R^2$]

19. Consider a part of a spherical surface of radius R, centred on the origin, given by $\theta = 5\pi/6$ to $\theta = \pi$. Suppose it has a uniform surface charge density σ_0. Find the electric field at the origin due to this charge. [**Ans.** $\dfrac{\sigma}{24\varepsilon_0}$]

20. A charge is distributed on the surface of a sphere of radius R with charge density $\sigma = \sigma_0 \sin^2\phi$ in the spherical polar coordinate system with the origin at the centre of the spherical surface. Find (a) the total charge on the surface, and (b) the electric field at the centre of the surface.
[**Ans.** (a) $2\sigma_0 \pi R^2$ (b) Zero]

21. Find the total charge contained in the sphere $\boldsymbol{r} = \boldsymbol{r}_0$ if the charge density is given by $\rho(\boldsymbol{r}) = kr^2 \sin^2\phi$, using spherical polar coordinates. [**Ans.** $\frac{2\pi k r_0 s}{5}$]

22. A hemispherical volume has charge distributed according to the charge density $\rho = Ar\sin\theta$ in spherical polar coordinates with the centre as the origin and the symmetry axis as the z-axis. Here A is a constant. (a) Give the SI unit of A. (b) Find the electric field at the centre. [**Ans.** (b) $\frac{AR^2}{12\varepsilon_0}$]

23. Charge is packed in a right circular cone with semivertical angle 60° and height h. The charge density is uniform and has a value of ρ_0. Find the magnitude of the electric field at the vertex of the cone.
[**Ans.** $\frac{\rho_0 h}{4\varepsilon_0}$]

24. A cylindrical surface of radius R and length $\sqrt{15}R$ has a uniform surface charge density of σ. Find the magnitude of the electric field at the centre of a circular periphery. [**Ans.** magnitude $= \frac{3\sigma}{8\varepsilon_0}$]

25. A cylinder of length L and radius $R = L/2$ is cut into half by a plane parallel to its length, as shown in Figure 1E.3. The half cylinder is given a uniform charge density ρ. Find the electric field at the centre of the plane face. [**Ans.** $\frac{PR}{\pi\varepsilon_0}\ln(1+\sqrt{2})$]

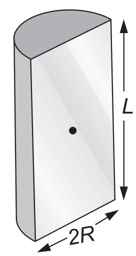


Fig. 1E.3

26. A semi-infinite wire along the z-axis and in the lower half carries a charge λ per unit length. Find the magnitude and direction of the electric field at a point in the x–y plane. [**Ans.** $\frac{\lambda}{4\pi\varepsilon_0 s}(\hat{s} + \hat{k})$]

27. Write the volume charge density corresponding to (i) a line charge along the z-axis with linear charge density λ, (ii) a charge q distributed in a circular loop of radius a, in the x–y plane with centre at the origin, (iii) a large plane surface, on the x–y plane, carrying a uniform surface charge density σ_0, and (iv) a surface charge density σ_0 at the cylindrical surface $s = R$.
[**Ans.** (i) $\lambda\delta(x)\delta(y)$ or $\frac{\lambda}{2\pi s}\delta(s)$, (ii) $\frac{q}{2\pi a}\delta(s-R)\delta(z)$, (iii) $\sigma_0\delta(z)$, (iv) $\sigma_0\delta(s-R)$]

28. (a) What is the volume charge density of an electric dipole consisting of a point charge $-q$ at (0, 0, $-a$) and $+q$ at (0, 0, a)? (b) What is the volume charge density of a uniform, infinitesimally thin spherical shell of radius R and total charge Q, centred on the origin?
[**Ans.** (a) $-q\delta(\boldsymbol{r}+a\hat{\boldsymbol{i}})+q\delta(\boldsymbol{r}-\boldsymbol{a})$, (b) $\frac{Q}{2\pi R^2}\delta(r-R)$]

❑

2 Gauss's Law

Electric charges produce an electric field. You know that the electric field produced at a point $\boldsymbol{r}$ due to a given charge distribution can be expressed as

$$\boldsymbol{E}(\boldsymbol{r}) = \int \frac{\rho(\boldsymbol{r}')\,d\tau'}{4\pi\varepsilon_0|\boldsymbol{r}-\boldsymbol{r}'|^3},$$

where $\rho(\boldsymbol{r}')$ is the volume charge density at $\boldsymbol{r}'$. This equation relates the charge distribution to the resulting electric field. The integration has to be performed over the volume containing all the charges. This is the general form of what we can call Coulomb's law for an electric field. Point charges, line charges and surface charges can also be expressed as volume charge distributions with the appropriate use of delta functions, and this equation can be used.

There is yet another beautiful relation between the electric field and the charge distribution, called *Gauss's law*. While Coulomb's law gives you a method of obtaining the electric field if the charge distribution is known, Gauss's law gives you a method of calculating the charge distribution if the electric field is known. In some cases you can also use Gauss's law to calculate the electric field from a given charge distribution and when you do this, the method turns out to be easier than the one using Coulomb's law. So what is this magical Gauss's law? Understanding it will require a knowledge of surface integration and divergence of a vector field. Perhaps you have studied these concepts in your mathematics course. If you haven't, you can refer to Appendices 1 and 4 before proceeding further.

2.1 Gauss's Law

Suppose the electric field at the position $\boldsymbol{r}$ is given by the function $\boldsymbol{E}(\boldsymbol{r})$. It is a vector field and we can calculate the divergence function $\nabla \cdot \boldsymbol{E}$ at any given point $\boldsymbol{r}$**.** Gauss's law states that this value will be the same as the volume charge density existing at that same point $\boldsymbol{r}$**,** divided by ε_0.

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}\cdot \tag{2.1}$$

Remember that both $\nabla \cdot \boldsymbol{E}$ and ρ are functions of space points. These are scalar quantities and hence are scalar fields. Evaluate each of them at the same point and Equation 2.1 will be valid.

How do we get this equation? In the formal treatment of electricity and magnetism, this equation is considered to be one of the basic laws of nature, not to be derived from anything else. However, we can pretend to derive it from Coulomb's law.

Consider a point charge q placed at the origin, and think of the electric field due to this charge alone. Then $\rho(\boldsymbol{r})$ is zero everywhere except at the origin. Since we have assumed a point charge, the charge density is infinity at the origin. In fact, the charge distribution may be represented by $\rho(\boldsymbol{r}) = q\delta^3(\boldsymbol{r})$. The electric field at a point P with position vector $\boldsymbol{r}$ due to the charge q at the origin is

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}},$$

using spherical polar coordinates. The unit vector $\hat{\boldsymbol{r}}$ gives the radially outward direction. The components in the $\hat{\boldsymbol{r}}, \hat{\theta}, \hat{\phi}$ directions are

$$E_r = \frac{q}{4\pi\varepsilon_0 r^2},$$

$$E_\theta = 0,$$

and $$E_\phi = 0.$$

Using the expression for divergence in spherical polar coordinates (Appendix 4),

$$\begin{aligned}\nabla \cdot \boldsymbol{E} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta E_\theta) + \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi} \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot \frac{q}{4\pi\varepsilon_0 r^2}\right) \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{q}{4\pi\varepsilon_0}\right) = \frac{1}{r^2}(0) \\ &= 0.\end{aligned}$$

It seems that $\nabla \cdot \boldsymbol{E}$ is zero everywhere. For all points except the origin, the charge density ρ is zero. So Gauss's law $\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$ is fine at all these points. But what happens at the origin? The charge density at the origin is infinity but $\nabla \cdot \boldsymbol{E}$ appears to be zero! There is a catch. In the last step of the derivation above, we ignored the presence of r^2 in the denominator and since $\frac{\partial}{\partial r}\left(\frac{q}{4\pi\varepsilon_0}\right) = 0$, we considered the whole expression to be zero. That is not allowed if $r = 0$. Also, cancelling r^2 in the numerator and denominator in the previous step is not allowed. So, the above derivation may not be valid for $r = 0$. How do you calculate $\nabla \cdot \boldsymbol{E}$ at the origin? Use the basic definition of divergence.

$$\nabla \cdot \boldsymbol{E} = \lim_{\Delta\tau \to 0} \frac{\oint \boldsymbol{E} \cdot d\boldsymbol{a}}{\Delta\tau} \cdot$$

To get the value of $\nabla \cdot \boldsymbol{E}$ at a given point $\boldsymbol{r}$, construct a closed surface enclosing the point $\boldsymbol{r}$ (Figure 2.1). Take a surface element da on this surface and construct the area vector $d\boldsymbol{a}$ with direction along the outward normal and magnitude equal to da. Evaluate $\boldsymbol{E} \cdot d\boldsymbol{a}$ at this element and integrate over the entire closed surface. This is $\oint \boldsymbol{E} \cdot d\boldsymbol{a}$. Divide this by the volume $\Delta\tau$ enclosed by the closed surface and let the volume shrink and take the limit $\Delta\tau \to 0$ and you get $\nabla \cdot \boldsymbol{E}$.

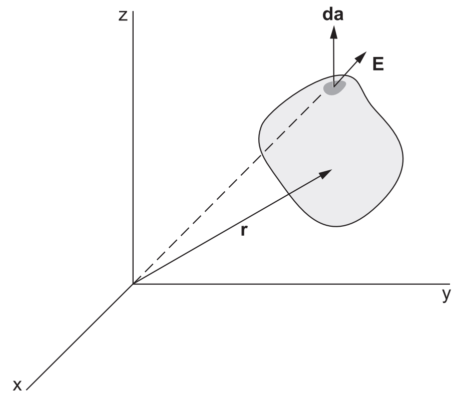


Fig. 2.1

In the present case, you need $\nabla \cdot \boldsymbol{E}$ at the origin. Draw a spherical surface of radius a with the centre at the origin (Figure 2.2). What is $\oint \boldsymbol{E} \cdot d\boldsymbol{a}$ over this surface? Consider any small portion of this spherical surface and the corresponding vector $d\boldsymbol{a}$ is along the $\hat{r}$ direction. So is the direction of the $\boldsymbol{E}$ vector. Thus

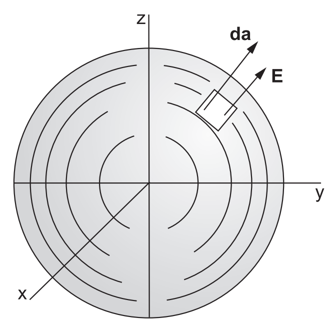


Fig. 2.2

$$\oint \boldsymbol{E} \cdot d\boldsymbol{a} = \int \frac{q}{4\pi\varepsilon_0 a^2} da$$

$$= \frac{q}{4\pi\varepsilon_0 a^2} \int da$$

$$= \frac{q}{4\pi\varepsilon_0 a^2} \times 4\pi a^2 = \frac{q}{\varepsilon_0} \cdot$$

So, $$\nabla \cdot \boldsymbol{E} = \lim_{\Delta\tau \to 0} \frac{\oint \boldsymbol{E} \cdot d\boldsymbol{a}}{\Delta\tau} = \lim_{a \to 0} \frac{q}{\varepsilon_0 \left(\frac{4}{3}\pi a^3\right)} = \infty.$$

Thus, $\nabla \cdot \boldsymbol{E} = 0$ at all points except the origin, and is infinity at the origin. This looks like a delta function. To see if it is really so, check that

$$\int_{\text{volume}} (\nabla \cdot \boldsymbol{E}) d\tau = \oint \boldsymbol{E} \cdot d\boldsymbol{a} \quad \text{(Gauss divergence theorem)}$$

$$= \frac{q}{\varepsilon_0} \cdot$$

So, $$\int \left[\frac{1}{q/\varepsilon_0}(\nabla \cdot \boldsymbol{E})\right] d\tau = 1$$

if the volume encloses the origin. Thus $\frac{1}{q/\varepsilon_0}(\nabla \cdot \boldsymbol{E})$ satisfies all the requirements of being the delta function $\delta^3(\boldsymbol{r})$ centred on the origin. It is zero everywhere except at the origin, at the origin it is infinity, and its integral over any volume enclosing the origin is 1. Hence,

$$\nabla \cdot \boldsymbol{E} = \frac{q}{\varepsilon_0}\delta^3(\boldsymbol{r}).$$

But $q\delta^3(\boldsymbol{r})$ is the volume charge density corresponding to the single charge q kept at the origin. So,

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$

even at the origin and hence everywhere.

The above derivation is made for a single point charge. As the field due to a charge distribution can be expressed as the vector sum of electric fields due to individual charges, this result is valid for any charge distribution.

It may seem that Gauss's law can be derived from Coulomb's law and so Coulomb's law is more fundamental than Gauss's law. However, this is not the case. We will show that Coulomb's law can also be derived from Gauss's law and so, on this account, any of the two may be taken as fundamental and the other derived. But Gauss's law scores higher on another account. As you know, charges are not the only source of an electric field; a changing magnetic field also produces an electric field. Coulomb's law gives no information about this second type of field. But Gauss's law is valid for this second variety of field also. In fact, $\nabla \cdot \boldsymbol{E}$ turns out to be zero for the field produced by a purely changing magnetic field.

EXAMPLE 2.1 **The electric field for a particular charge configuration is given as $\boldsymbol{E} = k(x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}})$ in a certain region. Find the charge distribution in that region.**

Solution $\boldsymbol{E} = k(x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}})$.

Thus, $E_x = kx,\ E_y = ky,\ E_z = kz.$

So, $$\nabla \cdot \boldsymbol{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = k\left[\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right] = 3k.$$

Thus, $\rho = \varepsilon_0 \nabla \cdot \boldsymbol{E} = 3\varepsilon_0 k.$

So the charge density in the given region is constant and its value is $3\varepsilon_0 k$.

2.2 Integral Form of Gauss's Law

For any given electric field $\boldsymbol{E}$ in space, $\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0} \cdot$ Consider a closed surface S enclosing a volume τ (Figure 2.3). Take a small area element on this surface. Construct the area vector $\boldsymbol{da}$ corresponding to this element. This vector has a magnitude equal to the area of the element and a direction perpendicular to the surface at this point, outward. Remember we have taken a closed surface and so there is a definite 'inside' and a definite 'outside'. The vector $\boldsymbol{da}$ is towards the outward normal.

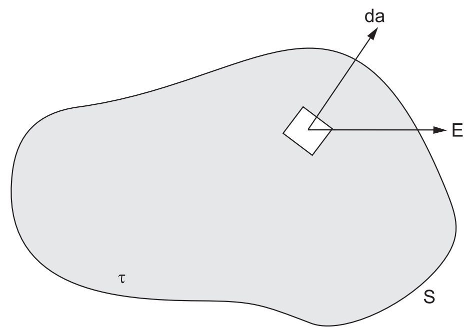


Fig. 2.3

Let the electric field at the location of this area element be $\boldsymbol{E}$**.** The scalar product $\boldsymbol{E} \cdot \boldsymbol{da}$ is the flux of the electric field through the area da. Integrate it over the entire closed surface S to get $\oint \boldsymbol{E} \cdot \boldsymbol{da}$. This is the total flux of the electric field through the closed surface.

By the Gauss divergence theorem,

$$\oint_S \boldsymbol{E} \cdot \boldsymbol{da} = \oint_\tau (\nabla \cdot \boldsymbol{E}) d\tau.$$

By Gauss's law, $\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0} \cdot$ So,

$$\oint_S \boldsymbol{E} \cdot \boldsymbol{da} = \frac{1}{\varepsilon_0} \oint_V \rho d\tau$$

or

$$\oint \boldsymbol{E} \cdot \boldsymbol{da} = \frac{q_{\text{en}}}{\varepsilon_0}, \tag{2.2}$$

where q_{en} is the net charge enclosed by the closed surface. Equation 2.2 is called the integral form of Gauss's law.

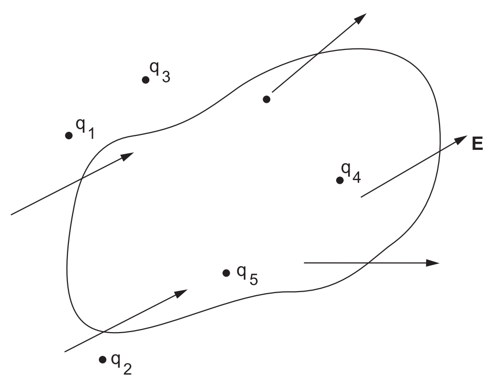


Fig. 2.4

What does Equation 2.2 suggest? In Figure 2.4, we have shown certain charges and the electric field. The electric field shown is the resultant electric field, due to all the charges that are there and also due to any changing magnetic field if present. The flux $\oint \boldsymbol{E} \cdot \boldsymbol{da}$ of the electric field is calculated for this field, that is the left-hand side of Equation 2.2. To write the right-hand side, we only look for the charges inside the closed surface—$q_4 + q_5$ in Figure 2.4. So the right-hand side will be $(q_4 + q_5)/\varepsilon_0$. Any charge outside the closed surface will contribute to the electric field $\boldsymbol{E}$ but not to $\oint \boldsymbol{E} \cdot d\boldsymbol{a}$. The flux of the resultant electric field through a closed surface is equal to the total charge inside the surface, divided by ε_0.

EXAMPLE 2.2 **Find the electric field $\boldsymbol{E}$ at a distance $\boldsymbol{r}$ from a point charge $\boldsymbol{q}$ using Gauss's law.**

Solution Let us take the origin at the position of the charge q. Suppose the field is needed at point P, at a distance r from q. What is the direction of the electric field $\boldsymbol{E}$ at P? Remember, we don't have to use Coulomb's law. The only charge in space is q at the origin and the electric field at P has a unique direction. What can this unique direction be? The only unique direction is the radial direction. Draw (mentally) a spherical surface through P centred on the origin. If the field at P is not radial, it will have a component along a tangent to this spherical surface (Figure 2.5). But there are so many tangents to the surface at this point and all are equivalent. So which of these many directions should the field choose for tangential component? None— the field can only be radial. If q is positive, the field should be in the $\hat{r}$ direction.

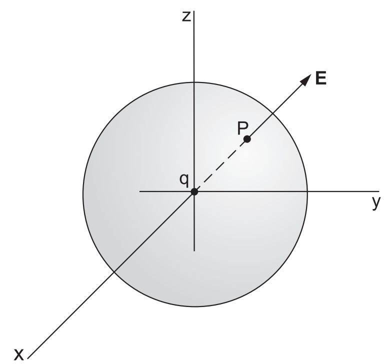


Fig. 2.5

All points of the spherical surface are at the same distance r from q. Thus, if the magnitude of the field at P is E at all points on this spherical surface, it will have the same value E. Thus $\boldsymbol{E} = E\hat{\boldsymbol{r}}$.

The flux of the electric field through the spherical surface is

$$\oint \boldsymbol{E} \cdot d\boldsymbol{a} = \oint (E\hat{\boldsymbol{r}}) \cdot (da\,\hat{\boldsymbol{r}}) = \oint E\,da$$

$$= E\oint da = E \cdot 4\pi r^2.$$

The charge enclosed by this closed surface is q. So, using Gauss's law, $\oint \boldsymbol{E} \cdot d\boldsymbol{a} = \dfrac{q_{\text{en}}}{\varepsilon_0}$,

$$4\pi r^2 E = \frac{q}{\varepsilon_0}$$

or $$E = \frac{q}{4\pi\varepsilon_0 r^2}$$

or $$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}}.$$

So you get Coulomb's law from Gauss's law.

EXAMPLE 2.3 **A charge q is placed on the axis of a circular disk of radius R. The distance of q from the centre of the disk is d. Find the flux of the electric field through this disk.**

Solution We can calculate the flux by expressing the field E at a point on the disk, and evaluating $\int \boldsymbol{E} \cdot d\boldsymbol{a}$. Let us do something different to make the calculation easy.

Take the position of q as the origin and the axis of the disk as the z-axis. Draw a spherical cap with the origin as the centre and the periphery of the given disk as the periphery of the cap (Figure 2.6). Consider the volume enclosed between the surface of the disk and the spherical cap. As there is no charge in this volume, the total flux through the closed surface so formed must be zero. This means the magnitude of the flux through the surface of the disk is the same as that through the spherical cap. So we will calculate the flux through the spherical cap only.

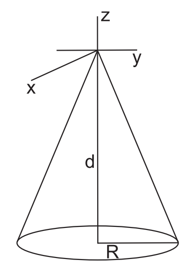


Fig. 2.6

With the origin and the z-axis as chosen, the area element at (r, θ, ϕ) on the cap is

$$d\boldsymbol{a} = R_0^2 \sin\theta \, d\theta \, d\phi \, \hat{\boldsymbol{r}},$$

where $R_0 = \sqrt{d^2 + R^2}$ is the radius of the sphere.

The electric field is

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0 R_0^2} \hat{\boldsymbol{r}}.$$

The flux through the area element is

$$d\Phi = \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{q R_0^2 \sin\theta \, d\theta \, d\phi}{4\pi\varepsilon_0 R_0^2}.$$

The flux through the cap is

$$\Phi = \int \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{q}{4\pi\varepsilon_0} \int_{\phi=0}^{2\pi} \int_{\theta=\pi-\theta_0}^{\pi} \sin\theta \, d\theta \, d\phi,$$

where $\theta_0 = \tan^{-1}\left(\frac{R}{d}\right)$.

So,
$$\Phi = \frac{q}{4\pi\varepsilon_0} 2\pi(1 - \cos\theta_0)$$
$$= \frac{q}{2\varepsilon_0}\left(1 - \frac{d}{\sqrt{R^2 + d^2}}\right).$$

In school, you may have studied Gauss's law in the form of Equation 2.2. If you are familiar with it, getting the differential form (Equation 2.1) is straightforward. It essentially involves reversing the mathematical steps given above. If

$$\oint \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{q_{en}}{\varepsilon_0},$$

using the Gauss divergence theorem,

$$\oint \boldsymbol{E} \cdot \boldsymbol{da} = \frac{\int \rho d\tau}{\varepsilon_0}$$

or
$$\int (\nabla \cdot \boldsymbol{E}) d\tau = \int \frac{\rho}{\varepsilon_0} d\tau.$$

Since this equation is valid for any volume, the integrand must be equal at all points.

Thus,
$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0} \cdot$$

2.3 Electric Field Due to Some Symmetric Charge Distributions

In Example 2.2, we used symmetry as a tool to argue that $\boldsymbol{E}$ should be along the $\hat{\boldsymbol{r}}$ direction and should have the same magnitude at all points on the spherical surface. This allowed us to evaluate $\oint \boldsymbol{E} \cdot \boldsymbol{da}$ on this surface quickly. There are some other charge distributions which also allow us to use similar symmetry arguments, and the flux over a properly chosen closed surface can be easily evaluated. Such a closed surface is often called a Gaussian surface. Consider three examples, representing three kinds of symmetrical charge distribution, which allow you to calculate the electric field using Gauss's law.

Electric field due to a uniformly charged sphere

A spherical volume of radius R contains a charge Q distributed uniformly throughout. What is the electric field at a point P? First suppose the point P is outside the sphere at a distance r from the centre of the spherical volume. Draw a spherical surface through P, concentric with the charge distribution (Figure 2.7). Take this as the Gaussian surface. As the charge is uniformly distributed in the spherical volume, the only unique direction through P in this arrangement is radial and hence the electric field will be radial. If the charge is positive, the field will be radially outward. This is true for any point in space.

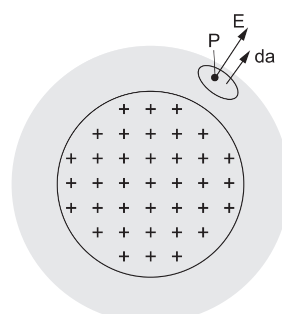


Fig. 2.7

All points on the Gaussian surface we have constructed are equidistant from the centre of the charge distribution. Look at the distribution from any point on this surface—it looks the same. So, the magnitude of the electric field E is also the same at all points on this surface. The area vector $\boldsymbol{da}$ for an elementary area is also radially outwards. Take the origin at the centre of the charge distribution. Then $\boldsymbol{E} = E\hat{\boldsymbol{r}}$ and $\boldsymbol{da} = da\hat{\boldsymbol{r}}$. Thus,

$$\oint \boldsymbol{E} \cdot \boldsymbol{da} = \oint (E\hat{\boldsymbol{r}}) \cdot (da\hat{\boldsymbol{r}})$$

$$= \oint E da = E \oint da = E \cdot 4\pi r^2.$$

The charge contained inside the Gaussian surface is Q. So by Gauss's law,

$$4\pi r^2 E = \frac{Q}{\varepsilon_0}$$

or $$E = \frac{Q}{4\pi\varepsilon_0 r^2}.$$

The electric field at P is

$$\boldsymbol{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}}. \tag{2.3}$$

This field is the same as that of a point charge Q placed at the origin. So to get the electric field at a point P outside the spherical charge distribution, you can assume that the entire charge is placed at the centre.

Next, consider a point P inside the charged sphere. Again draw a concentric spherical surface through P and use it as the Gaussian surface (Figure 2.8). All our arguments in the previous paragraphs regarding symmetry are valid here too. The field at all points on this Gaussian surface must have the same magnitude E and must be radial. So the flux is

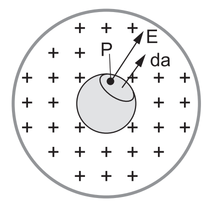


Fig. 2.8

$$\oint \boldsymbol{E} \cdot \boldsymbol{da} = 4\pi r^2 E$$

through this Gaussian surface. But the charge enclosed by this surface is not Q, as only part of the charge is inside this Gaussian surface. As the charge Q is uniformly spread over the spherical volume of radius R, the charge density everywhere in the sphere is

$$\rho = \frac{Q}{(4/3)\pi R^3}.$$

Hence, the charge enclosed by the Gaussian surface is

$$Q_{\text{en}} = \rho \cdot \frac{4}{3}\pi r^3 = \frac{Qr^3}{R^3}.$$

Thus, using Gauss's law,

$$4\pi r^2 E = \frac{Qr^3}{\varepsilon_0 R^3}$$

or
$$E = \frac{Qr}{4\pi\varepsilon_0 R^3}$$

or
$$\boldsymbol{E} = \frac{Q\boldsymbol{r}}{4\pi\varepsilon_0 R^3}. \tag{2.4}$$

As you can see, the magnitude of the electric field is proportional to r. At the centre, the field is zero. As you go away from the centre, the field increases and becomes maximum at $r = R$. Beyond that, Equation 2.3 works and the field decreases as $1/r^2$. Figure 2.9 shows the variation of the magnitude of the electric field with distance r. Notice that equations 2.3 and 2.4 give the same value of the field at $r = R$. We say that the electric field is continuous at $r = R$.

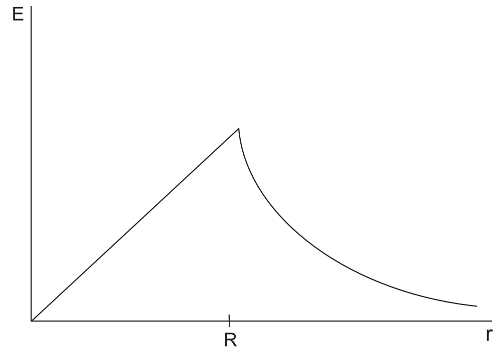


Fig. 2.9

You can express the electric field in terms of the charge density ρ. The charge density is

$$\rho = \frac{Q}{(4/3)\pi R^3}.$$

The field inside is

$$\boldsymbol{E}_{\text{in}} = \frac{Q\boldsymbol{r}}{4\pi\varepsilon_0 R^3} = \frac{\rho\boldsymbol{r}}{3\varepsilon_0} \tag{2.5}$$

and that outside is

$$\boldsymbol{E}_{\text{out}} = \frac{Q\boldsymbol{r}}{4\pi\varepsilon_0 r^3} = \frac{\rho\boldsymbol{r}R^3}{3\varepsilon_0 r^3}. \tag{2.6}$$

You can now realise the power of Gauss's law. If you were to use Coulomb's law to obtain the field at P due to a uniformly charged sphere, you would have to start with

$$\boldsymbol{E} = \int \frac{1}{4\pi\varepsilon_0}\frac{\rho(\boldsymbol{r}')(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}d\tau'.$$

This would be a very difficult integration to work out.

Spherically symmetric charge distribution

We considered the case of a uniformly charged sphere above. We will now discuss a spherically symmetric charge distribution. By this, we mean a charge distribution in which the charge density does not depend on θ or ϕ (using spherical polar coordinates with appropriate choice of origin). It may still depend on r. Go back and read the arguments as regards why the field had the same magnitude in the spherical Gaussian surface taken in the case of the uniformly charged sphere. From any point of this surface, the charge distribution looks identical. This is also valid if the charge density does not depend on θ and ϕ but depends on r. You can use the above method to get the field. Imagine a spherical surface through the given point, concentric with the charge distribution, and use it as the Gaussian surface. The flux of the electric field over this surface will be $4\pi r^2 E$. Calculate the charge contained inside the Gaussian surface and use Gauss's law to get E.

....

EXAMPLE 2.4 **A thin, spherical, hollow sphere has charge spread over the surface uniformly. The surface charge density is σ. Find the force on a small part of an area *da* on the surface due to the rest of the charge.**

Solution The charge on the small part is σda. The charge distribution is spherically symmetric if you take the origin at the centre. If you work out the electric field due to this sphere using the method described above, you will get

$$\boldsymbol{E} = 0, \text{ inside the sphere and}$$

$$= \frac{\sigma R^2}{\varepsilon_0 r^2}\hat{\boldsymbol{r}}, \text{ outside the sphere.}$$

So, on one side of the small part (just outside), the field is $\frac{\sigma}{\varepsilon_0}$, normal to the surface, and on the other side (just inside), it is zero. If you want to get the force on the small part from $\boldsymbol{F} = q\boldsymbol{E}$, which field should you use? It turns out that the proper field to be used is the average of the fields on the two sides. Try to prove this statement.

Thus, $\boldsymbol{F} = (\sigma da)\frac{1}{2}\left[0 + \frac{\sigma}{\varepsilon_0}\right]\hat{\boldsymbol{r}} = \frac{\sigma^2}{2\varepsilon_0} da\hat{\boldsymbol{r}}$.

Thus, the force per unit area is $\frac{\sigma^2}{2\varepsilon_0}$ in the outward direction.

EXAMPLE 2.5 **A sphere of radius *R* has charge distributed in its volume with a uniform charge density ρ. Find the force exerted by the charge in one hemisphere on the charge in the other hemisphere.**

Solution No charge distribution can exert a net force on itself. This is because forces exerted by any two parts in the distribution are equal in magnitude and opposite in direction. When you consider the force exerted by the whole charge distribution on itself, all forces cancel in pairs.

The force exerted by one hemisphere on the other is therefore the same as that exerted by the full sphere on the second hemisphere. This is because the second hemisphere does not exert any force on itself. Our strategy will be to take a volume element in this second hemisphere, write the electric field there due to the whole sphere and then get the force on this element. This force will be then integrated over the hemispherical volume to get the net force on it.

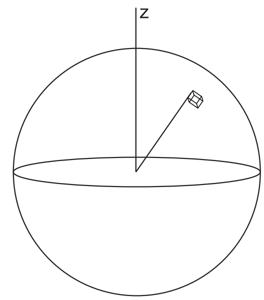


Fig. 2.10

Figure 2.10 shows the situation. Let us calculate the force on the upper hemisphere. The electric field at (r, θ, ϕ) (inside the upper hemisphere) is $\boldsymbol{E} = \dfrac{\rho \boldsymbol{r}}{3\varepsilon_0}\cdot$ The volume element $d\tau$ at this point is $d\tau = r^2 \sin\theta\, dr\, d\theta\, d\phi$. The force on the charge contained in this volume element is

$$d\boldsymbol{F} = \rho\, d\tau \boldsymbol{E} = \frac{\rho^2 r^3}{3\varepsilon_0} \sin\theta\, dr\, d\theta\, d\phi\, \hat{\boldsymbol{r}}.$$

From considerations of symmetry, the net force on the upper hemisphere will be along the z-direction. So you only have to integrate the z-component of the force $d\boldsymbol{F}$. The z-component of $\hat{\boldsymbol{r}}$ is $\cos\theta$. So the force on the upper hemisphere is

$$\boldsymbol{F} = \iiint \frac{\rho^2 r^3}{3\varepsilon_0} \sin\theta \cos\theta\, dr\, d\theta\, d\phi \text{ (within proper limits)}$$

$$= \frac{\rho^2}{3\varepsilon_0} \int_0^R r^3 dr \int_0^{\pi/2} \sin\theta \cos\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{\rho^2}{3\varepsilon_0} \frac{R^4}{4} \frac{1}{2} 2\pi = \frac{\pi \rho^2 R^4}{12\varepsilon_0}\cdot$$

Electric field due to an infinite, straight, line charge having a uniform linear charge density λ

Consider a charge distribution in which the charge is arranged along a straight line, the charge contained per unit length of the line being λ. You can call this a line charge. Also assume that the total length of the line charge is large. Call it an infinite line charge.

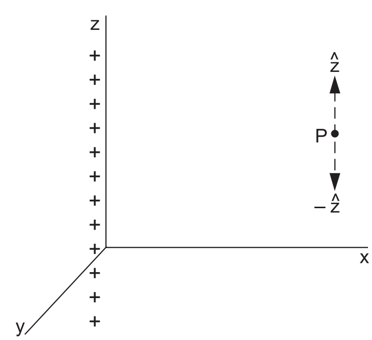


Fig. 2.11

Take the z-axis along the given line charge and use the cylindrical polar coordinate system (Figure 2.11). The field is to be obtained at a point P. What could be the direction of the electric field at P? You are given an infinite, straight, line charge of uniform density and no charge anywhere else. What is the unique direction from P along which this line charge can produce an electric field? Can it be parallel to the line charge? There are two such directions, one going up and one going down in Figure 2.11, that is, one along $\hat{z}$ and the other along the $-\hat{z}$ direction. Which of the two will you prefer? As the charge extends equally on both sides, there is no way you can prefer one over the other. This argument also shows that the electric field can have no component along the z-direction, as it will not be able to choose whether the component should be along the $+z$- or the $-z$-direction.

The same is the case with the ϕ-direction. If you draw a circle from P parallel to the x–y plane, with the centre on the z-axis, the $\hat{\phi}$-direction is tangential to this circle (Figure 2.12). Once again think of the $\hat{\phi}$- and the $-\hat{\phi}$-direction. The only charge is along the z-axis, extending to infinity on both sides and having a uniform linear charge density. With reference to this charge, how do you distinguish between the $+\hat{\phi}$ and $-\hat{\phi}$ directions? If the electric field due to this line charge has a component along the $+\hat{\phi}$ direction, why does it not have a component along the $-\hat{\phi}$ direction? The electric field cannot choose between $\hat{\phi}$ and $-\hat{\phi}$, and so there is no component along the $\hat{\phi}$ direction. So the field has to be along the $\hat{s}$ direction, perpendicularly away from the line charge if the charge is positive.

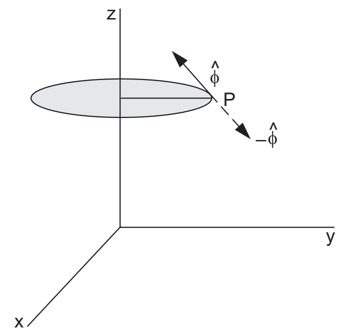


Fig. 2.12

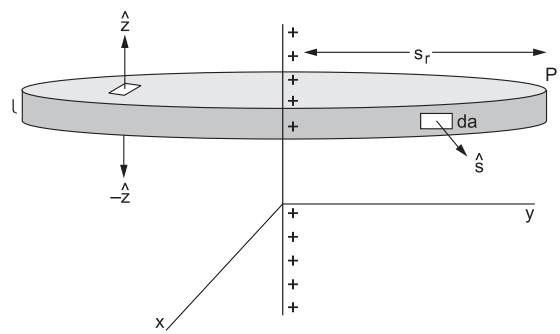


Fig. 2.13

Now choose a Gaussian surface. Imagine a cylindrical surface, coaxial with the line charge, of height l and passing through P (Figure 2.13). If the distance of point P from the line charge is s, so is the radius of this cylindrical surface. Add two plane surfaces to this cylindrical surface, one at the top and the other at the bottom, to close it from both sides. Thus you get a closed surface.

What is the flux of the electric field through this closed surface? Take a small area element da on the curved part. The perpendicular to this area is towards $\hat{\boldsymbol{s}}$; so the area vector is $da\hat{\boldsymbol{s}}$. The electric field here is also along $\hat{\boldsymbol{s}}$. The field is therefore $E\hat{\boldsymbol{s}}$. All points on the curved part of the Gaussian surface are at the same distance from the line charge and so the magnitude of the field at all these points will be the same. Thus,

$$\int_{\substack{\text{curved}\\ \text{part}}} \boldsymbol{E}\cdot \boldsymbol{da} = \int (E\hat{\boldsymbol{s}})\cdot(da\hat{\boldsymbol{s}}) = \int E\,da = E\int da = 2\pi s l E.$$

Is this the flux through the Gaussian surface we have taken? Yes and no. No, because we have only calculated flux through the curved parts and this itself is not a closed surface. We also have to look at the flux through the flat parts covering the cylindrical surface. And yes, because the flux through these flat parts is zero. This is because the area vector here will be either along the $\hat{\boldsymbol{z}}$ direction or the $-\hat{\boldsymbol{z}}$ direction. But the field is along the $\hat{\boldsymbol{s}}$ direction and so $\boldsymbol{E} \cdot d\boldsymbol{a}$ will be zero on the flat parts.

The flux through the closed Gaussian surface is therefore $2\pi slE$. What is the charge enclosed by this Gaussian surface? The height of this box is l and this length of the line charge is inside the box. The charge enclosed is therefore λl. Using Gauss's law,

$$2\pi slE = \frac{\lambda l}{\varepsilon_0}$$

or
$$E = \frac{\lambda}{2\pi\varepsilon_0 s}\cdot$$

Including the direction, the electric field at a distance s from an infinite line charge is

$$\boldsymbol{E} = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{\boldsymbol{s}}$$

or
$$\boldsymbol{E} = \frac{\lambda \boldsymbol{s}}{2\pi\varepsilon_0 s^2}\cdot \tag{2.7}$$

This equation is derived for an infinite wire. But in practice we don't have an infinite wire. Then what is the use of this equation? Actually most of the contribution to the electric field at the given point P comes from portions of the line charge close to P. So if you have quite a long wire, and use Equation 2.7, you don't make much of an error. How long should the wire be so that one can use Equation 2.7? That depends on how accurate you want the result to be. As a thumb rule, the length of the wire should be many times (say 10 times) s on both sides.

Cylindrically symmetric charge distribution

An infinite line charge is an example of what we call a cylindrically symmetric charge distribution. Using cylindrical coordinates, if the charge density does not depend on ϕ and z, we say that the charge distribution is cylindrically symmetric. It can still depend on s. In such cases, you can use the same procedure as described for an infinite line charge. Consider as a Gaussian surface a cylindrical box of length l, coaxial with the given charge distribution and passing through the point where the field is needed. The flux of the electric field will be $2\pi slE$. Calculate the charge inside the box and use Gauss's law.

EXAMPLE 2.6 **A long cylinder of radius R has a uniform charge density ρ. Find the electric field inside and outside the cylinder.**

Solution Take the z-axis along the axis of the cylinder. The electric field at any point has to be along the $\hat{s}$ direction. Also, it will depend only on s, the distance from the axis of the cylinder.

Case 1 Field inside the cylinder $(s < R)$.

Consider a point P at a distance s from the z-axis [Figure 2.14(a)]. Draw a cylindrical surface through P, of height l, coaxial with the cylinder. This surface together with the plane surfaces at the top and bottom form a closed surface. The flux of the electric field through this closed surface is

$$\oint \boldsymbol{E} \cdot \boldsymbol{da} = (2\pi sl)E,$$

where E is the magnitude of the electric field at distance s from the z-axis. The total charge contained in the volume enclosed by this closed surface is

$$q_{\text{in}} = \rho\pi s^2 l.$$

Thus, by Gauss's law,

$$(2\pi sl)E = \frac{\rho\pi s^2 l}{\varepsilon_0}$$

or $$E = \frac{\rho s}{2\varepsilon_0}$$

or $$\boldsymbol{E} = \frac{\rho \boldsymbol{s}}{2\varepsilon_0}.$$

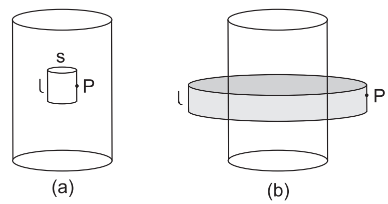


Fig. 2.14

Case 2 Field outside the cylinder $(s > R)$.

Again draw a closed surface in a similar way. Draw a cylindrical surface through P of height l, coaxial with the cylinder. This surface together with the plane surfaces at the top and the bottom form a closed surface. The flux through this surface is

$$\oint \boldsymbol{E} \cdot \boldsymbol{da} = (2\pi sl)E.$$

The total charge enclosed by the surface is

$$q_{\text{out}} = \rho\pi R^2 l.$$

Thus, by Gauss's law,

$$(2\pi sl)E = \frac{\rho\pi R^2 l}{\varepsilon_0}$$

or $$E = \frac{\rho R^2}{2\varepsilon_0 s}$$

or $$\boldsymbol{E} = \frac{\rho R^2 \boldsymbol{s}}{2\varepsilon_0 s^2}.$$

An infinite, uniform, plane charge layer

Suppose a charge layer is spread over the entire x–y plane with a uniform surface charge density σ. What is the electric field due to this layer at a point with $z > 0$ and with $z < 0$?

First consider a point P with $z > 0$, i.e., above the charge layer, take the x–y plane to be horizontal and the z-axis to be vertically upwards (Figure 2.15). What is the direction of the electric field at P? What is the unique direction from P with reference to the charge layer? The charge is spread uniformly all over the x–y plane. All directions parallel to the x–y plane are equivalent. The electric field cannot choose any of these directions for its component in the x–y plane. Thus the field has to be perpendicular to the charge layer. If the charge is positive, the field at P will be along the $+z$ direction. And the field below the x–y plane will be along the $-z$ direction if the charge is positive.

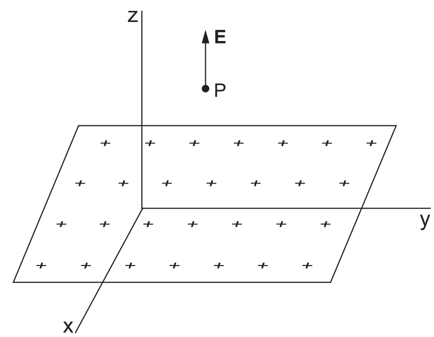


Fig. 2.15

How should you take the Gaussian surface to calculate the field at P, say at a distance z from the charge layer? Construct a surface of area ΔA parallel to the charge layer, and passing through the point P (Figure 2.16). Construct a similar area ΔA below the layer at the same distance. By joining the corresponding points of the two areas, make a cylinder. So the charge layer in the x–y plane goes right through the middle of this cylindrical box.

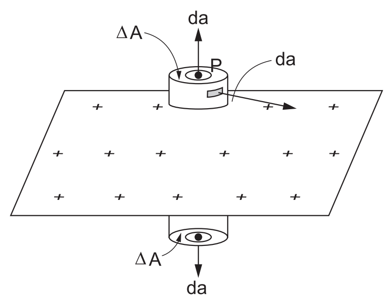


Fig. 2.16

The curved surface of the cylinder together with the two flat surfaces of area ΔA each, at the same distance z from the charge layer, make up the Gaussian surface. The electric field is either along the $+z$ direction or the $-z$ direction. Taking any small area on the curved part, the area vector $\boldsymbol{da}$ will be parallel to the x–y plane (Figure 2.16). Thus $\boldsymbol{E} \cdot \boldsymbol{da}$ will be zero. Thus $\int \boldsymbol{E} \cdot \boldsymbol{da}$ over the curved part is zero. Now come to the upper flat part ΔA. All the points on this area are at the same distance from the charge layer. Hence the magnitude of the field is the same. The direction of the electric field is also the same at all these points, i.e., along $+z$. The area vector is also along this direction. Thus,

$$\int_{\substack{\text{upper}\\ \text{flat part}}} \boldsymbol{E} \cdot \boldsymbol{da} = E\,\Delta A.$$

Finally look at the lower flat part ΔA. This area is at the same distance from the charge layer as the upper flat part. Hence the electric field at points on the lower flat part is also E. This field is along the $-z$ direction here and so is the area vector $\boldsymbol{da}$. Hence,

$$\int_{\substack{\text{lower} \\ \text{flat part}}} \boldsymbol{E} \cdot \boldsymbol{da} = E\,\Delta A.$$

The total flux of the electric field through the box is therefore

$$\begin{aligned}\oint \boldsymbol{E} \cdot \boldsymbol{da} &= \int_{\substack{\text{curved} \\ \text{part}}} \boldsymbol{E} \cdot \boldsymbol{da} + \int_{\substack{\text{upper} \\ \text{flat part}}} \boldsymbol{E} \cdot \boldsymbol{da} + \int_{\substack{\text{lower} \\ \text{flat part}}} \boldsymbol{E} \cdot \boldsymbol{da} \\ &= 0 + E\,\Delta A + E\,\Delta A = 2E\,\Delta A.\end{aligned}$$

How much charge is contained in the box? The area of the charge layer inside the box is also ΔA and hence the charge enclosed by the box is $\sigma\,\Delta A$.

Using Gauss's law,

$$2E\,\Delta A = \frac{\sigma\,\Delta A}{\varepsilon_0}$$

or

$$E = \frac{\sigma}{2\varepsilon_0}\cdot \tag{2.8}$$

Thus the electric field due to an infinite, uniform, plane charge layer with surface charge density σ is $E = \frac{\sigma}{2\varepsilon_0}$, perpendicular to the charge layer. For positive charge, the field is away from the layer and for negative charge it is towards the layer. We can express the field as

$$\boldsymbol{E} = \frac{\sigma}{2\varepsilon_0}\hat{\boldsymbol{n}}, \tag{2.9}$$

where $\hat{\boldsymbol{n}}$ is the unit vector from the field point perpendicular to the charge layer and away from it.

You may wonder why the field does not depend on the distance from the charge layer. The field 1 mm away from the charge layer is the same as that 100 mm away. What is the physical significance of this premise? If the point P is at a distance z from the charge layer, charges beyond several times of z do not contribute much. Thus if the point P is taken farther away from the layer, more distant charges will start contributing to the field. This extra contribution compensates for the increase in distance and the accompanying decrease in field due to the charges nearly below the point P.

Once again if a charge layer is spread over a large distance along all sides, large compared to the distance of the point where the field is needed, you can treat it as an infinite layer and use the equation $\boldsymbol{E} = (\sigma/2\varepsilon_0)\hat{\boldsymbol{n}}$.

Charge distribution with planar symmetry

If a charge density is independent of two Cartesian coordinates, such as x and y, it is said to have planar symmetry. A layer of infinite charge is an example of this kind of charge distribution. Wherever you have a charge distribution with planar symmetry, you can use Gauss's law to obtain the electric field. Assuming charge distribution to be independent of x and y, and symmetric about the x–y plane, do the following. Take an area ΔA passing through the given point, and parallel to the x–y plane. Take a similar area ΔA on the other side of the x–y plane, the two areas being mirror images of each other in the x–y plane. Join corresponding points of the two areas to make a cylindrical box. The flux through this box will be $2E\,\Delta A$. Calculate the charge contained in the box and use Gauss's law to find E.

2.4 How Gauss's Law is Useful?

We have discussed certain symmetric charge distributions and learnt how Gauss's law can be used to find the electric field due to these distributions. If the charge distribution is not symmetric, can Gauss's law still be used? Is Gauss's law valid for asymmetric charge distributions too? Answering the second question is very easy. Gauss's law is valid in all situations, whether or not the charge distribution is symmetric. Even if there are changing magnetic fields producing an electric field in addition to the field produced by the charges, Gauss's law is valid and the flux of the electric field through any closed surface is equal to the charge enclosed by that surface, divided by ε_0. The answer to the first question is not very straightforward. Calculation of the flux could become very difficult in some situations lacking symmetry. You still have $\oint \boldsymbol{E} \cdot d\boldsymbol{a} = q_{\text{en}}/\varepsilon_0$ but you may not be able to use it to obtain the field $\boldsymbol{E}$ at the given point. Even if the charge distribution is symmetric, you must choose the Gaussian surface according to the symmetry of the charge distribution to be able to get the flux integration easily. For example, if you have a uniformly charged sphere and, as a Gaussian surface, you take a spherical surface not concentric with the given charged sphere, you cannot write $\oint \boldsymbol{E} \cdot d\boldsymbol{a} = 4\pi r^2 E$ and the calculation will be difficult. So Gauss's law is always valid but not always useful.

Indeed you can use the principle of superposition. It is possible that the given charge distribution does not possess any of the three kinds of symmetry (spherical, cylindrical or planar) described above, but it can be seen as a combination of two or more charge distributions, each of which has some kind of symmetry. You can apply Gauss's law to calculate the electric field separately for each of these part distributions and then add the fields vectorially to get the resultant field.

<u>EXAMPLE 2.7</u> **Two spherical volumes of radii R_1 and R_2 have centres at C_1 and C_2. Some parts of these volumes overlap, as shown in Figure 2.17. The volume to the left of the overlap region contains positive charge with uniform charge density +ρ and the volume to the right of the overlap region contains negative charge with uniform charge density −ρ. There is no charge in the overlap region. Find the electric field at a point P in the overlap region.**

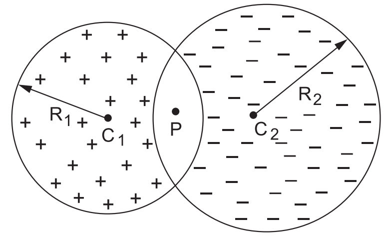


Fig. 2.17

Solution The charge distribution as given in the problem does not possess spherical, cylindrical or planar symmetry. However, it can be thought to be made of two parts—a uniformly charged sphere of radius R_1 with charge density +ρ (centred on C_1) and a uniformly charged sphere of radius R_2 with charge density −ρ (centred on C_2). In the overlap region, the two charge densities taken together will result in zero charge as given. Each of the two charged spheres in itself is a spherically symmetric charge distribution and the field due to this may be derived from Gauss's law.

First look at the sphere on the left. The point P is inside the sphere. You know the electric field due to a uniformly charged sphere at a point inside. If the centre of the sphere is taken as the origin, it is $\frac{\rho \boldsymbol{r}}{3\varepsilon_0}\cdot$ Here the centre is at C_1. So the field at P due to this sphere alone (Figure 2.17) is

$$\boldsymbol{E_1} = \frac{\rho}{3\varepsilon_0}\boldsymbol{C_1P}.$$

Next, look at the sphere on the right. The point P is inside this sphere. The sphere is uniformly charged with negative charge density −ρ. The field at P due to this sphere alone is

$$\boldsymbol{E_2} = -\frac{\rho}{3\varepsilon_0}\boldsymbol{C_2P}.$$

The net field at P due to the given charge distribution is

$$\begin{aligned}\boldsymbol{E} &= \boldsymbol{E_1} + \boldsymbol{E_2}\\ &= \frac{\rho}{3\varepsilon_0}(\boldsymbol{C_1P} - \boldsymbol{C_2P}) = \frac{\rho}{3\varepsilon_0}\boldsymbol{C_1C_2}.\end{aligned}$$

The net field is not affected by the position of P. It is the same at all points in the overlap region. It is along the direction $\boldsymbol{C_1C_2}$ and has the same magnitude.

Concepts at a Glance

1. Gauss's law: $\nabla\cdot\boldsymbol{E} = \rho/\varepsilon_0$, $\oint \boldsymbol{E}\cdot d\boldsymbol{a} = q_{\text{en}}/\varepsilon_0$.
2. The electric field due to a uniformly charged sphere is $\boldsymbol{E}_{\text{in}} = \frac{\rho \boldsymbol{r}}{3\varepsilon_0}$, $\boldsymbol{E}_{\text{out}} = \frac{\rho \boldsymbol{r}}{3\varepsilon_0}\frac{R^3}{r^3}\cdot$

3. For a long, uniformly charged line, $\boldsymbol{E} = \dfrac{\lambda \boldsymbol{s}}{2\pi\varepsilon_0 s^2}\cdot$

4. For a large, uniformly charged plane, $\boldsymbol{E} = \dfrac{\sigma}{2\varepsilon_0}\hat{\boldsymbol{n}}.$

5. Gauss's law is valid in all situations, though its integral form is useful only in the calculation of the electric field in certain symmetric situations.

EXERCISES

Based on Concepts

1. Consider two concentric spheres of radii R and $2R$. There is no charge in the space between them. If the flux of the electric field, $\int \boldsymbol{E}\cdot d\boldsymbol{a}$, through the inner sphere is Z, what will be the flux through the outer sphere?
2. Two charges are placed on the x-axis, at $x = 0$ and $x = a$. What will be the value of $\nabla \cdot \boldsymbol{E}$ at $x = a/3$?
3. A sphere of radius R has charge Q distributed uniformly in its volume. What will be the value of $\nabla \cdot \boldsymbol{E}$ at a point with $r = R/4$?
4. A sphere of radius R has a charge distributed uniformly throughout its volume. The electric field at $r = R/4$ is E_0. What will it be at $r = R/2$ and at $r = 2R$?
5. The electric field is zero everywhere around a point P (say up to 1 cm on all sides). Can you have a nonzero charge density at P?
6. Give an example where the electric field $\boldsymbol{E}(\boldsymbol{r})$ is zero at a point but $\nabla \cdot \boldsymbol{E}$ is not zero there.
7. The flux of the electric field through a closed surface is zero. Which of the following statements are necessarily true?
   (a) The electric field at any point is zero.
   (b) The electric field at any point on the surface is zero.
   (c) The electric field at any point outside the surface is zero.
   (d) There is no charge anywhere in the volume enclosed by the surface.
   (e) There is as much positive charge as negative charge in the volume enclosed by the surface.
8. Consider the volume between two concentric shells of different radii. What is the surface that forms the boundary of this volume? Can we apply $\oint \boldsymbol{E}\cdot d\boldsymbol{a} = q_{en}/\varepsilon_0$ to this volume? Which surface will you integrate on?
9. A point charge q is placed at a corner A of an equilateral triangle ABC of side L. What is the flux of the electric field through the triangle?
10. A point charge q is placed at the centre of a cube. What is the flux of the electric field through one face of the cube?

11. Positive charges q_1 and q_2 are placed at the origin and (0, 0, –1 m) respectively. Consider a hemispherical surface in the region $z > 0$, centred on the origin, with radius 0.5 m. Is the flux of the resultant electric field through this surface greater than, equal to or less than $q_1/(2\varepsilon_0)$?
12. The charge density at a point P and around it is zero. Show that if a charge q is placed at P, it cannot be in stable equilibrium for displacement in all directions.
13. The electric field in a region is given by $\boldsymbol{E} = k(x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}})$. Suggest one situation in which such a field can be produced.

Problems

1. The charge density ρ corresponding to a point charge q placed at the origin is given as $\rho = q\delta^3(\boldsymbol{r})$. Using this expression, and Gauss's law $\nabla \cdot \boldsymbol{E} = \rho/\varepsilon_0$, show that $\nabla \cdot \left(\dfrac{\hat{\boldsymbol{r}}}{r^2}\right) = 4\pi\delta^3(\boldsymbol{r})$.
2. The electric field in a region is given by $\boldsymbol{E} = Ar^2\hat{\boldsymbol{r}}$ using spherical polar coordinates. Find the charge density. [**Ans.** $4\varepsilon_0 Ar$]
3. The electric field in a region is given by $\boldsymbol{E} = \dfrac{k}{\varepsilon_0}r^2\hat{\boldsymbol{r}}$ in spherical polar coordinates. Find the charge contained in (a) the shell $a/2 < r < a$ and (b) the cone $0 < \theta < \pi/6$, $0 < r < a$ with a spherical cap.

   [**Ans.** (a) $\dfrac{15\pi ka^4}{4}$, (b) $(2-\sqrt{3})\pi a^4 k$]
4. Suppose the electric field is given by the expression $\boldsymbol{E}(\boldsymbol{r}) = \dfrac{Ae^{-\lambda r}\hat{\boldsymbol{r}}}{r}$ for $r < R$. Find the total charge in the region $r < R$. [**Ans.** $4\pi\varepsilon_0 A R e^{-\lambda R}$]
5. The electric field in a region $r > r_0$ is given as $\boldsymbol{E} = \dfrac{A}{r^3}(2\cos\theta\,\hat{\boldsymbol{r}} + \sin\theta\,\hat{\theta})$ in spherical polar coordinates. Find the charge density in the region. [**Ans.** Zero]
6. The electric field in a region $r_0 < r < 2r_0$, is given (in spherical coordinates) by $\boldsymbol{E} = -(A\hat{\boldsymbol{r}} + B\sin\theta\cos\varphi\,\hat{\varphi})/r$, where A and B are constants. What is the charge density in the region?

   [**Ans.** $\dfrac{1}{r^2}(A - B\sin\phi)$]
7. The electric field in a cylindrical region $a < s < 2a$ is given by $\boldsymbol{E} = \dfrac{\rho_0}{\varepsilon_0}s\hat{\boldsymbol{s}} + \dfrac{\rho_0}{\varepsilon_0}a\cos\phi\,\hat{\phi}$. Find the charge contained in a length L of this region. [**Ans.** $8\pi\rho_0 La^2$]
8. The electric field in a region is given by $\boldsymbol{E} = \dfrac{As^2}{3\varepsilon_0}\hat{\boldsymbol{s}}$ in cylindrical coordinates. Find the charge density in the region. [**Ans.** As]
9. A charged particle is released from rest from the origin in an electric field, and moves along the $\hat{i}$ direction, covering a distance x from the starting point in a time $t \propto \sqrt{x}$. Assuming only the electric force to act, show that the region is free of charge.
10. A plastic rod of length L is kept along the x-axis between $x = 0$ and $x = L$. It is given a charge density $\lambda(x) = Ax$. Find the flux of the electric field through a cube with its centre at the origin and having a length $L/2$. [**Ans.** $\dfrac{AL^2}{32\varepsilon_0}$]

11. A rectangular plate of length L and width $L/2$ is placed in a horizontal plane. A point charge q is located just above the centre of one of the longer edges at a distance $L/2$ from it. Find the flux of the electric field due to q, through the surface of the plate. [**Ans.** $\frac{q}{12\varepsilon_0}$]

12. A point charge q is placed at one of the corners of a cube of length L. Find the flux of the electric field through a face (a) not containing this charge and (b) containing this charge. [**Ans.** (a) $\frac{q}{24\varepsilon_0}$ (b) 0]

13. A charge q is placed at the origin. Find the flux of its electric field through the cylindrical surface $= R, -\frac{L}{2} < z < \frac{L}{2}\cdot$ [**Ans.** $\frac{qL}{\varepsilon_0\sqrt{L^2+4R^2}}$]

14. Find the flux of the electric field $\boldsymbol{E} = \frac{k\hat{\boldsymbol{r}}}{r^2}$ over the surface given by $r = R, 0 < \theta < \frac{\pi}{4}\cdot$ [**Ans.** $2\pi k\left(1-\frac{1}{\sqrt{2}}\right)$]

15. The electric field in a region is given by $\boldsymbol{E} = \frac{k}{s}\hat{\boldsymbol{s}}$ in cylindrical coordinates. Calculate the flux of this field through the surface given by $S = R, 0 < \phi < \frac{\pi}{2}, -\frac{L}{2} < z < \frac{L}{2}\cdot$ [**Ans.** $\frac{kL\pi}{2}$]

16. A spherical surface of radius R, centred on the origin, contains a surface charge density $\sigma = \sigma_0 \sin\theta$ in spherical polar coordinates. Find the electric flux due to this charge through other spherical surfaces, centred on the origin, of radii (a) $R/2$ and (b) $2R$. [**Ans.** (a) zero (b) $\frac{\pi^2\sigma_0 R^2}{\varepsilon_0}$]

17. The electric charge density in a hydrogen atom in the ground state is given by $\rho(\boldsymbol{r}) = -\frac{e}{\pi a_0^3}e^{-2r/a_0}$, where a_0 is the Bohr radius (53 pm) and a_0 is a constant. The proton is located at the origin. Find the electric field at $r = a_0$. [**Ans.** $\frac{5e}{4\pi\varepsilon_0 a_0^2 e'^2}$, where e' is the base of the natural logarithm]

18. Suppose the electric field in a region is given by $\boldsymbol{E} = kr^4\hat{\boldsymbol{r}}$ in spherical coordinates. (a) Find the charge density ρ. (b) Determine the total charge contained in a sphere of radius R centred on the origin using the integral form of Gauss's law. (c) Verify the result in part (b) by integrating the density ρ obtained in part (a). [**Ans.** (a) $6k\varepsilon_0 r^3$ (b) $4k\pi\varepsilon_0 R^6$]

19. Two point charges, q each, are placed in a uniformly charged sphere of radius R, having a total charge $-2q$ in such a way that the force on each of the two point charges is zero. Find the distance between the two point charges. [**Ans.** R]

20. A uniform volume charge distribution ρ exists in a spherical region of radius R, except for a spherical cavity of radius $R/2$, as shown in Figure 2E.1. Determine the electric field inside the cavity and the flux of the electric field through the left hemispherical surface of the cavity.

[**Ans.** $\frac{\rho R}{6\varepsilon_0}, \frac{\pi\rho R^3}{24\varepsilon_0}$]

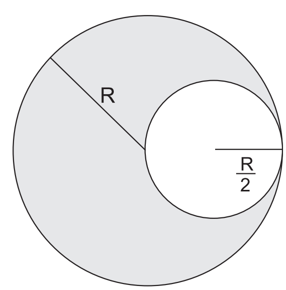


Fig. 2E.1

21. Consider a charge distribution with volume charge density given by

$$\rho(\boldsymbol{r}) = \frac{Q}{4\pi R^2}\delta(r - R).$$

Find the corresponding electric field. [**Ans.** Zero for $r < R$ and $\dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}}$ for $r > R$]

22. An infinite plane slab of thickness d carries a uniform volume charge density ρ in it. Assuming the surfaces of the slab to be at $z = \pm d/2$, find the electric field as a function of z. [**Ans.** $E_{\text{in}} = \dfrac{\rho z}{\varepsilon_0}$, $E_{\text{out}} = \dfrac{\rho d}{2\varepsilon_0}$]

23. A plane charge layer with a uniform surface charge density σ is placed near an unknown volume charge distribution. The layer coincides with the x–y plane. P(a, a, ε) and P′(a, a, $-\varepsilon$) are points close to the surface. If the electric field at P is $E_1\hat{\boldsymbol{i}} + E_2\hat{\boldsymbol{j}} + E_3\hat{\boldsymbol{k}}$, find the field at P′. [**Ans.** $E_1\hat{\boldsymbol{i}} + E_2\hat{\boldsymbol{j}} + \left(E_3 - \dfrac{\sigma}{\varepsilon_0}\right)\hat{\boldsymbol{k}}$]

24. An infinite slab of thickness d, placed with the central plane along the y–z plane, carries a charge density $\rho = \rho_0\left(1 - \dfrac{|x|}{d}\right)$. Find the electric field everywhere. [**Ans.** $\dfrac{\rho d}{4\varepsilon_0}$ outside, $\dfrac{\rho}{\varepsilon_0}\left(x - \dfrac{x^2}{2d}\right)$ inside]

25. A spherical surface of radius R carries a charge Q distributed uniformly on the entire surface. How much charge should be kept at the centre so that there is no electrostatic force on any part of the surface? [**Ans.** $Q/2$]

26. A spherical cavity of radius a is cut inside a long, charged solid cylinder of radius $b(b > a)$ carrying an otherwise uniform charge density ρ_0. The axis of the cylinder is taken to be the z-axis and the centre of the cavity is at the origin. Find the electric field inside the cavity and express your results in Cartesian coordinates. [**Ans.** $\dfrac{\rho}{6\varepsilon_0}(x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} - 2z\hat{\boldsymbol{k}})$]

27. Find an expression for $\boldsymbol{E}(x, y, z)$ due to charge distributions

(a) $\rho(x, y, z) = \dfrac{A}{a^2 + x^2}$ (b) $\rho(x, y, z) = \dfrac{A}{a^2 + x^2 + y^2}$ (c) $\rho(x, y, z) = \dfrac{A}{a^2 + x^2 + y^2 + z^2}$.

[**Ans.** (a) $\dfrac{A}{a\varepsilon_0}\tan^{-1}\left(\dfrac{x}{a}\right)$ (b) $\dfrac{A}{\varepsilon_0 \cdot 2\sqrt{x^2 + y^2}}\ln\left(\dfrac{a^2 + x^2 + y^2}{a^2}\right)$ (c) $\dfrac{4\pi A}{\varepsilon_0}\left(\dfrac{1}{r} - \dfrac{a}{r^2}\tan^{-1}\dfrac{r}{a}\right)$]

28. A spherical volume of radius R has a uniform charge density ρ_0. Consider a cubical volume of edge length $\dfrac{R}{4}$ with its centre at a distance of $\dfrac{R}{2}$ from the centre of the sphere. Find the force on the charge contained in this cube due to the rest of the charge in the sphere. [**Ans.** $\dfrac{\rho^2 L^4}{128\varepsilon_0}$]

29. A spherical surface of radius R carries a charge with uniform surface charge density σ. Consider an area on this surface given by $0 < \theta < \pi/6$. Find the fore on this area due to the rest of the spherical surface.

[**Ans.** $\dfrac{2 - \sqrt{3}}{2\varepsilon_0}\sigma^2 R^2\hat{\boldsymbol{n}}$]

30. In Figure 2E.2, S is a spherical surface (no material) enclosing one of the plates of a parallel-plate capacitor. The capacitor is being charged by a current $I(t)$. Find the rate at which the flux of the electric field through the spherical surface changes. Do not neglect the fringing of the field. [**Ans.** $-\dfrac{I(t)}{\varepsilon_0}$]

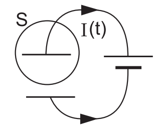


Fig. 2E.2

❑

3 Electric Potential and Potential Energy

3.1 Electric Potential Energy

You must have learnt about potential energy when you were in school. A stretched or a compressed spring has elastic potential energy, a stone close to the earth's surface has gravitational potential energy, and so on. Whenever two or more objects exert forces on each other that are conservative, you can define the potential energy of the group of objects corresponding to these forces. The earth attracts the stone and the stone attracts the earth. Corresponding to this interaction, the earth–stone system has a potential energy. Various parts of a stretched spring pull or push each other and, corresponding to these forces, the spring has a potential energy.

Not all forces correspond to potential energy. A box sliding on a table exerts a frictional force on the table and the table exerts a frictional force on the box. But these forces do not correspond to anything like 'frictional potential energy'. Frictional forces are not conservative.

The electrostatic forces between charged particles are conservative and one can define the electrostatic potential energy or electric potential energy corresponding to these forces. How do we do so?

Let us start with a system of two charges q_1 and q_2. Let us also assume that when the two charges are far from each other, the potential energy of this system is zero. Now let us bring the charges closer and put them at a separation r. To be specific, suppose the charge q_1 is permanently fixed at the origin, and q_2 is on the x-axis, initially at a large distance (infinite) from q_1. Let us exert a force on q_2 of the right amount and in the right direction so that q_2 moves very slowly without any appreciable acceleration and finally put it at a separation r from q_1. How much work did we do?

Suppose, during the transit, at a certain time, the charge q_2 is at a distance x from q_1. The Coulomb force on q_2 at this time is

Fig. 3.1

$$F_1 = \frac{q_1 q_2}{4\pi\varepsilon_0 x^2}\hat{\boldsymbol{i}}$$

in the direction away from q_1 (assuming the same sign for q_1 and q_2). You have to apply the same amount of force on q_2, towards q_1, in order to ensure that there is no acceleration in q_2. Thus the force you apply is

$$\boldsymbol{F} = -\frac{q_1 q_2}{4\pi\varepsilon_0 x^2}\hat{\boldsymbol{i}}$$

and hence the work you do over a small displacement $\boldsymbol{dr}$ when q_2 moves a distance dx is

$$\begin{aligned} dW = \boldsymbol{F}\cdot\boldsymbol{dr} &= \left(-\frac{q_1 q_2}{4\pi\varepsilon_0 x^2}\hat{\boldsymbol{i}}\right)\cdot(dx\,\hat{\boldsymbol{i}}) \\ &= -\frac{q_1 q_2}{4\pi\varepsilon_0 x^2}dx. \end{aligned}$$

The net work you do in decreasing the separation from infinity to r is

$$W = \int_{\infty}^{r} -\frac{q_1 q_2}{4\pi\varepsilon_0 x^2}dx = \frac{q_1 q_2}{4\pi\varepsilon_0 r}\cdot$$

If you do positive work on a system, its energy is increased by the same amount. You have done work W on the system (the charge q_1 was kept fixed, the work done on it was zero, so the work done on q_2 was the same as that on the system 'q_1 and q_2'). Therefore, the energy of this system must have increased by this amount. Initially, the separation between the charges was large and the potential energy was taken to be zero. So the potential energy, after the separation, was reduced to r,

$$U = \frac{q_1 q_2}{4\pi\varepsilon_0 r}\cdot \tag{3.1}$$

Equation 3.1 gives the potential energy of a pair of charges q_1, q_2 at a separation r. We had put several restrictions in order to obtain this expression. We had assumed that q_1 was fixed and only q_2 was moved. We assumed the two charges to have the same sign. We moved q_2 along a straight line. All these restrictions may be removed. Let two charges q_1 and q_2 (of any signs) be at a large separation from each other and be brought to a given separation, in any way, ensuring that they are at rest to begin with and also at the end. Then the work done will always be the same. In fact, this kind of statement is true for all conservative forces, and the Coulomb force is a conservative force. Equation 3.1 gives the expression of potential energy of the pair of charges at a separation r, independent of how they are brought closer.

If the signs of q_1 and q_2 are opposite, the potential energy of the system will be negative. It is still given by Equation 3.1. If the charge q_1 is at position $\boldsymbol{r_1}$ and q_2 is at position $\boldsymbol{r_2}$, the separation between them will be $|\boldsymbol{r_1} - \boldsymbol{r_2}|$ and the potential energy will be expressed as

$$U = \frac{q_1 q_2}{4\pi\varepsilon_0\,|\boldsymbol{r_1} - \boldsymbol{r_2}|}\cdot \tag{3.2}$$

If you wish to express the potential energy of a system of more than two point charges, you have to write the potential energy terms for all possible pairs of charges and add them. Thus if there are three charges q_1, q_2, q_3 at $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, respectively (Figure 3.2), there are three possible pairs (q_1, q_2), (q_3, q_1), (q_2, q_3) and so the potential energy of the system will be

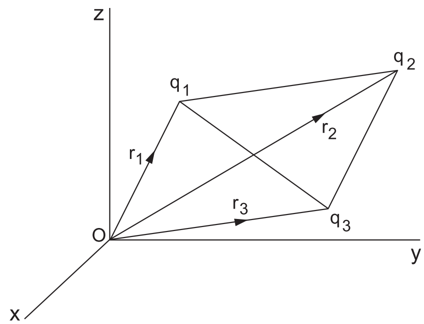


Fig. 3.2

$$U = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|} + \frac{q_1 q_3}{|\mathbf{r}_1 - \mathbf{r}_3|} + \frac{q_2 q_3}{|\mathbf{r}_2 - \mathbf{r}_3|}\right].$$

In general, the potential energy of a system of N point charges may be expressed as

$$U = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1,\, j\neq i}^{N}\frac{1}{4\pi\varepsilon_0}\frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|}. \tag{3.3}$$

The factor of $\frac{1}{2}$ appears because when you vary i from 1 to N and j from 1 to N, you count each pair twice. For example, the pair (q_2, q_3) is counted when $i = 2$ and $j = 3$ and also when $i = 3$ and $j = 2$. The factor of $\frac{1}{2}$ makes the correction. The condition $j \neq i$ is needed so that you always have two different charges in the pair. You do not express the potential energy of a point charge q_i with itself. You assume that the point charges are given readymade to you, and you are only placing them at proper positions. Equation 3.3 can also be written as

$$U = \sum_{i=j+1}^{N}\sum_{j=1}^{N}\frac{1}{4\pi\varepsilon_0}\frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|}. \tag{3.4}$$

EXAMPLE 3.1 **Find the potential energy of a system of four charges 1 nC, 4 nC, –2 nC, –1 nC kept in order at the four vertices of a square of side 1 cm.**

Solution The situation is shown in Figure 3.3. There are six pairs of charges and so the potential-energy expression will have six terms.

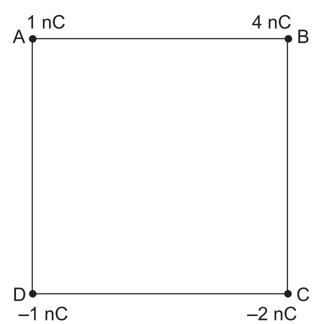


Fig. 3.3

$$U = \frac{1}{4\pi\varepsilon_0}\left[\frac{(1\,\text{nC})(4\,\text{nC})}{\text{AB}} + \frac{(1\,\text{nC})(-2\,\text{nC})}{\text{AC}} + \frac{(1\,\text{nC})(-1\,\text{nC})}{\text{AD}}\right.$$

$$\left. + \frac{(4\,\text{nC})(-2\,\text{nC})}{\text{BC}} + \frac{(4\,\text{nC})(-1\,\text{nC})}{\text{BD}} + \frac{(-2\,\text{nC})(-1\,\text{nC})}{\text{CD}}\right]$$

$$= 9\times 10^{9}\times 10^{-18}\left[\frac{4}{10^{-2}} - \frac{2}{\sqrt{2}\times 10^{-2}} - \frac{1}{10^{-2}} - \frac{8}{10^{-2}} - \frac{4}{\sqrt{2}\times 10^{-2}} + \frac{2}{10^{-2}}\right]\text{J}$$

$$= 9\times10^{-7}\left[-3-\frac{6}{\sqrt{2}}\right]\text{J} = -27(\sqrt{2}+1)\times10^{-7}\text{ J}.$$

The zero of potential energy

While formulating Equations 3.1, 3.2 and 3.3, we have assumed that the potential energy of the charge system is zero when all the charges are separated from each other by large distances. We have done this by choice—there is no compulsion. We can take the potential energy of an infinitely separated charge system to be U_0, if we like. Then, in all the expressions for potential energy, we have to add U_0. Ultimately, it is the change in potential energy that is important and the constant U_0 added in all configurations does not matter.

3.2 Electric Potential Due to a Given Charge Distribution

Suppose there is a charge distribution consisting of charges $q_1, q_2, q_3, \ldots, q_N$ placed at $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \ldots, \boldsymbol{r}_N$ respectively. The charges are all fixed at their places. The potential energy of this system is

$$U = \frac{1}{2}\sum_i \sum_{j\neq i} \frac{1}{4\pi\varepsilon_0}\frac{q_i q_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}.$$

Suppose we bring yet another charge q from far away and place it at a certain point $\boldsymbol{r}$. The N charges q_i are all fixed at their respective positions. By bringing the charge q, we have formed new pairs and hence the potential energy has changed. What is this extra potential energy? The new pairs are $(q, q_1), (q, q_2), (q, q_3), \ldots, (q, q_N)$. And so, the increase in potential energy due to bringing the charge q from infinity to $\boldsymbol{r}$ is

$$\Delta U = \sum_{i=1}^{N} \frac{1}{4\pi\varepsilon_0}\frac{q q_i}{|\boldsymbol{r} - \boldsymbol{r}_i|}.$$

The quantity

$$V(\boldsymbol{r}) = \frac{\Delta U}{q}$$

is called the potential at $\boldsymbol{r}$ due to the given charge distribution. For the charge distribution that we are discussing,

$$V(\boldsymbol{r}) = \sum_{i=1}^{N} \frac{q_i}{4\pi\varepsilon_0\ |\boldsymbol{r} - \boldsymbol{r}_i|}. \tag{3.5}$$

For a single point charge q placed at $\boldsymbol{r}'$, the potential at the point $\boldsymbol{r}$ will be

$$V(\boldsymbol{r}) = \frac{q}{4\pi\varepsilon_0|\boldsymbol{r} - \boldsymbol{r}'|}. \tag{3.6}$$

Note that although the potential V itself is a scalar quantity, it is a function of the space point represented by the position vector $\boldsymbol{r}$. It is an example of a scalar field. If you know the potential V at a point $\boldsymbol{r}$ due to a given charge distribution, you can immediately tell what will be the increase in electric potential energy if you place a new charge q at that point. The increase will be qV. As the increase in potential energy is also equal to the work done on the charge in bringing it from infinity to that point, the potential can also be defined as follows.

The potential at a point, due to a given charge distribution, is equal to the work done per unit charge when a charge is brought from infinity and placed at that point, without disturbing the given charge distribution.

As $V = \Delta U/q$, its SI unit is J/C, also called the volt, denoted by V.

The charge q used in this definition is not part of the charge distribution that produces the potential. It is sometimes called test charge. As long as the given charge distribution is fixed, the magnitude of the test charge has no significance as the potential is independent of it.

Often we are interested in finding the change in potential energy as a charge q is taken from one point to another without disturbing any other charge. If V_1 and V_2 are the potentials at these points, the change in potential energy will be $\Delta U = q(V_2 - V_1)$.

EXAMPLE 3.2 **Three charges 1 nC, –2 nC and –3 nC are placed at (1 cm, 0, 0), (0, 1 cm, 0) and (0, 0, 1 cm) respectively. (a) Find the potential at the origin. (b) If a charge of 4 nC is brought from a large distance to the origin, what will be the change in the potential energy?**

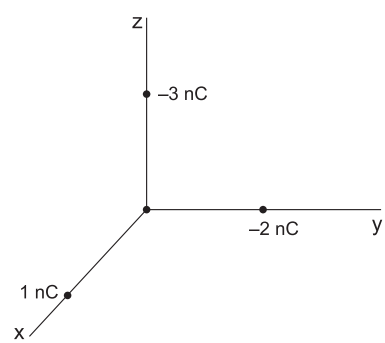


Fig. 3.4

Solution (a) The situation is shown in Figure 3.4. The potential at the origin is

$$\frac{1}{4\pi\varepsilon_0}\left[\frac{1\text{ nC}}{1\text{ cm}} + \frac{-2\text{ nC}}{1\text{ cm}} + \frac{-3\text{ nC}}{1\text{ cm}}\right]$$

$$= 9 \times 10^9 \times \frac{(-4 \times 10^{-9})}{10^{-2}}\text{ J/C}$$

$$= -3600\text{ J/C}.$$

(b) The increase in potential energy when a charge of 4 nC is placed at the origin is

$$\Delta U = qV = (4 \times 10^{-9}\text{ C})(-3600\text{ J/C})$$

$$= -1.44 \times 10^{-5}\text{ J}.$$

So, the potential energy is decreased by 1.44×10^{-5} J.

Look at Equation 3.5. It gives the potential at a point $\boldsymbol{r}$ due to a charge distribution having point charges, that is, due to a discrete charge distribution. You can modify this expression to get the potential due to a continuous charge distribution. Suppose there is a charge distribution described by the volume charge density $\rho(\boldsymbol{r}')$ in a given volume. In place of $\boldsymbol{r}_i$, you can write $\boldsymbol{r}'$ to show the location of the source charge (which produces the potential). Take a small volume element $d\tau'$ at $\boldsymbol{r}'$ (Figure 3.5). The charge contained in this volume is $\rho(\boldsymbol{r}')\,d\tau'$. The charge q_i in Equation 3.5 is to be replaced by $\rho(\boldsymbol{r}')\,d\tau'$. The summation over the given discrete charges is to be replaced by the integration over the volume containing the charge distribution. Equation 3.5 is then modified as

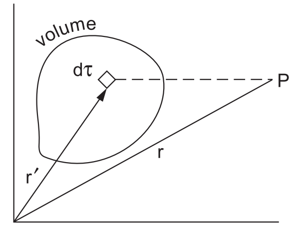


Fig. 3.5

$$V(\boldsymbol{r}) = \int_{\text{volume}} \frac{\rho(\boldsymbol{r}')\,d\tau'}{4\pi\varepsilon_0\,|\boldsymbol{r}-\boldsymbol{r}'|}, \tag{3.7}$$

which gives the potential at the point $\boldsymbol{r}$ due to the volume charge distribution $\rho(\boldsymbol{r}')$.

If you have a surface charge distribution characterized by the surface charge density $\sigma(\boldsymbol{r}')$, the expression for potential at $\boldsymbol{r}$ will be

$$V(\boldsymbol{r}) = \int_{\text{surface}} \frac{\sigma(\boldsymbol{r}')\,da'}{4\pi\varepsilon_0\,|\boldsymbol{r}-\boldsymbol{r}'|}, \tag{3.8}$$

where da' is a small surface element on the surface at point $\boldsymbol{r}'$, carrying the charge distribution. The surface integration has to be performed over the whole of the surface charge. For a line charge, the corresponding expression is

$$V(\boldsymbol{r}) = \int_{\text{line}} \frac{\lambda(\boldsymbol{r}')\,dl'}{4\pi\varepsilon_0|\boldsymbol{r}-\boldsymbol{r}'|}, \tag{3.9}$$

where $\lambda(\boldsymbol{r}')$ gives the linear charge density, dl' is a line element on the linear charge at point $\boldsymbol{r}'$, and the line integration has to be performed over the line carrying the charge.

EXAMPLE 3.3 **A thin, spherical shell of radius R carries a charge Q distributed uniformly over its surface. Find the potential due to this charge at a point inside the shell, at a distance r from the centre of the shell.**

Solution Take the centre O of the shell as the origin and the line OP as the z-axis, where P is the point where the potential is required. The Cartesian coordinates of P are (0, 0, r) and so

$$\boldsymbol{r} = \boldsymbol{OP} = r\hat{\boldsymbol{k}}.$$

Take a surface element at the spherical coordinates (R, θ, ϕ) on the surface of the shell where the charge is present. The area of the element is

$$da' = R^2 \sin\theta' \, d\theta' \, d\phi'$$

and the position vector of the location of this element is

$$\boldsymbol{r}' = R\sin\theta'\cos\phi'\,\hat{\boldsymbol{i}} + R\sin\theta'\sin\phi'\,\hat{\boldsymbol{j}} + R\cos\theta'\,\hat{\boldsymbol{k}}.$$

The distance of the field point P from the element da' is

$$|\boldsymbol{r} - \boldsymbol{r}'| = [(-R\sin\theta'\cos\phi')^2 + (-R\sin\theta'\sin\phi')^2 + (r - R\cos\theta')^2]^{1/2}$$

$$= [R^2 + r^2 - 2Rr\cos\theta']^{1/2}.$$

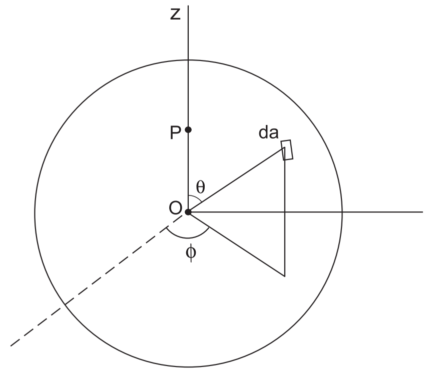


Fig. 3.6

The surface charge density $\sigma = \dfrac{Q}{4\pi R^2}$ is the same everywhere on the surface of the shell.

Now you have all the information and are ready to write the potential at P. It is

$$V = \int \frac{\sigma(\boldsymbol{r}')\,da'}{4\pi\varepsilon_0 |\boldsymbol{r} - \boldsymbol{r}'|}$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{\sigma R^2 \sin\theta' \, d\theta' \, d\phi'}{4\pi\varepsilon_0 [R^2 + r^2 - 2Rr\cos\theta']^{1/2}}$$

$$= \frac{2\pi\sigma R^2}{4\pi\varepsilon_0} \int_{\theta=0}^{\pi} \frac{\sin\theta' \, d\theta'}{[R^2 + r^2 - 2Rr\cos\theta']^{1/2}}. \qquad \text{(i)}$$

To evaluate the integral, write

$$[R^2 + r^2 - 2Rr\cos\theta']^{1/2} = t$$

or $\qquad R^2 + r^2 - 2Rr\cos\theta' = t^2.$

You can see that t is the distance of the source point (R, θ', ϕ'), where the charge element is taken, from the point P, where the potential is to be evaluated. Then

$$2Rr\sin\theta' \, d\theta' = 2t\,dt$$

or $\qquad \sin\theta' \, d\theta' = \dfrac{t\,dt}{Rr}.$

So, $\qquad \displaystyle\int \frac{\sin\theta' \, d\theta'}{[R^2 + r^2 - 2Rr\cos\theta']^{1/2}} = \int \frac{t\,dt}{Rrt} = \frac{t}{Rr}.$

When $\theta' = 0$, $t = R - r$ and when $\theta' = \pi$, $t = R + r$. Why did we write $t = R - r$ for $\theta' = 0$, and not $r - R$? We did so because $t = [R^2 + r^2 - 2Rr\cos\theta']^{1/2}$ has to be positive. This is the distance between the field point and the source point, and has to be positive.

So, from (i),

$$V = \frac{2\pi\sigma R^2}{4\pi\varepsilon_0 Rr} [t]_{R-r}^{R+r} = \frac{\sigma R}{2\varepsilon_0 r} [(R + r) - (R - r)]$$

$$= \frac{\sigma R}{\varepsilon_0} = \frac{Q}{4\pi\varepsilon_0 R} \cdot$$

This is independent of r. So, at all points inside the spherical shell, the potential is the same.

3.3 Relation between *E* and *V*

We have discussed the procedures to calculate the electric field $\boldsymbol{E}(\boldsymbol{r})$ and the potential $V(\boldsymbol{r})$ due to a given charge distribution. How are the vector field $\boldsymbol{E}$ and the scalar field V related to each other? It turns out that if $\boldsymbol{E}(\boldsymbol{r})$ is given, you can get $V(\boldsymbol{r})$, and if $V(\boldsymbol{r})$ is given, you can get $\boldsymbol{E}(\boldsymbol{r})$, without any need to know about the charge distribution. What we will now discuss requires the use of vector operations called gradient and curl. If you are not comfortable with these, you can read Appendix 4 before moving to the next section.

E from *V*

The potential at $\boldsymbol{r}$ due to a charge q at the origin is

$$V(\boldsymbol{r}) = \frac{q}{4\pi\varepsilon_0|\boldsymbol{r}|} = \frac{q}{4\pi\varepsilon_0 r} \cdot$$

What is the gradient of this function $V(\boldsymbol{r})$? $V(\boldsymbol{r})$ is a function of the distance r from the origin. Using spherical polar coordinates,

$$\nabla V(\boldsymbol{r}) = \frac{\partial V}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}$$

$$= -\frac{q}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}} = -\frac{q\boldsymbol{r}}{4\pi\varepsilon_0|\boldsymbol{r}|^3} \cdot$$

But the electric field at $\boldsymbol{r}$ due to this charge is $\frac{q\boldsymbol{r}}{4\pi\varepsilon_0|\boldsymbol{r}|^3} \cdot$ This means that

$$\boldsymbol{E} = -\nabla V$$

for this charge.

Changing the origin does not change $\boldsymbol{E}$ or V as these fields depend only on the charges producing them. So the relation between them will also be the same if the origin is changed. This means that even if the charge is placed at a point other than the origin, the relation $\boldsymbol{E} = -\nabla V$ is valid.

Now consider a charge distribution with several charges $q_1, q_2, q_3, \ldots$. The field $\boldsymbol{E}$ at $\boldsymbol{r}$ due to this charge distribution is

$$\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2 + \boldsymbol{E}_3 + \ldots$$

and the potential is

$$V = V_1 + V_2 + V_3 + \ldots$$

where $\boldsymbol{E}_i$ and V_i are the electric field and potential at $\boldsymbol{r}$ due to the charge q_i.

$$\nabla V = \nabla V_1 + \nabla V_2 + \nabla V_3 + \ldots$$

$$= -\boldsymbol{E}_1 - \boldsymbol{E}_2 - \boldsymbol{E}_3 - \ldots = -(\boldsymbol{E})$$

or $$\boldsymbol{E} = -\nabla V \tag{3.10}$$

for the given charge distribution.

Equation 3.10 itself can be taken as the definition of electric potential. Any scalar function $V(\boldsymbol{r})$ that can give us the electric field on taking its gradient and multiplying by –1 is a valid potential. Is this a unique function for a given charge distribution? Not exactly.

$$\nabla V(r) = \nabla\{V(r) + V_0\},$$

where V_0 is a constant. Thus if $V(\boldsymbol{r})$ is a valid potential for a given charge distribution, $V(\boldsymbol{r}) + V_0$ is also a valid potential. So potential is arbitrary to a constant.

For a charge distribution confined to a finite volume, it is customary to take the potential at infinity to be zero. Again this is your choice and you have the right to choose $V = 0$ at any other point of your liking. However, there are advantages in choosing the potential to be zero at infinity and we advise you to respect this tradition. If the charge distribution itself extends to infinity, choosing $V = 0$ at a finite point may be more advantageous.

EXAMPLE 3.4 **The electric potential in the region $s > a$ is given by $V = k \ln \frac{s}{a}$, where s denotes distance from the z-axis. Find the electric field in this region.**

Solution We are using cylindrical polar coordinates. In this system,

$$\nabla V = \frac{\partial V}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{\partial V}{\partial z}\hat{z}.$$

$$\boldsymbol{E} = -\nabla V = -k\nabla\left(\ln \frac{s}{a}\right)$$

$$= -k\frac{\partial}{\partial s}\left(\ln \frac{s}{a}\right)\hat{s} = -\frac{k}{s}\hat{s}.$$

The electric field is towards the z-axis and its magnitude is inversely proportional to the distance s from the z-axis.

EXAMPLE 3.5 **Using the fact that $\boldsymbol{E} = -\nabla V$, show that**

$$\nabla\left|\frac{1}{\boldsymbol{r}-\boldsymbol{r}'}\right| = -\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3},$$

where $\boldsymbol{r}'$ is a fixed point.

Solution Suppose a charge q is kept at the position $\boldsymbol{r}'$. The electric field and the potential at the point $\boldsymbol{r}$ due to this charge are

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0}\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \quad \text{and} \quad V = \frac{q}{4\pi\varepsilon_0}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\cdot$$

But $\boldsymbol{E} = -\nabla V$.

So, $$\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} = -\nabla\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

or $$\nabla\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = -\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\cdot$$

As you know, the gradient of a scalar function at a given point tells you how fast the function is changing as you move a little from the given point. The direction of ∇V gives the direction in which V increases at the fastest rate. Thus the direction of $(-\nabla V)$ gives the direction in which V decreases at the fastest rate. This means the electric field has a direction along which the potential decreases at the fastest rate. keep this in mind. If you move along the direction of the electric field, the electric potential will decrease and if you move opposite to the direction of the electric field, the electric potential will increase.

$$\boldsymbol{E} = -\nabla V.$$

Thus, $$E_x\hat{\boldsymbol{i}} + E_y\hat{\boldsymbol{j}} + E_z\hat{\boldsymbol{k}} = -\frac{\partial V}{\partial x}\hat{\boldsymbol{i}} - \frac{\partial V}{\partial y}\hat{\boldsymbol{j}} - \frac{\partial V}{\partial z}\hat{\boldsymbol{k}}$$

or $$E_x = -\frac{\partial V}{\partial x},\ E_y = -\frac{\partial V}{\partial y},\ E_z = -\frac{\partial V}{\partial z}\cdot \tag{3.11}$$

If you move a small distance dx in the x-direction, look at the change dV in the potential and calculate $-(dV)/(dx)$, you will get the x-component of the electric field. The case is similar for the y- and z-directions. But any direction can be chosen as the x-, y-, z-direction. Thus, if you move a small distance dl in any direction, look at the change in potential dV and calculate $-(dV)/(dl)$—it will give you the component of the electric field in that direction.

$$E_l = -\frac{\partial V}{\partial l}\cdot \tag{3.12}$$

V from *E*

Suppose you are at a point P where there exists an electric field $\boldsymbol{E}$. Move to a nearby point Q.

Suppose the displacement vector $\boldsymbol{dl}$ makes an angle θ with the direction of the electric field. The component of $\boldsymbol{E}$ along $\boldsymbol{dl}$ is $E\cos\theta$. So,

$$E\cos\theta = -\frac{dV}{dl}, \quad \text{where } dV = V(Q) - V(P)$$

or $$dV = -Edl\cos\theta$$

or $$dV = -\boldsymbol{E}\cdot\boldsymbol{dl}. \tag{3.13}$$

Consider any two points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ (Figure 3.7). Connect them by any arbitrary path. Then the integral of $\boldsymbol{E}\cdot\boldsymbol{dl}$ from $\boldsymbol{r} = \boldsymbol{r}_1$ to $\boldsymbol{r} = \boldsymbol{r}_2$ along this path is

$$\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{E}\cdot\boldsymbol{dl} = -\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} dV$$

$$= -[V(\boldsymbol{r}_2) - V(\boldsymbol{r}_1)]$$

or $$V(\boldsymbol{r}_2) - V(\boldsymbol{r}_1) = -\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{E}\cdot\boldsymbol{dl}. \tag{3.14}$$

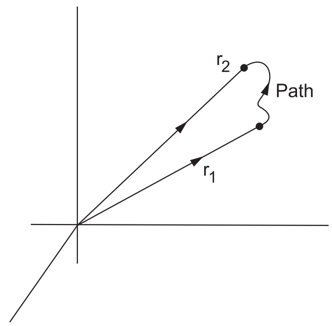


Fig. 3.7

Let us take $\boldsymbol{r}_1$ as a point at an infinite distance from the given charge distribution. Using Equation 3.14,

$$V(\boldsymbol{r}_2) - V(\infty) = -\int_{\infty}^{\boldsymbol{r}_2} \boldsymbol{E}\cdot\boldsymbol{dl}.$$

Take $V(\infty)$ to be zero. So,

$$V(\boldsymbol{r}_2) = -\int_{\infty}^{\boldsymbol{r}_2} \boldsymbol{E}\cdot\boldsymbol{dl}$$

or $$V(\boldsymbol{r}) = -\int_{\infty}^{\boldsymbol{r}} \boldsymbol{E}\cdot\boldsymbol{dl}. \tag{3.15}$$

The point where $V = 0$ is taken to be zero is called the reference point. We have taken the reference point at infinity. In general,

$$V(\boldsymbol{r}) = -\int_{\substack{\text{reference}\\ \text{point}}}^{\boldsymbol{r}} \boldsymbol{E}\cdot\boldsymbol{dl}. \tag{3.16}$$

You can choose any path from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$ to evaluate the integral in Equation 3.12, and you will get the same result, that is, $V(\boldsymbol{r}_2) - V(\boldsymbol{r}_1)$. The line integral $\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{E} \cdot \boldsymbol{dl}$ depends only on the end points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, and not on the path connecting $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. Such a field is said to be conservative and its curl is zero. Thus the electrostatic field $\boldsymbol{E}$ is conservative and has zero curl everywhere.

What if you take a closed loop and perform $\oint \boldsymbol{E} \cdot \boldsymbol{dl}$ on it? Using Equation 3.14, and taking $\boldsymbol{r}_1 = \boldsymbol{r}_2$ for the closed loop, $\oint \boldsymbol{E} \cdot \boldsymbol{dl} = V(\boldsymbol{r}) - V(\boldsymbol{r}) = 0$.

EXAMPLE 3.6 **The electric field in a region is given by $\boldsymbol{E} = \dfrac{p_0}{4\pi\varepsilon_0 r^3}[2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\theta}]$ in spherical polar coordinates, where p_0 is a constant. Find the potential function corresponding to this field taking the potential at infinity to be zero.**

Solution Let P be a general point (r, θ, ϕ) as shown in Figure 3.8. Let us move from infinity to this point along the line OP. In any short displacement along this path,

$$\boldsymbol{dl} = dr\,\hat{\boldsymbol{r}}.$$

Why didn't we write $\boldsymbol{dl} = -dr\,\hat{\boldsymbol{r}}$ when we were approaching the origin, i.e, moving along the $-\hat{\boldsymbol{r}}$ direction? Because, as we come towards the origin, dr itself is negative and so $\boldsymbol{dl} = dr\,\hat{\boldsymbol{r}}$ is in the $(-\hat{\boldsymbol{r}})$ direction.

$$\boldsymbol{E} \cdot \boldsymbol{dl} = \frac{p_0}{4\pi\varepsilon_0 r^3} 2\cos\theta\, dr$$

or
$$-\int_{\infty}^{r} \boldsymbol{E} \cdot \boldsymbol{dl} = -\frac{2p_0\cos\theta}{4\pi\varepsilon_0}\int_{\infty}^{r}\frac{dr}{r^3}$$

$$= \frac{p_0\cos\theta}{4\pi\varepsilon_0 r^2}.$$

Thus,
$$V(r, \theta, \phi) = \frac{p_0\cos\theta}{4\pi\varepsilon_0 r^2}.$$

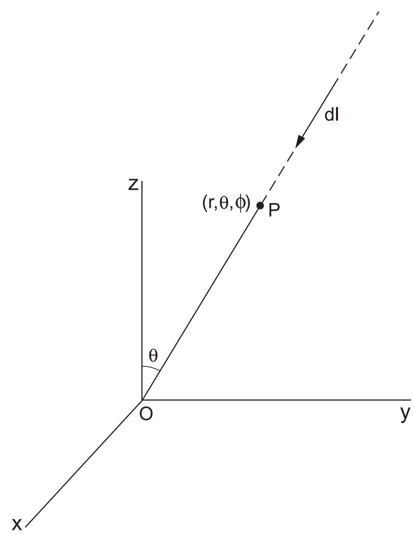


Fig. 3.8

In Example 3.6, the θ-component of $\boldsymbol{E}$ had no bearing on the calculation of V. If you replace $\sin\theta\hat{\theta}$ by $2\sin\theta\hat{\theta}$ in the expression for $\boldsymbol{E}$, you will still get the same potential function. How can you have two different electric field functions corresponding to the same potential? Find the answer to this puzzle yourself!

Curl of *E*

The electric field $\boldsymbol{E}$ due to a given charge distribution can be expressed as the gradient of a scalar function $(-V)$. The curl of the gradient of any scalar function is identically zero. Hence,

$$\nabla \times \boldsymbol{E} = -\nabla \times \nabla V = 0$$

for an electrostatic field.

The electric field produced by a changing magnetic field is not curl-free. Not only does $\int \boldsymbol{E} \cdot \boldsymbol{dl}$ depend on the end points, it depends on the actual path taken too. Thus we don't define potential corresponding to such an electric field.

3.4 Equipotential Surfaces and Field Lines

For a given charge distribution, the potential at a point is a specific function of the coordinates of that point. If you put this function equal to a constant V_0, you will get a surface. All points of this surface will have the same potential V_0. Such a surface is called an *equipotential surface*. By putting different values of V_0, one gets a family of equipotential surfaces for the same charge distribution.

For example, consider a point charge at the origin. The potential V at the point (r, θ, ϕ) is $V = \dfrac{q}{4\pi\varepsilon_0 r}\cdot$ If we put this expression equal to V_0, we get

$$\frac{q}{4\pi\varepsilon_0 r} = V_0$$

or $$r = \frac{q}{4\pi\varepsilon_0 V_0}.$$

This gives us a spherical surface of radius $\dfrac{q}{4\pi\varepsilon_0 V_0}$, centred on the origin. This is an equipotential surface. By taking different values of V_0, we get different spherical surfaces, each of which is an equipotential surface.

Consider a point P on an equipotential surface. Draw a small line dl from P on this surface. Don't worry if the surface is not plane and dl does not completely fall on the surface. We are talking about an infinitesimally small line, which should be tangential to the surface. As you move on the equipotential surface, the potential V remains the same. So the change in potential, dV, is zero. In other words, $dV/dl = 0$.

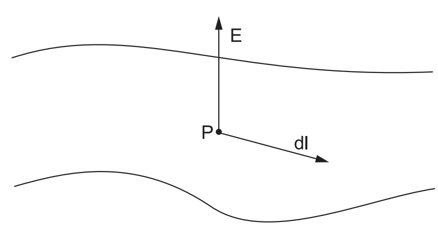


Fig. 3.9

But $-dV/dl$ is the component of the electric field at P along the direction of dl. So the electric field at P has no component along this direction. Draw an infinitesimally small line from P in any direction tangential to the equipotential surface, the component of the field in this direction is zero. Thus the field at P is normal to the equipotential surface through P. Electric field lines from points on an equipotential surface are always perpendicular to this surface.

3.5 Potential Energy of a Continuous Charge Distribution

The potential energy of a discrete charge distribution is given as

$$U = \frac{1}{2}\sum_i \sum_{j \neq i} \frac{q_i q_j}{4\pi\varepsilon_0 |\boldsymbol{r}_i - \boldsymbol{r}_j|}$$

$$= \frac{1}{2}\sum_i q_i \sum_{j \neq i} \frac{q_j}{4\pi\varepsilon_0 |\boldsymbol{r}_i - \boldsymbol{r}_j|}.$$

The quantity $\sum_{j \neq i} \frac{q_j}{4\pi\varepsilon_0 \, |\boldsymbol{r}_i - \boldsymbol{r}_j|}$ is the potential at the point $\boldsymbol{r}_i$ due to all charges except q_i. Expressing this as V_i, the potential energy of the charge distribution becomes

$$U = \frac{1}{2}\sum_i q_i V_i. \tag{3.17}$$

It is from here that we can formulate a useful expression for the electric potential energy of a continuous charge distribution. Suppose we have a continuous charge distribution described by a volume charge density $\rho(\boldsymbol{r})$. Then in place of q_i, we will write $\rho(\boldsymbol{r})d\tau$ and in place of summation over i, we will have integration over the volume of the charge distribution. Then

$$U = \frac{1}{2}\int \rho(\boldsymbol{r})\, V(\boldsymbol{r})\, d\tau. \tag{3.18}$$

For a surface charge distribution, this equation will be

$$U = \frac{1}{2}\int \sigma(\boldsymbol{r})\, V(\boldsymbol{r})\, da \tag{3.19}$$

and for a linear charge distribution, it will be

$$U = \frac{1}{2}\int \lambda(\boldsymbol{r})\, V(\boldsymbol{r})\, dl. \tag{3.20}$$

The volume over which you should integrate the expression in Equation 3.18 need not be the smallest one containing the charges. The only requirement is that the charge distribution be contained in the volume of integration. In fact, you can consider all space to be the volume of integration. Places where charges do not exist will any way have $\rho(\boldsymbol{r}) = 0$ and there will be no contribution from them.

There is a major difference between Equations 3.17 and 3.18. Equation 3.17 gives the change in potential energy when charges q_i are brought from infinity and assembled. The charges q_i themselves are given readymade to us. In Equation 3.18, the change in energy corresponds to the case when infinitesimal charges $\rho d\tau$ are brought from infinity and assembled at the desired position. These pieces $\rho d\tau$ are infinitesimal, no lumps are given to us readymade, so the two quantities are really different. However, each is significant and correct in its own situation.

Equation 3.18 has a reference to the charge distribution $\rho(\boldsymbol{r})$ and also to the potential field $V(\boldsymbol{r})$. We can altogether remove any reference to the charges and express the potential energy of the charge distribution in terms of the fields alone. The potential energy is

$$U = \frac{1}{2}\int \rho(\boldsymbol{r})\, V(\boldsymbol{r})\, d\tau.$$

Using Gauss's law ($\nabla \cdot \boldsymbol{E} = \rho/\varepsilon_0$), we get

$$U = \frac{1}{2}\int \varepsilon_0 (\nabla \cdot \boldsymbol{E})\, V d\tau. \tag{i}$$

$$\nabla \cdot (\boldsymbol{E}V) = (\boldsymbol{E} \cdot \nabla V) + (\nabla \cdot \boldsymbol{E})V \qquad \text{[from vector calculus]}$$

So $$(\nabla \cdot \boldsymbol{E})\, V = \nabla \cdot (\boldsymbol{E}V) - (\boldsymbol{E} \cdot \nabla V)$$
$$= \nabla \cdot (\boldsymbol{E}V) + E^2.$$

So, from (i), $$U = \int \frac{\varepsilon_0}{2}\left[E^2 + \nabla \cdot (\boldsymbol{E}V)\right] d\tau$$
$$= \int \frac{\varepsilon_0}{2} E^2 d\tau + \int \frac{\varepsilon_0}{2} \nabla \cdot (\boldsymbol{E}V)\, d\tau. \tag{ii}$$

What is the volume on which this integration is to be performed? Any volume that contains the whole charge distribution will do. Now use Gauss's divergence theorem to rewrite the second term in (ii).

$$\int \nabla \cdot (\boldsymbol{E}V)\, d\tau = \oint_{\text{surface}} \boldsymbol{E}V \cdot d\boldsymbol{a}$$

So $$U = \int_{\text{volume}} \frac{\varepsilon_0}{2} E^2 d\tau + \oint_{\text{surface}} \frac{\varepsilon_0}{2} \boldsymbol{E}V \cdot d\boldsymbol{a}. \tag{3.21}$$

Consider a closed surface completely enclosing the given charge distribution. Any closed surface that encloses all the charges given will do. Evaluate $\frac{\varepsilon_0}{2}\int E^2 d\tau$ over the volume and $\frac{\varepsilon_0}{2}\oint \boldsymbol{E}V \cdot d\boldsymbol{a}$ over the closed surface itself. Their sum will be the total electric potential energy.

What happens if we consider all space to be the volume of integration. The closed surface then extends to infinity in all directions. Look at the surface integration $\oint \boldsymbol{E}V \cdot d\boldsymbol{a}$. For a finite charge distribution, $\boldsymbol{E}$ falls off with distance as $\frac{1}{r^2}$ or faster for large values of r. The potential V falls off with distance as $\frac{1}{r}$ or faster. Thus $\boldsymbol{E}V$ will fall off with distance as $\frac{1}{r^3}$ or faster. Area increases with distance as r^2. Thus the integral will fall off as $\frac{1}{r}$ or faster as we increase the size of the volume. The surface integration $\oint \boldsymbol{E}V \cdot d\boldsymbol{a}$ will become zero if we take all space as the volume for integration in the first term so that r extends to infinity. Thus

$$U = \int\limits_{\substack{\text{all}\\\text{space}}} \frac{1}{2}\varepsilon_0 (E^2)\, d\tau. \tag{3.22}$$

Both Equations 3.18 and 3.22 give expressions for the same quantity, that is, electrostatic potential energy corresponding to a given charge distribution. If you look at Equation 3.18, the contribution to the energy seems to come only from places where $\rho \neq 0$, that is, where you have charge. From places where there is no charge, there is no contribution to the energy $U = \frac{1}{2}\int \rho(r)\, V(r)\, d\tau$. The energy seems to be concentrated in the charges.

On the other hand, if you look at Equation 3.22, all regions where $E \neq 0$ contribute to energy. A finite charge distribution creates a field everywhere and $U = \frac{1}{2}\varepsilon_0 \int E^2 d\tau$ gets a contribution from all the places where $E \neq 0$ even if $\rho = 0$. So the energy seems to be concentrated in the field.

Where does the potential energy reside? In the charges or in the fields? It turns out that both interpretations are equally valid and in electrostatic situations, you cannot decide between the two.

But you should mentally picture the energy to be associated with the electric field. We will often call it electric field energy. You can interpret Equation 3.22 as follows. In the volume $d\tau$ where an electric field exists, the electric field energy is $dU = \frac{1}{2}\varepsilon_0 E^2 d\tau$. The total energy of the field is obtained by integrating over the entire volume where the field exists. The quantity $\frac{dU}{d\tau} = \frac{1}{2}\varepsilon_0 E^2$ is called the electric field density.

<u>EXAMPLE 3.7</u> **Find the electric potential energy corresponding to a charge Q spread uniformly over a spherical surface of radius R.**

Solution For a uniformly charged spherical surface, the electric field is zero inside and

$$\boldsymbol{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}}$$

outside. The total potential energy for this charge distribution is

$$U = \int_{\substack{\text{all}\\\text{space}}} \frac{1}{2}\varepsilon_0 E^2 d\tau$$

$$= \frac{\varepsilon_0}{2}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=R}^{\infty}\left(\frac{Q}{4\pi\varepsilon_0 r^2}\right)^2 r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{Q^2}{32\pi^2\varepsilon_0}\left[\int_R^{\infty}\frac{dr}{r^2}\int_0^{\pi}\sin\theta\, d\theta\int_0^{2\pi}d\phi\right]$$

$$= \frac{Q^2}{32\pi^2\varepsilon_0}\left[\left|-\frac{1}{r}\right|_R^{\infty}\ \left|-\cos\theta\right|_0^{\pi}\ \left|\phi\right|_0^{2\pi}\right]$$

$$= \frac{Q^2}{8\pi\varepsilon_0 R}.$$

3.6 Boundary Conditions Satisfied by an Electrostatic Field

Consider a surface and let $\hat{\boldsymbol{n}}$ be the unit vector along the normal to it from a point P on the surface (Figure 3.10). The surface could be imaginary or that of a material body. Think of two points P_1 and P_2 very close to the surface, on opposite sides of it, and on the normal from P. The point P_1 is on the same side on which normal $\hat{\boldsymbol{n}}$ was drawn and P_2 is on the opposite side. Let the distance between P_1 and P_2 be h. Let the electric field (due to some stationary charge distribution) at P_1 be $\boldsymbol{E_1}$ and that at P_2 be $\boldsymbol{E_2}$. We are looking for a relation between $\boldsymbol{E_1}$, $\boldsymbol{E_2}$ and any surface charge density that may exist on the surface near P.

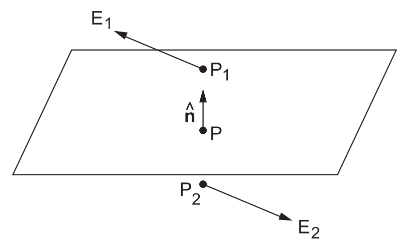


Fig. 3.10

(a) Relation between the normal components

Draw a surface of area Δa passing through P_1, parallel to the given surface. Draw a similar area through P_2 and join the corresponding points to form a cylindrical box (Figure 3.11) of height h. Apply Gauss's law on this box. The box has two flat surfaces, on the opposite sides of the given surface, and a curved surface. The flux of the electric field through the flat surface through P_1 is

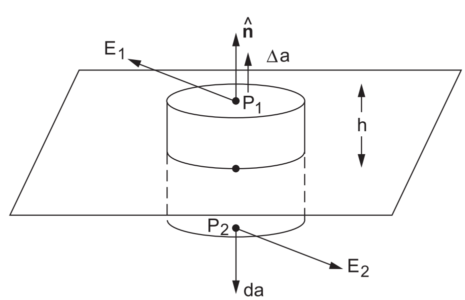


Fig. 3.11

$$\Phi_1 = \boldsymbol{E_1} \cdot \Delta \boldsymbol{a_1} = \boldsymbol{E_1} \cdot (\Delta a)\hat{\boldsymbol{n}} = E_{1n}\Delta a,$$

where E_{1n} is the normal component (component towards $\hat{n}$) of the electric field $\boldsymbol{E_1}$ at P_1. Now calculate the flux through the other flat surface, the one going through P_2. The area vector is $\Delta \boldsymbol{a_2} = \Delta a(-\hat{\boldsymbol{n}})$ and so the flux is

$$\Phi_2 = \boldsymbol{E_2} \cdot \Delta \boldsymbol{a_2} = \boldsymbol{E_2} \cdot (\Delta a)(-\hat{\boldsymbol{n}}) = -E_{2n}\Delta a,$$

where E_{2n} is the component of $\boldsymbol{E_2}$ along $\hat{\boldsymbol{n}}$.

The flux through the curved surface will tend to vanish if you take $h \to 0$, i.e., if the points P_1 and P_2 are very close to the surface. This is because the whole area of this curved surface will tend to zero. Then the flux through the cylindrical box will be

$$\Phi = E_{1n}\Delta a - E_{2n}\Delta a.$$

If there is a surface charge density σ at P, supposed to be uniform over the area Δa around P, the charge contained in the cylindrical box is $\sigma\Delta a$. Thus Gauss's law gives

$$E_{1n}\Delta a - E_{2n}\Delta a = \frac{\sigma\Delta a}{\varepsilon_0}$$

or

$$E_{1n} - E_{2n} = \frac{\sigma}{\varepsilon_0}. \tag{3.23}$$

If there is a surface charge density on the given surface, the normal component of the electric field is discontinuous across the surface by the amount σ/ε_0. If the surface does not contain any surface charge, the normal component has to be continuous.

Keep track of the direction of the normal. You have to call one side of the surface Side 1 and the other Side 2. The normal is to be drawn in Side 1. Components are to be taken in this direction. In Equation 3.23, remember to subtract E_{2n} (component in Side 2) from E_{1n} (component in Side 1).

(b) Relation between the tangential components

Again consider the same situation as shown in Figure 3.10. There is a surface which may have a surface charge density. P_1 and P_2 are two points on the opposite sides of and very close to the surface. The electric field is $\boldsymbol{E_1}$ at P_1 and $\boldsymbol{E_2}$ at P_2. From P_1, draw a line AB of length Δl parallel to the given surface (Figure 3.12). Can you draw more than one such line? Yes, you can draw infinite lines from P_1, all parallel to the given surface. Pick any one of them. Put an arrow on it. Let the unit vector in this direction be $\hat{\boldsymbol{t}}$. Then, $\boldsymbol{AB} = \Delta l\hat{\boldsymbol{t}}$. Draw a line CD of equal length Δl from P_2, parallel to the first line as shown in Figure 3.12. Then, $\boldsymbol{CD} = -\Delta l\hat{\boldsymbol{t}}$. Join the corresponding end points to make

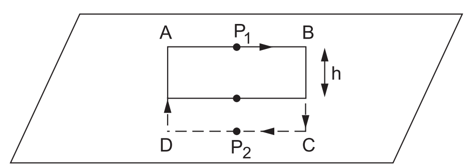


Fig. 3.12

the rectangle *ABCD*. Half the rectangle will be on one side of the given surface and the other half will be on the other side. Put arrows on the four arms of the rectangle to make a closed loop for line integration. We will evaluate $\oint \boldsymbol{E} \cdot \boldsymbol{dl}$ on this rectangular loop.

For electrostatic fields, $\oint \boldsymbol{E} \cdot \boldsymbol{dl} = 0$. $\int \boldsymbol{E} \cdot \boldsymbol{dl}$ on the line *AB* is

$$\boldsymbol{E_1} \cdot \Delta l \hat{\boldsymbol{t}} = E_{1t} \Delta l,$$

where E_{1t} is the component of $\boldsymbol{E}$ at P_1 along the direction of $\hat{\boldsymbol{t}}$. t stands for tangential. E_{1t} is a tangential component (parallel to the given surface) of the electric field at P_1. The value of $\int \boldsymbol{E} \cdot \boldsymbol{dl}$ on the line *CD* is

$$\boldsymbol{E_2} \cdot \Delta l(-\hat{\boldsymbol{t}}) = -E_{2t} \Delta l.$$

The lengths BC and DA are small (h). As P_1 and P_2 are assumed to be very close to the given surface, this length tends to zero and so does $\int \boldsymbol{E} \cdot \boldsymbol{dl}$ on these lengths. So

$$\oint \boldsymbol{E} \cdot \boldsymbol{dl} = E_{1t} \Delta l - E_{2t} \Delta l. \tag{3.24}$$

But $\oint \boldsymbol{E} \cdot \boldsymbol{dl} = 0$ for electrostatic fields. Thus

$$E_{1t} = E_{2t}.$$

The component of the electric field (produced by stationary charges) in any direction parallel to the given surface is continuous across the surface, be it charged or uncharged.

EXAMPLE 3.8 **Verify the boundary conditions on an electric field created by an infinite, plane layer of charge with surface charge density σ.**

Solution Suppose the charge is distributed on the x–y plane (Figure 3.13). Let us call the side $z > 0$ Side 1. Draw a normal to the surface on Side 1. This normal $\hat{\boldsymbol{n}}$ will be the same as the unit vector $\hat{\boldsymbol{k}}$. The electric field due to this layer of charge will be $\boldsymbol{E} = \frac{\sigma}{2\varepsilon_0}\hat{\boldsymbol{k}}$ on Side 1 and $-\frac{\sigma}{2\varepsilon_0}\hat{\boldsymbol{k}}$ on Side 2. The tangential component in both cases is zero. Thus

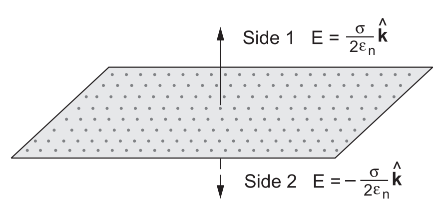


Fig. 3.13

$$E_{1t} = E_{2t}.$$

Hence, the boundary condition on the tangential component is satisfied.

$$E_{1n} = \boldsymbol{E}_1 \cdot \hat{\boldsymbol{n}} = \left(\frac{\sigma}{2\varepsilon_0}\hat{\boldsymbol{k}}\right) \cdot \hat{\boldsymbol{k}} = \frac{\sigma}{2\varepsilon_0}$$

and $E_{2n} = \boldsymbol{E}_2 \cdot \hat{\boldsymbol{n}} = \left(-\frac{\sigma}{2\varepsilon_0}\hat{\boldsymbol{k}}\right)\cdot\hat{\boldsymbol{k}} = -\frac{\sigma}{2\varepsilon_0}\cdot$

So, $E_{1n} - E_{2n} = \frac{\sigma}{2\varepsilon_0} - \left(-\frac{\sigma}{2\varepsilon_0}\right) = \frac{\sigma}{\varepsilon_0}\cdot$

Thus the boundary condition on the normal component is also satisfied.

Concepts at a Glance

1. The electrostatic field is a conservative field and can expressed as $\boldsymbol{E} = -\nabla V$.
2. The potential due to given charge distribution is $V(r) = \int \frac{\rho(\boldsymbol{r}')\,d\tau'}{4\pi\varepsilon_0[\boldsymbol{r}-\boldsymbol{r}']}\cdot$
3. For a finite charge distribution, the potential is taken to be zero at large distances from the distribution.
4. A system of charges has potential energy $U = \frac{1}{2}\int \rho(\boldsymbol{r})\,V(\boldsymbol{r})\,d\tau$.
5. For continuous charge distributions, the electric field can be assumed to have energy with density
$$u_E = \frac{1}{2}\varepsilon_0 E^2.$$
6. The tangential component of the electric field in an electrostatic situation across any given surface is continuous.
7. The normal component of the electric field across a surface jumps by $\frac{\sigma}{\varepsilon_0}$ if σ is the surface charge density there.

EXERCISES

Use standard references for potential and potential energy unless stated otherwise.

Based on Concepts

1. For which of the following forces can you define potential energy and for which can you not?
   (a) The force between the different parts of a stretched spring
   (b) The gravitational force between two stars
   (c) The viscous force between water and a ball when the ball moves through it
2. Which of the following is true?
   (a) The potential energy of a pair of charges is $\frac{q_1 q_2}{4\pi\varepsilon_0 r}\cdot$ Therefore, at infinite separation, the electric potential energy of the pair is zero.
   (b) The electric potential energy of a pair of charges is taken to be zero at infinite separation. Therefore, the potential energy of this pair is given by $\frac{q_1 q_2}{4\pi\varepsilon_0 r}\cdot$

3. A charge q_1 is kept on the highest point A of a sphere of radius R. Another charge q is kept initially at the lowest point B of the sphere and is later shifted to a point C of the sphere where the radius makes an angle θ with BA (Fig. 3E.1).
   (a) Has the electric potential energy of the two-charge system changed?
   (b) Has the electric potential at the centre of the sphere changed?

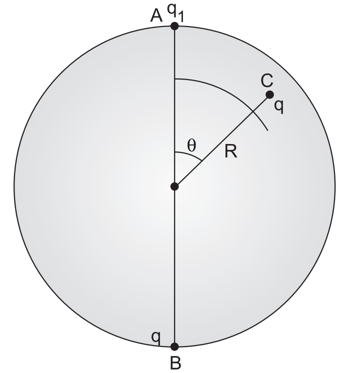


Fig. 3E.1

4. The x–y plane happens to be an equipotential surface for a given charge distribution. What are the possible directions of the electric field at the origin?
5. Two identical, positive, point charges, q each, are placed at $(-a, 0, 0)$ and $(a, 0, 0)$. Sketch the potential $V(x)$ as a function of x as one goes from $x = -2a$ to $x = 2a$.
6. Two point charges $+q$ and $-q$ (q is positive) are placed at $(-a, 0, 0)$ and $(a, 0, 0)$. Sketch the potential $V(z)$ as a function of z as one moves from $z = -2a$ to $z = 2a$.
7. Show that $\int E^2 d\tau$ evaluated over the space outside a closed surface is equal to $\int_S V\boldsymbol{E}\cdot d\boldsymbol{a}$ over this surface. Here $\boldsymbol{E}$ is the electric field and V is the electric potential. Assume a continuous charge distribution.
8. The electric field in a region $s < a$ is given by $\boldsymbol{E} = \frac{k}{s}\hat{\phi}$ in cylindrical coordinates. Can you produce such a field with stationary charges? Can you find the charge distribution in the region containing this field?

Problems

1. Two point charges (+5.0 mC and –5 mC) are kept at (–10 cm, 0, 0) and (10 cm, 0, 0) in Cartesian coordinates. At which points is the potential due to this charge distribution zero?
2. Four point charges (q each) are placed on a circle of radius R. What is the potential due to this charge distribution at the centre of the circle? [**Ans.** $\frac{q}{\pi\varepsilon_0 R}$]
3. A ring of radius r has charge q distributed nonuniformly on it. Find the work done to take a charge q_0 from the centre of the ring to infinity. [**Ans.** $\frac{-q}{4\pi\varepsilon_0 r}$]
4. The potential at the origin is found to be 20 V. As one goes along the positive x-direction, the potential decreases at a constant rate and becomes 18.5 V at a distance of 3 cm from the origin. If one moves along the positive y-direction, the potential increases at a constant rate and becomes 21.5 V at a distance of 3 cm from the origin. No change is observed if one moves along the z-axis. Find the electric field at the origin. [**Ans.** $-0.5\hat{\boldsymbol{i}} - 0.5\hat{\boldsymbol{j}}$]

5. The electric potential in a region is given by $V(x, y, z) = A(3x + 4y)$ where A is a constant. Find the angle made by the electric field at the origin with the x-axis. [**Ans.** $\cos^{-1}\left(\frac{-3}{5}\right)$]

6. The electrical potential due to a configuration of charges is given by the expression $V = A\frac{e^{-\lambda r}}{r}$ where A and λ are constants. Find (a) the electric field, (b) the charge density and (c) the total charge in the space corresponding to this potential. The potential is called screened Coulomb's potential and is often used to approximate the potential seen by the outer electron in an atom. [**Ans.** (b) $\varepsilon_0\left[4\pi A\delta^3(\boldsymbol{r}) - \frac{A\lambda^2}{r}e^{-\lambda r}\right]$]

7. The electric potential in a cylindrical region with a rectangular cross section (Figure 3E.2) is given by

$$V(x, y, z) = \frac{k/\varepsilon_0}{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}.$$

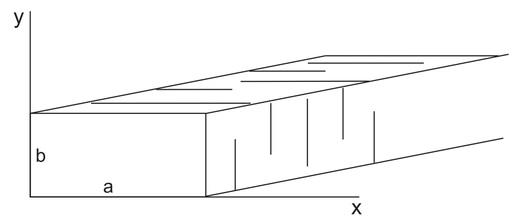


Fig. 3E.2

   (a) Find the electric field in the cylindrical region.
   (b) Evaluate the curl of the electric field obtained and verify that it is zero.
   (c) Find the charge density in the cylindrical region.
   (d) Find the charge contained at a depth d in the region by integrating over the charge density and also by finding the flux through the bounding surface. [**Ans.** (d) $5kabd/\pi^2$]

8. Charge q is distributed uniformly on a circular wire of radius a. Find the electric potential at a point on the axis at a distance r from the centre, and, consequently, the electric field on this axis. [**Ans.** $\frac{q}{4\pi\varepsilon_0\sqrt{a^2 + r^2}}$]

9. A conical surface of height h and semi-vertical angle $\pi/4$ radian carries a uniform surface charge density σ_0. Find the potential at the vertex. [**Ans.** $\frac{h\sigma}{2\varepsilon_0}$]

10. A disk of radius R carries a charge Q distributed uniformly on one of its surfaces. Find the electric potential due to this charge at a point on the axis of the disk at a distance z from its centre. [**Ans.** $\frac{Q}{2\pi\varepsilon_0 R^2}(\sqrt{R^2 + z^2} - z)$]

11. The electric field in a region is given as $\boldsymbol{E} = (20\text{ V/m})\hat{z}$. If the potential at (1 m, 1 m, 1 m) is 20 V, what is the potential at (2 m, 2 m, 2 m)? [**Ans.** Zero]

12. Give the potential due to a spherical shell of radius R carrying a uniform surface density σ_0 as a function of the distance r from the centre for all r. [**Ans.** Outside, $\frac{\sigma_0 R^2}{\varepsilon_0 r}$]

13. The electric field varies in a region as $\boldsymbol{E} = \frac{A}{r^2}\hat{\boldsymbol{r}}$ in spherical polar coordinates. Find the potential difference $V_A - V_B$, where $\boldsymbol{A}$ is the point $\left(R, \frac{\pi}{3}, \frac{\pi}{2}\right)$ and B is the point $\left(2R, \frac{2\pi}{3}, \pi\right)$ in spherical polar coordinates. [**Ans.** $\frac{A}{2R}$]

14. Find the electric potential due a uniformly charged sphere of radius R carrying a charge density ρ.

[**Ans.** $\frac{\rho}{2\varepsilon_0}\left(R^2 - \frac{r^2}{3}\right)$ for $r < R$]

15. Suppose the entire x–y plane carries a uniform surface charge density σ_0. Write a suitable potential function $V(x, y, z)$ for this charge distribution. Where do you take the potential to be zero?

[**Ans.** $\frac{-\sigma}{2\varepsilon_0}(z - z_0)$, at $z = z_0$]

16. The electric field at points on the z-axis is given by $\boldsymbol{E} = \frac{Az}{\sqrt{z^2 + L^2}}\hat{\boldsymbol{k}}$. Find the change in potential as one moves from (0, 0, L) to (0, 0, 2L). [**Ans.** $-AL(\sqrt{5} - \sqrt{2})$]

17. Find the electric potential everywhere due to a hollow, cylindrical surface carrying a uniform surface charge density σ_0. Take the potential at the surface to be zero. [**Ans.** $-\frac{\sigma_0 R}{s}\ln\left(\frac{s}{R}\right)$, $s > R$]

18. Three charges, each of 2.5 nC, are kept at the vertices of an equilateral triangle of edge 10 cm. How much work has to be done to put them at the vertices of an equilateral triangle of edge 15 cm?

[**Ans.** -5.6×10^{-7} J]

19. Two particles, each of mass 120 g and charge 12 nC, are released from rest at a separation of 10 cm. Assuming only the Coulomb forces act, find the speed of each particle at the instant the separation between them becomes 20 cm. [**Ans.** $6\sqrt{5}$ m/s]

20. Three charges $q, -q, -q$ are kept on a straight line, and the distance between the negative charges is x. Find the location of the positive charge so that the net work done by the electric forces to take the three charges far apart is zero. [**Ans.** $\frac{x(1+\sqrt{2})}{2}$ from the nearest charge]

21. Three charges $q, -q, -q$ are kept at the vertices of an isosceles triangle, q being at the intersection of the equal sides. The total electrostatic energy is zero. Find the angle between the equal sides.

[**Ans.** $\cos^{-1}\left(\frac{1}{4}\right)$]

22. Find the electrostatic field energy of two concentric, thin, metallic, spherical shells of radii R and 2R, the inner one having charge q and the outer one, 2q. The charges distribute uniformly on each surface.

[**Ans.** $\frac{5q^2}{8\pi\varepsilon_0 R}$]

23. Find the energy stored in a uniformly charged sphere of radius r and total charge q in the following ways. (a) Build the distribution layer by layer and calculate the work done. (b) Use field energy density and integrate over the whole space. [**Ans.** $\frac{3Q^2}{20\pi\varepsilon_0 R}$]

24. Find the electrostatic energy corresponding to a charge Q spread uniformly over the surface of a sphere of radius R using $U = \frac{1}{2}\int \sigma V\, da$. [**Ans.** $\frac{Q^2}{8\pi\varepsilon_0 R}$]

25. Find the electrostatic energy corresponding to a charge Q spread uniformly over the surface of a sphere of radius R using $U = \frac{1}{2}\varepsilon_0\int E^2 d\tau + \frac{1}{2}\varepsilon_0\int \boldsymbol{E}\cdot V\boldsymbol{da}$ and taking a concentric surface of radius 2R for the integrations. [**Ans.** $\frac{Q^2}{8\pi\varepsilon_0 R}$]

26. A point charge of +12 mC is kept at the origin and a negative charge of –24 mC is kept at (10 cm, 0, 0). Find the equation of the surface at which the electric potential is zero. What is the nature of this surface? [**Ans.** Spherical, centre at $\left(\frac{10}{3}\text{ cm}, 0\right)$ and radius $\frac{20}{3}$ cm]

27. Two long line charges situated at $x=-a$, $y=0$, and $x=+a$, $y=0$ carry linear charge densities $-\lambda$ and $+\lambda$ respectively. Find the equation of the equipotential surface corresponding to the potential $\frac{\lambda}{4\pi\varepsilon_0}$.

[**Ans.** Cylindrical surface with radius $\frac{2ae}{(e^2-1)}$]

28. Consider $\boldsymbol{E}=k(y\hat{\boldsymbol{i}}+x\hat{\boldsymbol{j}})$. Is this a possible electrostatic field? Can you obtain the charge density ρ using Gauss's law? If you can, do so. [**Ans.** Yes, zero]

29. Which of the following functions may represent an electrostatic field? (a) $As\hat{\phi}$ in cylindrical coordinates. (b) $A(y\hat{\boldsymbol{i}}-x\hat{\boldsymbol{j}})$ in Cartesian coordinates (c) $Ar\hat{\boldsymbol{r}}$ in spherical coordinates. [**Ans.** $Ar\hat{\boldsymbol{r}}$]

30. Using the fact that $\nabla\times\nabla V=0$ for any scalar field, show that $\nabla\times\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$ for a given $\boldsymbol{r}'$ is zero.

31. A charge $-\frac{q}{2}$ is placed at $(a, 0, 0)$ and another charge q is placed at $(4a, 0, 0)$. (a) Find the point (x_0, y_0, z_0) where the electric field is zero. (b) Find the potential V_0 at this point. (c) What are $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$ at (x_0, y_0, z_0)? (d) Is $\frac{\partial^2 V}{\partial x^2}$ at (x_0, y_0, z_0) positive, negative or zero? (e) Is $\frac{\partial^2 V}{\partial y^2}$ at (x_0, y_0, z_0) positive, negative or zero? [**Ans.** (a) $x_0=-(3\sqrt{2}-2)a$, $y_0=0$, $z_0=0$]

32. The x–y plane contains a surface charge density $\sigma_0=2\,\text{nC/m}^2$. There are other charges also in the region above and below the x–y plane. The electric field at a point P_1 just above a point P on the plane is $\boldsymbol{E_1}=(25\text{ V/m})(3\hat{\boldsymbol{i}}+4\hat{\boldsymbol{k}})$. Find the field at a point P_2 just below P. [**Ans.** $(75\text{ V/m})\hat{\boldsymbol{i}}-(116\text{ V/m})\hat{\boldsymbol{k}}$]

33. The electric field outside the sphere $r=R$ is given as $\boldsymbol{E}=\frac{k}{r^3}(2\cos\theta\,\hat{\boldsymbol{r}}+\sin\theta\,\hat{\theta})$ and that inside the sphere as $\mathbf{E}=-E_0\hat{\boldsymbol{k}}$. (a) Find the relation between k, E_0 and R. (b) Find the surface charge density at $r=R$, $\theta=0$.

[**Ans.** (a) $E_0R^3=k$, (b) $3\varepsilon_0E_0$]

34. Calculate the electric potential due to a uniformly charged disk (surface charge density σ, radius a) at its centre and at a point on its periphery. If the charges are free to move, what kind of redistribution do you expect?

[**Ans.** $\frac{\sigma a}{2\varepsilon_0}$ at the centre and $\frac{\sigma a}{3\varepsilon_0}$ at the periphery]

35. Two circular disks, each of radius R, carry uniform surface charge densities $+\sigma$ and $-\sigma$. They are kept parallel to each other at a separation d, the line joining their centres being perpendicular to their surfaces as shown in Fig. 3E.3. Assume $d \ll R$.

(a) Calculate the potential at the centre of the disk carrying positive charge. (b) Find the expression for the electric field at a distance $x\,(x \gg R)$ above the centre of the positively charged disk on the line joining the two centres. [**Ans.** (a) $\frac{\sigma d}{2\varepsilon_0}$ (b) $\frac{\sigma R^2 d}{2\varepsilon_0 x^3}$]

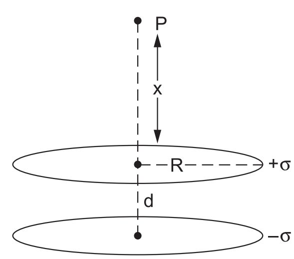


Fig. 3E.3

36. There is an electric field $\boldsymbol{E_1} = A\hat{\boldsymbol{r}}$ in the spherical region $r < R$ where A is a constant. There are no charges outside the spherical region. (a) Find the electric field for $r > R$. (b) Find the charge density for $r < R$. Is there a net point charge at the origin? (c) Find the total electrostatic energy.

[**Ans.** (c) $\frac{8}{3}\lambda\varepsilon_0 R^3 A^3$]

37. Suppose the electric field of a charge q kept at the origin were given by $\boldsymbol{E} = \dfrac{q\boldsymbol{r}}{4\pi\varepsilon_0 r^{3+\delta}}$ where δ is a very small number compared to 1. (a) Write an expression for the electric potential due to this charge. (b) A spherical surface of radius R contains a uniformly distributed charge Q. Find the potential due to this distribution, both inside and outside the surface. (c) Two concentric, metallic spherical shells of radii a and b ($a < b$) are connected by a thin wire so that they are at the same potential. A total charge Q_0 is given to the system. Find the ratio of the charges Q_b/Q_a where Q_b and Q_a are the charges on the outer and inner shells respectively. [**Ans.** (c) $\frac{\delta}{a-b}[2a \ln 2a - (a+b)\ln (a+b) + (b-a)\ln (b-a)]$]

❑

4 Multipole Expansion of Electric Potential

4.1 Expansion of Electric Potential in Powers of (1/*r*)

You know how to get the electric potential (and field) due to a given charge distribution. For certain charge distributions, the mathematics is simple, the integrations can be evaluated, and the expressions for potential can be obtained. However, in many cases, the exact expression for the potential due to a finite charge distribution is difficult to obtain. In these cases, an approximate expression can be written for points far from the given charge distribution. In this section, we will talk about the methods to get such expressions.

Suppose a charge is distributed in a given volume τ', and the volume charge density is $\rho(\boldsymbol{r}')$ where $\boldsymbol{r}'$ denotes a point inside that volume (Figure 4.1). Take the origin to be somewhere in the volume containing the charge. Let $d\tau'$ be a small volume element at $\boldsymbol{r}'$ and P be a point far from the charge distribution, the position vector of P being $\boldsymbol{r}$. Then the potential at P due to this charge distribution is

$$V(\boldsymbol{r}) = \int \frac{\rho(\boldsymbol{r}')\,d\tau'}{4\pi\varepsilon_0\,|\boldsymbol{r}-\boldsymbol{r}'|} \tag{i}$$

where integration has to be performed over the volume in which the charge is distributed. If α be the angle between $\boldsymbol{r}$ and $\boldsymbol{r}'$,

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{1}{[r^2 + r'^2 - 2rr'\cos\alpha]^{1/2}}$$

$$= \frac{1}{r}\left[1 + \left(\frac{r'^2}{r^2} - \frac{2r'}{r}\cos\alpha\right)\right]^{-1/2}.$$

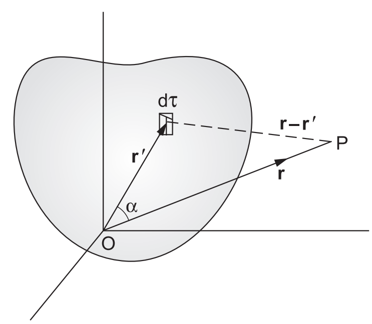


Fig. 4.1

If the point P is far from the charge distribution, $\frac{r'}{r}$ will be small compared to 1 and so will be the quantity within parentheses. Using the binomial theorem,

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}=\frac{1}{r}\left[1-\frac{1}{2}\left(\frac{r'^2}{r^2}-\frac{2r'}{r}\cos\alpha\right)+\frac{(-1/2)(-3/2)}{2}\left(\frac{r'^2}{r^2}-\frac{2r'}{r}\cos\alpha\right)^2+\ldots\right]$$

$$=\frac{1}{r}\left[1+\frac{r'}{r}\cos\alpha-\frac{r'^2}{2r^2}+\frac{3}{8}\frac{4r'^2}{r^2}\cos^2\alpha+\ldots\right]$$

$$=\frac{1}{r}\left[1+\frac{r'\cos\alpha}{r}+\frac{1}{2}\left(\frac{r'}{r}\right)^2(3\cos^2\alpha-1)+\ldots\right]. \tag{ii}$$

Check that we have written all the terms explicitly up to $(r'/r)^2$ within square brackets and that any term not written explicitly is of the order of $(r'/r)^3$ or higher. Using expression (ii) in (i),

$$V(\boldsymbol{r})=\left[\frac{\int\rho(\boldsymbol{r}')\,d\tau'}{4\pi\varepsilon_0 r}+\frac{\int\rho(\boldsymbol{r}')\,r'\cos\alpha\,d\tau'}{4\pi\varepsilon_0 r^2}+\frac{\int\rho(\boldsymbol{r}')\,r'^2(3\cos^2\alpha-1)\,d\tau'}{4\pi\varepsilon_0 r^3}+\ldots\right]. \tag{4.1}$$

Remember the integration has to be done over the volume containing the charge, and r is a constant for this integration.

Looking at the r-dependence of $V(\boldsymbol{r})$, the first term in this expansion varies as $1/r$, the second term, as $1/r^2$ and the third, as $1/r^3$. For points far from the charge distribution, r is large and so $1/r >> 1/r^2 >> 1/r^3$. Equation 4.1 is called the *multipole expansion* of the potential and the three terms, successively, are called *monopole, dipole* and *quadrupole moment* contributions to the potential.

4.2 Monopole Contribution

The monopole contribution to the potential $V(\boldsymbol{r})$ is

$$V_m=\frac{1}{4\pi\varepsilon_0}\frac{\int\rho(\boldsymbol{r}')\,d\tau'}{r}=\frac{Q}{4\pi\varepsilon_0 r} \tag{4.2}$$

where Q is the total charge in the distribution. If $Q\neq 0$, this term dominates over all the other terms and all these higher-order terms can be neglected for faraway points.

The physical explanation for monopole contribution is very simple. From a large distance, the whole charge distribution looks like a point charge and hence the potential is like that of a single charge Q.

Now you know why we choose the origin to be inside the volume containing the charge. Then the maximum of r' typically gives the size of the charge distribution and r gives the distance of the point P "from the charge distribution". Thus, for faraway points, $r >> r'$.

The total charge Q is called the *monopole moment* of the given charge distribution.

4.3 Dipole Contribution

What happens if the total charge in the charge distribution is zero? Then there is as much positive charge in the distribution as negative charge. In this case, the monopole contribution is zero and we cannot stop here. We must look at the next term in the multipole expansion. The next term in Equation 4.1 is

$$V_d(\boldsymbol{r}) = \frac{\int \rho(\boldsymbol{r}')\, r' \cos\alpha\, d\tau'}{4\pi\varepsilon_0 r^2}. \tag{4.3}$$

This contribution to the potential varies as $1/r^2$, and is known as the dipole contribution.

Now $r' \cos\alpha = \boldsymbol{r}' \cdot \hat{\boldsymbol{r}}$, where $\hat{\boldsymbol{r}}$ is the unit vector in the direction of the field point P from the origin O. Thus, Equation 4.3 may be written as

$$V_d = \frac{1}{4\pi\varepsilon_0 r^2}\left(\int \rho(\boldsymbol{r}')\, \boldsymbol{r}'\, d\tau'\right) \cdot \hat{\boldsymbol{r}}.$$

We have written $\hat{\boldsymbol{r}}$ outside the integral, because it does not depend on the location of $d\tau'$.

The quantity in brackets has no reference to the point P and is purely a property of the charge distribution. This quantity is called the *dipole moment* of the charge distribution, and is denoted by the symbol $\boldsymbol{p}$. Thus,

$$\boldsymbol{p} = \int \rho(\boldsymbol{r}')\, \boldsymbol{r}'\, d\tau'$$

and the dipole contribution to the potential at the point P is

$$V_d = \frac{\boldsymbol{p} \cdot \hat{\boldsymbol{r}}}{4\pi\varepsilon_0 r^2}. \tag{4.4}$$

The dipole moment $\boldsymbol{p}$ of a charge distribution is a vector quantity, having a particular direction and a particular magnitude. The SI unit of the dipole moment is the coulomb metre, abbreviated as C m.

As usual, we have used dashed quantities to describe the location of the charges and undashed quantities to give the location of the point where the potential is to be calculated. If we only have to calculate the dipole moment of a given charge distribution, we can use only one kind of symbol and naturally, we will use undashed symbols. In such cases, we will take the volume element to be $d\tau$ (instead of $d\tau'$) and the position vector of the volume element to be $\boldsymbol{r}$ (instead of $\boldsymbol{r}'$). The dipole moment of the distribution is then

$$\boldsymbol{p} = \int \rho(\boldsymbol{r})\, \boldsymbol{r}\, d\tau. \tag{4.5}$$

Choice of the origin

Equation 4.5 seems to suggest that the dipole moment of a charge distribution depends on the choice of the origin. If we change the origin, the position vector $\boldsymbol{r}$ of each point will change and hence the integral $\int \rho \boldsymbol{r}\, d\tau$ is likely to change. It turns out that if the total charge in the charge distribution is zero, the dipole moment is independent of the choice of the origin. Indeed if the total charge is not zero, the monopole contribution will dominate for large distances and the dipole contribution will not remain important any more. So for situations in which the dipole contribution is important, we can use the definition given in Equation 4.5 with any choice of origin.

Electric dipole

If asked what an electric dipole is, most of you will answer that it is a collection of two point charges $+q$ and $-q$ placed at a small separation. That is how we generally introduce the electric dipole to school students. It is true that two point charges $+q$ and $-q$ placed at a small separation make up an electric dipole. However, any finite charge distribution having no net charge but having a nonzero dipole moment can also be called an electric dipole for faraway points. Let us calculate the dipole moment of the two-charge dipole using the general definition given by Equation 4.5. Suppose a charge $-q$ is placed at $\boldsymbol{r}_1$ and $+q$, at $\boldsymbol{r}_2$ (Figure 4.2). The expression for electric dipole moment is $\boldsymbol{p} = \int \rho(\boldsymbol{r})\boldsymbol{r}\, d\tau$. For a discrete charge distribution, we can replace $\rho d\tau$ by q_i, $\boldsymbol{r}$ by $\boldsymbol{r}_i$ and integration by $\sum_i$. Then,

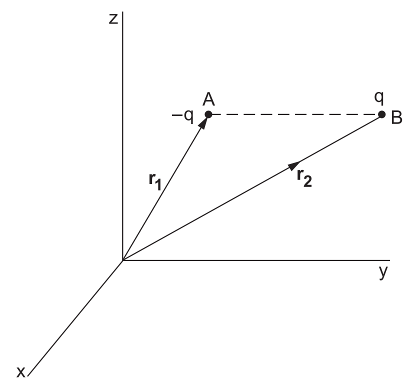


Fig. 4.2

$$\boldsymbol{p} = \sum_i q_i \boldsymbol{r}_i = (-q)\boldsymbol{r}_1 + (q)\boldsymbol{r}_2$$
$$= q(\boldsymbol{r}_2 - \boldsymbol{r}_1) = q\boldsymbol{AB}.$$

The magnitude of the dipole moment of this system is the magnitude of the charge q multiplied by the distance between the charges. The direction of the dipole moment is from the negative charge to the positive charge.

You can have a surface charge distribution or a line charge distribution with a total charge of zero. You can express the dipole moments of such distributions by replacing $\rho(\boldsymbol{r})d\tau$ in Equation 4.5 with $\sigma(\boldsymbol{r})da$ and $\lambda(\boldsymbol{r})dl$ in the two cases.

As electric dipole moment is a vector quantity, it adds like a vector quantity. Suppose you are given a charge distribution with a net charge zero. If you can divide it into two or more charge distributions, each with known electric dipole moment, you can get the net dipole moment of the given distribution by adding the individual dipole moments vectorially.

EXAMPLE 4.1 **Find the dipole moment of the group of three charges $-q, -q, 2q$, placed at the vertices of an equilateral triangle of edge length a.**

Solution Let the three charges be placed at points A, B, C as shown in Figure 4.3. Take the origin at A, the x-axis along AB and the y-axis in the plane of the triangle. The position vectors of A, B and C are $\boldsymbol{r}_A = 0$, $\boldsymbol{r}_B = a\hat{\boldsymbol{i}}$ and $\boldsymbol{r}_C = \frac{a}{2}\hat{\boldsymbol{i}} + \frac{\sqrt{3}}{2}a\hat{\boldsymbol{j}}$. Then the dipole moment is

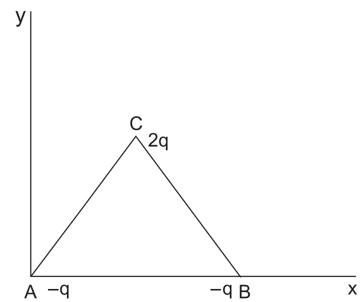


Fig. 4.3

$$p = \sum_i q_i \boldsymbol{r}_i = (-q)(0) + (-q)(a\hat{\boldsymbol{i}}) + 2q\left(\frac{a}{2}\hat{\boldsymbol{i}} + \frac{\sqrt{3}}{2}a\hat{\boldsymbol{j}}\right)$$

$$= q\sqrt{3}\,a\hat{\boldsymbol{j}}.$$

You can also solve this problem in a different way. The charges $-q$ at A and $+q$ at C make a dipole of dipole moment $q\boldsymbol{AC}$. The charges $-q$ at B and $+q$ at C make a dipole of dipole moment $q\boldsymbol{BC}$. The given distribution is a combination of these two. Thus the dipole moment of the given distribution is

$$q\boldsymbol{AC} + q\boldsymbol{BC} = q\left(\frac{a}{2}\hat{\boldsymbol{i}} + \frac{\sqrt{3}}{2}a\hat{\boldsymbol{j}}\right) + q\left(-\frac{a}{2}\hat{\boldsymbol{i}} + \frac{\sqrt{3}}{2}a\hat{\boldsymbol{j}}\right) = q\sqrt{3}\,a\hat{\boldsymbol{j}}.$$

EXAMPLE 4.2 **A charge is distributed on a spherical surface, and the surface charge density is $\sigma = \sigma_0 \cos\theta$ (spherical coordinates, origin at the centre). (a) Show that the monopole moment is zero. (b) Find the dipole moment for this distribution.**

Solution The charge lies on the spherical surface. In spherical coordinates, the area element is $da = R^2 \sin\theta\, d\theta\, d\phi$.

(a) Monopole moment $= \int \sigma\, da$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} (\sigma_0 \cos\theta) R^2 \sin\theta\, d\theta\, d\phi$$

$$= 2\pi\sigma_0 R^2 \int_0^{\pi} \cos\theta \sin\theta\, d\theta = 0.$$

(b) Dipole moment $\boldsymbol{p} = \int \sigma(\boldsymbol{r})\boldsymbol{r}\, da$

$$= \int \sigma_0 \cos\theta\, \boldsymbol{r}\, da = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} (\sigma_0\cos\theta)(\boldsymbol{r}) R^2 \sin\theta\, d\theta\, d\phi.$$

It is clear that positive charges are distributed on the "upper half" ($\theta < \pi/2$) and that the "lower half" ($\theta > \pi/2$) has negative charge. The charge density is independent of ϕ. Thus, the dipole moment will be directed towards the positive z-axis. Thus we only have to evaluate p_z. The z-component of $\boldsymbol{r}$ is $R\cos\theta$. So,

$$p_z = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}(\sigma_0\cos\theta)(R\cos\theta)R^2\sin\theta\, d\theta\, d\phi$$

$$= 2\pi\sigma_0 R^3\int_0^{\pi}\cos^2\theta\sin\theta\, d\theta$$

$$= 2\pi\sigma_0 R^3\left[-\frac{\cos^3\theta}{3}\right]_0^{\pi}$$

$$= \frac{4}{3}\pi\sigma_0 R^3.$$

Therefore, the dipole moment is $\boldsymbol{p} = \frac{4}{3}\pi R^3\sigma_0\hat{\boldsymbol{k}}$.

EXAMPLE 4.3 **Show that the charge distribution in the previous example can be obtained by taking two uniformly charged spheres of radius R each, one containing a charge density $+\rho$ and the other a charge density $-\rho$, and putting them with an infinitesimal separation d between the centres along the z-axis where $\rho d = \sigma_0$.**

Solution Take the origin at the centre of the negatively charged sphere. Place the positively charged sphere in such a way that its centre is a distance d above (along the z-axis) the origin, where d is an infinitesimally small quantity. In the whole of the region where the two spheres overlap, the net charge density is zero. At the surface, there is a small layer where the spheres do not overlap and there is a net charge density. On the upper hemisphere surface, a positive charge appears and on the lower hemispherical surface, there is a negative charge (Figure 4.4). The thickness of this charge layer is different at different places. Take a small surface area ΔA at the position (R, θ, ϕ). How much is the charge on this area? Everywhere, the positive sphere is shifted by the same distance d in the upward direction (that is, in the z-direction). So, the thickness of the layer at (R, θ, ϕ), perpendicular to the surface, is $d\cos\theta$. Thus, the volume of the charged layer at ΔA is $(\Delta A)d\cos\theta$ and the charge in this layer is $(\Delta A)\rho d\cos\theta$. The surface charge density at (R, θ, ϕ) is, therefore, $\sigma = \rho d\cos\theta = \sigma_0\cos\theta$, where $\sigma_0 = \rho d$.

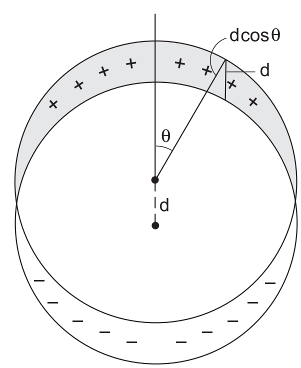


Fig. 4.4

Electric field due to an electric dipole

Consider a charge distribution having no monopole moment and having a dipole moment $\boldsymbol{p}$. Take the z-axis along the dipole moment and the origin somewhere within the charge distribution (Figure 4.5). Let P be a distant point with position vector $\boldsymbol{r}$. Using spherical coordinates, the potential at P is

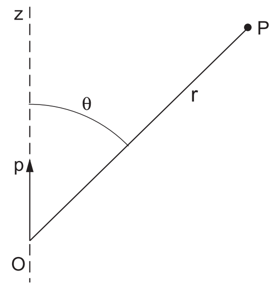


Fig. 4.5

$$V(\boldsymbol{r}) = \frac{\boldsymbol{p}\cdot\hat{\boldsymbol{r}}}{4\pi\varepsilon_0 r^2} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}. \tag{4.6}$$

The electric field at P is

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r}) &= -\nabla V(\boldsymbol{r}) \\ &= -\frac{\partial}{\partial r}\left(\frac{p\cos\theta}{4\pi\varepsilon_0 r^2}\right)\hat{\boldsymbol{r}} - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{p\cos\theta}{4\pi\varepsilon_0 r^2}\right)\hat{\theta} \\ &= \frac{p}{4\pi\varepsilon_0}\left[2\frac{\cos\theta}{r^3}\hat{\boldsymbol{r}} + \frac{\sin\theta}{r^3}\hat{\theta}\right] \\ &= \frac{p}{4\pi\varepsilon_0 r^3}[2\cos\theta\,\hat{\boldsymbol{r}} + \sin\theta\,\hat{\theta}].\end{aligned} \tag{4.7}$$

Remember that this equation should only be used for points far away from the charge distribution having the dipole moment. If you take a point P_1 on the z-axis itself, $\theta = 0$ and

$$\boldsymbol{E} = \frac{2p}{4\pi\varepsilon_0 r^3}\hat{\boldsymbol{r}}. \tag{4.8}$$

The direction of the field is along the direction of the dipole moment. If you take a point P_2 with OP_2 perpendicular to the dipole moment, $\theta = \pi/2$ and

$$\boldsymbol{E} = \frac{p}{4\pi\varepsilon_0 r^3}\hat{\theta}. \tag{4.9}$$

Note that this direction is opposite to the direction of the dipole moment in this case.

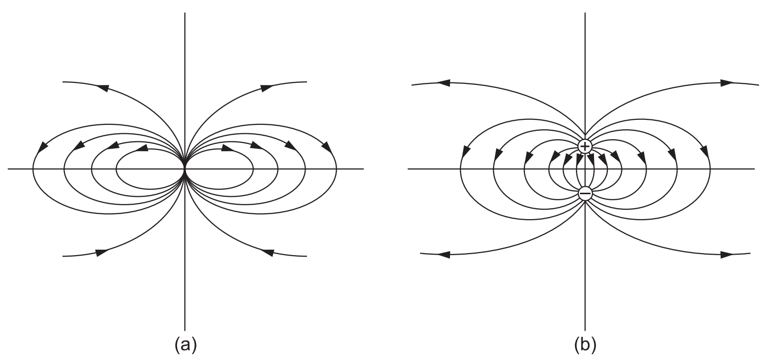


Fig. 4.6

In both the cases, the field is parallel to the dipole moment. Figure 4.6(a) qualitatively shows the electric field lines of an ideal electric dipole. An ideal dipole is one whose size is vanishingly small. We only specify the point where the dipole is situated and not its size. Then any distance from the dipole is a large distance and you can use the equation of dipole field and dipole potential. A real dipole has some size and the equation for the field or the potential, derived above, can be used only at distances large compared to the size of the dipole. Figure 4.6(b) shows the field lines due to a real dipole made of two point charges $-q$, q.

You can write the expression (Equation 4.7) for electric field due to the dipole in terms of quantities independent of the choice of coordinate axes. You have

$$\boldsymbol{p} = p\hat{\boldsymbol{k}} = p(\cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\theta})$$

$$\cos\theta = \boldsymbol{p}\cdot\hat{\boldsymbol{r}}.$$

Using these equations, you can find $\cos\theta\,\hat{\boldsymbol{r}}$ and $\sin\theta\,\hat{\theta}$ in terms of $\boldsymbol{p}$ and $\boldsymbol{p}\cdot r$.

The field is $$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0 r^3}[3(\boldsymbol{p}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{p}]. \tag{4.10}$$

EXAMPLE 4.4 **Find the electric field due to a surface charge distribution on the spherical surface $r = R$, described by the surface charge distribution $\sigma = \sigma_0 \cos\theta$ where spherical polar coordinates are used.**

Solution The distribution can be thought of as a superposition of two uniformly charged spheres of radius R having volume charge densities $+\rho$ and $-\rho$, and separated by an infinitesimally small distance d along the z-axis (Figure 4.7) with $\sigma_0 = \rho d$ (Example 4.3). The negative sphere is displaced in the negative z-direction so that its displacement is $\boldsymbol{d} = -d\hat{\boldsymbol{k}}$. In the overlap region, the field is uniform and is given by

$$\boldsymbol{E}_{\text{in}} = \frac{\rho\boldsymbol{d}}{3\varepsilon_0} = -\frac{\sigma_0}{3\varepsilon_0}\hat{\boldsymbol{k}}. \tag{i}$$

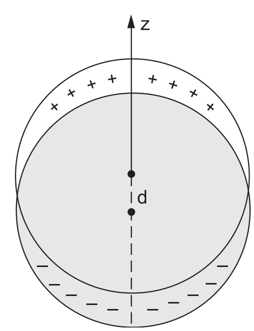


Fig. 4.7

This overlap region is the entire sphere of radius R, because d is infinitesimally small. So the field in the whole of the inside of the sphere is given by (i).

For points outside ($r > R$), the sphere of uniform charge density ρ can be replaced by a point charge $Q = \frac{4}{3}\pi R^3\rho$ placed at its centre. Similarly, the sphere of uniform charge density $-\rho$ can be replaced by a point charge $-Q = -\frac{4}{3}\pi R^3\rho$ placed at its centre. So the field due to the given charge distribution $\sigma = \sigma_0 \cos\theta$, for $r > R$, is the same as that due to two point charges Q and $-Q$ placed at a vanishingly small separation d. But these two charges form a dipole of dipole moment

$$\boldsymbol{p} = Qd\hat{\boldsymbol{k}} = \frac{4}{3}\pi R^3 \rho d\hat{\boldsymbol{k}}$$
$$= \frac{4}{3}\pi R^3 \sigma_0 \hat{\boldsymbol{k}}.$$

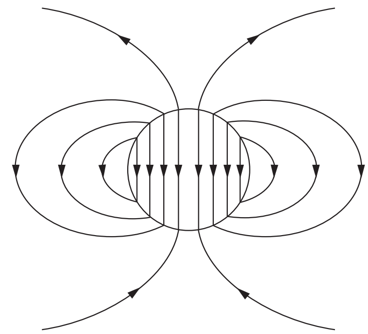


Fig. 4.8

Considering these two point charges, the points outside the surface $r = R$ are at large distances from the dipole. Thus the field outside the sphere is given by the dipole field

$$\boldsymbol{F}_{\text{out}} = \frac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\theta})$$
$$= \frac{R^3\sigma_0}{3\varepsilon_0 r^3}(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\theta}). \qquad \text{(ii)}$$

Figure 4.8 qualitatively shows the electric field lines.

EXAMPLE 4.5 **The electric field due to an electric dipole is $E = \frac{p_0}{4\pi\varepsilon_0 r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})$. Find the equation of the electric field lines on the x–z plane in terms of plane polar coordinates and also in Cartesian coordinates.**

Solution A field line is shown in Figure 4.9. At a point P, the electric field is along the tangent to the line. The tangent itself is along $\vec{dr}$ as you move an infinitesimal distance from P on the line. $\vec{E} = E_r\hat{r} + E_0\hat{\theta}$ and $\vec{dr} = dr\hat{r} + rd\theta\hat{\theta}$ are along the same direction. So,

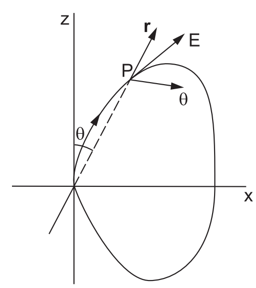


Fig. 4.9

$$\frac{E_r}{E_\theta} = \frac{dr}{rd\theta}$$

or $$\frac{dr}{rd\theta} = 2\cot\theta$$

or $$\frac{dr}{r} = 2\cot\theta d\theta$$

or $$r = A\sin^2\theta.$$

In Cartesian coordinates,

$$\sqrt{x^2+z^2} = A\left(\frac{y}{r}\right)^2 = A\frac{x^2}{x^2+z^2}$$

or $$(x^2+z^2)^3 = Cx^4.$$

Force on an electric dipole placed in an electric field

The force on a point charge q placed in an electric field $\boldsymbol{E}$ is $q\boldsymbol{E}$. What will be the force on an electric dipole placed in such an electric field? If the electric field is uniform over the extent of the charge distribution, the answer is very simple. The force is zero.

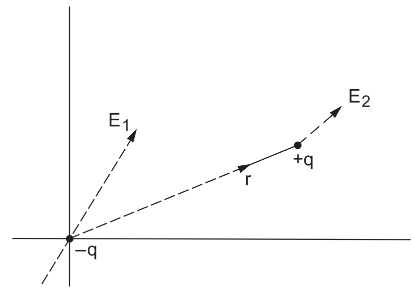


Fig. 4.10

$$\begin{aligned}\boldsymbol{F} &= \int \rho \boldsymbol{E}\, d\tau \\ &= \boldsymbol{E}\int \rho\, d\tau \\ &= 0\end{aligned}$$

as the total charge is zero for an electric dipole. What if the electric field is nonuniform? The force is nonzero and neither its expression nor its derivation is simple. Let us first work out the expression for a two-charge dipole, charges $+q$ and $-q$ being separated by a small distance d. Let us take the origin to be at the charge $-q$. The other charge $+q$ is at the position $\boldsymbol{r}$, where $\boldsymbol{r}$ is an infinitesimally small vector (Figure 4.10).

Let us represent the fields at the positions of the two charges $-q$ and $+q$ by E_1 and E_2. The force on the dipole is

$$\begin{aligned}\boldsymbol{F} &= (-q)\boldsymbol{E_1} + (+q)\boldsymbol{E_2} \\ &= q(\boldsymbol{E_2} - \boldsymbol{E_1}) \\ &= q d\boldsymbol{E}\end{aligned}$$

where $d\boldsymbol{E}$ is the change in the electric field as one goes a small distance away from the origin and reaches the point $\boldsymbol{r}$. Let us write

$$d\boldsymbol{E} = dE_x \hat{\boldsymbol{i}} + dE_y \hat{\boldsymbol{j}} + dE_z \hat{\boldsymbol{k}}.$$

Using the basic property of the gradient function $(\nabla\phi)\cdot \boldsymbol{dl} = d\phi$,

$$dE_x = (\nabla E_x)\cdot \boldsymbol{r},\ \ dE_y = (\nabla E_y)\cdot \boldsymbol{r} \text{ and } dE_z = (\nabla E_z)\cdot \boldsymbol{r}.$$

So, $$\boldsymbol{dE} = [(\nabla E_x)\cdot \boldsymbol{r}]\hat{\boldsymbol{i}} + [(\nabla E_y)\cdot \boldsymbol{r}]\hat{\boldsymbol{j}} + [(\nabla E_z)\cdot \boldsymbol{r}]\hat{\boldsymbol{k}}$$

or $$\begin{aligned}\boldsymbol{F} = q\boldsymbol{dE} &= [(\nabla E_x)\cdot (q\boldsymbol{r})]\hat{\boldsymbol{i}} + [(\nabla E_y)\cdot (q\boldsymbol{r})]\hat{\boldsymbol{j}} + [(\nabla E_z)\cdot (q\boldsymbol{r})]\hat{\boldsymbol{k}} \\ &= [\boldsymbol{p}\cdot \nabla E_x]\hat{\boldsymbol{i}} + [\boldsymbol{p}\cdot \nabla E_y]\hat{\boldsymbol{j}} + [\boldsymbol{p}\cdot \nabla E_z]\hat{\boldsymbol{k}}.\end{aligned} \tag{4.11}$$

This is the expression for the force. It is symbolically given by

$$\boldsymbol{F} = (\boldsymbol{p}\cdot\nabla)(E_x\hat{\boldsymbol{i}} + E_y\hat{\boldsymbol{j}} + E_z\hat{\boldsymbol{k}}) = (\boldsymbol{p}\cdot\nabla)\boldsymbol{E}. \tag{4.12}$$

Expression of force for a general dipole

Suppose there is a charge distribution confined to a very small volume and described by a volume charge density $\rho(\boldsymbol{r})$. The net charge $\int \rho d\tau$ is zero but there is a nonzero dipole moment $\boldsymbol{p} = \int \rho(\boldsymbol{r})\boldsymbol{r}\, d\tau$. Thus the charge distribution is an electric dipole. An external electric field $\boldsymbol{E}(\boldsymbol{r})$ exists in the region. By external, we mean the field produced by charges other than the charge distribution $\rho(\boldsymbol{r})$ making up the dipole. Let us calculate the force due to this external field on the dipole charge distribution.

The field at $\boldsymbol{r}$ is

$$
\begin{aligned}
\boldsymbol{E}(\boldsymbol{r}) &= \boldsymbol{E}(0) + \boldsymbol{dE}(\boldsymbol{r}) \\
&= \boldsymbol{E}(0) + dE_x(\boldsymbol{r})\,\hat{\boldsymbol{i}} + dE_y(\boldsymbol{r})\,\hat{\boldsymbol{j}} + dE_z(\boldsymbol{r})\,\hat{\boldsymbol{k}} \\
&= \boldsymbol{E}(0) + (\nabla E_x \cdot r)\,\hat{\boldsymbol{i}} + (\nabla E_y \cdot r)\,\hat{\boldsymbol{j}} + (\nabla E_z \cdot r)\,\hat{\boldsymbol{k}}.
\end{aligned}
$$

The force on the charge $\rho d\tau$ in a small element $d\tau$ at $\boldsymbol{r}$ is

$$
\boldsymbol{dF} = \boldsymbol{E}(0)\,\rho\, d\tau + (\nabla E_x \cdot \rho \boldsymbol{r}\, d\tau)\,\hat{\boldsymbol{i}} + (\nabla E_y \cdot \rho r d\tau)\,\hat{\boldsymbol{j}} + (\nabla E_z \cdot \rho \boldsymbol{r}\, d\tau)\,\hat{\boldsymbol{k}}.
$$

Assuming that the volume occupied by the charge distribution is so small that ∇E_x, ∇E_y, ∇E_z may be treated as constants, the net force is

$$
\begin{aligned}
\boldsymbol{F} &= \boldsymbol{E}(0)\int \rho\, d\tau + \left[\nabla E_x \cdot \int \rho \boldsymbol{r}\, d\tau\right]\hat{\boldsymbol{i}} + \left[\nabla E_y \cdot \int \rho \boldsymbol{r}\, d\tau\right]\hat{\boldsymbol{i}} + \left[\nabla E_z \cdot \int \rho \boldsymbol{r}\, d\tau\right]\hat{\boldsymbol{k}} \\
&= (\boldsymbol{p} \cdot \nabla E_x)\,\hat{\boldsymbol{i}} + (\boldsymbol{p} \cdot \nabla E_y)\,\hat{\boldsymbol{j}} + (\boldsymbol{p} \cdot \nabla E_z)\,\hat{\boldsymbol{k}} \\
&= (\boldsymbol{p} \cdot \nabla)(E_x\hat{\boldsymbol{i}} + E_y\hat{\boldsymbol{i}} + E_z\hat{\boldsymbol{k}})
\end{aligned}
$$

or $$\boldsymbol{F} = (\boldsymbol{p} \cdot \nabla)\boldsymbol{E},$$

which is the same as Equation 4.12. A uniform electric field does not exert force on a dipole but a nonuniform field does.

EXAMPLE 4.6 **An electric dipole of dipole moment $p\hat{\boldsymbol{i}}$ is placed on the x-axis at $x\hat{\boldsymbol{i}}$ while a point charge q is placed at the origin. Using $F = (\boldsymbol{p} \cdot \nabla)\boldsymbol{E}$, find the force on the dipole due to the electric field of the point charge.**

Solution The force is

$$
\boldsymbol{F} = (\boldsymbol{p} \cdot \nabla)\boldsymbol{E}.
$$

It is clear that the force will have only an x-component, given by

$$
F = p\hat{\boldsymbol{i}} \cdot \left(\frac{\partial E_x}{\partial x}\hat{\boldsymbol{i}}\right) = p\frac{\partial E_x}{\partial x}\cdot
$$

The electric field due to the point charge is

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0}\frac{(x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}})}{(x^2 + y^2 + z^2)^{3/2}}.$$

Thus, $$\left.\frac{\partial E_x}{\partial x}\right|_{y=z=0} = \frac{q}{4\pi\varepsilon_0}\left(\frac{-2}{x^3}\right).$$

The force on the dipole is, therefore,

$$\boldsymbol{F} = -\frac{2qp}{4\pi\varepsilon_0 x^3}\hat{\boldsymbol{i}}.$$

You could have derived the force more easily by taking the field of the dipole at the origin and writing $\boldsymbol{F} = q\boldsymbol{E}$, and then using Newton's third law.

Torque on an electric dipole in a uniform electric field

A uniform electric field does not exert a force on an electric dipole. But it does exert a torque on the dipole. Consider a two-charge dipole consisting of charges $-q$ and $+q$ separated by a small distance d. Suppose the two charges are rigidly connected so that the separation between them remains the same. (The two charges may be situated at the ends of a tiny speck of material, the thickness of the speck defining the separation between the charges.) Let $-q$ and $+q$ be located at $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ (Figure 4.11). An external field $\boldsymbol{E}$ exists in the region. Assume the field $\boldsymbol{E}$ to be uniform.

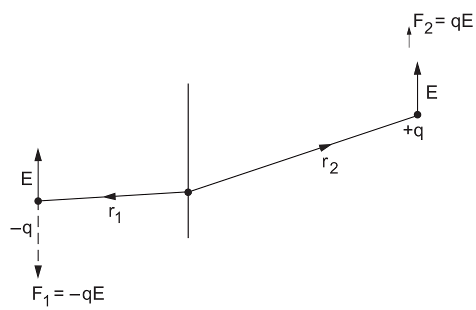


Fig. 4.11

The force on $-q$ is $\boldsymbol{F}_1 = -q\boldsymbol{E}$ and that on $+q$ is $\boldsymbol{F}_2 = q\boldsymbol{E}$. The torque of $\boldsymbol{F}_1$ about the origin is

$$\Gamma_1 = \boldsymbol{r}_1 \times \boldsymbol{F}_1 = -q\boldsymbol{r}_1 \times \boldsymbol{E}$$

and that of $\boldsymbol{F}_2$ about the origin is

$$\Gamma_2 = \boldsymbol{r}_2 \times \boldsymbol{F}_2 = q\boldsymbol{r}_2 \times \boldsymbol{E}.$$

The net torque on the dipole is

$$\Gamma = \Gamma_1 + \Gamma_2 = q(\boldsymbol{r}_2 - \boldsymbol{r}_1) \times \boldsymbol{E}.$$

But $(\boldsymbol{r}_2 - \boldsymbol{r}_1)$ is the vector from the charge $-q$ to the charge $+q$. Thus $q(\boldsymbol{r}_2 - \boldsymbol{r}_1) = \boldsymbol{p}$, the dipole moment, and hence the torque on the dipole due to the electric field is

$$\Gamma = \boldsymbol{p} \times \boldsymbol{E}. \tag{4.13}$$

You can easily do the mathematics for a general dipole given by a charge density $\rho(\boldsymbol{r})$ and get the same expression for the torque.

Energy of a dipole in an electric field

Suppose a two-charge dipole is placed in an external uniform electric field $\boldsymbol{E}$. The charge $-q$ is at A and $+q$ is at B as shown in Figure 4.12. The electric field $\boldsymbol{E}$ is produced by charges other than these two. What is the energy corresponding to the interaction of $-q$, $+q$ with these other charges producing $\boldsymbol{E}$? The electric field $\boldsymbol{E}$ corresponds to a potential function V. Let us take the x-axis to be along the direction of $\boldsymbol{E}$. The potential can then be expressed as $V = V_0 - Ex$. Using $\boldsymbol{E} = -\nabla V$, check that this potential gives a uniform electric field E along the x-direction. Let the potential at A be V_A and that at B be V_B. The energy of $-q$, $+q$ in the electric field $\boldsymbol{E}$ is

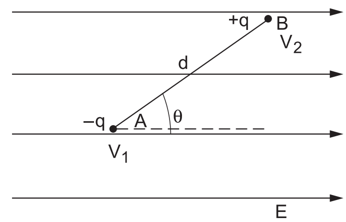


Fig. 4.12

$$\begin{aligned} U &= (-q)V_A + (+q)V_B \\ &= q(V_B - V_A) \\ &= q[(V_0 - Ex_B) - (V_0 - Ex_A)] \\ &= -qE(x_B - x_A) \\ &= -qE(AB)\cos\theta \end{aligned}$$

where θ is the angle between the direction of the dipole moment and the electric field. The quantity $q(AB)$ is the magnitude of the dipole moment $\boldsymbol{p}$. The energy of the dipole in the field $\boldsymbol{E}$ is, therefore,

$$U = -pE\cos\theta$$

or $$U = -\boldsymbol{p}\cdot\boldsymbol{E}. \tag{4.14}$$

The energy is zero if the dipole moment is oriented perpendicular to the field. However, the energy is minimum when $\cos\theta = 1$, i.e., $\theta = 0$. In this case, the dipole can stay in stable equilibrium.

Equation 4.14 is valid for any kind of electric dipole placed in a uniform electric field $\boldsymbol{E}$.

Please note that we are neither considering the energy of the dipole itself nor that of the external charge distribution that has created the field $\boldsymbol{E}$. These are given readymade. Only when the dipole is placed in the field is the extra energy of interaction given by the above equation.

4.4 Quadrupole Contribution

If the monopole and dipole contributions are both zero, you need to consider the third term in the multipole expansion given in Equation 4.1. This term is

$$V_q(\boldsymbol{r}) = \frac{\int \rho(\boldsymbol{r})\, r'^2 (3\cos^2\alpha - 1)\, d\tau'}{4\pi\varepsilon_0 r^3}.$$

A definition of the quadrupole moment of the charge distribution, independent of any reference to the field point (the point where the electric field is being calculated), is possible. We will not discuss the derivation but will give you the result. The monopole moment of a charge distribution is the total charge Q, which is just a scalar. The dipole moment $\boldsymbol{p}$ is a vector quantity so you have three components p_x, p_y, p_z. The quadrupole moment Q has nine components and is called a tensor quantity of rank two. Suppose you have a charge distribution given by the charge density $\rho(\boldsymbol{r})$. Set up a Cartesian coordinate system and choose the volume element $d\tau$ at the position $\boldsymbol{r}$ with coordinates (x, y, z). The nine components of the quadrupole moment of this charge distribution are written as follows.

$$Q_{xx} = \int (2x^2 - y^2 - z^2)\, \rho(\boldsymbol{r})\, d\tau$$

$$Q_{yy} = \int (2y^2 - x^2 - z^2)\, \rho(\boldsymbol{r})\, d\tau$$

$$Q_{zz} = \int (2z^2 - x^2 - y^2)\, \rho(\boldsymbol{r})\, d\tau$$

$$Q_{xy} = Q_{yx} = \int 3xy\, \rho(\boldsymbol{r})\, d\tau$$

$$Q_{xz} = Q_{zx} = \int 3xz\, \rho(\boldsymbol{r})\, d\tau$$

$$Q_{yz} = Q_{zy} = \int 3yz\, \rho(\boldsymbol{r})\, d\tau.$$

The nine components are often arranged in the form of a matrix.

$$\begin{pmatrix} Q_{xx} & Q_{xy} & Q_{xz} \\ Q_{yx} & Q_{yy} & Q_{yz} \\ Q_{zx} & Q_{zy} & Q_{zz} \end{pmatrix}.$$

As usual, if you have a linear charge distribution specified by a linear charge density $\lambda(\boldsymbol{r})$, $\rho d\tau$ is replaced by λdl in the integration above. Similarly, for a surface charge distribution, $\rho d\tau$ is replaced by σda. If the charge distribution is discrete, integration is replaced by summation over i and $\rho d\tau$ is replaced by q_i.

All the nine components of quadrupole moment can be expressed as $Q_{ij} = \int (3r_i r_j - r^2 \delta_{ij})\, \rho d\tau$. Quadrupole moment components, in general, depend on the origin. But if the monopole moment and the dipole moment are both zero, the quadrupole moment does not depend on the choice of the origin.

Once you know these nine components, the contribution to the potential at $\hat{r}$ is

$$V_q = \frac{1}{8\pi\varepsilon_0 r^3}\sum_i \sum_j (\hat{r})_i(\hat{r})_j Q_{ij}. \tag{4.15}$$

Here i and j take values x, y, z, and $\hat{r}$ is the unit vector along the direction from the origin to the point P where the potential is needed. The quantity $(\hat{r})_i$ is the ith component of this unit vector. For example, $(\hat{r})_x$ denotes the x-component of this unit vector, that is, $\cos\alpha$ if α is the angle between $\boldsymbol{r}$ and the x-axis. The case is similar for $(\hat{r})_y$ and $(\hat{r})_z$.

The quadrupole contribution becomes important only when the monopole and dipole moments are zero. A charge distribution of this type (when monopole and dipole are zero) is itself sometimes called a quadrupole. The easiest way to construct a quadrupole is to take two dipole distributions and place them close by with their dipole moments in the opposite directions. The net dipole moment of the distribution will then become zero and the distribution will most likely behave like a quadrupole.

EXAMPLE 4.7 **Three charges $-q$, $2q$ and $-q$ are placed on the x-axis, $2q$ at the origin and the other two charges on the two sides of the origin at a distance a from it (Figure 4.13).**

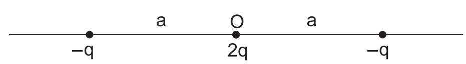


Fig. 4.13

(a) Obtain all the nine components of the quadrupole moment. (b) Find the potential due to this charge distribution at a point P on the x–y plane at a large distance r from the origin, OP making an angle of 45° with the x-axis.

Solution (a) The coordinates of the three charges are $(-a, 0, 0)$, $(0, 0, 0)$ and $(a, 0, 0)$.

$$Q_{xx} = \sum_i (2x_i^2 - y_i^2 - z_i^2)q_i$$

$$= \sum_i 2x_i^2 q_i = 2[(-q)a^2 + (2q)\cdot 0 + (-q)a^2] = -4qa^2.$$

$$Q_{yy} = \sum_i (2y_i^2 - x_i^2 - z_i^2)q_i$$

$$= \sum_i -q_i \cdot x_i^2 = [qa^2 + qa^2] = 2qa^2.$$

$$Q_{zz} = \sum_i (2z_i^2 - x_i^2 - y_i^2)q_i$$

$$= \sum_i -q_i x_i^2 = 2qa^2.$$

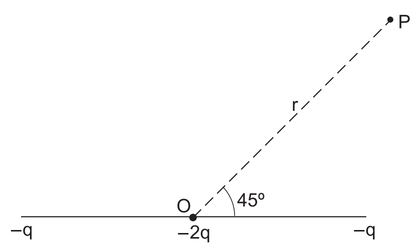


Fig. 4.14

$$Q_{xy} = Q_{yx} = \sum_i 3x_i y_i q_i = 0.$$

$$Q_{xz} = Q_{zx} = \sum_i 3x_i z_i q_i = 0.$$

$$Q_{yz} = Q_{zy} = \sum_i 3y_i z_i q_i = 0.$$

If we arrange the nine components in a matrix in standard fashion, we get

$$Q = \begin{pmatrix} -4qa^2 & 0 & 0 \\ 0 & 2qa^2 & 0 \\ 0 & 0 & 2qa^2 \end{pmatrix}.$$

(b) The unit vector along the direction of ***OP*** is

$$\hat{\boldsymbol{r}} = (\cos 45°)\,\hat{\boldsymbol{i}} + (\sin 45°)\,\hat{\boldsymbol{j}} = \frac{1}{\sqrt{2}}\hat{\boldsymbol{i}} + \frac{1}{\sqrt{2}}\hat{\boldsymbol{j}}.$$

So,

$$(\hat{\boldsymbol{r}})_x = (\hat{\boldsymbol{r}})_y = \frac{1}{\sqrt{2}},\ (\hat{\boldsymbol{r}})_z = 0.$$

The monopole and dipole moments of this distribution are zero. So the dominant contribution for large $\boldsymbol{r}$ is made only by the quadrupole moment.

$$V_q = \frac{1}{8\pi\varepsilon_0 r^3}\left[\sum_i \sum_j (\hat{\boldsymbol{r}})_i (\hat{\boldsymbol{r}})_j Q_{ij}\right]$$

$$= \frac{1}{8\pi\varepsilon_0 r^3}[(\hat{\boldsymbol{r}})_x(\hat{\boldsymbol{r}})_x Q_{xx} + (\hat{\boldsymbol{r}})_y(\hat{\boldsymbol{r}})_y Q_{yy}].$$

Each of the other seven terms is zero because Q_{ij} or $(\hat{\boldsymbol{r}})_i$ or $(\hat{\boldsymbol{r}})_j$ is zero in the present situation. Thus,

$$V_q = \frac{1}{8\pi\varepsilon_0 r^3}\left[\left(\frac{1}{2}\right)(-4qa^2) + \left(\frac{1}{2}\right)(2qa^2)\right]$$

$$= -\frac{qa^2}{8\pi\varepsilon_0 r^3}.$$

Note that the potential drops with distance as $1/r$ for a monopole, as $1/r^2$ for a dipole and as $1/r^3$ for a quadrupole. Likewise, the electric field drops with distance as $1/r^2$ for a monopole, as $1/r^3$ for a dipole and as $1/r^4$ for a quadrupole.

Quadrupole moment as a measure of deviation of charge distribution from spherical symmetry

Suppose you have a spherically symmetric charge distribution. This means the charge density $\rho(\vec{r})$ depends only on $\boldsymbol{r}$ and not on θ or ϕ with a suitably chosen origin. Then, all the directions are equivalent and hence

$$Q_{xx} = Q_{yy} = Q_{zz}.$$

$$Q_{zz} = \int \rho(\boldsymbol{r})(3z^2 - r^2)\,d\tau$$

$$= \int\int\int \rho(\boldsymbol{r})(3r^2\cos^2\theta - r^2)\,r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_0^\infty \rho(r)\,r^4\,dr \int_0^\pi (3\cos^2\theta - 1)\sin\theta\, d\theta \int_0^{2\pi} d\phi.$$

Check that $\int_0^\pi (3\cos^2\theta - 1)\sin\theta\, d\theta = 0.$

Hence, $\quad Q_{xx} = Q_{yy} = Q_{zz} = 0.$

Also, for any point (x, y, z) in the distribution, you have a point $(x, y, -z)$ with the same charge density ρ as the distance of these points from the origin is the same.

$$Q_{zx} = \int 3zx\rho(\boldsymbol{r})\,d\tau.$$

The contribution from the volume element at (x, y, z) and $(x, y, -z)$ is

$$3zx\rho(\boldsymbol{r})\,d\tau + 3(-z)\,x\rho(\boldsymbol{r})\,d\tau = 0.$$

As the whole volume can be divided into such pairs,

$$Q_{zx} = 0.$$

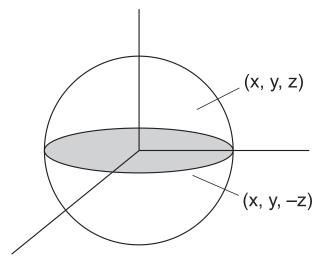


Fig. 4.15

Similarly all other components Q_{xy}, Q_{yz}, etc., are zero. The quadrupole moment of a spherically symmetric charge distribution is zero.

Suppose you have an oblate-shaped charge distribution symmetric about the z-axis. It is like our earth, flattened in the polar regions and bulged in the equatorial region. Also assume that the charge density ρ is uniform everywhere in the distribution. What is Q_{zz}?

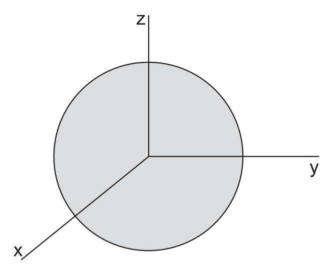


Fig. 4.16

$$Q_{zz} = \int \rho(\boldsymbol{r})(3z^2 - r^2)\,d\tau$$

$$= \rho\left[\int (2z^2 - x^2 - y^2)\,d\tau\right]$$

$$= \rho\left[\int 2z^2 d\tau - \int x^2 d\tau - \int y^2 d\tau\right]$$

$$= 2\rho\left[\int z^2 d\tau - \int x^2 d\tau\right].$$

As the charge distribution is flattened in the z-direction, the integral $\int z^2 d\tau$ will be smaller than $\int x^2 d\tau$. In an extreme case, you can think of a disk on the x–y plane where $\int z^2 d\tau = 0$ but $\int x^2 d\tau$ will have positive value.

Thus, $\quad Q_{zz} < 0.$

If you have a charge distribution of prolate shape, $Q_{zz} > 0$.

Electric quadrupole moments of nuclei can be obtained experimentally from the measurement on the γ-rays that they emit. From these moments, physicists get to know about the shape of nuclei.

If the quadrupole moment of a charge distribution is not zero, it indicates deviation from spherical symmetry.

Concepts at a Glance

1. For a finite charge distribution, the potential at a point far away can be expressed as a sum of terms varying as $1/r$, $1/r^2$, $1/r^3$, and so on. These are known as monopole, dipole, quadrupole terms, and so on.
2. The monopole moment of a charge distribution is its total charge.
3. The dipole moment of a charge distribution is $\boldsymbol{p} = \int \rho \boldsymbol{r} d\tau$.
4. If the monopole moment of a charge distribution is zero, the dipole moment is independent of the choice of origin.
5. If a small charge distribution around the origin with zero net charge has dipole moment $\boldsymbol{p} = p\hat{\boldsymbol{k}}$, the potential at a point (r, θ, ϕ) far away is $V = \dfrac{p\cos\theta}{4\pi\varepsilon_0 r^2}$, and the electric field is $E = \dfrac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta\,\hat{\boldsymbol{r}} + \sin\theta\,\hat{\theta})$.
6. The quadrupole moment of a charge distribution is given by a tensor of rank 2 having nine components.
7. The quadrupole moment of a spherically symmetric charge distribution is zero.

EXERCISES

Based on Concepts

1. The oxygen atom has larger electronegativity than hydrogen. As a result, in a water molecule, oxygen pulls the electron clouds more than the hydrogen atoms. Because of this, the water molecule acquires a dipole moment of about 6.5×10^{-30} cm. Is the water molecule (HOH) linear or bent?
2. Sketch the shapes of a few equipotential surfaces of an electric dipole consisting of charges $-q$ and $+q$ separated by a small distance d. Can you have a finite equipotential surface enclosing both the charges?
3. Can three point charges make up an electric dipole, without having monopole moment?
4. Two point charges $-q$ and $+q$ are placed at $(-a, 0, 0)$ and $(a, 0, 0)$. The electric potential at a point $(x, y, 0)$ due to these charges is found to be 40 V. Here $x >> a$ and $y >> a$. What would be the potential at $(2x, 2y, 0)$?
5. Consider the situation of the above problem. Suppose two more charges $-2q$ and $+2q$ are placed at points $(a/2, 0, 0)$ and $(-a/2, 0, 0)$ respectively. If the potential at point $(x', y', 0)$ is found to be 40 V, what would be the potential at $(2x', 2y', 0)$? Again assume $x' >> a$, $y' >> a$.
6. An electric dipole is placed at the origin with its dipole moment vector in the x–y plane. The electric field at the point $(0, a, 0)$ is found to be $E_0\hat{i}$. What is the direction of the dipole moment?
7. An electric dipole of dipole moment 1.0×10^{-4} C m is placed at the origin with its dipole moment along the positive x-direction. Find the potential at the point (0, 20 cm, 0) due to this dipole.
8. An electric dipole is constructed by placing charges $+q$ and $-q$ at $(a, 0, 0)$ and$(-a, 0, 0)$ respectively. What are the values of $\nabla \cdot \boldsymbol{E}$ and $\nabla \times \boldsymbol{E}$ at (a) $(a, 0, 0)$, (b) $(0, 0, 0)$ and (c) $(-a, 0, 0)$?

Problems

1. An electric dipole has dipole moment 2.0×10^{-5} C m. Find the magnitude of the electric field due to this dipole at a point 10 cm from the dipole in a direction making an angle of 30° with the direction of the dipole moment. [**Ans.** 5.85×10^{8} V/m]
2. An electric dipole is placed at the origin with its dipole moment vector in the x–y plane. The electric field at the point $(0, a, 0)$ is found to be $E_0\hat{\boldsymbol{i}}$. Find the electric field at $(a, 0, 0)$. [**Ans.** $-2E_0\hat{\boldsymbol{i}}$]
3. Four charges $q, -2q, 3q$ and $-2q$ are put on the x-axis at $(-2a, 0, 0), (-a, 0, 0), (0, 0, 0)$ and $(a, 0, 0)$ respectively. Find the approximate electric potential at the point $(x, y, 0)$, which is at a large distance from this charge distribution. [**Ans.** $\dfrac{-2qax}{4\pi\varepsilon_0(x^2+y^2)^{3/2}}$]
4. A dipole of dipole moment $p\boldsymbol{k}$ is placed at the origin. Find the force on the dipole if a point charge q is placed (a) on the x-axis at a distance r from the origin, and (b) on the z-axis at a distance r from the origin. [**Ans.** (a) $\dfrac{pq}{4\pi\varepsilon_0 r^3}\hat{\boldsymbol{i}}$ (b) $\dfrac{-2pq}{4\pi\varepsilon_0 r^3}\hat{\boldsymbol{j}}$]

5. An ideal dipole is placed at the centre of a spherical surface of radius R. The maximum potential on this surface due to the dipole is V_0. The points of the spherical surface where the potential is $V_0/2$ form a closed curve. Find the length of the periphery of this curve. [**Ans.** $\sqrt{3}\pi R$]

6. An ideal dipole of dipole moment $p_0\hat{k}$ is placed at the origin. Consider the surface $r^2 = A\cos\theta$ in spherical polar coordinates. What is the shape of this surface? Does it extend from $\theta = 0$ to π? Show that this is an equipotential surface except for points close to $\theta = \pi/2$. What is the value of the potential on this surface? Why can you not use this potential for points close to $\theta = \pi/2$? [**Ans.** $\frac{p}{4\pi\varepsilon_0 A}$]

7. An ideal dipole of dipole moment $\boldsymbol{p} = (2.0\times 10^{-10}\ \text{cm})\hat{\boldsymbol{i}}$ is kept at the origin. Find the work done in moving a point charge of 15 mC from the point (1 cm, 0, 0) to (0, 1 cm, 0). [**Ans.** 270 J]

8. An ideal dipole of dipole moment 1.0×10^{-6} C m is placed at the centre of a square ABCD of edge length 10 cm. The dipole moment is along the direction of AB. Find the work done in moving a 5-mC charge from (a) A to B (b) B to C (c) C to D and (d) D to A.

[**Ans.** (a) $9000\sqrt{2}$ J, (b) 0, (c) $-9000\sqrt{2}$ J, (d) 0]

9. A spherical surface of radius R carries a charge with surface charge density $\sigma = \sigma_0\cos\theta$. Find the maximum potential difference between any two points on the surface. [**Ans.** $\frac{2\sigma_0 R}{3\varepsilon_0}$]

10. Charge is assembled on a spherical surface of radius R. Taking the centre of the spherical surface to be the origin and using spherical coordinates, the final surface charge density is $\sigma = \sigma_0\cos\theta$. (a) Find the potential at the centre. (b) Find the potential at the surface as a function of θ. (c) Find the work done to assemble this distribution by bringing charge from infinity in infinitesimal amounts.

[**Ans.** (c) $2\pi\frac{\sigma_0^2 R^3}{9\varepsilon_0}$]

11. A spherical surface of radius R carries a charge with surface charge density $\sigma = \sigma_0\cos\theta$. Find the electric field just outside the sphere (a) on the z-axis, (b) on the x-axis, and (c) on the y-axis. Check that they satisfy the boundary conditions across the surface. [**Ans.** (a) $\frac{2\sigma_0}{3\varepsilon_0}\hat{\boldsymbol{k}}$ (b) $-\frac{\sigma_0}{3\varepsilon_0}\hat{\boldsymbol{k}}$ (c) $-\frac{\sigma_0}{3\varepsilon_0}\hat{\boldsymbol{k}}$]

12. Consider a spherical surface of radius R centred on the origin. The upper half, that is $0 < \theta < \pi/2$, contains a charge $+q$ distributed uniformly over it and the lower half, that is $\pi/2 < \theta < \pi$, carries a charge $-q$ distributed uniformly over it. Find the electric potential at points far away from the surface.

[**Ans.** $V = \frac{qR\cos\theta}{4\pi\varepsilon_0 r^3}$]

13. Two ideal electric dipoles, each having dipole moments $p_0\hat{\boldsymbol{k}}$, are placed at $(0, 0, -d)$ and $(0, 0, d)$. Calculate the magnitude of the force these dipoles exert on each other. [**Ans.** $\frac{3p_0^2}{32\pi\varepsilon_0 d^4}$]

14. (a) A ring of radius R is kept in the x–y plane with its centre at the origin. It carries a linear charge density $\lambda(\varphi) = \lambda_0\cos\varphi$, where the angle φ is measured from a fixed radius taken as the x-axis. (a) Find the electric monopole and dipole moments of this ring. (b) Using spherical polar coordinates, find the electric potential at a point far away from the ring. [**Ans.** (a) 0, $\lambda\lambda_0 R$ (b) $\frac{\lambda_0 R\sin\theta\cos\phi}{4\varepsilon_0 r^2}$]

15. Find the first nonvanishing multipole moment of the following charge distributions.

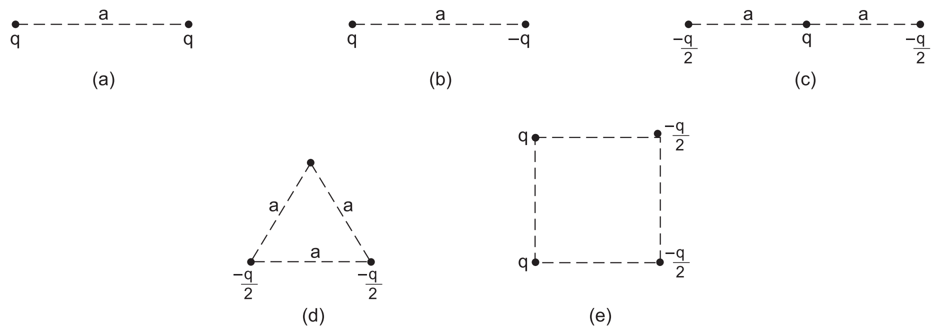


Fig. 4E.1

[**Ans.** (a) monopole, $2q$ (b) dipole, qa towards left (c) quadrupole, $Q_{11} = -2qa^2$, $Q_{22} = Q_{33} = -qa^2$, other components zero (d) dipole, $\dfrac{-qa\sqrt{3}}{2}$ (e) monopole, q.]

16. Four charges are put at the corners of a square of side a as shown in the figure. Find all the nine components of the quadrupole moment of this charge configuration.

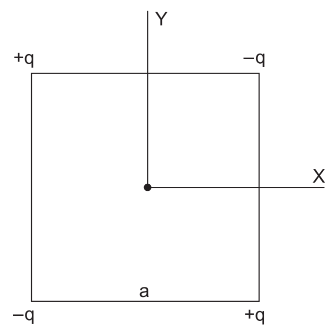


Fig. 4E.2

17. A spherical volume of radius R has charge density $\oint(\boldsymbol{r}) = \dfrac{kr(R-2r)}{r^2}\sin\theta$ with its centre at the origin and a particular diameter as the z-axis. (a) Show that the monopole moment and the dipole moment are zero. (b) Calculate all components of the quadrupole moment tensor. (c) Calculate the potential at $(0, 0, z)$ and $(0, 0, y)$ where y and z are very large compared to R.

18. A solid sphere of radius R centred on the origin has a charge density $\rho(\boldsymbol{r}) = k_0 \dfrac{R}{r^2}(R-2r)\cos\theta$. Find the first nonzero multipole moment of the charge distribution. [**Ans.** dipole moment $\dfrac{2k\pi R^4}{9}$]

19. A ring of radius a contains uniformly distributed charge Q. Take the axis of the ring as the z-axis and the centre of the ring as the origin. Find the expression for the potential at large distances from the ring up to the order $1/r^3$. [**Ans.** $\dfrac{1}{r} - \dfrac{a^2}{4r^3}(3\cos^2\theta - 1)$]

20. Consider two circular disks with radii R, kept parallel to each other at a separation d which is much smaller than R. The line joining the centres of the disks is perpendicular to the surfaces of the disks

as shown in the figure. The upper disk carries a uniform positive surface charge density σ and the lower disk carries a uniform negative surface charge density $-\sigma$. Consider a rectangular loop of sides x and $2x$ as shown in the figure. Here $x >> R$.

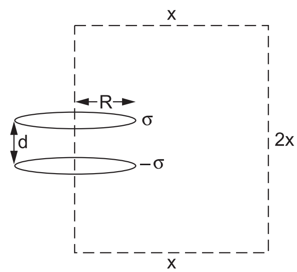


Fig. 4E.3

(a) What is the fallacy in the following argument? As $d << R$, the electric field is $\frac{\sigma}{\varepsilon_0}$ between the plates and zero outside, barring some fringing near the ends. Thus $\oint \boldsymbol{E} \cdot \boldsymbol{dl}$ gets a contribution only from the part of the loop between the plates, which is equal to $\sigma d/\varepsilon_0$. Hence, $\oint \boldsymbol{E} \cdot \boldsymbol{dl} \neq 0$ in violation of the electrostatic nature of the field.

(b) Resolve the fallacy quantitatively by explicitly integrating $\oint \boldsymbol{E} \cdot \boldsymbol{dl}$ over the loop and showing that it is zero.

❑

5 Conductors and Electric Fields

5.1 Conductors, Insulators and Semiconductors

What is an electrical conductor? Put simply, it is a material that conducts electricity. But at present, we are not talking about currents or the conduction of electricity. We are still discussing electrostatics, which describes a situation where all charges are stationary. In this context too, conductors have a very special behaviour and hence we have devoted a whole chapter to them. For the present, call a material a conductor if it contains a great many *free electrons*. What is a *free electron*?

Any given chunk of material is made up of atoms, and atoms are made of neutrons, protons and electrons. All the neutrons and protons of an atom are packed in a very small volume called the nucleus. Electrons are attached to the nucleus and arranged in orbitals. Not all electrons are bound to the nucleus by the same force. You have core electrons which are bound to the nucleus very tightly and then there are valence electrons in the outer orbit which are bound to the nucleus only weakly. In some solids, the valence electrons are so weakly bound to the nucleus that the interactions between the neighbouring atoms themselves break the bonds. Then these electrons keep moving from one atom to another randomly. Such electrons, which can move almost freely in the whole of the material, are called free electrons. In some materials, each atom donates one or more electrons to the group of free electrons and, as a result, the number of free electrons in the given sample becomes enormously large. Such materials are called conductors. All metals are conductors.

Insulators are materials in which even the valence electrons are tightly bound to the corresponding nuclei. You don't have free electrons in any significant number.

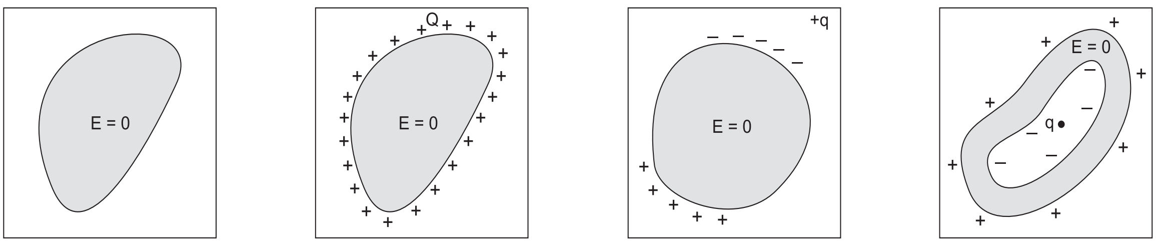
Some materials are classified as semiconductors. At zero kelvin, all electrons in a semiconductor are bound to the respective nuclei but at higher temperatures some of them become free. The number of free electrons is not as enormously large as in conductors but is still significant. There are also some orbitals with missing electrons in these materials. Such empty orbitals are called *holes* or *vacancies* and in many respects they act as free positive charges (the way free electrons are free negative charges). Most importantly, one can control the number of free electrons and holes, and this makes semiconductors very useful for manufacturing electronic equipment. However, this chapter is devoted to conductors and from now on, we will focus on them.

5.2 Some Properties of Conductors

(a) The electric field inside the material of a conductor is zero in electrostatic situations.

A conductor has a large number of free electrons everywhere in its volume. If there is an electric field inside the material of the conductor, it will exert forces on the free electrons. Due to these forces, the electrons will move in a particular direction. This will not be an electrostatic situation. If the free electrons stop moving (except for the random motion that is always there with zero average velocity) and all the charges are stationary, there can be no electric field inside the material of the conductor. Indeed, we will assume that there are no nonelectrostatic forces on the electrons.

This result is true irrespective of whether a given conductor is neutral or is given a charge. It is also true irrespective of what charges exist in the space outside the conductor. Also, the electric field inside the material of the conductor will be zero even if you place charges in any cavity (an empty space) the conductor may have. Figure 5.1 shows some such situations.



(a) Uncharged conductor (b) Charged conductor (c) Charge outside a conductor (d) Conductor with cavity carrying a charge

Fig. 5.1

Remember that the length scales considered in all these discussions are much larger than the sizes of atoms. We consider a small volume in the material, which still contains thousands of atoms, and then look at the average electric field. This field is zero inside a metal. But if you just consider a point close to a nucleus, you indeed have a large electric field. As you move inside the metal from one atom to another, the electric field fluctuates very rapidly. We are not talking about this fluctuating field. In this book, we will always talk about the average over a small volume. When we say "electric field at a point in the material", you can think of a small volume around that point, and the field is the average field in this volume.

(b) Charge density inside a conductor is zero.

As we have said, the electric field inside a conductor is zero everywhere—$\boldsymbol{E} = 0$, and hence $\nabla \cdot \boldsymbol{E} = 0$. As $\nabla \cdot \boldsymbol{E} = \rho/\varepsilon_0$, we have $\rho = 0$. Thus no charge can reside inside the material of the conductor. Once again, remember the length scale. Indeed there are protons and electrons inside the conductor. The average in any tiny volume will be zero. However, there is no restriction on the surface of the conductor—charges may reside on it. For example, if you put a charge Q on a conductor, it will spread over the surface. Or if you put a conductor near a charge Q, positive- and negative-charge densities appear on the surface.

What happens to the divergence of the electric field at the surface? You may have an electric field just outside the surface, and it becomes suddenly zero as you cross the surface and come just inside the surface. Thus $\nabla \cdot \boldsymbol{E}$ is infinity at the surface. Indeed, you only have a surface charge density and if you express a surface charge density in terms of volume charge density, you have to bring in the Dirac delta function, which tends to infinity at the surface and the equation $\nabla \cdot \boldsymbol{E} = \rho/\varepsilon_0$ is still satisfied.

Actually, the charge on a metal is spread over a few atomic thicknesses. We will treat this whole layer as the surface of the metal, and talk in terms of surface charge density.

Thus, if you take any volume in the material of the conductor, the net charge in it will be zero. Charge can only reside on the surface of a conductor.

What happens if we look at the centre of the nucleus of an atom inside the conductor? We see only positive charges (protons) in the immediate vicinity. How can we say that the charge density is zero? As always, we don't go to the length scale of angstroms and femtometres. When we say "charge density ρ at a point", we mean $dq/d\tau$ evaluated over a small volume element $d\tau$ that still contains several thousand or more atoms. For such a volume inside the material, $dq/d\tau = 0$. The charge density mentioned in this book is a kind of average charge density. When we refer to the charge contained in any volume inside the material of the conductor, the volume we consider is always large enough to contain at least several thousand atoms.

(c) The whole conductor has the same potential.

As the electric field is zero everywhere in the conductor, the potential must remain constant throughout the body of the conductor. This is because any variation in the potential would accompany a nonzero electric field according to the equation $\boldsymbol{E} = -\nabla V$.

5.3 Charge Distribution on a Conductor

Suppose you give a positive charge q to a conductor, and there is no charge anywhere else. Initially you put the charge at a particular point A [Figure 5.2(a)]. Maybe you rub an inflated rubber balloon on your jeans and touch the rubbed portion of the balloon to a point of the conductor and thus transfer a charge q from the balloon to that point of the conductor. How will the charge q be distributed on the whole surface of the conductor?

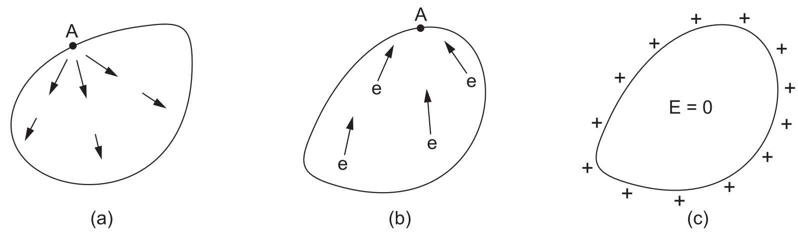


Fig. 5.2

When the charge q is placed at A, it creates an electric field everywhere according to the rule $\boldsymbol{E} = \dfrac{q(\boldsymbol{r} - \boldsymbol{r}_A)}{4\pi\varepsilon_0 |\boldsymbol{r} - \boldsymbol{r}_A|^3}$. The field is created inside the conductor also. (Remember, the charges are not yet settled.) This field will exert forces on the free electrons in the direction opposite to the field, that is, towards A. Remember that electrons are negatively charged. When electrons rush towards A, positive charges appear at the sites from where the electrons move out. The electrons reaching A will neutralize part of the charge there. Gradually, a charge distribution will appear on the surface, the instantaneous electric field guiding the further movement of electrons. The movement stops only when the electric field due to the surface charge distribution becomes zero everywhere in the material of the conductor. The electrostatic situation then prevails. How long do we have to wait before the charge distribution becomes static and the electric field becomes zero inside? For typical conductors, it is surely less than a nanosecond. If you don't mind neglecting time intervals like this, you can say that the charge immediately spreads on the surface making the field inside zero.

Fine, the charge q distributes itself on the surface of the conductor in such a way that the field everywhere inside the conductor is zero. But is this distribution unique? Can we have two ways of distributing q on the surface, each of which gives a zero field everywhere in the conductor? No, there is only one way of doing so. Conductors are very choosy. If you give the total charge, the conductor will go for just one kind of distribution as long as you do not change the geometry of the surface or the external charges. How do we justify this statement? Wait till the next chapter.

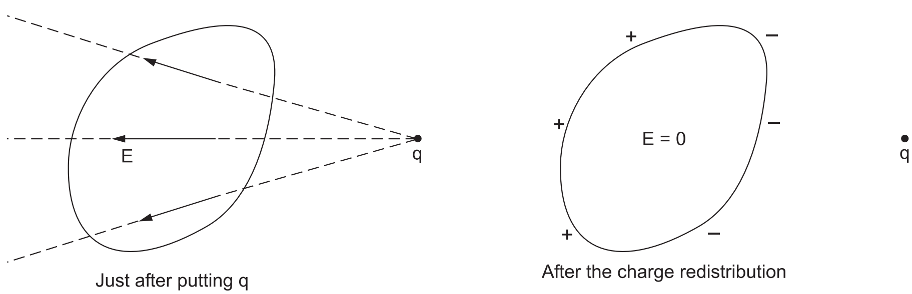


Fig. 5.3

The story is similar if you place the conductor in an external field. If you place a point charge q in front of the conductor, an electric field is created everywhere, including the space inside the conductor (Figure 5.3). But this field causes the free electrons to move and settle at the surface to give a surface charge distribution. This surface charge together with the external point charge q makes the field everywhere inside the material zero.

As the conductor is neutral and no charge is given to it, the total charge remains zero. Negative charges appear at certain places and positive charges at certain others.

5.4 A Charge Placed in a Cavity in a Conductor

Suppose you have a conductor and there is a cavity in it. So it has an outer as well as an inner surface. In the cavity, say we somehow manage to put a point charge $+q$, suspended in mid-air, having no contact with the conductor (Figure 5.4). What kind of charge distribution will be there on the surface or surfaces of the conductor?

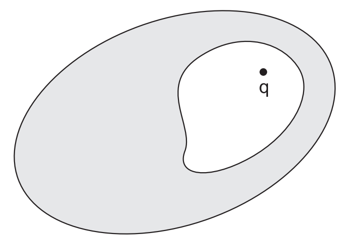


Fig. 5.4

Charges will appear on both the surfaces. The total charge appearing on the inner surface will be $-q$. How do we say that? Take a Gaussian surface going entirely through the conductor and enclosing the cavity. The electric field at all the points of this Gaussian surface is zero, because these points are all inside the material of the conductor. Thus the flux of the electric field through this surface is zero and hence the total charge enclosed is zero. But the charge $+q$ in the cavity is inside the Gaussian surface. So you must have a total charge $-q$ on the inner surface of the conductor.

But the conductor itself is neutral. We have not put any charge on it, while putting $+q$ in the cavity. So if charge $-q$ appears on the inner surface, charge $+q$ must appear on the outer surface.

The distribution of $-q$ on the inner surface depends on the shape and size of the surface together with the location of $+q$ in the cavity. The distribution is such that the electric field due to the point charge $+q$ and the surface charge on the inner surface becomes zero at all points in the conductor. In fact, it also works out that the field due to these charges ($+q$ in the cavity and $-q$ on the inner surface) is zero everywhere beyond the cavity, even outside the conductor.

The charge $+q$ on the outer surface of the conductor distributes itself in a fashion that depends on the shape and size of the outer surface. It turns out that it does not depend on the geometry of the cavity and placement of the point charge q in it. If you change the position of the charge in the cavity, the outer surface will have the same charge distribution.

The electric field due to the charge distributed on the outer surface is zero everywhere inside the conductor, including the cavity, but not outside the conductor. Thus, the field in the cavity is solely determined by the charges on the inner surface and the charge in the cavity. And the field outside the conductor is solely due to the charge on the outer surface of the conductor. In Figure 5.5, we again show a conductor with a cavity which contains a charge q in it. $\boldsymbol{E_1}$ is the field due to the point charge q in the cavity and the charge $-q$ on the inner surface. The field $\boldsymbol{E_2}$ is due to only the charge $+q$ on the outer surface of the conductor. Check that the expressions for $\boldsymbol{E_1}$ and $\boldsymbol{E_2}$ match what we have just described. The following discussion may help you understand this better.

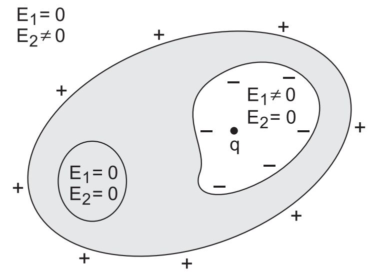


Fig. 5.5

Suppose we have two conductors A and B (Figure 5.6). A has no cavity and the outer surface has the same shape and size as the conductor in Figure 5.5. Let us put a charge $+q$ on this conductor, which spreads on the surface. Let the charge distribution on the surface be represented by the surface charge density $\sigma_2(\boldsymbol{r})$. This charge distribution must produce zero field everywhere in the conductor A.

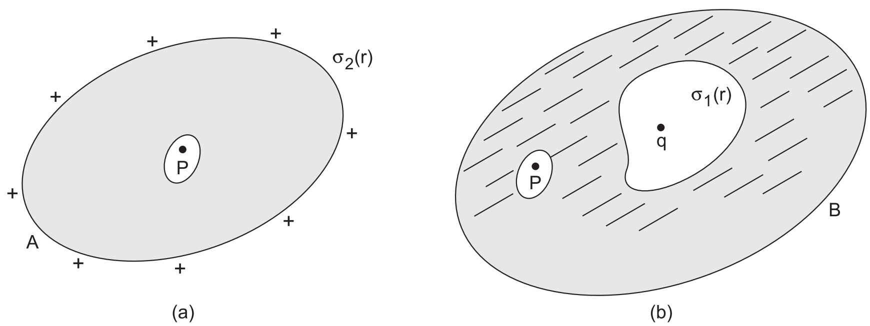


Fig. 5.6

The conductor B is very large and has a cavity of the same shape and size as the one in Figure 5.5. Let us put a point charge $+q$ in the cavity at the same location as in Figure 5.5. As a result, a charge $-q$ appears on the inner surface of the conductor B and $+q$ appears on the outer surface. Let the charge distribution on the inner surface be represented by a surface charge density $\sigma_1(\boldsymbol{r})$.

The field at any point in the material of the conductor B, such as at P (Figure 5.6), is zero. The outer surface of the conductor B is far away and the field from those faraway charges must be vanishingly small. This means the field due to the point charge q and $\sigma_1(\boldsymbol{r})$ must be zero at all points beyond the cavity.

Now come to the original conductor with the cavity, the one we started with in Figure 5.5. Its outer surface is the same as that of conductor A of Figure 5.6(a) and its cavity is the same as that in conductor B of Figure 5.6(b). Let us put a point charge $+q$ in the cavity, in the same way as in Figure 5.5, $\sigma_1(\boldsymbol{r})$ on the inner surface and $\sigma_2(\boldsymbol{r})$ on the outer surface. This distribution is shown in Figure 5.7. Take any point P in the conductor outside the cavity. As discussed above, the charge

q in the cavity and $\sigma_1(\boldsymbol{r})$ on the inner surface will produce a zero field here. So will $\sigma_2(\boldsymbol{r})$. Thus, the net field due to the distribution shown in Figure 5.7 will be zero at all points in the conductor.

So we have depicted one possible charge distribution on the conductor, which ensures zero field in the conductor. We have to believe, once again, that with the same external charge distribution and the same charge given to the conductor, the charge on the conductor cannot be distributed in more than one way to ensure zero electric field everywhere within. Here we put a charge q in the cavity at the given place and tell you the total charge on the conductor, which is zero. This ensures that there is only one way of distributing the charge on the conductor. Figure 5.7 gives one such distribution and this must be the only possible distribution.

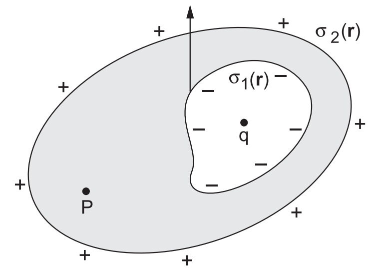


Fig. 5.7

Thus the charges will be distributed in such a way that the charge on the outer surface produces zero field everywhere inside the outer surface (including the cavity), and the point charge q together with the charge on the inner surface produces zero field everywhere outside the cavity (including the space outside the conductor).

5.5 Electric Field Just outside a Conductor

Consider a conductor with a certain charge distribution on its surface (Figure 5.8). The charge distribution might have come about due to the charge given to the conductor and/or charges kept outside the conductor. Consider a point P just outside the conductor. Suppose the surface charge density at the surface, in front of P, is σ. Now consider a point P_1 just inside the surface, opposite to P. The electric field will be zero at P_1 as this point is in the material of the conductor. Suppose the field at P is $\boldsymbol{E}$, which can be expressed as $\boldsymbol{E} = \boldsymbol{E}_\mathrm{n} + \boldsymbol{E}_\mathrm{t}$ where $\boldsymbol{E}_\mathrm{n}$ and $\boldsymbol{E}_\mathrm{t}$ are components of $\boldsymbol{E}$ normal and tangential to the surface there. The tangential component of the electric field (electrostatic case) across a surface is always continuous. Hope you remember the boundary conditions on the electrostatic field, which we have derived earlier.

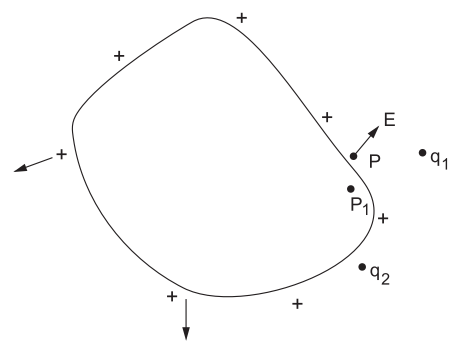


Fig. 5.8

The electric field at P_1 is zero and hence its tangential component is also zero. As the tangential component should be continuous across the surface, it should also be zero at P. The electric field just outside a conductor has to be normal to the surface of the conductor.

Now come to the normal component of the electric field at P. It may be discontinous if the surface has a charge density. If the surface charge density on the conductor close to P is σ, the normal component will be discontinuous by the amount σ/ε_0. As the electric field just inside the surface is zero, its normal component is also zero. Thus

$$E_n - 0 = \frac{\sigma}{\varepsilon_0},$$

giving
$$E_n = \frac{\sigma}{\varepsilon_0} \cdot$$

But the field has no tangential component. So,

$$\boldsymbol{E} = E_n \hat{\boldsymbol{n}} = \frac{\sigma}{\varepsilon_0}\hat{\boldsymbol{n}}.$$

Thus the electric field just outside the surface of a conductor is σ/ε_0 in the normal direction. Make sure you understand that the electric field $\frac{\sigma}{\varepsilon_0}\hat{\boldsymbol{n}}$ is due to all the charges, including those placed outside the conductor, if any. It is the net electric field existing just outside the surface of a conductor. The unit vector $\hat{\boldsymbol{n}}$ is in the direction of the outward normal. If σ is positive, the field just outside is outwards, and if σ is negative, the field is inwards.

5.6 Electrostatic Force on a Small Area of a Conductor

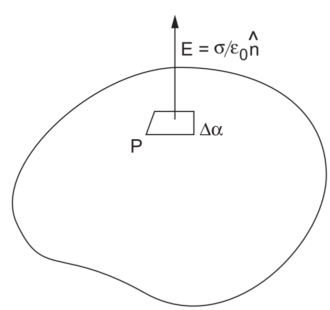


Fig. 5.9

Think of a conductor which has a surface charge density σ at a particular point P (Figure 5.9). Construct a small surface area Δa containing P. The charge on this area is $\sigma\Delta a$. What is the force on this charge $\sigma\Delta a$ due to all the other charges present? The field on one side of $\sigma\Delta a$ is $\boldsymbol{E} = \frac{\sigma}{\varepsilon_0}\hat{\boldsymbol{n}}$ and that on the other side is zero. Can we use the equation $\boldsymbol{F} = q\boldsymbol{E}$ to get this force? And if we do so, which electric field should we use? The fields on the two sides of $\sigma\Delta a$ are different. Should we use the average of the two? That will be $\frac{1}{2}\left(\frac{\sigma}{\varepsilon_0}\hat{\boldsymbol{n}} + 0\right) = \frac{\sigma}{2\varepsilon_0}\hat{\boldsymbol{n}}$. Interestingly, taking the average turns out to give the correct result. Let us analyze the situation more logically.

Let us look at the entire surface of the conductor except the patch $\sigma\Delta a$. Suppose the field at P due to the rest of the conductor is $\boldsymbol{E}_{rest}$. Then the force on $\sigma\Delta a$ will be $(\boldsymbol{E}_{rest})\sigma\Delta a$. To get the field due to the rest of the charges only, we must subtract the field due to $\sigma\Delta a$ from the total field. At points very close to the patch Δa, the field due to this patch will be $\frac{\sigma}{2\varepsilon_0}$ away from the layer. For points just outside the surface, the field will be $\frac{\sigma}{2\varepsilon_0}\hat{\boldsymbol{n}}$ whereas for points just inside, this field will be $-\dfrac{\sigma}{2\varepsilon_0}\hat{\boldsymbol{n}}$. These fields and the patch are shown in Figure 5.10 in an enlarged version. The field due to the rest of the charges is obtained as follows.

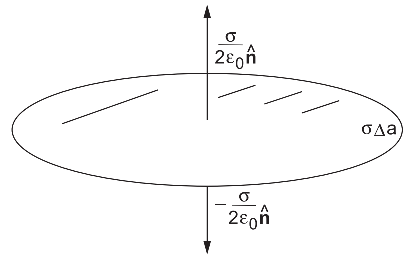


Fig. 5.10

For points just outside,

$$\boldsymbol{E}_{\text{rest}} = \frac{\sigma}{\varepsilon_0}\hat{\boldsymbol{n}} - \frac{\sigma}{2\varepsilon_0}\hat{\boldsymbol{n}} = \frac{\sigma}{2\varepsilon_0}\hat{\boldsymbol{n}}.$$

For points just inside,

$$\boldsymbol{E}_{\text{rest}} = 0 - \left(-\frac{\sigma}{2\varepsilon_0}\hat{\boldsymbol{n}}\right) = \frac{\sigma}{2\varepsilon_0}\hat{\boldsymbol{n}}.$$

So there is no discontinuity in the field due to the rest of the charges. The charge $\sigma\Delta a$ finds itself in a field $\left(\dfrac{\sigma}{2\varepsilon_0}\hat{\boldsymbol{n}}\right)$ (due to the rest of the charges) from both sides.

Thus the force on the charge $\sigma\Delta a$ is

$$\Delta\boldsymbol{F} = (\sigma\Delta a)\left(\frac{\sigma}{2\varepsilon_0}\hat{\boldsymbol{n}}\right)$$

$$= \frac{\sigma^2}{2\varepsilon_0}\Delta a\,\hat{\boldsymbol{n}}$$

or

$$\frac{\Delta\boldsymbol{F}}{\Delta a} = \frac{\sigma^2}{2\varepsilon_0}\hat{\boldsymbol{n}}. \tag{5.1}$$

This quantity, force per unit area, is sometimes called electrostatic pressure.

EXAMPLE 5.1 **A conducting sphere of radius R_1 is surrounded by a concentric, conducting spherical shell of radius R_2. The inner sphere is given a charge Q_1 and the outer shell is given a charge Q_2. (a) How will the charge distribute itself on the surfaces? (b) Taking the potential at infinity to be zero, what is the potential of the inner sphere and that of the outer shell? (c) Find the force per unit area on the outer surface of the shell.**

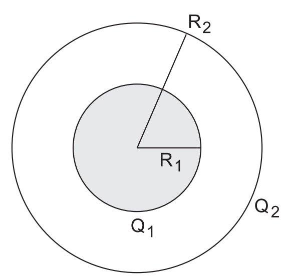


Fig. 5.11

Solution (a) The charge Q_1 will be distributed uniformly on the surface of the inner sphere. An equal charge $-Q_1$ must appear on the facing surface, i.e., the inner surface of the shell (for the shell, the whole of the inside is like a cavity and you have a charge Q_1 in this cavity). As the shell contains a net charge Q_2, the charge appearing on the outer surface of the shell will be $Q_1 + Q_2$. Because of spherical symmetry, the charges will distribute themselves uniformly over the respective surfaces.

(b) The potential due to the charge Q_1 placed on the inner sphere will be

$$V_1 = \frac{Q_1}{4\pi\varepsilon_0 R_1} \text{ at } r = R_1$$

and $$V_2 = \frac{Q_1}{4\pi\varepsilon_0 R_2} \text{ at } r = R_2.$$

The potential due to the charge Q_2 placed on the outer shell will be

$$V' = \frac{Q_2}{4\pi\varepsilon_0 R_2} \text{ at } r = R_1 \text{ and also at } r = R_2.$$

This is because the potential inside a spherical shell due to a uniformly distributed charge on it is the same everywhere.

Thus the potential of the inner sphere is

$$V = V_1 + V' = \frac{Q_1}{4\pi\varepsilon_0 R_1} + \frac{Q_2}{4\pi\varepsilon_0 R_2}$$

and that of the outer shell is

$$V = V_2 + V' = \frac{Q_1}{4\pi\varepsilon_0 R_2} + \frac{Q_2}{4\pi\varepsilon_0 R_2}.$$

(c) The charge density on the outer surface of the shell is

$$\sigma = \frac{Q_1 + Q_2}{4\pi R_2^2}.$$

So the force per unit area on this surface is

$$\frac{\Delta \boldsymbol{F}}{\Delta a} = \frac{\sigma^2}{2\varepsilon_0}\hat{\boldsymbol{n}} = \frac{(Q_1 + Q_2)^2}{32\pi^2 \varepsilon_0 R_2^4}\hat{\boldsymbol{r}}.$$

5.7 Capacitors

Two conductors placed close to each other make up a capacitor. One of the conductors is given a positive charge (say $+Q$) and the other, an equal negative charge ($-Q$). Depending on the shapes of the conductors and the way they are placed, charges will be distributed on their surfaces and will create an electric field in space. This will result in a potential difference between the conductors, given by

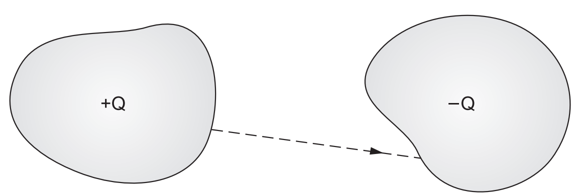


Fig. 5.12

$$V = V_2 - V_1 = -\int_1^2 \boldsymbol{E} \cdot d\boldsymbol{l}$$

where the integration can be performed on any line joining the first conductor to the second. The electric field and the potential are created by the charges distributed on the surfaces of the conductors and hence are proportional to Q for the given shape and positioning of the conductors. Thus you can write

$$Q = CV,$$

where C is a constant for the given pair of conductors in the given geometry. This constant is called the capacitance of the capacitor formed by these two conductors. The SI unit of capacitance is coulomb/volt, which is written as farad, or F.

To understand the basics, we will consider three kinds of capacitors with different types of geometry.

Parallel-plate capacitors

In such capacitors, two large conducting plates are placed parallel to and in front of each other. Suppose the separation between the facing surfaces is d and the surface area of each plate is A. If you put a charge $+Q$ on one plate and $-Q$ on the other, the charge densities on the two plates will be $\sigma = \pm Q/A$ and the field at any point P between the plates will be

$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A}$$

in a direction perpendicular to the plates. The potential difference between the plates will be

$$V = Ed = \frac{Qd}{\varepsilon_0 A}\cdot$$

Hence, the capacitance is

$$C = \frac{Q}{V} = \frac{\varepsilon_0 A}{d}\cdot \tag{5.2}$$

We have used the expression for electric field (σ/ε_0) that is valid for infinite charge layers, whereas here the charge is spread only on a finite area. The field will not be given by $\frac{\sigma}{\varepsilon_0}$ near the edges of the capacitor and its direction will also be different. Figure 5.13 qualitatively shows the electric field lines. The lines bend near the edges. However, if the plates are large enough, the expression for capacitance given by Equation 5.2 can be used to a good approximation.

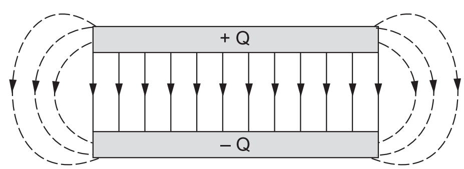


Fig. 5.13

Spherical capacitors

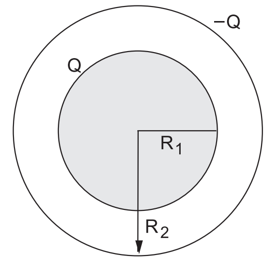


Fig. 5.14

A spherical capacitor is made by placing a spherical conductor concentrically within a larger spherical shell. Suppose the radius of the inner conductor is R_1 and that of the outer one is R_2 (Figure 5.14). These are the radii of the facing surfaces of the two conductors. Suppose the inner conductor is given a charge $+Q$ and the outer one, $-Q$.

The situation is the same as that in Example 5.1, with $Q_1 = Q$ and $Q_2 = -Q$. There will be no charge on the outer surface of the outer conductor. You can get the potential difference between the two by using the expressions derived for the potentials. However, let us calculate it afresh using the electric field in the space between the conductors.

The field in the gap between the two conductors is given by

$$\boldsymbol{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}}$$

and the potential difference between the conductors is

$$\begin{aligned} V = V_1 - V_2 &= -\int_2^1 \boldsymbol{E}\cdot d\boldsymbol{r} \\ &= -\int_{R_2}^{R_1} \frac{Q}{4\pi\varepsilon_0 r^2}dr \\ &= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{Q(R_2 - R_1)}{4\pi\varepsilon_0 R_1 R_2}. \end{aligned}$$

The capacitance is, therefore,

$$C = \frac{Q}{V} = \frac{4\pi\varepsilon_0 R_1 R_2}{R_2 - R_1}. \tag{5.3}$$

Cylindrical capacitors

A cylindrical capacitor is made by putting a cylindrical conductor of radius R_1 coaxially within a larger cylindrical shell of radius R_2. You can refer to Figure 5.14 in this case too. If the length of the cylinders is large as compared to their radii, you can use the equation for the electric field due to charged, infinitely long cylinders. If the length of the cylinders is L and the charges given to the inner and outer conductors are $+Q$ and $-Q$, the field in the gap between them (not close to the ends) will be

$$E = \frac{Q/L}{2\pi\varepsilon_0 s}\hat{s}$$

where s is the distance from the common axis of the cylinders, and cylindrical coordinates are used. This is the same as that due to a long, straight, line charge having a charge density $\lambda = Q/L$. The potential difference between the two cylinders is

$$V = V_1 - V_2 = -\int_{R_2}^{R_1} \boldsymbol{E} \cdot d\boldsymbol{l}$$

$$= -\frac{Q/L}{2\pi\varepsilon_0}\int_{R_2}^{R_1}\frac{ds}{s} = \frac{Q/L}{2\pi\varepsilon_0}\ln\frac{R_2}{R_1}.$$

Hence, the capacitance is

$$C = \frac{2\pi\varepsilon_0 L}{\ln (R_2/R_1)}. \tag{5.4}$$

5.8 Energy Stored in a Capacitor

Consider a capacitor with two conductors having charges $+Q$ and $-Q$. Depending on the geometrical arrangement of the conductors, the charges will spread in particular fashions on the surfaces of the conductors. We want to know the electrostatic energy of the charge configuration.

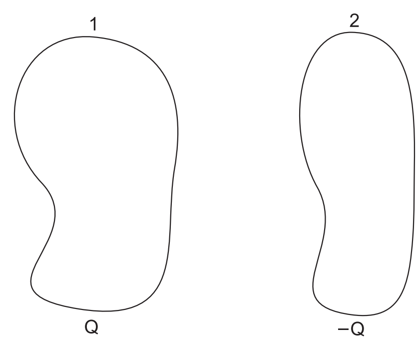


Fig. 5.15

Consider the charges $+Q$, $-Q$ being built up on the conductors in bits. Initially the two conductors are uncharged. We transfer a charge dq from conductor 2 to conductor 1. We keep transferring such charges dq till the total charge transferred is Q. Finally, conductor 1 will acquire a charge $+Q$ and conductor 2, a charge $-Q$.

Consider the stage when the charge q has been transferred. The two conductors have charges $+q$ and $-q$ and so the potential difference between them at this time is

$$V_1 - V_2 = \frac{q}{C},$$

where C is the capacitance of the system.

The next charge quantum dq is taken from conductor 2 to conductor 1 and hence the electrostatic energy is increased by $dq(V_1 - V_2)$. The total energy U stored in the capacitor when the charges are Q and $-Q$ is, therefore, given by

$$U = \int_0^Q dq\,(V_1 - V_2) = \int_0^Q \frac{q}{C} dq$$

$$= \frac{Q^2}{2C} = \frac{1}{2}QV = \frac{1}{2}CV^2.$$

These are general expressions applicable to any kind of capacitor.

5.9 Experiments

The capacitance of a capacitor can be easily measured. There are instruments called *L-C-R* meters which can be connected to the two plates of a capacitor and the capacitance can be read from the panel. They can also measure the inductance of a coil and the resistance of a resistor. You must be familiar with these quantities from your school physics classes. You can measure capacitance using some multimeters too.

Buy capacitors of different kinds. Read the capacitance values mentioned on the unit. What are the typical values? They are generally in microfarads. The SI unit farad is a really large one. You don't see capacitors of capacitance 1 farad around.

Can you make your own capacitor? Very easily. One suggestion is as follows. Take two steel tumblers, put one inside the other, separated by a piece of paper or a polythene. Measure the capacitance.

Concepts at a Glance

1. Conductors have a large number of electrons that can move almost freely inside the conductor but cannot come out on their own. These are called free electrons.
2. The electric field inside a conductor is zero in an electrostatic situation, irrespective of any charge put on or outside it.
3. Any charge given to a conductor distributes itself on its surface. The charge density inside a conductor is zero in an electrostatic situation.
4. The electric field just outside the conductor is perpendicular to the surface. Its magnitude is σ/ε_0 where σ is the surface charge density on the conductor close to that point.
5. The electrostatic force on a small part of a charged conductor due to the rest of the conductor is $\sigma^2/2\varepsilon_0$ per unit area.

EXERCISES

Based on Concepts

1. Why is wood an insulator when it also has a large number of electrons?
2. Give an argument to show that there can be no electric field inside a conductor in an electrostatic situation.
3. A thin, circular disk of radius R is made up of a conducting material. A charge Q is given to it, which spreads on the two surfaces. Will the surface charge density be uniform? If not, where will it be minimum?
4. A point charge Q is placed near a metallic block (Figure 5E.1). Charges are induced on the block as shown in the figure. Suppose A is the point on the block closest to Q and B is the point farthest from Q. Which of the two potentials, V_A and V_B, will be larger? Or will they be the same?

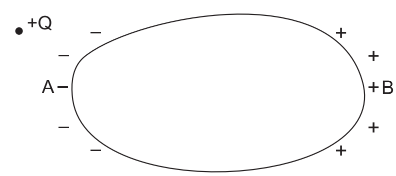


Fig. 5E.1

5. A charge distribution creates an electric field of $\boldsymbol{E}_0$ at a point A. A metallic sphere is placed with its centre at A, without disturbing the charge distribution. As a result, a certain charge density $\sigma(\boldsymbol{r})$ is induced on the surface of the sphere. What is the electric field at the point A only due to the charges appearing on the surface?
6. Consider the charge density at the centre of a copper atom in a copper wire and that at the centre of a carbon atom in a plastic wire. Are these both zero, both nonzero, or one zero and the other nonzero? If the last is true, which is zero and which is nonzero?
7. You must have learnt Ohm's law in school. The current through a conducting wire is said to be proportional to the potential difference between the ends of the wire. But in this chapter you learned that a conductor has the same potential everywhere. Then how can there be a potential difference between the ends?
8. A metallic block contains a cavity. A charge Q is placed on the conductor in one case and in the cavity in another case. A, B, C are three points, A outside the block, B in the block, and C in the cavity. At which of these points will the electric field be different in the two cases? [**Ans.** C]

Problems

1. A metallic sphere of radius R is given a charge Q. What is $\nabla \cdot \boldsymbol{E}$ as a function of $\boldsymbol{r}$? [**Ans.** $\dfrac{Q}{4\pi R^2}\delta(r-R)$]
2. The electric field just outside the surface of a conductor has magnitude 25 V/m and points away from the surface. What is the surface charge density on the conductor. [**Ans.** 2.2×10^{-10} C/m^2]
3. A large metallic plate is placed along the plane $x = y$. The electric field at a point $(\varepsilon, 0, 0)$ has an x-component $E_x = 2.0$ V/m where ε is a small positive quantity. Find E_y and E_z at this point. [**Ans.** $E_y = -2.0$ V/m, $E_z = 0$]

4. A metallic body is given a charge Q. A point charge q is placed at some distance from the body. The surface charge density at a point P on the surface is σ_0. The line joining P to the point charge q has length d and is normal to the surface at P. The unit vector in this direction is $\hat{n}$. Find the electric field just outside P. [**Ans.** $\frac{\sigma_0}{\varepsilon_0}\hat{\boldsymbol{n}}$]

5. A conductor of irregular shape is surrounded by a conducting spherical shell of radius R. The inner conductor is given a charge q.
   (a) What will be the charges induced on the inner and the outer surfaces of the shell?
   (b) Will the charge induced on the inner surface of the shell be distributed uniformly on this surface?
   (c) Will the charge induced on the outer surface of the shell be distributed uniformly on this surface?
   (d) What will be the electric field due to this system at a distance r from the centre of the shell, r being larger than R? [**Ans.** (d) $\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}}$]

6. A spherical cavity of radius $R/4$ is carved out from a conducting sphere of radius R. The centre of the sphere is at the origin and that of the cavity is on the x-axis at $x = -R/2$. A charge q is placed at the centre of the cavity.
   (a) Find the surface charge density σ_1 on the surface of the cavity.
   (b) Find the surface charge density σ_2 on the outer surface of the conductor.
   (c) Find the electric field everywhere. [**Ans.** (a) $-\frac{4q}{\pi r^2}$, (b) $\frac{q}{4\pi R^2}$]

7. Consider the situation described in the previous problem. In addition to the charge in the cavity, another charge q' is placed on the x-axis at $x = 2R$.
   (a) Will σ_1 remain the same or will it change?
   (b) Will σ_2 remain the same or will it change?
   (c) Will the field inside the cavity remain the same or will it change?
   (d) Will the field outside the sphere remain the same or will it change?
   [**Ans.** (a) will remain the same, (b) will change]

8. Consider the situation in the previous problem. The charge in the cavity is moved a little.
   (a) Will σ_1 remain the same or will it change?
   (b) Will σ_2 remain the same or will it change?
   (c) Will the field inside the cavity remain the same or will it change?
   (d) Will the field outside the sphere remain the same or will it change? [**Ans.** (b) will remain the same]

9. A conducting sphere of radius a is surrounded by a concentric, thick conducting shell of inner and outer radii b and c. A charge q is given to the inner sphere.
   (a) Find the surface charge density on each of the three surfaces.
   (b) Find the electric potential as a function of distance from the centre and plot it. Take the potential at infinity to be zero.
   (c) Find the electric field as a function of r and plot its magnitude.

10. Suppose the outer shell of a spherical capacitor has a very large radius (say infinity) and the inner sphere has a radius R. (a) What is the capacitance of this capacitor? (b) A charge Q is given to the capacitor. Find the energy of the capacitor using $U = \frac{Q^2}{(2C)}$. [**Ans.** (a) $4\pi\varepsilon_0 R$]

11. Find the energy of a charged metallic shell of radius R carrying a uniformly distributed charge Q. Do you get the same result as in the previous question?

12. The inner sphere of a spherical capacitor is grounded ($V = 0$) and the outer shell is given a charge Q. The radii of the two surfaces are R and $3R$.
    (a) Find the charge appearing on the inner sphere.
    (b) Find the capacitance $C = Q/V$, where V is the potential difference appearing between the two conductors. [**Ans.** (a) $-Q/3$ (b) $\frac{9}{8} \times 4\pi\varepsilon_0 R$]

13. A parallel-plate capacitor is made by placing plates of area A each at a separation d. A charge Q is given to one plate and $-Q$ to the other plate. How much work has one to do to increase the separation between the plates to $2d$? [**Ans.** $\frac{Q^2 d}{2\varepsilon_0 A}$]

14. Two large metallic plates are kept parallel to each other at a small separation. A charge q_1 is put on one plate and a charge q_2, on the other plate (Figure 5E.2). Neglecting the thickness of the plates, find (a) the charge appearing on each of the four surfaces 1, 2, 3 and 4, and (b) the electric field everywhere.

    [**Ans.** (a) $\frac{q_1 + q_2}{2}, \frac{q_1 - q_2}{2}, \frac{q_2 - q_1}{2}, \frac{q_1 + q_2}{2}$ respectively]

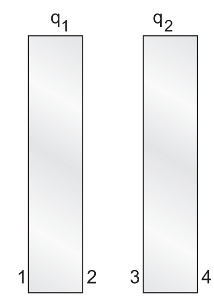


Fig. 5E.2

15. A capacitor is formed by two conductors in the shape of coaxial cylindrical strips as indicated in Figure 5E.3. The radii of the strips are a and b, and the arcs make an angle ϕ at the centre. Find the capacitance per unit length of the capacitor. [**Ans.** $\frac{\varepsilon_0 \phi}{\ln (b/a)}$]

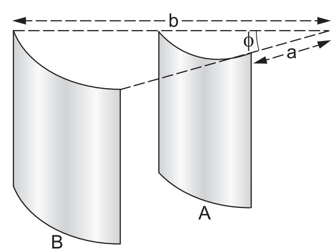


Fig. 5E.3

❑

6 Poisson's and Laplace's Equations

For a given charge distribution, you can express the potential as $V(\boldsymbol{r}) = \int \frac{\rho(\boldsymbol{r}')\,d\tau'}{4\pi\varepsilon_0|\boldsymbol{r}-\boldsymbol{r}'|}\cdot$ Potential and charge distributions are also related through a differential equation, called Poisson's equation, which gives many more interesting results than one can get from the integral expression for the potential. Especially if you have metallic boundaries in the region where the induced charges are not known, Poisson's equation and its special case, called Laplace's equation, are very useful. This chapter will make you familiar with such interesting results.

6.1 Poisson's and Laplace's Equations

In an electrostatic situation, the electric field $\boldsymbol{E}(\boldsymbol{r})$ can be expressed in terms of a potential function $V(\boldsymbol{r})$. The relation between the two is

$$\boldsymbol{E} = -\nabla V.$$

Also, the electric field is related to the charge density as

$$\nabla \cdot \boldsymbol{E} = \rho/\varepsilon_0.$$

Combining the two,

$$\nabla \cdot (-\nabla V) = \rho/\varepsilon_0$$

or

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}\cdot \tag{6.1}$$

This equation is called *Poisson's equation* in electrostatics. At places where there is no charge, it becomes

$$\nabla^2 V = 0, \tag{6.2}$$

which is called *Laplace's equation*. These two equations are very useful for solving electrostatic problems if the potentials are known at certain surfaces bounding a volume.

If you are not familiar with the "del square" operator, ∇^2, see Appendix 4. The expressions for $\nabla^2 V$ in different coordinate systems are reproduced here for ready reference.

$$\text{Cartesian: } \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \tag{6.3}$$

$$\text{Spherical polar: } \nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial^2 V}{\partial r^2}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} \tag{6.4}$$

$$\text{Cylindrical polar: } \nabla^2 V = \frac{1}{\rho_r}\frac{\partial}{\partial\rho_r}\left(\rho_r \frac{\partial V}{\partial\rho_r}\right) + \frac{1}{\rho_r^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2} \tag{6.5}$$

Laplace's and Poisson's equations are "partial differential equations". Solving partial differential equations needs specialized methods. You can familiarize yourself with Laplace's and Poisson's equations through a few simple examples.

EXAMPLE 6.1 **Two infinitely large charge layers are uniformly spread parallel to each other, one along the plane $z = 0$ and the other along the plane $z = z_0$. As a result, the whole of the plane $z = 0$ is at a constant potential $V = V_1$ and the whole of the plane $z = z_0$ is at a constant potential $V = V_2$. There is no charge anywhere else. Find the potential in the region between the charge layers.**

Solution The region between the charge layers, that is, $0 < z < z_0$, is charge-free. So, the potential in this region satisfies Laplace's equation

$$\nabla^2 V = 0$$

or

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{i}$$

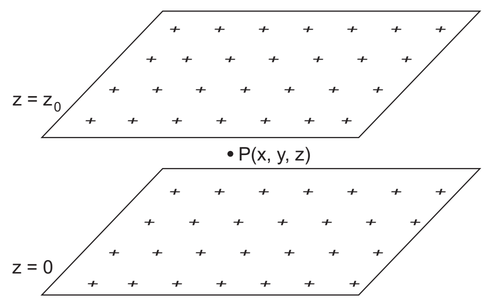


Fig. 6.1

As the whole of the plane $z = 0$ is at the same potential and so is the entire plane $z = z_0$, nothing changes if we shift along the x- or y-direction. The situation seen from (x, y, z) is the same as that seen from (x', y', z'). So V does not depend on x or y. Thus,

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial y^2} = 0.$$

Hence, from (i),

$$\frac{\partial^2 V}{\partial z^2} = 0.$$

As V can only be a function of z, this partial differentiation can be replaced by total differentiation. So,

$$\frac{d^2V}{dz^2} = 0$$

or
$$\frac{dV}{dz} = c_1$$

or
$$V = c_1 z + c_2.$$

Using the given conditions $V = V_1$ at $z = 0$ and $V = V_2$ at $z = z_0$, you get the values of c_1 and c_2.

These are $c_2 = V_1$ and $c_1 = \dfrac{V_2 - V_1}{z_0}$.

Thus,

$$V = \frac{V_2 - V_1}{z_0} z + V_1.$$

We chose Example 6.1 just to illustrate the use of Laplace's equation. In fact you can solve the problem using much simpler methods. The charge density on each of the two planes is uniform. Let the charge density at $z = 0$ be σ_1 and that at $z = z_0$ be σ_2. The electric field at P due to the layer at $z = 0$ is $\dfrac{\sigma_1}{2\varepsilon_0}\hat{\boldsymbol{k}}$ and that due to the layer at $z = z_0$ is $-\dfrac{\sigma_2}{2\varepsilon_0}\hat{\boldsymbol{k}}$. The net electric field is $\dfrac{\sigma_1 - \sigma_2}{2\varepsilon_0}\hat{\boldsymbol{k}}$ everywhere in the region between the layers. As the field is uniform, the potential changes at a uniform rate with z. This rate is $(V_2 - V_1)/z_0$ and so in distance z, it changes by $\dfrac{(V_2 - V_1)}{z_0} z$, giving the solution.

Let us now solve another simple example, using Poisson's equation.

EXAMPLE 6.2 **Charge is distributed with uniform density ρ_0 in a spherical volume of radius R. Assuming the potential to be $V = V_0$ at $r = R$, find the potential in the interior of this volume.**

Solution As the potential is constant over the surface $r = R$, and the charge density is uniform, nothing depends on θ or ϕ. The potential $V(\boldsymbol{r})$ must be a function of r only and hence only the first term in the RHS of Equation 6.4 will be nonzero. Poisson's equation for the region $r \le R$ is, therefore,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = -\frac{\rho_0}{\varepsilon_0}.$$

As V is a function of only r, partial differentiation can be replaced by total differentiation. Thus,

$$\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = -\frac{\rho_0 r^2}{\varepsilon_0}$$

or $$r^2 \frac{dV}{dr} = -\frac{\rho_0 r^3}{3\varepsilon_0} + c_1$$

or $$\frac{dV}{dr} = -\frac{\rho_0 r}{3\varepsilon_0} + \frac{c_1}{r^2}$$

or $$V = -\frac{\rho_0 r^2}{6\varepsilon_0} - \frac{c_1}{r} + c_2.$$

V has to be finite everywhere in the region as the charge itself is finite. Also, the origin is in the region where this equation is valid. In order to keep the potential finite at the centre of the sphere, c_1 must be zero. Thus

$$V = -\frac{\rho_0 r^2}{6\varepsilon_0} + c_2.$$

As $V = V_0$ at $r = R$, $c_2 = V_0 + \frac{\rho_0 R^2}{6\varepsilon_0}\cdot$ This gives

$$V = V_0 + \frac{\rho_0}{6\varepsilon_0}(R^2 - r^2).$$

Try to solve the problem by writing the expression for the electric field in the sphere and integrating it within proper limits to get the potential difference. Check if you get the same result. So, by now you should know in what kind of problems you should use Laplace's equation. You can use this equation if the potential is known at the boundaries of a change-free region and you want to know the potential in the whole of the region. We will come to a more general set of such problems but before that, let us discuss a very interesting property of the solution of Laplace's equation.

6.2 The Potential at the Centre of a Sphere in a Charge-free Region

Consider an imaginary spherical surface of radius R and a charge q outside this surface. The potential due to the charge q is given by $V(\boldsymbol{r})$. We will show that

$$\frac{1}{4\pi R^2} \oint_{\substack{\text{spherical}\\ \text{surface}}} V da = V_{\text{centre}}. \tag{6.6}$$

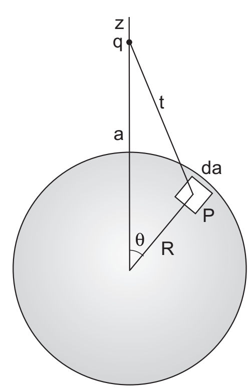


Fig. 6.2

Take a small surface element da at a point P on the spherical surface and let the value of the potential be V at the location of this element (Figure 6.2). Calculate Vda. Note that only the magnitude of da is used and not the area vector $\boldsymbol{da}$. The total of Vda over the entire surface is $\oint Vda$. Dividing by the total surface area $4\pi R^2$, you get the "average potential" over the spherical surface. What we are asserting in Equation 6.6 is that this average potential is equal to the potential at the centre of the spherical surface.

Let us take the centre of the spherical surface as the origin and the line joining the centre to the charge q as the z-axis (Figure 6.2). Let the distance of the charge q from the centre be a. The point P on the surface of the sphere has coordinates (R, θ, ϕ). The surface element here has an area $da = R^2 \sin\theta \, d\theta \, d\phi$. Let the distance of q from the point P be t. Then $t^2 = a^2 + R^2 - 2aR\cos\theta$. The potential at da due to q is

$$V(R, \theta, \phi) = \frac{q}{4\pi\varepsilon_0 t},$$

and so

$$\oint V da = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{qR^2 \sin\theta \, d\theta \, d\phi}{4\pi\varepsilon_0 t}$$

$$= \frac{2\pi q R^2}{4\pi\varepsilon_0} \int_{\theta=0}^{\pi} \frac{\sin\theta \, d\theta}{t}. \tag{i}$$

As

$$a^2 + R^2 - 2aR\cos\theta = t^2,$$

$$\sin\theta \, d\theta = \frac{tdt}{aR}.$$

Check from Figure 6.2 that when $\theta = 0$, the point P is closest to q, and $t = a - R$. When $\theta = \pi$, the point P is farthest from q, and $t = a + R$. Thus equation (i) becomes

$$\oint V \, da = \frac{qR^2}{2\varepsilon_0 aR} \int_{a-R}^{a+R} \frac{t \, dt}{t}$$

$$= \frac{qR}{2\varepsilon_0 a} [t]_{a-R}^{a+R}$$

$$= \frac{qR}{2\varepsilon_0 a} \times 2R = \frac{qR^2}{\varepsilon_0 a}.$$

Or

$$\frac{1}{4\pi R^2} \oint V da = \frac{1}{4\pi R^2} \times \frac{qR^2}{\varepsilon_0 a}$$

$$= \frac{q}{4\pi\varepsilon_0 a} = V_{\text{centre}},$$

showing that Equation 6.6 is valid for a point charge.

Now consider an arbitrary charge distribution outside the spherical surface. The above equation holds for each charge or charge element. The net potential at a given point is equal to the sum of the potentials due to all the charges in the charge distribution. So, Equation 6.6 also holds for the net potential. Thus, for any charge distribution, the average potential at the surface of a sphere drawn in a charge-free region is equal to the potential at the centre. The sphere has to be drawn in a charge-free region as is clear from the derivation above. Find out for yourself exactly where we have used the fact that q is outside the sphere in this derivation.

The potential in a charge-free region is a solution of Laplace's equation. The fact described in Equation 6.6 results in a very special property of the solution of Laplace's equation, described in the next section.

6.3 Absence of an Absolute Maximum or a Minimum

Consider a region terminated by a boundary surface. Suppose a function $V(\boldsymbol{r})$ satisfies Laplace's equation $\nabla^2 V = 0$ for all points in the region. To be specific, suppose the region is charge-free and V represents electric potential.

We assure you that there is no point in the region (don't include points on the boundary surface) where the potential $V(\boldsymbol{r})$ can take a maximum or a minimum value. How do we say that?

Take any point P in the region (not on the boundary). Imagine a small spherical surface with P as the centre (Figure 6.3). If the potential V_P at P is maximum in its neighbourhood, potentials at all points on the surface of the small surrounding sphere will be smaller than V_P. But in a charge-free region, the average of the potential over the surface of a sphere is equal to the potential at the centre. If the potential at all the points on the surface is less than V_P, the average cannot be equal to V_P, and hence our assumption that the potential at P is maximum in its neighbourhood is wrong. You can convince yourself in a similar fashion that the potential at P cannot be minimum in its neighbourhood.

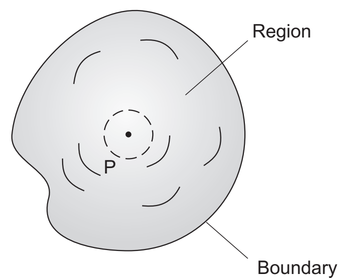


Fig. 6.3

This means if the potential increases as you go in one direction from P, it must decrease in some other direction. It cannot increase in all directions from P. Nor can it decrease in all directions from P.

If you look at the values of V in the whole of the region (don't include the boundary), you will not be able to find the highest value of the function. The point with the highest value has to be a local maximum. Hence, the highest value must occur only at the boundary. Similarly you don't have the lowest value of V anywhere in the region. The lowest value must also occur only at the boundary.

A related amazing result, known as Ernshaw's theorem, is that a charge cannot stay in stable equilibrium under electrostatic forces alone. For a particle to be in stable equilibrium under conservative forces, it should be situated at a minimum of the potential energy function. In case of electrostatic forces on a charge q, the potential energy is $U = qV$. As V does not have a local maximum or minimum in a charge-free region, the charge q will not find a position where it can stay in stable equilibrium.

You may find the charge in equilibrium due to electrostatic forces but not in stable equilibrium for displacements in all directions. For motion in some directions, it may behave as if it is in stable equilibrium but then in some other directions, it will behave as if it is in unstable equilibrium.

6.4 Uniqueness Theorem with Laplace's Equation

Suppose the values of a function $f(x)$ are given at $x = 2$ and $x = 5$. Let $f(x) = 5$ at $x = 2$ and $f(x) = 10$ at $x = 4$. Can you tell what are the values of $f(x)$ at all x between 2 and 4, that is, for $2 < x < 4$? We have shown the conditions in Figure 6.4. The range $2 < x < 4$ defines a region, and $x = 2$ and $x = 4$ are the boundaries of this region. We are talking about only one variable x. We have given you the values of $f(x)$ only at the boundaries and are asking you what the values of $f(x)$ are everywhere in the region.

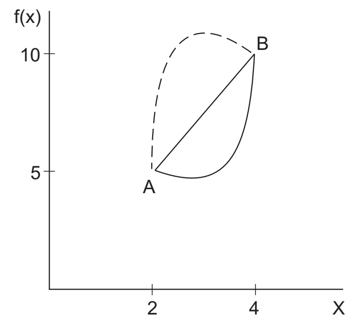


Fig. 6.4

As you can see, there is no unique set of values of $f(x)$ at the intermediate points. You can construct many functions $f(x)$ which give you $f(x = 2) = 5$ and $f(x = 4) = 10$, and which have different values at a given x inside the region $2 < x < 4$. In Figure 6.4, you can join the points A and B by a straight line, a circular arc, a parabolic curve, and so on. Each of these will give you an $f(x)$ with the same values at the boundary but different values at points in the region. The values at the boundary do not uniquely determine the function in the region.

Remember that the function $f(x)$ satisfies Laplace's equation in one dimension, that is,

$$\frac{d^2 f(x)}{dx^2} = 0$$

and also satisfies the boundary conditions $f(x = 2) = 5$, $f(x = 4) = 10$. Since $\dfrac{d^2 f}{dx^2} = 0$, $f(x) = ax + b$. Putting the boundary conditions, $f(x) = \dfrac{5}{2}x$. So the function $f(x)$ is uniquely determined in the whole of the region.

This property is valid for functions of space points also. Suppose $V(\boldsymbol{r})$ is a function of space points (written as a function of x, y, z or r, θ, ϕ, or ρ_r, ϕ, z, or in some other coordinates), which satisfies Laplace's equation in a region, and the values of $V(\boldsymbol{r})$ at all points on the boundary of the region are given. Then the function $V(\boldsymbol{r})$ in the region is uniquely determined. Let us justify this statement.

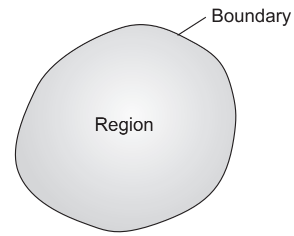


Fig. 6.5

Consider a region bounded by a surface. Suppose the values of a function are given at all points of this boundary surface. Also assume the function satisfies Laplace's equation in the region. Suppose $V_1(\boldsymbol{r})$ and $V_2(\boldsymbol{r})$ are two such functions. Both of them satisfy Laplace's equation in the region. Also, at each of the points on the boundary surface, both assume the same value. Define a function

$$V_3(\boldsymbol{r}) = V_1(\boldsymbol{r}) - V_2(\boldsymbol{r}).$$

Now,
$$\nabla^2 V_3 = \nabla^2 (V_1 - V_2)$$
$$= \nabla^2 V_1 - \nabla^2 V_2 = 0 - 0 = 0.$$

So $V_3(\boldsymbol{r})$ also satisfies Laplace's equation. What are the values of $V_3(\boldsymbol{r})$ at the points on the boundary surface? Since $V_1(\boldsymbol{r})$ and $V_2(\boldsymbol{r})$ have the same value at each such point, $V_3(\boldsymbol{r})$ must be zero here. At all the points on the boundary surface, $V_3(\boldsymbol{r}) = 0$.

A solution of Laplace's equation cannot have a maximum or minimum in the whole of the region (don't include boundary points) where Laplace's equation is valid. Any maximum or minimum has to occur only at the boundary. But at all points on the boundary, V_3 is zero. So, throughout the inside of the region, V_3 has to be zero. Therefore,

$$V_3(\boldsymbol{r}) = 0 \qquad \text{everywhere in the region}$$

or
$$V_1(\boldsymbol{r}) = V_2(\boldsymbol{r}) \qquad \text{everywhere in the region.}$$

This shows that you cannot have two different functions $V_1(\boldsymbol{r})$ and $V_2(\boldsymbol{r})$ satisfying Laplace's equation and the same boundary condition. So if a solution of Laplace's equation exists in the region, it has to be unique. This is one version of the *uniqueness theorem*. Call it uniqueness theorem 1. Let us state the uniqueness theorem one last time.

If the potential at all the points on the boundary surface of a charge-free region is given, the potential in the region is uniquely determined.

EXAMPLE 6.3 **Show that the electric potential in the empty space surrounded by a conducting shell must be uniform and independent of the charge put on or outside the shell.**

Solution Whatever charge is put on the conductor or outside, the potential at all points of the shell must be the same, say V_0. This is because the electric field in the material of a conductor is always zero in an electrostatic situation. Now look at the empty space inside the shell (Figure 6.6).

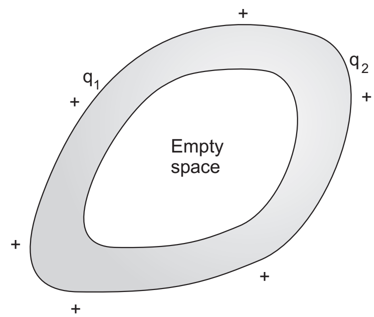


Fig. 6.6

As there is no charge inside, the potential here satisfies Laplace's equation $\nabla^2 V(\boldsymbol{r}) = 0$. This region is bounded by the inner surface of the shell and the potential at all points on this boundary surface is given (V_0). By the uniqueness theorem, there can be only one function $V(\boldsymbol{r})$ which is a solution of $\nabla^2 V(\boldsymbol{r}) = 0$ in the empty space surrounded by the shell and which assumes the value $V = V_0$ everywhere on the boundary surface. But it is very simple to guess this function, it is $V(\boldsymbol{r}) = V_0$ everywhere. Check that it satisfies Laplace's equation in the empty space and is V_0 at the boundary. Hence the electric potential inside the empty space is uniform and therefore the electric field here is zero.

There is another kind of boundary condition that ensures a unique solution of Laplace's equation in the given region.

If the normal component of the electric field is specified on all points on the boundary surface of a charge-free region, the potential in the region is uniquely specified (up to a constant).

The proof needs a little mathematical manipulation. Once again, suppose there are two functions $V_1(\boldsymbol{r})$ and $V_2(\boldsymbol{r})$ satisfying the conditions given above. So,

$$\nabla^2 V_1(\boldsymbol{r}) = 0, \quad \nabla^2 V_2(\boldsymbol{r}) = 0.$$

In terms of electric field, $\nabla \cdot \boldsymbol{E}_1 = 0, \quad \nabla \cdot \boldsymbol{E}_2 = 0,$

where $\quad \boldsymbol{E}_1 = -\nabla V_1, \quad \boldsymbol{E}_2 = -\nabla V_2.$

Also, $E_{1\perp} = E_{2\perp}$ at the surface.

As usual, consider the function $V_3(\boldsymbol{r}) = V_1(\boldsymbol{r}) - V_2(\boldsymbol{r})$.

Let $\quad \boldsymbol{E}_3 = -\nabla V_3.$

$$\boldsymbol{E}_3 = \boldsymbol{E}_1 - \boldsymbol{E}_2.$$

The conditions on $\boldsymbol{E}_3$ are (a) $\nabla \cdot \boldsymbol{E}_3 = 0$ everywhere and (b) $E_{3\perp} = 0$ at the surface.

Now write the expression for $\nabla \cdot (V_3 \boldsymbol{E}_3)$. It happens to be very simple:

$$\nabla \cdot (V_3 \boldsymbol{E}_3) = V_3 \nabla \cdot \boldsymbol{E}_3 + \boldsymbol{E}_3 \cdot (\nabla V_3).$$

Not every time will you get such a simple expression for an operation of ∇ on products of two functions. This time you have been lucky.

This equation becomes

$$\nabla \cdot (V_3 \boldsymbol{E}_3) = -E_3^2$$

because $\nabla \cdot \boldsymbol{E}_3 = 0$ everywhere and $\nabla V_3 = -\vec{\boldsymbol{E}}_3$.

Integrating over the region,

$$\int_{\text{region}} \nabla \cdot (V_3 \boldsymbol{E}_3) d\tau = \int_{\text{region}} -E_3^2 d\tau$$

or
$$\int_{\text{boundary}} V_3 \boldsymbol{E}_3 \cdot \boldsymbol{da} = \int_{\text{region}} -E_3^2 d\tau$$

or
$$\int_{\text{boundary}} V_3 E_{3\perp}\, da = \int_{\text{region}} -E_3^2 d\tau .$$

But the normal component of the electric field at the boundary is already given. Thus both $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ must have the same value of normal component at the surface. So, $E_{3\perp}$ must be zero on the surface and hence the surface integral on the left-hand side must be zero. This gives

$$\int_{\text{region}} E_3^2 d\tau = 0.$$

And hence, $E_3 = 0$ everywhere. This tells us that

$$\boldsymbol{E}_1 = \boldsymbol{E}_2 \text{ and hence } V_1(\boldsymbol{r}) = V_2(\boldsymbol{r}) \text{ with a given reference for zero potential.}$$

6.5 Uniqueness Theorem for Solutions of Poisson's Equation

The uniqueness theorem we have stated pertains to the solution of Laplace's equation. Thus, in this form, it is valid for charge-free regions. It can be extended to solutions of Poisson's equation too.

Suppose you have a region bounded by a surface. Charge is distributed in the region, the distribution being given by $\rho(\boldsymbol{r})$. The potential in the region satisfies the equation

$$\nabla^2 V = -\frac{\rho(\boldsymbol{r})}{\varepsilon_0} . \qquad \text{(i)}$$

There may be other charges outside the region that we shall not specify. Also, suppose we give you the values of the potential V at all the points on the boundary surface. So we are looking for a function $V(\boldsymbol{r})$ which satisfies equation (i) for the whole of the region with the given charge distribution $\rho(\boldsymbol{r})$ and which assumes the given values at the points of the boundary surface. We assure you that not more than one such function can exist.

As earlier, suppose there are two functions $V_1(\boldsymbol{r})$ and $V_2(\boldsymbol{r})$, both of which satisfy equation (i) with the same $\rho(\boldsymbol{r})$, and assume the same value at each point of the boundary surface. Define

$$V_1(\boldsymbol{r}) = V_2(\boldsymbol{r})$$

and
$$V_3(\boldsymbol{r}) = V_1(\boldsymbol{r}) - V_2(\boldsymbol{r}).$$

Then,

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = \frac{\rho(\boldsymbol{r})}{\varepsilon_0} - \frac{\rho(\boldsymbol{r})}{\varepsilon_0} = 0.$$

So, $V_3(\boldsymbol{r})$ satisfies Laplace's equation in the entire region. Also, $V_3 = V_2 - V_1 = 0$ at all the points on the boundary, as both functions $V_1(\boldsymbol{r})$ and $V_2(\boldsymbol{r})$ are supposed to have the same value at any point on the boundary surface. Thus, $V_3(\boldsymbol{r}) = 0$ in the whole of the region, as this is a function which satisfies Laplace's equation in the region and has a value of 0 at all points on the boundary. This gives $V_1(\boldsymbol{r}) = V_2(\boldsymbol{r})$ everywhere. So we arrive at another uniqueness theorem.

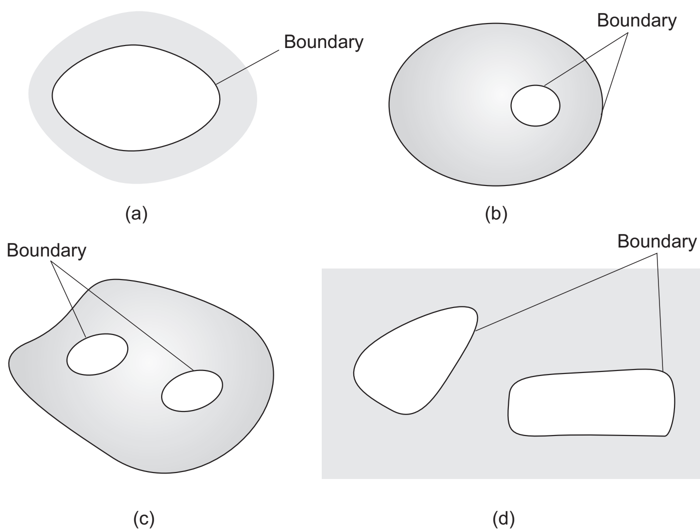


Fig. 6.7

If the potential at all the points on the boundary surface of a region and the charge distribution in the region are given, the potential in the region is uniquely determined.

In this chapter, we have used the words "region" and "boundary" several times. Normally, we have taken a closed surface as the boundary and the volume enclosed as the region. You can also take the volume outside a closed surface as the region [Figure 6.7(a)]. The closed surface again forms the boundary of this region as the region terminates here, but you also have a boundary at infinity as the region is extended up to infinity outside the closed surface. Figure 6.7 shows some more regions with their boundaries. Here, the shaded parts form the region and the solid lines denote the boundaries. In parts (a) and (d), the boundary also exists at infinity. In (b) and (c), the boundaries are only those shown in the figure.

All our theorems proved above are valid for any such region.

6.6 Unique Distribution of Charge on a Conductor

Consider a conductor with no charge on it. Also suppose there is no charge outside. What is the surface charge density on the conductor?

This question seems rather silly. If there is no charge anywhere, neither on the conductor, nor outside it, obviously there is no surface charge density anywhere on the conductor. But how are you so sure that you do not have positive charges at some places on the surface, negative charges at other places, giving a total of zero on the conductor? Suppose we divide the surface into 100 patches and put positive and negative charges on alternate patches. Why can this distribution not give us a zero field everywhere inside the conductor? If you really think about it, may be you won't find an obvious, simple argument. Let us work out the arguments.

Look at the region outside the conductor. It is bounded by a surface at infinity and by the surface S of the conductor (Figure 6.8). If there is some charge distribution on S, it will produce a potential $V(\boldsymbol{r})$ in the region outside. However, there is no charge anywhere outside S and so $V(\boldsymbol{r})$ must satisfy Laplace's equation in this region. The potential at all points on S must be the same, say V_1, as a conductor is always equipotential. So, the potential is specified at the boundary.

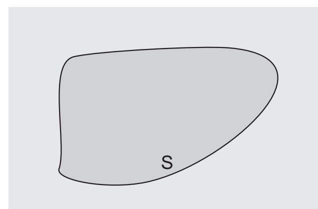


Fig. 6.8

$$V(\boldsymbol{r}) = V_1 \text{ (constant) at S}$$

$$V(\boldsymbol{r}) = 0 \text{ at } \infty.$$

Suppose V_1 is positive. In a charge-free region where $V(\boldsymbol{r})$ is a solution of Laplace's equation, the potential cannot have a maximum value. The maximum value must occur only at the boundary. So everywhere in the region outside, the potential must be less than V_1.

Now consider a closed surface S_1 enclosing the conductor, very close to S (Figure 6.9). At every point on S_1, the potential must be less than V_1. This means that the electric field just outside is away from the conductor everywhere. But this implies you only have a positive surface charge density on S, which is impossible as the conductor itself is neutral.

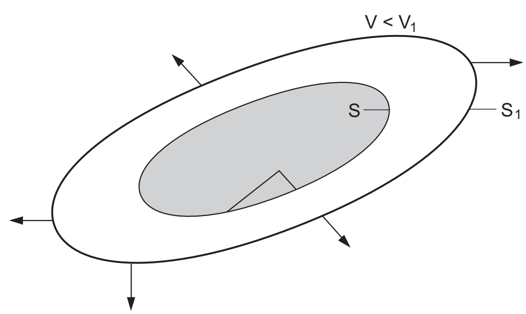


Fig. 6.9

Similarly, if V_1 is negative, the field just outside S will go towards S, leading to a negative surface charge density everywhere, which is again impossible.

So V_1 can be neither positive nor negative—it has to be zero. The potential is zero at infinity and zero at S. It cannot have a maximum or a minimum value in the region as maximum and minimum values occur only at the boundaries for solutions of Laplace's equation. So $V(\mathbf{r}) = 0$ everywhere outside S. Thus the electric field is zero everywhere outside S and hence the surface charge density is zero everywhere on S. You cannot have positive and negative charge densities on the surface of a conductor if no charges are put on the conductor or outside it.

There is a simple argument to prove this assertion.

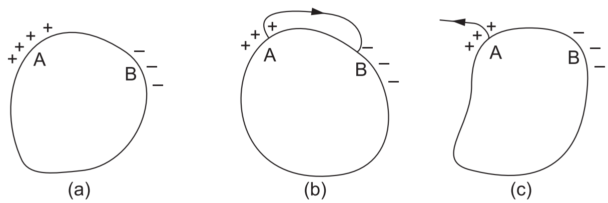


Fig. 6.10

Suppose no net charge is put on or outside a conductor, which has some positive charge at A and some negative charge at B [Figure 6.10(a)]. Where will the electric field lines originating from A go? No lines can go into the conductor as the field inside a conductor is zero. If a line terminates at a negative charge on the conductor at B, as suggested in Figure 6.10(b), the potential at B will be smaller than that at A. This is because, as you go along the electric field, the potential decreases. So if you go along a field line from A to B, you will reach a lower potential. But the conductor has to be equipotential. There is no negative charge outside the conductor. Thus these field lines originating from A cannot terminate at any point outside—they will have to go to infinity as suggested in Figure 6.10(c).

A similar argument holds for the negative charge at B. The electric field lines must terminate at B. But from where do they come? From infinity. The potential at infiniy is taken to be zero. If we come from infinity to the point B along a field line, the potential at B must be less than zero as we are moving along the direction of the electric field. But if we come from infinity to A along a field line, the potential at A must be more than zero as we are moving against the electric field. But the potentials at A and B must be the same as the whole conductor is equipotential. Thus you cannot have positive and negative charges separated on the conductor in the given situation.

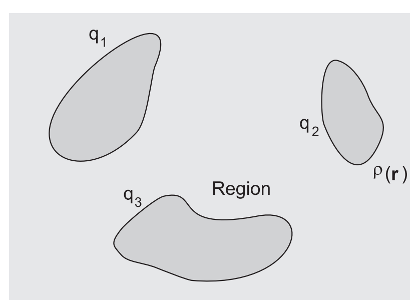


Fig. 6.11

We can generalize this finding and it is left to you to provide the right arguments for the following statement.

If a conductor is given a charge q and the charge distribution $\rho(\boldsymbol{r})$ outside the conductor is given, there is a unique way of distribution of q on the surface, and the electric potential is uniquely determined.

You can generalize further. Suppose you have several conductors which are given some charges. Consider the region outside the conductors. Some charges may also exist in this region (Figure 6.11).

If the total charge on each conducting surface, and the charge density $\rho(\mathbf{r})$ in the region outside the conductors are given, there is a unique way of distributing the given charges on the conductors and there is a unique potential function in the region.

6.7 Charge Distribution on a Conducting Sphere in an Otherwise Uniform Electric Field

Think of a conducting sphere of radius R, with its centre at the origin, placed in an electric field. The original electric field is produced by some sources and is uniform everywhere, given as $\boldsymbol{E}_{\text{ext}} = E_0\hat{\boldsymbol{k}}$ (Figure 6.12). What is the charge distribution on the sphere and how does it modify the electric field?

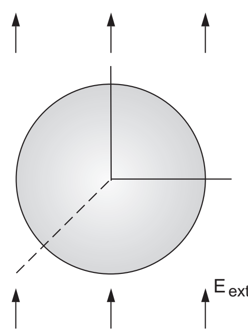


Fig. 6.12

The electric field inside the sphere must be zero as the sphere is conducting and the field is electrostatic. This means the charge distribution on the surface should produce an electric field $\boldsymbol{E}' = -E_0\hat{\boldsymbol{k}}$ everywhere inside the sphere. This ensures that the net electric field $\boldsymbol{E}_{\text{ext}} + \boldsymbol{E}'$ is zero inside. But we know one distribution on the spherical surface that can produce this field $\boldsymbol{E}' = -E_0\hat{\boldsymbol{k}}$ and that is $\sigma = \sigma_0 \cos\theta$, with a proper value of σ_0. We have found the electric field due to such a charge distribution in previous chapters. This charge distribution gives a field $\boldsymbol{E} = -\dfrac{\sigma_0}{3\varepsilon_0}\hat{\boldsymbol{k}}$. Thus choosing $\sigma_0 = 3\varepsilon_0 E_0$ will produce the required field and will ensure that the net field inside the sphere is zero everywhere.

But from the uniqueness theorem, there is only one kind of charge distribution on the conducting sphere, with a total charge of zero and a given external charge distribution. So the charge density appearing on the surface will be $\sigma = 3\varepsilon_0 E_0 \cos\theta$. The field outside the sphere, because of this distribution, will be the same as that produced by a dipole of dipole moment

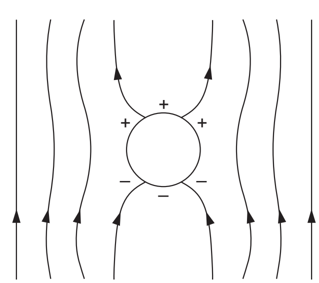


Fig. 6.13

$$\boldsymbol{p} = \frac{4}{3}\pi R^3 \sigma_0 \hat{\boldsymbol{k}} = 4\pi\varepsilon_0 R^3 E_0 \hat{\boldsymbol{k}}.$$

This field is

$$\boldsymbol{E}'' = \frac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\theta}) = \frac{E_0 R^3}{r^3}(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\theta}).$$

The net field for $r > R$ is thus

$$\boldsymbol{E} = E_0\hat{\boldsymbol{k}} + \frac{E_0 R^3}{r^3}(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\theta}).$$

The field lines are sketched in Figure 6.13.

6.8 Solution of Laplace's Equation

As stated earlier, the general application of Laplace's equation is to get the potential function in a region. If the potential is known at all points of the boundary surfaces, there are three classes of these problems depending on the kind of the boundary surfaces. For example, you may have concentric spherical surfaces as the boundary. In such a case, one would naturally like to use spherical coordinates. Similarly, you may have a situation in which Cartesian coordinates or cylindrical coordinates are more appropriate. We will describe the methods of solving Laplace's equation corresponding to the three commonly used coordinate systems.

Separation of variables in Cartesian coordinates

If the boundary surfaces are planes parallel to the coordinate planes, one uses Cartesian coordinates to solve Laplace's equation. The potential is specified at the boundary surfaces and we look for solutions of the equation which satisfy these boundary conditions.

Laplace's equation is

$$\frac{\partial^2}{\partial x^2}V(x, y, z) + \frac{\partial^2}{\partial y^2}V(x, y, z) + \frac{\partial^2}{\partial z^2}V(x, y, z) = 0. \tag{i}$$

A very effective general method of solving this equation is to start with a trial solution

$$V(x, y, z) = X(x)Y(y)Z(z), \tag{ii}$$

where $X(x)$ is a function of x only, $Y(y)$ is a function of y only and $Z(z)$ is a function of z only. In general, the function $V(x, y, z)$ will not be in this form but can be written as a sum of such product functions. Putting (ii) in Laplace's equation (i),

$$YZ\frac{d^2X}{dx^2} + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} = 0.$$

Dividing by XYZ,

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0$$

or $$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=-\frac{1}{Z}\frac{d^2Z}{dz^2}.$$

The LHS does not depend on z and the RHS does not depend on x or y. This means each of these sides is independent of x, y and z. In other words, each of these is a constant. Thus $C_3=\frac{1}{Z}\frac{d^2Z}{dz^2}$ is a constant. Similarly, you can show that $C_1=\frac{1}{X}\frac{d^2X}{dx^2}$ and $C_2=\frac{1}{Y}\frac{d^2Y}{dy^2}$ are also constants.

As the sum of the three terms is zero, two of these will have the same sign and the third will be of opposite sign. Assume that $\frac{1}{X}\frac{d^2X}{dx^2}$ and $\frac{1}{Y}\frac{d^2Y}{dy^2}$ are negative and $\frac{1}{Z}\frac{d^2Z}{dz^2}$ is positive. Then you can write

$$\frac{1}{X}\frac{d^2X}{dx^2}=-k_1^2$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2}=-k_2^2.$$

And, $$\frac{1}{Z}\frac{d^2Z}{dz^2}=(k_1^2+k_2^2).$$

Thus, $$X(x)=A_1\sin k_1x+B_1\cos k_1x \tag{i}$$

$$Y(y)=A_2\sin k_2y+B_2\cos k_2y \tag{ii}$$

$$Z(z)=A_3e^{-k_3y}+B_3e^{-k_3y} \tag{iii}$$

where $$k_3=\sqrt{k_1^2+k_2^2}.$$

The first two are oscillatory functions and the third is of the exponential type. Thus a solution is

$$V(x,y,z)=[A_1\sin k_1x+B_1\cos k_1x][A_2\sin k_2y+B_2\cos k_2y][A_3e^{-k_3y}+B_3e^{-k_3y}]. \tag{6.7}$$

You had assumed C_1 and C_2 to be negative and C_3 to be positive. The boundary conditions decide which of the three constants will be positive and which will be negative. In other words, which of the three functions X, Y, Z will be oscillatory and which will be of the exponential type is decided by the boundary conditions. Also, these boundary conditions allow only some selected sets of k_1, k_2, k_3. A proper combination of $V(x, y, z)$ with these allowed k_1, k_2, k_3 gives the final solution. The example below will clarify the procedures further.

....

EXAMPLE 6.4 **A rectangular parallelepiped has surfaces $x = 0$, $x = a$, $y = 0$, $y = b$, $z = 0$ and $z = c$. The first five surfaces are conducting and maintained at zero potential. The surface $z = c$ is nonconducting and is maintained at a potential**

$$V_0(x, y) = V_1(x)V_2(y),$$

where $$V_1(x) = \sqrt{V_0}\left(\sin\frac{\pi x}{a} + \sin\frac{2\pi x}{a}\right)$$

and $$V_2(y) = \sqrt{V_0}\left(\sin\frac{\pi y}{b} + \sin\frac{2\pi y}{b}\right).$$

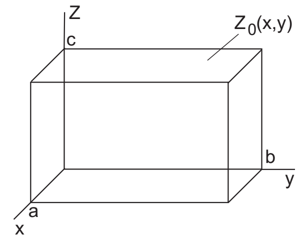


Fig. 6.14

Find the expression for potential inside the rectangular parallelepiped.

Solution Expressing the potential as

$$V(x, y, z) = X(x)Y(y)Z(z),$$

the functions X, Y, Z must be either oscillatory or exponential as described in the preceding section (Section 6.8). The potential approaches zero at $x = 0$ and $x = a$. The exponential-type function $Ae^{kx} + Be^{-kx}$ will not be zero at the two values of x and hence $X(x)$ should be oscillatory. Similarly, $Y(y)$ should be oscillatory. Thus, the product function will be

$$V(x, y, z) = (A_1 \sin k_1 x + B_1 \cos k_1 x)(A_2 \sin k_2 y + B_2 \cos k_2 y)(A_3 e^{k_3 z} + B_3 e^{-k_3 z}).$$

At $x = 0$, $V = 0$ for all y, z. This gives $B_1 = 0$. Also, at $x = a$, $V = 0$ for all y, z. Then,

$$k_1 = \frac{n_1 \pi}{a},$$

where n_1 is a positive integer. Similarly, the potential is zero at $y = 0$ and $y = b$ for all x and z. Then you have $B_2 = 0$ and $k_2 = \frac{n_2 \pi}{b}\cdot$ The potential is zero at $z = 0$. This gives $A_3 + B_3 = 0$

or $$A_3 e^{k_3 z} + B_3 e^{-k_3 z} = 2A_3 \frac{e^{k_3 z} - e^{k_3 z}}{2} = 2A_3 \sinh k_3 z.$$

The product function is, therefore,

$$V(x, y, z) = A \sin\frac{n_1 \pi x}{a} \sin\frac{n_2 \pi x}{b} \sinh k_3 z.$$

Remember two things. First, $k_3^2 = k_1^2 + k_2^2 = \pi^2\left[\left(\frac{n_1}{a}\right)^2 + \left(\frac{n_2}{b}\right)^2\right]\cdot$ So the expression for $V(x, y, z)$ written above has only two parameters n_1 and n_2 which can take positive integral values. The second thing to remember is that the above expression gives a family of solutions with n_1 and n_2 taking different positive integral values and the actual potential could be a linear combination of such product terms. The constant A can be different for different values of n_1 and n_2. Thus, $V(x, y, z)$ may be written as

$$V(x, y, z) = \sum_{n_1}\sum_{n_2} A_{n_1, n_2} \sin\frac{n_1 \pi x}{a} \sin\frac{n_2 \pi y}{b} \sinh k_3 z. \quad \text{(i)}$$

The task that remains is to calculate A_{n_1,n_2}. For this we will use the boundary condition at $z = c$ where $V(x, y, z) = V_0(x, y)$ as given in the problem.

Thus,
$$V_0(x, y) = \sum_{n_1}\sum_{n_2} A_{n_1,n_2} \sin\frac{n_1\pi x}{a} \sin\frac{n_2\pi y}{b} \sinh k_3 c. \tag{ii}$$

Getting A_{n_1,n_2} from this equation is a typical problem involving Fourier series analysis. The given function $V(x, y, c)$ is expanded as a linear combination of orthogonal functions $\sin\frac{n_1\pi x}{a}\sin\frac{n_2\pi y}{b}$ and we have to find the expansion coefficients. The orthogonality conditions are

$$\int_{x=0}^{a} \sin\frac{n_1\pi x}{a}\sin\frac{m\pi x}{a}dx = \frac{a}{2}\delta_{n_1 m}$$

$$\int_{y=0}^{b} \sin\frac{n_2\pi y}{b}\sin\frac{l\pi y}{b}dy = \frac{b}{2}\delta_{n_2 l}.$$

Multiply both sides of equation (ii) by $\sin\frac{m\pi x}{a}\sin\frac{l\pi y}{b}$ and integrate.

$$\int_{x=0}^{a}\int_{y=0}^{b} V_0(x, y)\sin\frac{m\pi x}{a}\sin\frac{l\pi y}{b}dx\,dy$$

$$= \sum_{n_1}\sum_{n_2} A_{n_1,n_2}\sinh k_3 c \int_{x=0}^{a}\sin\frac{n_1\pi x}{a}\sin\frac{m\pi x}{a}dx \int_{y=0}^{b}\sin\frac{n_2\pi y}{b}\sin\frac{l\pi y}{b}dy$$

$$= \sum_{n_1}\sum_{n_2} A_{n_1,n_2}(\sinh k_3 c)\frac{ab}{4}\delta_{n_1 m}\delta_{n_2 l}$$

$$= A_{ml}\frac{ab}{4}\sinh\left(\pi c\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{l}{b}\right)^2}\right)$$

or
$$A_{ml} = \frac{4}{(ab)\sinh\left(\pi c\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{l}{b}\right)^2}\right)}\int_{x=0}^{a} V_1(x)\frac{\sin m\pi x}{a}dx \int_{y=0}^{b} V_2(y)\sin\frac{l\pi y}{b}dy,$$

where $V_1(x)$ and $V_2(y)$ are as given in the problem.

$$\int_{x=0}^{a} V_1(x)\sin\frac{m\pi x}{a}dx = \int_{x=0}^{a}\sqrt{V_0}\left(\sin\frac{\pi x}{a} + \sin\frac{2\pi x}{a}\right)\sin\frac{m\pi x}{a}dx = \frac{a\sqrt{V_0}}{2}(\delta_{1m} + \delta_{2m})$$

and
$$\int_{y=0}^{b} V_2(y)\sin\frac{l\pi y}{b}dy = \int_{y=0}^{b}\sqrt{V_0}\left(\sin\frac{\pi y}{b} + \sin\frac{2\pi y}{b}\right)\sin\frac{l\pi y}{b}dy = \frac{b}{2}\sqrt{V_0}(\delta_{1l} + \delta_{2l}).$$

Thus $$A_{m,l} = \frac{1}{\sinh\left[\pi c\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{l}{b}\right)^2}\right]} V_0(\delta_{1m} + \delta_{2m})(\delta_{1l} + \delta_{2l}).$$

So, for all $m > 2$ and $l > 2$, $A_{m,l} = 0$.

$$A_{1,1} = \frac{V_0}{\sinh\left[\pi c\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}\right]}$$

$$A_{1,2} = \frac{V_0}{\sinh\left[\pi c\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{2}{b}\right)^2}\right]}$$

$$A_{2,1} = \frac{V_0}{\sinh\left[\pi c\sqrt{\left(\frac{2}{a}\right)^2 + \left(\frac{1}{b}\right)^2}\right]}$$

$$A_{2,2} = \frac{V_0}{\sinh\left[2\pi c\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}\right]}.$$

Now you can figure out $V(x, y, z)$ from (i)—it will have four terms, corresponding to $n_1 = 1, 2$ and $n_2 = 1, 2$. Put values of $A_{1,1}$, $A_{1,2}$, $A_{2,1}$ and $A_{2,2}$ and the function $V(x, y, z)$ is ready.

Separation of variables in spherical coordinates

Sometimes the boundary surfaces are spherical or can easily be described in spherical coordinates. If the potential is given on such a boundary of a charge-free region, you can separate Laplace's equation in spherical coordinates r, θ, ϕ to get the potential inside the region. Take a look at the steps.

To start with, look for the solutions which can be expressed as a product of three functions $R(r)$, $\Theta(\theta)$ and $\Phi(\phi)$, each depending on only one spherical coordinate.

Then $$V(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi). \quad \text{(i)}$$

Laplace's equation in spherical coordinates is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0.$$

By substituting $V(r, \theta, \phi)$ from (i) in this equation, one can separately find differential equations for R, Θ and Φ. However, we will consider only those cases where the potential is independent of ϕ. Such a situation is said to have azimuthal symmetry.

In this case,

$$V(r, \theta, \phi) = R(r)\Theta(\theta)$$

and Laplace's equation is

$$\Theta\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R}{r^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = 0.$$

Dividing by $R(r)\Theta(\theta)$,

$$\frac{1}{R}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = 0$$

or

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = C,$$

where C is a constant.

The r equation is $\quad \frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = RC$

or

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - CR = 0. \tag{ii}$$

The θ equation is

$$\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -C. \tag{iii}$$

It turns out that this equation has finite solutions at $\theta = 0$ and $\theta = \pi$ only if C is of the form $C = l(l+1)$, where l is a non-negative integer. As we are interested in the electric potential $V(x, y, z)$, we must look for only those solutions which are finite at all space points in the given region. Thus we write $C = l(l+1)$ with the understanding that l is either zero or a positive integer. Equation (ii) is expressed as

$$r^2\frac{dR}{dr^2} + 2r\frac{dR}{dr} - l(l+1)R = 0.$$

There are standard ways of getting a solution of this second-order differential equation, and the general solution (having two arbitrary constants) is

$$R = Ar^l + \frac{B}{r^{l+1}}\cdot \tag{iv}$$

If the origin is in the region, B must be zero to keep R finite at the origin. Similarly, if the region extends to infinitely large values of r, A must be zero. However, we will retain both terms in (iv). Equation (iii) is

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + l(l+1)\Theta = 0.$$

This is called Legendre's equation and its solutions are polynomials in $\cos\theta$. These polynomials are called Legendre polynomials and are written as $P(\cos\theta)$. The polynomials for some values of l are as follows.

$$
\begin{array}{ll}
l = 0 & P_0(\cos\theta) = 1 \\
l = 1 & P_1(\cos\theta) = \cos\theta \\
l = 2 & P_2(\cos\theta) = \dfrac{3\cos^2\theta - 1}{2} \\
l = 3 & P_3(\cos\theta) = \dfrac{5\cos^3\theta - 3\cos\theta}{2}
\end{array}
$$

Combining $R(r)$ and $\Theta(\theta)$, the product-type solution of Laplace's equation with azimuthal symmetry is

$$V(r, \theta, \phi) = \left(Ar^l + \frac{B}{r^{l+1}}\right)P_l(\cos\theta).$$

The actual solution could be a linear combination of such product terms. The constants A and B could be different for different values of l, and so we will write these as A_l and B_l. The general solution is, then,

$$V(r, \theta, \phi) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta). \tag{6.8}$$

The values of A_l and B_l are obtained from the potential given at the boundary surface.

EXAMPLE 6.5 **The potential on a spherical surface of radius R is $V = V_0 \cos\theta$ in spherical coordinates with the origin at the centre. There are no charges anywhere except for the surface, where $r = R$. Find the potential inside and outside this surface.**

Solution Use Equation (6.8) to denote the potential. For $r < R$, the origin is included in the region. Thus, the $\frac{1}{r^{l+1}}$ term will not be there in the expression for the potential. So

$$V(r, \theta, \phi) = \sum_l A_l r^l P_l(\cos\theta).$$

At $r = R$, $V = V_0 \cos\theta$.

Thus, $$V_0\cos\theta = \sum_l A_l R^l P_l(\cos\theta).$$

But $\cos\theta = P_l(\cos\theta)$. This shows that all A_l except A_1 are zero. And $A_1 = \frac{V_0}{R}$.

Thus, inside the given spherical surface,

$$V = \frac{V_0}{R} r\cos\theta.$$

The region $r > R$ includes large (infinite) values of r. Thus r^l terms will not be present in the expression for V. Thus

$$V = \sum_l \frac{B_l}{r^{l+1}} P_l(\cos\theta).$$

At the surface, $r = R$, and the potential is $V = V_0 \cos\theta$. Thus,

$$V_0 P_1(\cos\theta) = \sum_l \frac{B_l}{R^{l+1}}(\cos\theta).$$

This shows that for all l, except $l = 1$, B_l is zero. And,

$$V_0 = \frac{B_1}{R^2} \quad \text{or} \quad B_1 = V_0 R^2.$$

Thus, the potential outside $r = R$ is $V = \dfrac{V_0 R^2}{r^2}\cos\theta$.

This is a familiar situation. It corresponds to the surface charge density $\sigma = \sigma_0 \cos\theta$, where the electric field inside the surface is $\dfrac{\sigma}{3\varepsilon_0}$ (constant) and is in the negative z-direction. The potential is then $V = \dfrac{\sigma_0}{3\varepsilon_0} z = \dfrac{\sigma_0}{3\varepsilon_0} r\cos\theta$. Outside the sphere, it is a dipole field given by $V = \dfrac{p\cos\theta}{4\pi\varepsilon_0 r^2}$ with $p = \dfrac{4}{3}\pi R^3 \sigma_0$.

In the above example, the coefficients could be calculated just by inspecting $V(R, \theta)$. In general, you can get A_l and B_l from $V(R, \theta)$ using the orthogonality relations of Legendre polynomials.

$$\int_{-1}^{1} P_l(x) P_m(x)\, dx = \frac{2}{2l+1}\delta_{lm}.$$

Putting $x = \cos\theta$,

$$\int_{\theta=0}^{\pi} P_l(\cos\theta) P_m(\cos\theta) \sin\theta d\theta = \frac{2}{2l+1}\delta_{lm}.$$

Also,
$$V_0(R, \theta) = \sum_{l=0}^{\infty}\left(A_l R^l + \frac{B_l}{R^{l+1}}\right) P_l(\cos\theta).$$

If the potential at $r = R$ is given as $V = V_0(R, \theta)$, multiplying both sides by $P_m(\cos\theta)\sin\theta$ and integrating,

$$\int_0^{\pi} V(R, \theta) P_m(\cos\theta) \sin\theta d\theta = \sum_{l=0}^{\infty}\left(A_l R^l + \frac{B_l}{R^{l+1}}\right)\int_0^{\pi} P_l(\cos\theta) P_m(\cos\theta) \sin\theta d\theta$$

$$= \sum_{l=0}^{\infty}\left(A_l R^l + \frac{B_l}{R^{l+1}}\right)\cdot \frac{2}{2l+1}\delta_{lm} = \frac{2}{2m+1}\left(A_m R^m + \frac{B_m}{R^{m+1}}\right)\cdot$$

If you are looking for the potential in the region $r < R$,

$$B_m = 0 \text{ and } A_m = \frac{2m+1}{2R^m}\int_0^{\pi} V(R,\theta)P_m(\cos\theta)\sin\theta d\theta.$$

If you are looking for the potential in the region $r > R$, which extends up to infinity, A_l will be zero. In this case,

$$B_m = \frac{2m+1}{2}R^{m+1}\int_0^{\pi} V(R,\theta)P_m(\cos\theta)\sin\theta d\theta.$$

Separation of variables in cylindrical coordinates

Suppose the potential on a cylindrical surface is given and you need to get the potential either inside or outside this surface. If this region is charge-free, you can write Laplace's equation in cylindrical coordinates and solve it by the method of separation of variables. The equation is

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{dz^2} = 0.$$

Assuming that the cylindrical surface is long and the region of interest is away from the edges, the potential can be taken to be independent of z. Then

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} = 0.$$

Write $V(s, \phi, z) = S(s)\Phi(\phi)$, substitute in the above equation and divide by $\frac{S\Phi}{s^2}\cdot$ You get

$$\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0.$$

Thus,
$$\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right) = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = n^2, \tag{i}$$

where n^2 is a positive constant. Taking this constant to be positive is essential as will be clear after solving the ϕ equation.

$$\frac{d^2\Phi}{d\phi^2} = -n^2\Phi$$

or
$$\Phi = A\cos n\phi + B\sin n\phi.$$

As (s, ϕ, z) and $(s, 2\pi + \phi, z)$ represent the same point, $\Phi(\phi) = \Phi(2\pi + \phi)$. This shows that if $n \neq 0$, it must be an integer.

If $n = 0$, $\Phi = A$ and the Φ part is just a constant.

If n^2 is negative, Φ will exponentially increase with ϕ which is unphysical.

Taking negative integers for n does not give any new term as it only changes the sign of B. From equation (i), the s-equation is

$$s\frac{d}{ds}\left(s\frac{dS}{ds}\right) = n^2 S$$

or
$$s^2\frac{d^2 S}{ds^2} + s\frac{dS}{ds} - n^2 S = 0.$$

This is a second-order differential equation and its general solution (having two arbitrary constants) is

$$S = Cs^n + Ds^{-n}.$$

Once again if your region includes $s = 0$, D will be zero and if it includes $s = \infty$, C will be zero for $n \neq 0$ as can be verified by direct substitution.

If $n = 0$, the s-equation is $\frac{d}{ds}\left(s\frac{dS}{ds}\right) = 0$

or $$s\frac{dS}{ds} = E$$

or $$dS = E\frac{ds}{s}$$

or $$S = E\ln\frac{s}{R} + F.$$

We have written $\frac{s}{R}$ to maintain dimensional consistency. This only introduces a constant which is absorbed in F, which is arbitrary. Thus a general solution for the potential is

$$V(s, \phi, z) = \left(E\ln\frac{s}{R} + F\right) + \sum_{n=1}^{\infty}(C_n s^n + D_n s^{-n})(A_n \cos n\phi + B_n \sin n\phi).$$

From the given boundary conditions, one gets the appropriate constants. Suppose the potential $V(R, \phi)$ at the boundary surface $s = R$ is independent of ϕ. Then all coefficients A_n and B_n will be zero and the function is of the $\ln\frac{s}{s_0}$ type, or a constant. The function $\cos n\phi$ is an even function and $\sin n\phi$ is an odd function. If $V(R, \phi)$ is an even function of ϕ, only $\cos n\phi$ terms will be there in the $V(s, \phi, z)$ expression. Similarly, if $V(R, \phi)$ is an odd function of ϕ, only $\sin n\phi$ terms will be there. These tips will help you determine the constants.

EXAMPLE 6.6 **A long conducting pipe of radius R is placed with its axis along the z-axis. Charge distributions at very large distances produce a uniform electric field $E = E_0 \hat{i}$, which gets modified by the pipe. Find the electric potential outside the pipe.**

Solution For the region outside the pipe, the boundary surface is the surface $s = R$ together with $s = r_0$ where $r_0 >> R$. For points far away from the pipe, the electric field is $\boldsymbol{E} = E_0 \hat{\boldsymbol{i}}$. Take the potential at $s = R$ as zero. The potential at (s, ϕ, z) far away from the pipe can be written as $V = -Ex = -E_0 s \cos\phi$.

So the boundary conditions are $V = 0$ at $s = R$ and $V = -E_0 s \cos\phi$ for $s >> R$. You have to apply these boundary conditions to the solution of Laplace's equation

$$V(s, \phi, z) = \left(E \ln \frac{s}{R} + F\right) + \sum_{n=1}^{\infty} (C_n s^n + D_n s^{-n})(A_n \cos n\phi + B_n \sin n\phi). \tag{i}$$

Boundary condition for large *s*

Outside the pipe, there are points with large values of s. But you should not conclude that s^n terms will be zero. This is because we have not chosen $V = 0$ for $s = \infty$. Rather, for large s, the potential $V = -E_0 s \cos\phi$ is itself infinite for $\phi \neq \pi/2$ or $3\pi/2$. For large values of s, the terms s^{-n} tend to zero. Thus, from (i), the potential for large s is

$$V(s, \phi, z) = \left(E \ln \frac{s}{R} + F\right) + \sum_{n=1}^{\infty} C_n s^n (A_n \cos n\phi + B_n \sin n\phi).$$

Comparing with $V(s, \phi, z) = -E_0 s \cos\phi$,

$$E = 0, \;\; F = 0, \;\; C_1 A_1 = -E_0 \tag{ii}$$

$$B_n = 0 \text{ for all } n, C_n = 0 \text{ for all } n \neq 1.$$

Using these values in (i),

$$V(s, \phi, z) = -E_0 s \cos\phi + \sum_{n=1}^{\infty} D_n s^{-n} A_n \cos n\phi. \tag{iii}$$

Boundary condition at *s* = *R*

At $s = R$, $V = 0$.

Thus $$0 = -E_0 R \cos\phi + \sum_{n=1}^{\infty} (D_n A_n) R^{-n} \cos n\phi.$$

This shows that $D_n A_n = 0$ for all $n \neq 1$. Also, $0 = -E_0 R \cos\phi + \dfrac{D_1 A_1}{R} \cos\phi$

or $$D_1 A_1 = E_0 R^2.$$

Using these, the final expression for $V(s, \phi, z)$ is [from (iii)],

$$V(s, \phi, z) = -E_0 s \cos\phi + \frac{E_0 R^2}{s} \cos\phi$$

or $$V(s, \phi, z) = -E_0 \left[s - \frac{R^2}{s}\right] \cos\phi.$$

Concepts at a Glance

1. Poisson's equation is $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$ and Laplace's equation is $\nabla^2 V = 0$. These are satisfied by the electrostatic potential V.
2. The average of potential over a spherical surface is equal to the potential at the centre of the sphere.
3. In a charge-free region, you cannot have a maximum or a minimum of the potential.
4. The potential is completely determined at all points in a charge-free region if it is specified at all points on its boundary surface.
5. If the potential at the boundary surface of a region and the charge distribution in the region are specified, the potential at all points in the region is completely determined.
6. If there are certain conductors in a region, and the total charge on each conductor and the charge distribution in the region are specified, the surface charge distribution on each of the conductors is uniquely determined.

EXERCISES

Based on Concepts

1. A function f satisfies the equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -\frac{\partial^2 f}{\partial z^2}$$

in a region. The function varies from 3 units to 5 units on a closed surface. Can there be a point in the volume enclosed by the surface where the value of f is (a) 4 units, (b) 6 units and (c) zero?

2. A charge is in equilibrium at a point A under the purely electrostatic forces of some point charges. When slightly displaced in a particular direction and then released, it goes away from A. Show that if the charge is slightly displaced from A along at least one direction and then released, it will go towards A.
3. A spherical surface of radius R carries a total charge Q distributed nonuniformly on its surface. What is the value of $\oint V da$ evaluated over the whole surface, where V is the electric potential?

Problems

1. Two charges q and $2q$ are placed at $(a, 0, 0)$ and $(2a, 0, 0)$ respectively. Find $\oint V da$ over a spherical surface of radius $a/2$ centred at the origin. [**Ans.** $\frac{2aq}{\varepsilon_0}$]
2. A charge of 10 nC is placed at the origin. Find $\oint V da$ integrated over the surface of the sphere of radius 2 cm centred at (5 cm, 5 cm, 0). [**Ans.** 126 V m^2]

3. The potential in a charge-free region depends only on x and not on y or z. The values of the potential at $x = 5$ cm and at $x = 6$ cm are 6 V and 9 V. Find the potential at the origin. **[Ans.** –9 V]

4. Evaluate $\nabla^2 V$ for $V(\boldsymbol{r}) = K|z|$, where K is a constant and spherical polar coordinates are used. **[Ans.** $2\varepsilon_0 K\delta(z)$]

5. A conducting sphere is placed in an otherwise uniform electrical field $E_0\hat{k}$. Find the maximum electric field. **[Ans.** $3E_0$]

6. A uniform charge density ρ exists between planes $x = 0$ and $x = d$. The planes themselves are maintained at a potential $V = 0$. Write Poisson's equation in Cartesian coordinates and solve it to get the potential (a) inside the charge distribution and (b) outside the charge distribution. **[Ans.** (a) $\dfrac{\rho x}{2\varepsilon_0}(d - x)$ (b) $\dfrac{\rho d|x|}{2\varepsilon_0}$]

7. Two large metallic plates are placed at a small separation. They are maintained at potentials 0 (cathode) and V_0 (anode). The cathode is heated so that it emits electrons at negligible speed, which accelerate towards the other plate. Let $V(x)$, $\rho(x)$ and $u(x)$ denote the potential, charge density and speed of the electrons at x. In the steady state, a constant current I flows in the region.

   Suppose the plates are so large that the edge effects can be neglected. Then V, ρ and u are all functions of x alone.

   (a) Write Poisson's equation for the region between the plates.

   (b) What is the speed u of the electrons at point x where the potential is $V(x)$?

   (c) In the steady state, current I is independent of x. What is the relation between ρ and u?

   (d) Use this information to obtain a differential equation for V by eliminating ρ and u. Solve this equation to get V as a function of x.

   (e) Show that $I = KV_0^{3/2}$ and find the constant K. This equation is known as the Child–Langmuir equation. **[Ans.** (a) $V = V_0\left(\dfrac{x}{d}\right)\dfrac{4}{3}$]

8. Consider a cylindrical surface $s = R$. The potential at this surface is given by $V(R, \phi, z) = -\dfrac{\lambda}{\varepsilon_0}\cos\phi$. If there is no charge inside the cylinder bounded by this surface, find the potential at $\left(\dfrac{R}{2}, \phi, z\right)\cdot$ **[Ans.** $\dfrac{-\lambda}{2\varepsilon_0}\cos\phi$]

9. The potential on a spherical surface $r = R$ is given by $V(r, \theta, \phi) = \dfrac{\lambda}{\varepsilon_0 R\sqrt{5 - 4\cos\theta}}\cdot$ There is no charge inside the spherical volume. Find the potential at the origin. **[Ans.** $\dfrac{\lambda}{2\varepsilon_0 R}$]

10. The potential at all points on a spherical surface $r = R$ is $V = V_0\cos^2\theta$. Find the potential outside the sphere if there is no charge in this region. **[Ans.** $\dfrac{V_0}{3}\left[\dfrac{R}{r} + \left(\dfrac{R}{r}\right)^3(3\cos^2\theta - 1)\right]$]

11. The potential at points on a spherical surface is given by $V = V_0(4\cos^3\theta - 3\cos\theta)$. Find the potential outside the sphere if there is no charge in this region. **[Ans.** $\dfrac{V_0}{5}\left[20\left(\dfrac{R}{r}\right)^4\cos^3\theta - 3\left\{\left(\dfrac{R}{r}\right)^2 + 4\left(\dfrac{R}{r}\right)^4\right\}\cos\theta\right]$]

❑

7 Method of Images

If the charge distribution everywhere in space is given, we can, in principle, calculate the electric field and potential using Coulomb's law or Gauss's law, or by solving Poisson's equation. However, when conductors are present, the charge distributions on their surfaces are often not known except in some very symmetric situations. Even if a spherical conductor is placed near a point charge, it is not easy to work out the charge density on its surface. If you have an ellipsoidal conductor and put a charge q on it, you need special skills to know how the charge q will be distributed on the surface.

The method of images helps to obtain the field and potential due to certain charged conductors and also the surface charge distributions on such conductors. This method too can only be used for certain specific geometries of the conductor and is not useful in general cases. But in situations where it is usable, it gives results with stunning simplicity.

7.1 Basic Theory

Suppose there is a charge distribution because of which there is an equipotential surface with potential V_0. To be specific, we have shown some point charges in Figure 7.1 representing the charge distribution and a surface S that has the same potential V_0 at all points, due to these charges. The charges q_1 and q_2 are inside S and the other charges are outside.

Consider the region outside S. It is bounded by the surface S and the surface at infinitely. The charge distribution in this region is given and so is the potential at the boundary. At the surface S it is V_0 and at the surface at infinity it is zero. With all this information, we can understand that the potential function is unique in this region. This potential is due to all the charges.

Now imagine a conducting block with its outer surface identical to the equipotential surface S. Suppose the charges inside S (q_1, q_2 in Figure 7.1) are removed, the conducting block is placed to fit in S exactly and a charge Q is given to it. All the charges outside S (q_3, q_4, q_5 in Figure 7.1) are kept unchanged at their places.

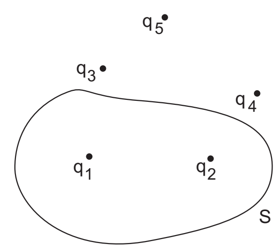


Fig. 7.1

The conducting block has to be equipotential. Its potential can be changed by changing the charge Q put on it. Adjust the value of Q to make the potential of the conductor equal to V_0.

Look at the region outside S. The charge distribution here is not changed by our removing q_1, q_2, fitting a conducting block in S and putting Q on it. The potential at S is also not changed as it is V_0 in both cases. The potential at infinity is still zero. So, neither the charge distribution in the region nor the potential at the boundary is changed. With these specifications, only one potential function $V(\boldsymbol{r})$ can exist. This guarantees that the potential in the region outside has not changed during our operations. And if the potential does not change, the electric field does not change either.

How much charge Q should we put to make the potential on the conductor V_0? In Figure 7.2, we have shown the two situations separately. Take a closed surface near S (outside it) and apply Gauss's law in each case.

For the first case $\oint \boldsymbol{E} \cdot \boldsymbol{da} = \frac{q_{en}}{\varepsilon_0} = \frac{q_1 + q_2}{\varepsilon_0}$ and in the second case $\oint \boldsymbol{E} \cdot \boldsymbol{da} = \frac{Q}{\varepsilon_0}$. But $\boldsymbol{E}$ is the same in the two cases and so is the surface of integration. Thus $\oint \boldsymbol{E} \cdot \boldsymbol{da}$ is also the same in the two cases. We then have $Q = q_1 + q_2$.

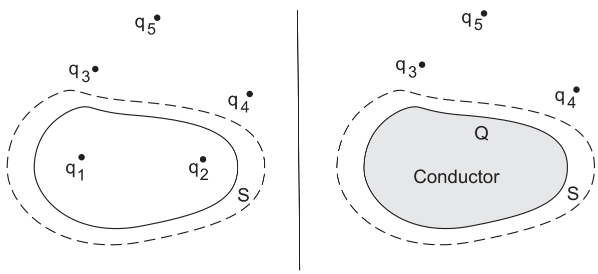


Fig. 7.2

So, the recipe to use the method of images is ready. Start with a charge distribution and locate an equipotential surface S due to this distribution. Get a conducting block made with its surface identical to the equipotential surface S chosen. Make this block fit exactly in S and transfer any charge inside S to this conductor. Don't disturb the charges outside S. By doing all this, you will not change the potential and field outside S.

When we say conducting block, it need not be a solid block, it can be a conducting shell with its outer surface having the same shape and size as the equipotential surface S. No charge is to be put in the empty space within the shell. The charge originally enclosed by S should be transferred to the shell.

As an example of this procedure, suppose a charge q is placed at some point and no other charge is placed anywhere else. The equipotential surfaces are spherical, all centred on the charge.

Consider a spherical surface S of radius R (Figure 7.3). If we bring a conductor, spherical in shape with radius R, place it to fit exactly in S and transfer the charge q on it, the field outside should not change according to our recipe developed. What does this mean? This means that if you take a spherical conductor and put a charge q on it, the electric field outside it will be the same as that produced by an equal charge q placed at the centre. Of course you know this from what you have studied in school. This example should give you confidence in the method of images we have discussed.

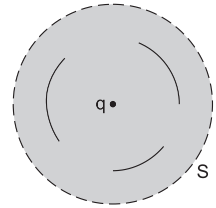


Fig. 7.3

7.2 A Charge *q* near a Large, Charged Conducting Plane

Consider two point charges q and $-q$ placed a distance $2d$ apart (Figure 7.4). To be specific, let their positions be (0, 0, d) and (0, 0, $-d$). Figure 7.4(a) shows some of the electric field lines due to these charges. The equipotential surfaces will be perpendicular to the field lines. Draw curves cutting the field lines perpendicularly. We have shown some intersections of the equipotential surfaces in the plane of the drawing, in Figure 7.4(b).

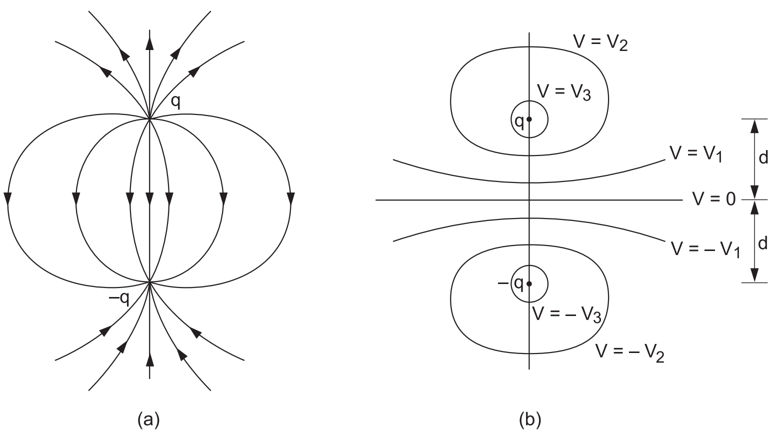


Fig. 7.4

Close to the charge $-q$, the effect of $+q$ will be small and so the equipotential surfaces will be nearly spherical, centred on the charge $-q$. One such surface is shown in the figure. The potential $V = -V_3$ here will be a large negative quantity. If you move away from $-q$ gradually, the effect of $+q$ will start appearing. The equipotential surfaces will still enclose $-q$, but will deviate more and more from the spherical shape and the potential will increase. Thus in Figure 7.4(b), $-V_1 > -V_2 > -V_3$. Similar equipotential surfaces will exist enclosing the charge $+q$. Here the potential will be positive. Now look at the plane bisecting the line joining the two charges, that is, $z = 0$. Here the potentials due to the two charges add up to zero. This plane itself is an equipotential surface with $V = 0$.

We can choose any of these equipotential surfaces and work using our recipe. However, the most useful of all these equipotential surfaces is the plane $z = 0$, which is an equipotential with $V = 0$. You can think of this plane as the limiting case of the equipotential surfaces enclosing $-q$, as the potential is taken close to zero. Fill the whole of the space $z \le 0$ by a conducting block and transfer the charge $-q$ to this block. By doing this, you will not change the potential or the field in the region $z > 0$. The field in the region $z > 0$ due to charges $+q$ at $(0, 0, d)$ and $-q$ at $(0, 0, -d)$ is the same as the field due to a charge $+q$ at $(0, 0, d)$ and a conducting block having charge $-q$ and filling the entire region $z < 0$. Knowing this fact, we can claim to have a solution to the following problem.

Consider a separate problem. The space $z \le 0$ is filled with a conducting block having charge $-q$. A charge q is placed at $(0, 0, d)$. What is the electric field $\boldsymbol{E}(\boldsymbol{r})$ in the space $z > 0$ (Figure 7.5)?

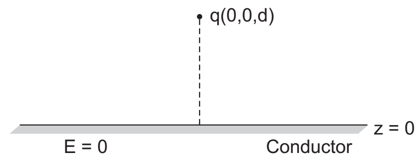


Fig. 7.5

The field will be the same as that due to $+q$ at $(0, 0, d)$ and $-q$ at $(0, 0, -d)$. This means that the charge $-q$ spread over the conducting surface at $z = 0$ is equivalent to a single charge $-q$ placed at $(0, 0, -d)$. The field at (x, y, z) with $z > 0$ will be

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0}\left[\frac{x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + (z-d)\hat{\boldsymbol{k}}}{\{x^2 + y^2 + (z-d)^2\}^{3/2}} - \frac{x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + (z+d)\hat{\boldsymbol{k}}}{\{x^2 + y^2 + (z+d)^2\}^{3/2}}\right]. \tag{7.1}$$

Indeed, for $z < 0$, the field is zero. The field lines are shown in Figure 7.6, and are identical to those shown in Figure 7.4 for $z > 0$.

It is not necessary to have a solid conducting block filling the whole of the space $z < 0$. You can have a conducting plate of any thickness with the top surface placed at $z = 0$. Finally, you also need not have an infinite plate with its surface extending over the entire plane $z = 0$. The dimensions of the plate should be large compared to the distance d and to the distances where the field is needed. Thus a charge $+q$ placed close to the centre of a conducting plate will give the same electric field (almost) as given above.

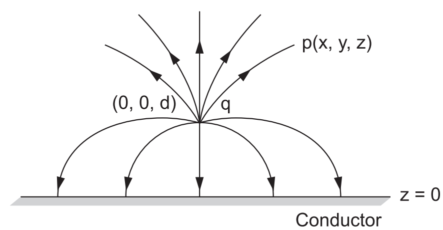


Fig. 7.6

Surface Charge Density on the Conducting Surface

The electric field just outside $z = 0$, from Equation 7.1, is

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0}\left[\frac{(x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} - d\hat{\boldsymbol{k}}) - (x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + d\hat{\boldsymbol{k}})}{(x^2 + y^2 + d^2)^{3/2}}\right]$$

$$= -\frac{2qd}{4\pi\varepsilon_0(x^2+y^2+d^2)^{3/2}}\hat{\boldsymbol{k}}.$$

The surface charge density at $(x, y, 0)$ on the surface of the conductor, facing the charge q, is

$$\sigma = \varepsilon_0 E = -\frac{qd}{2\pi(x^2+y^2+d^2)^{3/2}}.$$

Writing $x^2+y^2=a^2$,

$$\sigma = -\frac{qd}{2\pi(a^2+d^2)^{3/2}}.$$

Right below the charge q you have the origin where $x = 0$, $y = 0$ and hence $a = 0$. The charge density is maximum here in magnitude.

$$\sigma_0 = -\frac{q}{2\pi d^2}.$$

As you go away from this point on the conducting surface, the surface charge density decreases in magnitude. The charge in an annular ring of inner radius a and outer radius $a + da$ (Figure 7.7), centred on the origin, is $\sigma(2\pi a da)$. The total charge on the surface is

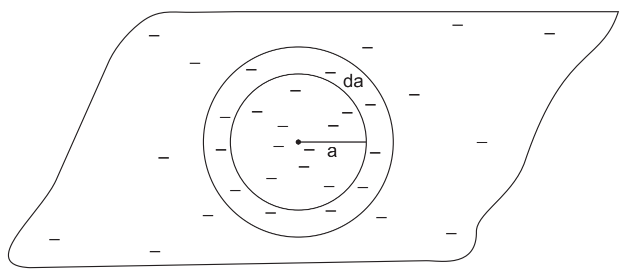


Fig. 7.7

$$\int_{a=0}^{\infty} \sigma(2\pi a da)$$

$$= \int_{a=0}^{\infty} \frac{-qd2\pi a}{2\pi(a^2+d^2)^{3/2}} da = -q$$

as expected.

Grounding

In many senses, the earth behaves like a huge conductor. Most of its surface carries a small negative charge density, while up in the atmosphere you have positive charge. In this model, the earth has got a negative potential, V_{earth}. There is an electric field near the surface of the earth in the fair-weather region, due to this negative charge, which of course gets modified by the presence of materials.

Earthing or grounding a conductor is an operation which ensures that the conductor is in good electrical contact with the earth and the potential of the conductor is the same as that of the earth. Commonly, a large metallic plate is buried deep into the soil with certain chemicals put around it, and wires firmly connected to this plate at one end are brought to the laboratory (or house) using electrical fittings. The conductor to be grounded is connected to this earth wire in the laboratory so that it gets connected to the buried metal plate and hence to the earth. Charges flow from the conductor to the earth or from the earth to the conductor and finally the potential of the conductor becomes the same as that of the earth. As the total charge on the earth's surface is very large, transfer of charge from or to the conductor on grounding does not change the potential V_{earth} of the earth to any significant extent.

If there is a neutral conductor and no charge is put in the vicinity, the potential of the conductor is the same as V_{earth}. If you put a charge on the conductor or near it, the potential changes to $V_{\text{earth}} + V'$, where V' is the potential produced by the extra charge. When you ground the conductor, again its potential becomes V_{earth} due to the flow of charge between the earth and the conductor. Generally, we do not mention V_{earth} and talk of the potential V' created by the charge distribution other than the earth's charge.

In the example of a charge q near a large conducting surface, one has to put a charge $-q$ on it. There is a simple way of doing it. You just ground the conductor. Charge $-q$ will appear on the plate and its potential will become zero with reference to the earth's potential.

Image charge

The whole charge distribution on the conducting, plane surface in front of the point charge q is equivalent to a single charge $-q$ placed on the opposite side of the surface. The equivalent charge $-q$ is called the *image charge* of the charge q, in the conducting plane surface. Obviously, the word comes from optics. When a light source A is placed in front of a mirror, the light received at point P consists of the direct light from A and the light reflected by the mirror (Figure 7.8). However, the light reflected by the mirror may be assumed to come from another light source at A′, which is the image of A in the mirror. But this is valid only for the front of the mirror. Behind the mirror (the lower half in the figure), the light intensity is zero.

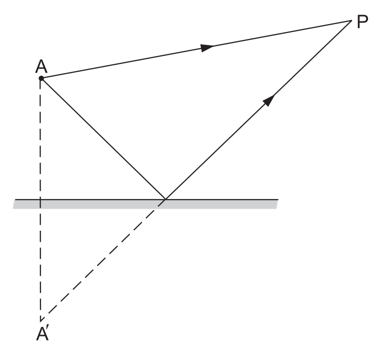


Fig. 7.8

Similarly, in the problem of an electric charge q in front of a plane conductor, the effect of the conductor turns out to be the same as that of the image charge $-q$, on the side on which charge q lies. For the other side, the field is zero.

Do not extend the analogy too far by using equations like $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$ for curved surfaces.

Force between the charge and the plane

When a point charge q is placed at a distance d from a grounded conducting plane, a charge distribution appears on the facing surface of the plane, which we call induced charge. If the point charge is positive, the induced charge is negative. This induced charge attracts the point charge. How much is this force of attraction?

The force on q may be expressed as

$$\boldsymbol{F} = q\boldsymbol{E},$$

where $\boldsymbol{E}$ is the electric field at the position of q, produced by the induced charge on the conducting plane. But this field is the same as that produced by the image charge $-q$. As the distance of the image charge from the point charge q is $2d$, the electric field at the position of q due to the induced charge will be

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0 (2d)^2}\hat{\boldsymbol{n}},$$

where $\hat{n}$ is the unit vector towards the plane. The force of attraction by the conducting plane on the point charge is, therefore,

$$F = \frac{q^2}{16\pi\varepsilon_0 d^2}\cdot$$

Electrostatic energy of the point charge–conducting plane system

Suppose a conducting plane is grounded and a point charge is brought slowly from infinity and is placed at a distance d from the plane. How much work is done in this process?

When the distance of the point charge from the plane is x, the force of attraction is $F = \dfrac{q^2}{16\pi\varepsilon_0 x^2}\cdot$ Thus the person bringing the charge must exert this much force at this distance in the opposite direction in order to move the charge slowly. Therefore the work done in bringing the charge from infinity to the desired point is

$$W = -\int_{\infty}^{d} F dx = -\int_{\infty}^{d} \frac{q^2}{16\pi\varepsilon_0 x^2} dx$$

$$= \frac{q^2}{16\pi\varepsilon_0}\left[\frac{1}{x}\right]_{\infty}^{d} = \frac{q^2}{16\pi\varepsilon_0 d}\cdot$$

This is the electrostatic energy of the charge configuration q and the induced charge.

You can also get this result using a different argument. When a charge q is placed in front of the conducting plate at a distance d, the electric field is the same as that due to charges $q, -q$ at a separation of $2d$. But this is true only for half the space, that is $z > 0$. For $z < 0$, the field is zero. The electrostatic energy for the $+q, -q$ combination at a separation of $2d$ is $-\frac{q^2}{8\pi\varepsilon_0 d}\cdot$ In the present case, the field is only in half the space, so the energy will be $-\frac{q^2}{16\pi\varepsilon_0 d}\cdot$

EXAMPLE 7.1 **A point charge of 12 nC is placed in front of a large, grounded metallic plate at a distance of 1 cm. What is the electric field at a point situated at a distance of 3 cm from the plate and 2 cm from the point charge?**

Solution It is clear that the given point lies on the perpendicular from the point charge on the plate (Figure 7.9). The field at this point P is the same as that due to the given charge 12 nC and its image charge –12nC in the plate. The field is

$$\frac{12\ \text{nC}}{4\pi\varepsilon_0(2\ \text{cm})^2} - \frac{12\ \text{nC}}{4\pi\varepsilon_0(4\ \text{cm})^2}$$

$$= 12\times10^{-9}\times9\times10^9\left[\frac{1}{4}-\frac{1}{16}\right]\times10^4\ \text{V/m}$$

$$\approx 2.0\times10^5\ \text{V/m}.$$

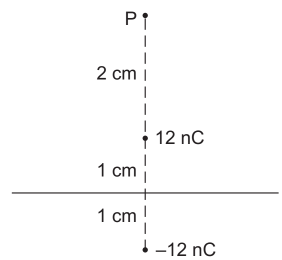


Fig. 7.9

7.3 A Point Charge Placed in Front of a Conducting Sphere

Consider two point charges q_1 and q_2 placed at a separation d. Take the position of q_1 as the origin and that of q_2 as $(d, 0, 0)$ as shown in Figure 7.10. The potential at (x, y, z) is

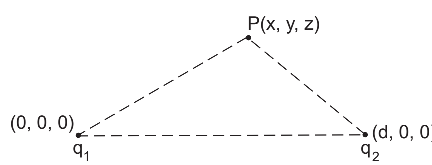


Fig. 7.10

$$V(x, y, z) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1}{(x^2+y^2+z^2)^{1/2}} + \frac{q_2}{[(x-d)^2+y^2+z^2]^{1/2}}\right].$$

Let us look for the equipotential surface with $V(x, y, z) = 0$. For this surface,

$$\frac{q_1}{(x^2+y^2+z^2)^{1/2}} = \frac{-q_2}{[(x-d)^2+y^2+z^2]^{1/2}}.$$

This shows that q_1 and q_2 must be of opposite signs if an equipotential surface with $V = 0$ exists.

Also

$$q_1^2[(x-d)^2+y^2+z^2] = q_2^2[x^2+y^2+z^2]$$

or $$(q_1^2-q_2^2)(x^2+y^2+z^2)-2q_1^2dx+q_1^2d^2=0$$

or $$x^2+y^2+z^2-2\frac{q_1^2d}{q_1^2-q_2^2}x+\frac{q_1^2d^2}{q_1^2-q_2^2}=0. \tag{i}$$

The equation of the sphere of radius R with centre at $(a, 0, 0)$ is

$$(x-a)^2+y^2+z^2=R^2$$

or $$x^2+y^2+z^2-2ax+(a^2-R^2)=0.$$

Comparing with (i), you can see that the surface represented by (i) is a sphere with its centre at $(a, 0, 0)$ and radius R, where

$$a=\left(\frac{q_1^2d}{q_1^2-q_2^2}\right) \tag{ii}$$

and $$a^2-R^2=\frac{q_1^2d^2}{q_1^2-q_2^2}. \tag{iii}$$

Take $q_1^2>q_2^2$. Then, from (ii) and (iii), $a>d$ and also $a>R$. Thus the centre C of the sphere is on the line joining q_1 and q_2, and falls beyond q_2. The sphere will enclose q_2 but not q_1 (Figure 7.11).

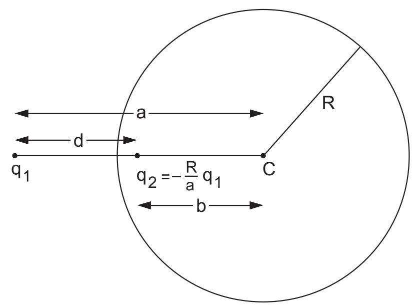


Fig. 7.11

You can also argue in the following way. The potential will be zero on the x-axis at a point between q_1 and q_2 and at a point beyond q_2 (as $q_1>-q_2$). The sphere will enclose q_2 but not q_1.

$$R^2=\left(\frac{q_1^2d}{q_1^2-q_2^2}\right)^2-\frac{q_1^2d^2}{q_1^2-q_2^2}=\frac{q_1^2q_2^2d^2}{(q_1^2-q_2^2)^2}$$

or $$R=-\frac{q_1q_2d}{(q_1^2-q_2^2)}. \tag{iv}$$

Why did we take the negative sign? Because q_1 and q_2 are of opposite signs and the radius has to be positive. Remember, we have taken $q_1^2>q_2^2$.

The distance of q_2 from C is

$$b=a-d=\frac{q_1^2}{q_1^2-q_2^2}d-d=\frac{q_2^2}{q_1^2-q_2^2}d.$$

Using (ii) and (iv), this becomes

$$b = \frac{R^2}{a}. \tag{7.2}$$

From (ii) and (iv),

$$\frac{R}{a} = -\frac{q_2}{q_1}$$

or

$$q_2 = -\left(\frac{R}{a}\right)q_1. \tag{7.3}$$

Let us summarize these results in a slightly different way. Consider a spherical surface of radius R. A charge q is placed at a distance a ($a > R$) from the centre of the surface and another charge $\left(-\frac{R}{a}q\right)$ is placed on the line joining the charge q and the centre, at a distance $\frac{R^2}{a}$ from the centre. In such a situation, the spherical surface will be an equipotential surface with $V = 0$.

Use the image method. Construct a spherical conductor of radius R and put it in place of the spherical surface shown above. Put charge $-\frac{R}{a}q$ on it. In fact, you don't need to put this charge—just ground the conducting sphere. To maintain the potential at zero (with reference to the earth), charge $\left(-\frac{R}{a}q\right)$ will flow from the ground to the sphere. The charge q remains at its place. By doing this operation, you do not change the electric field or potential outside the sphere.

Stating the same thing in reverse, suppose you put a charge q in front of a grounded conducting sphere at a distance a from its centre. Charges appear on the sphere and are distributed on the surface of the sphere. To get the electric field or potential outside the sphere, you can replace the conducting sphere (that is, the charge distribution appearing on it) by a single image charge $-\frac{R}{a}q$, put at a distance $\frac{R^2}{a}$ from the centre (towards q as in Figure 7.12).

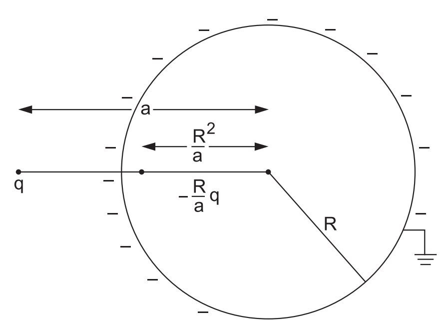


Fig. 7.12

Note carefully that the total induced charge appearing on the outer surface, when the point charge q is kept in front of it, is $-\frac{R}{a}q$ and not $-q$.

EXAMPLE 7.2 **A conducting sphere of radius 12 cm is grounded. A point charge of 24 nC is placed in front of it at a distance of 48 cm from the centre of the sphere. Find the value and the location of the image charge that can replace the charge distribution on the surface.**

Solution The radius of the sphere is $R = 12$ cm. The distance of the point charge from the centre is $a = 48$ cm. The value of the image charge is

$$q' = -\frac{R}{a}q = -\frac{12}{48} \times 24 \text{ nC} = -6 \text{ nC}.$$

Its distance from the centre is

$$b = \frac{R^2}{a} = \frac{(12 \text{ cm})^2}{(48 \text{ cm})} = 3 \text{ cm}.$$

So it is 3 cm from the centre, on the line joining the centre to the given charge.

Charge in front of a neutral spherical conductor

What happens if you do not ground the conducting sphere and do not give any charge to it? Put the charge q in front of it as before. In this case also, surface charge will appear on the sphere to make the electric field zero everywhere inside the sphere. Indeed the sphere remains neutral. So you have positive charge density at some places and negative charge density at some other places. The potential is not zero in this case. Can you use an image or images to replace this sphere and get the electric field outside?

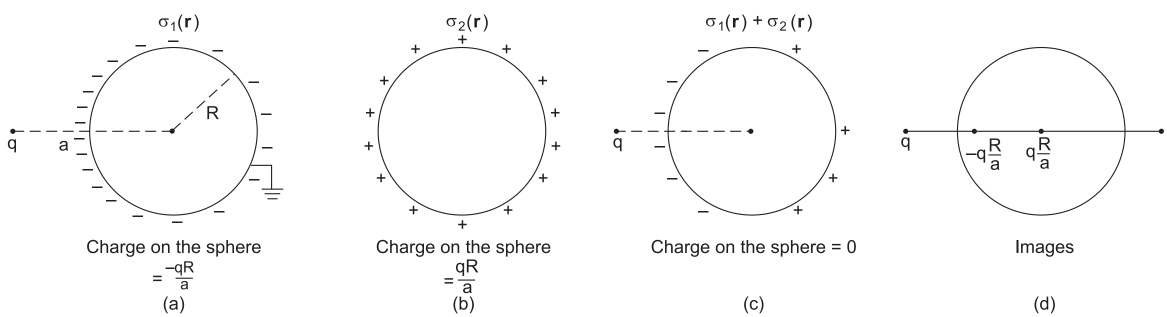


Fig. 7.13

The problem is quite simple. Suppose $\sigma_1(r)$ is the charge distribution appearing on the surface when we ground the sphere [Figure 7.13(a)]. The total charge $\oint \sigma_1(r)da$ on the surface will be $-\frac{R}{a}q$. This $\sigma_1(r)$ together with the point charge q outside will make the electric field zero everywhere inside the conductor. Now suppose you do not ground the sphere, do not have an external charge and distribute a charge $+\frac{R}{a}q$ uniformly on the surface of the spherical conductor [Figure 7.13(b)]. Call this distribution $\sigma_2(r)$. This distribution alone can make the electric field zero everywhere inside the spherical conductor. Thus charges $q, \sigma_1(r)$ and $\sigma_2(r)$ together can make the electric field zero everywhere inside the conductor. But $\int [\sigma_1(\boldsymbol{r}) + \sigma_2(\boldsymbol{r})]\,da = 0$. So you have obtained one surface charge distribution, with total charge zero on the conductor, which in the presence of charge q makes the field zero everywhere inside the conductor [Figure 7.13(c)]. But there can be only one way in which a given charge (zero here) can be distributed on the surface in the presence of the given external charge distribution. So when a charge q is placed in front of a neutral conducting sphere, the surface charge density on the sphere is $\sigma_1(\boldsymbol{r}) + \sigma_2(\boldsymbol{r})$ where $\sigma_1(\boldsymbol{r})$ is the charge density that would appear if the sphere were grounded and $\sigma_2(\boldsymbol{r})$ is the charge density obtained by distributing the charge $+\frac{R}{a}q$ uniformly on the surface. The charge distribution corresponding to $\sigma_1(\boldsymbol{r})$ can be replaced by a single point charge $-\frac{R}{a}q$ at a distance $\frac{R^2}{a}$ from the centre and the charge corresponding to $\sigma_2(\boldsymbol{r})$ can be replaced by a single point charge $+\frac{R}{a}q$ placed at the centre of the sphere, for calculating the field or potential outside.

So the neutral conducting sphere can be replaced by two image charges, $-\frac{R}{a}q$ and $\frac{R}{a}q$, the first one at a distance $\frac{R^2}{a}$ from the centre and the other at the centre itself [Figure 7.13(d)].

7.4 Charged Ellipsoidal Conductor

Consider a line charge of length $2d$ spread from $(-d, 0, 0)$ to $(+d, 0, 0)$. Let the linear charge density λ be constant. Consider an ellipse given by

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - d^2} = 1. \tag{i}$$

Writing $a^2 - d^2$ as b^2,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Each point of this ellipse may be written as $(a\cos\theta, b\sin\theta)$ for some value of θ (Figure 7.14). Consider a small element dx on the x-axis at $(x, 0, 0)$. The charge on this element is λdx and the potential due to this element at $(a\cos\theta, b\sin\theta)$ is

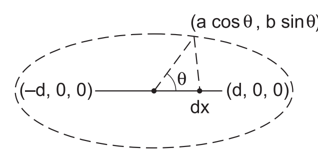


Fig. 7.14

$$dV = \frac{\lambda dx}{4\pi\varepsilon_0[(a\cos\theta - x)^2 + (b\sin\theta)^2]^{1/2}}.$$

The potential due to the whole of the line charge is

$$V = \int_{-d}^{d} \frac{\lambda dx}{4\pi\varepsilon_0[(a\cos\theta - x)^2 + (b\sin\theta)^2]^{1/2}}.$$

Solve this integration carefully, use $d = \sqrt{a^2 - b^2}$. You will get

$$V = \frac{\lambda}{4\pi\varepsilon_0} \ln \frac{a+d}{a-d} \cdot \tag{7.4}$$

This is independent of θ. This means that at all points on this ellipse, the potential is the same. Now rotate the ellipse about the x-axis. You get an ellipsoidal surface. As rotating about the x-axis does not change the geometry of Figure 7.14, the whole of the ellipsoid will have the same potential.

By changing a in equation (i), you get a family of ellipsoids, each of which corresponds to a different constant value of V.

We have got a path to deal with charged conducting ellipsoids. Suppose a conducting ellipsoid of revolution about the x-axis is given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and a charge q is put on it. For the points outside, this ellipsoid of revolution is equivalent to a line charge of length $2d = 2\sqrt{a^2 - b^2}$ kept centrally on the major axis and carrying the charge q uniformly distributed on it.

EXAMPLE 7.3 **A conductor is made in the shape of an ellipsoid of revolution with major axis 2*a* and minor axis 2*b*. It is given a charge *q*. Find the electric field at a point P on a perpendicular bisector of the major axis at a distance *a* from the centre.**

Solution The given point P is outside the ellipsoid as $a > b$. The conducting ellipsoid may be replaced by a line charge of length $2d = 2\sqrt{a^2 - b^2}$ placed on the major axis with its centre at the centre of the ellipsoid, carrying a uniformly distributed charge q. The electric field at the perpendicular bisector, at a distance a, will be in the direction of OP where O is the centre of the ellipsoid [Figure 7.15(a)]. To get the electric field at P due to the line charge, take an element dx on the line charge at a distance x from the centre and write the electric field at P

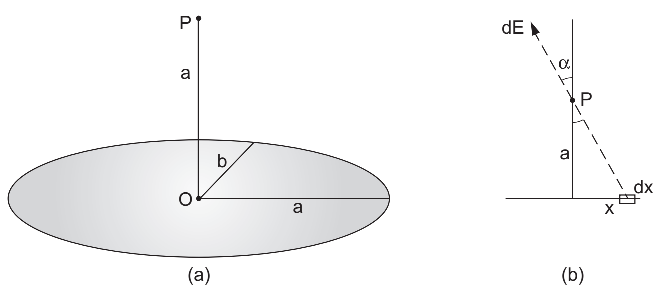


Fig. 7.15

due to this dx [Figure 7.15(b)]. We have done such calculations earlier. The charge on this dx is $dq = \frac{q}{2d}dx$ and hence the field due to this element is

$$dE = \frac{\frac{q}{2d}dx}{4\pi\varepsilon_0(a^2+x^2)}.$$

Its component in the direction of OP is

$$\frac{qdx}{8\pi\varepsilon_0 a(a^2+x^2)} \times \frac{a}{\sqrt{a^2+x^2}} = \frac{qa}{8\pi\varepsilon_0 d}\frac{dx}{(a^2+x^2)^{3/2}}.$$

The field at P will be

$$\frac{qa}{8\pi\varepsilon_0 d}\int_{-d}^{d}\frac{dx}{(a^2+x^2)^{3/2}} = \frac{q}{4\pi\varepsilon_0 ad\sqrt{a^2+d^2}}.$$

Writing $\quad d = \sqrt{a^2-b^2}$,

$$E = \frac{q}{4\pi\varepsilon_0 a\sqrt{(a^2-b^2)(2a^2-b^2)}}.$$

This is the required field.

7.5 A Line Charge in Front of a Conducting Cylinder

Consider two line charges, having linear charge densities $-\lambda$ and $+\lambda$, placed parallel to each other with a separation d between them. Let the line charges be at $x = -\frac{d}{2}$ and $x = +\frac{d}{2}$ respectively. Figure 7.16 shows the cross section in the x–y plane. Consider a point P$(x, y, 0)$.

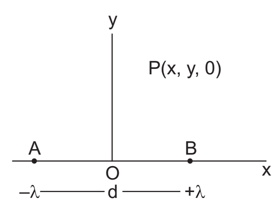


Fig. 7.16

Taking zero at the origin, the potential here is

$$V(x, y, 0) = -\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{AP}{AO} + \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{BP}{OB}$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{BP}{AP} = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{\sqrt{\left(x-\frac{d}{2}\right)^2+y^2}}{\sqrt{\left(x+\frac{d}{2}\right)^2+y^2}}$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\ln\frac{\left(x-\frac{d}{2}\right)^2+y^2}{\left(x+\frac{d}{2}\right)^2+y^2}. \qquad \text{(i)}$$

On what curve is this potential constant? It will be constant if

$$\frac{\left(x-\frac{d}{2}\right)^2+y^2}{\left(x+\frac{d}{2}\right)^2+y^2}=c$$

or $$\left(x-\frac{d}{2}\right)^2+y^2=c\left[\left(x+\frac{d}{2}\right)^2+y^2\right]$$

or $$x^2(1-c)+y^2(1-c)-(1+c)xd+\frac{d^2}{4}(1-c)=0$$

or $$x^2+y^2-\left(\frac{1+c}{1-c}\right)xd+\frac{d^2}{4}=0.$$

Or $$x^2-axd+y^2+\frac{d^2}{4}=0,\text{ where } a=\frac{1+c}{1-c}\text{ is another constant.}$$

Or $$\left(x-\frac{da}{2}\right)^2+y^2=\frac{d^2}{4}(a^2-1)\,. \tag{ii}$$

The constant a is related to the potential through equation (i).

$$V=\frac{\lambda}{4\pi\varepsilon_0}\ln c$$

or $$c=e^{4\pi\varepsilon_0 V/\lambda}.$$

This gives $$a=\frac{1+e^{4\pi\varepsilon_0 V/\lambda}}{1-e^{4\pi\varepsilon_0 V/\lambda}}\,. \tag{iii}$$

Equation (ii) gives a circle with centre at $\left(\frac{da}{2},0,0\right)$ and radius $\frac{d}{2}\sqrt{a^2-1}$. Two of these circles for different values of a are shown in Figure 7.17. As the line charges are infinitely long in the z-direction, the potential will not depend on z. The equipotential surfaces are therefore circular cylinders with cross-sections as shown in Figure 7.17. Now you can solve a problem based on these calculations. Suppose a line charge with uniform linear charge density λ is placed in front of an infinitely long conducting circular cylinder with linear charge density λ. You can calculate the potential and field everywhere using the method of images.

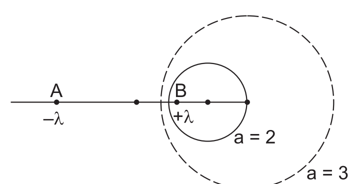


Fig. 7.17

....

Concepts at a Glance

1. With a given charge distribution, if a conductor is placed with its surface identical to an equipotential surface, and all the charges inside this equipotential surface are transferred to the conductor, the electric field outside the conductor remains unchanged.
2. The image of a point charge q placed in front of a large conducting surface is $-q$ at the same distance on the other side of the conducting surface.
3. For a point charge q placed at a distance d from the centre of a grounded conducting sphere of radius R, the image charge is $q' = -q\frac{R}{d}$ at a distance $\frac{R^2}{d}$ from the centre.
4. If the conducting sphere is isolated (not grounded) in the above situation, an extra $+q\frac{R}{d}$ image charge has to be placed at the centre of the sphere.

EXERCISES

Based on Concepts

1. A charge q is placed in front of a large, thin conducting plate. The plate is not grounded, and is neutral. How much is the total charge on the surface facing q and how much charge is on the other side? Neglect the charge on the side faces.
2. A charge q is placed at $(0, 0, d)$ and a large, grounded, conducting plate is placed on the x–y plane. What is the electric potential at $(0, 0, -d/2)$?
3. A large, neutral, metallic plate is placed on the x–y plane and a charge q is placed at a distance d from it. What will be the force on the charge? Will there be a difference in the force if the plate is grounded?
4. Can we have an equipotential surface enclosing an electric dipole? No other charge exists anywhere.
5. A charge Q is placed in front of a neutral conducting sphere at a distance d from its centre. What is the potential of the sphere?
6. Will the image of an electric dipole with no monopole moment in a grounded, conducting sphere be a dipole with no monopole moment?

Problems

1. A large, grounded metallic plate is placed on the x–y plane. Two point charges, q each, are placed at $(0, 0, d)$ and $(0, 0, 2d)$. Find the force on the charge at $(0, 0, d)$. **[Ans.** $\frac{49}{36}\frac{q^2}{4\pi\varepsilon_0 d^2}\hat{k}$**]**
2. Two point charges, q each, are placed in front of a large, grounded, conducting plate. The distance between the charges is d and that between each charge and the plate is $d/2$ (Figure 7E.1). Find the electrostatic energy of the configuration. **[Ans.** $\frac{q^2}{4\sqrt{2}\,\varepsilon_0 d}$**]**

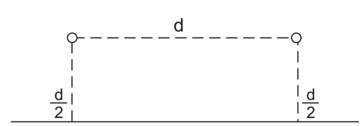


Fig. 7E.1

3. A charge q is placed at a distance d from a large, grounded, conducting plane. Find the charge appearing on the surface inside a circular area of radius d centred on the foot of the perpendicular from the charge on the plane. [**Ans.** $-q\dfrac{\sqrt{2}-1}{\sqrt{2}}$]

4. A charge q is placed at a height h from a grounded, infinite conducting plate. P is the foot of the perpendicular from the charge to the plate. Find the radius of the circular area on the plate that contains a charge $-q/2$ on the plate and has its centre at P. [**Ans.** $\sqrt{3}\,d$]

5. A large, grounded metallic plate is placed on the x–y plane. A charge q is placed at (0, 0, d) and another charge $2q$ is placed at (0, 0, $-d$). Find the field at (0, 0, $3d$). [**Ans.** $\dfrac{3q}{64\pi\varepsilon_0 d^2}\hat{k}$]

6. A particle having charge q and mass m is placed at a distance x_0 from a large, grounded conducting plane. It is released from here and moves under the force applied by the induced charges.
   (a) Find the acceleration at the time of release.
   (b) Find the speed of the particle just before it reaches the plane.
   (c) Find the time taken by the particle to reach the plane. [**Ans.** (c) $\dfrac{\pi x_0}{q}\sqrt{2\pi\varepsilon_0 m x_0}$]

7. A long line charge at $y = 0$, $z = d$ has a uniform linear charge density λ. Let the x–y plane represent a large conducting plane maintained at potential $V = 0$. Find the surface charge density on this surface, that is at (x, y, 0). [**Ans.** $-\dfrac{\lambda d}{\pi(y^2+d^2)}$]

8. A point charge 12 mC is kept at a distance of 15 cm from the centre of a grounded, conducting sphere of radius 10 cm. How much is the total charge induced on the sphere? [**Ans.** –4 mC]

9. A conducting sphere of radius R is kept with its centre at the origin, and is grounded. A point charge q is kept at (0, 0, $2R$). Find the electric field at (a) the origin, (b) (0, 0, $-2R$). [**Ans.** (a) zero, (b) $\dfrac{7}{1600\pi\varepsilon_0 R^2}\hat{k}$]

10. A point charge q is placed at a distance $5R$ from the centre of a neutral conducting sphere of radius R. Find the force of attraction exerted by the point charge on the sphere. [**Ans.** $\dfrac{49q^2}{288000\pi\varepsilon_0 R^2}$]

11. A conducting spherical shell of inner and outer radii a and b is kept with its centre at the origin. It is grounded and a point charge Q is kept at a distance $4b$ from the centre. Find the potential at the midpoint between the centre and the charge. [**Ans.** $\dfrac{11q}{240\pi\varepsilon_0 b}$]

12. A charge q is placed at a distance $2R$ from the centre of a metallic sphere of radius R carrying a charge $2q$. Find the electric field at a point on the line joining the centre to the charge q, at a distance $3R/2$ from the centre. [**Ans.** $\dfrac{61q}{72\pi\varepsilon_0 R^2}$ towards the centre]

13. A charge q is kept at a distance $2R$ from the centre of a grounded, conducting sphere of radius R. Find the surface charge density induced on the sphere at the point closest to the charge q. [**Ans.** $\dfrac{3q}{4\pi R^2}$]

14. A conducting spherical shell of inner and outer radii a and b is grounded. A charge q is placed inside the shell at a distance c from the centre of the shell. Find the electric field just inside the shell on the line joining the charge to the centre. [**Ans.** $\dfrac{q^2}{4\pi\varepsilon_0 a^2}\left(1+\dfrac{rc^2}{a(a-c)^2}\right)$]

15. Two point charges q and $-q$ are placed at $(0, 0, z_0)$ and $(0, 0, -z_0)$ respectively. A neutral spherical conductor of radius $R\left(R = \frac{z_0}{2}\right)$ is placed with its centre at the origin. Find the surface charge density induced on the sphere at a point where the radius vector makes an angle of 60° with the z-axis.

[**Ans.** $\frac{-q}{392z^2}(49\sqrt{3} - 9\sqrt{7})$]

16. A point charge q is placed inside a grounded, hollow, conducting sphere of radius R, at a distance $R/2$ from its centre. Find the surface charge density on the sphere. Evaluate the total charge appearing on the sphere by explicitly integrating the surface charge density on the surface of the sphere.

[**Ans.** Total charge $= -2q$]

17. Consider the situation of the above problem. After electrostatic equilibrium is achieved, the grounding connection is removed. Will there be a change in the charge density? How much charge should now be put on the sphere so that the surface charge density on the sphere at the point closest to the point charge q becomes zero? [**Ans.** No, $-6q$]

18. A grounded metallic sphere of radius R is surrounded by a concentric ring of radius $2R$. The ring carries a charge q uniformly distributed along its length. Find the electric potential at a point on the axis of the ring at a distance $5R$ from the centre. [**Ans.** $\frac{q}{4\pi\varepsilon_0 R}\left(\frac{1}{\sqrt{29}} + \frac{101}{4}\right)$]

19. Two charges q and $-q$ are placed on the two sides of a conducting sphere of radius R. Each of the charges is at a distance d from the centre. The two charges and the centre of the sphere are on the same straight line. Take the centre of the sphere as the origin and the line joining the charges as the z-axis. The positive direction of the z-axis is from q to $-q$.
   (a) Find the electric field everywhere.
   (b) What will be the field in the limit $d \to \infty, q \to \infty, q/d^2$ remaining finite, say equal to $2\pi\varepsilon_0 E_0$. This gives you the electric field where a conducting sphere is kept in an otherwise uniform external electric field. Find the field at (r, θ, ϕ) with $r = 2R$. [**Ans.** $\frac{E_0}{8}(10\cos\theta\,\hat{\boldsymbol{r}} - 7\sin\theta\,\hat{\theta})$]

20. A physical dipole with charges q, $-q$ and separation d is placed at a distance r from an uncharged conducting sphere of radius a. The dipole moment points toward the centre of the sphere. Determine the image charges and their positions.

[**Ans.** There will be three image charges. The one at the centre will be qad/r^2.]

21. An electric dipole of dipole moment p is kept at the centre of a spherical cavity of radius R inside a conductor. Take the origin at the centre of the cavity and the z-axis along the direction of the dipole moment. Find the electric field inside the cavity and the surface charge density induced on the cavity surface. [**Ans.** $\left\{\frac{p}{4\pi\varepsilon_0 R^3}\left[\left(1 + \frac{2R^3}{r^3}\right)\cos\theta\,\hat{\boldsymbol{r}} + \left(\frac{R^3}{r^3} - 1\right)\sin\theta\,\hat{\theta}\right], \frac{-3p}{4\pi R^3}\cos\theta\right\}$]

22. Two long, cylindrical conductors, each of radius R, are placed in such a way that their axes are parallel and a distance d apart ($d > 2R$). Find the capacitance of the system. [**Ans.** $\frac{\pi\varepsilon_0}{\ln(d/R)}$]

23. A metallic body in the shape of an ellipsoid is given a charge Q. The ellipsoid can be obtained by rotating the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the x-axis. Here $a > b$. Find the electric field at $(d, 0, 0)$, where d is much larger than a and b. [**Ans.** $\frac{q}{4\pi\varepsilon_0(d^2+b^2-a^2)}$]

24. The surface of a metallic body is given by the equation

$$\frac{1}{|\boldsymbol{r}-a\hat{\boldsymbol{i}}|}+\frac{1}{|\boldsymbol{r}+a\hat{\boldsymbol{i}}|}=\frac{1}{4a}\cdot$$

(a) Sketch the surface on the x–y plane.

(b) A charge q is given to this metallic body. Find the electric field at the point $(7a, 0, 0)$ and at $(3a, 0, 0)$.

[**Ans.** (a) $\frac{100q}{18432\pi\varepsilon_0 a^2}$ (b) zero]

25. The surface of a conductor can be obtained by rotating the curve (drawn on the x–y plane)

$$\frac{1}{\sqrt{x^2+y^2}}+\frac{1}{\sqrt{(x-a)^2+y^2}}=k$$

about the x-axis. A charge q is put on this conductor. Find the electric field everywhere.

[**Ans.** For outside, $\frac{q}{8\pi\varepsilon_0}\left[\frac{\vec{r}}{r^3}+\frac{\vec{r}-\vec{a}}{|\vec{r}-\vec{a}|^3}\right]$, inside zero]

26. A thin, spherical shell of radius a and a fixed charge density $\sigma=\sigma_0\cos\theta$ (θ measured with respect to the z-axis) is kept concentrically inside a spherical metal shell of inner radius $b(>a)$ (Figure 7E.2).

Find (a) the induced surface charge density (which ensures $\boldsymbol{E}=0$ in the metal) on the inner surface of the metal shell, and (b) the electric field in the region $r < b$.

[**Ans.** (a) $-\sigma_0\cos\theta$ (b) $\frac{\sigma_0}{360}\left\{\left(\frac{2R^3}{r^3}+1\right)\cos\theta\,\hat{\boldsymbol{r}}+\left(\frac{R^3}{r^3}-1\right)\sin\theta\,\hat{\theta}\right\}$]

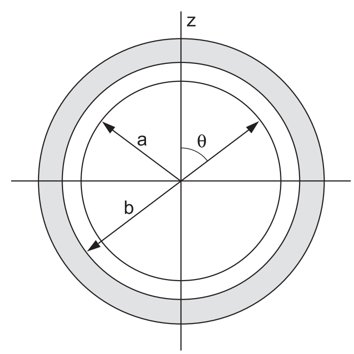


Fig. 7E.2

❑

8 Dielectrics and Polarization

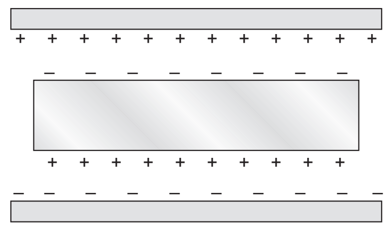


Fig. 8.1

As you know, materials can be grouped into three broad categories, namely, conductors, insulators and semiconductors, depending on how they behave in an electric field. In the earlier chapters, we have talked in some detail about the behaviour of conductors in an electric field. Conductors have a large number of free electrons, which can easily move from one part of the material to another, and this gives rise to many interesting properties. The materials in the broad category of insulators, also called dielectric materials or dielectrics, do not contain a significant number of free electrons. All electrons, even the outer ones, are firmly bound to the atoms or molecules of the material. Each electron in a dielectric knows which molecule it belongs to and there are no homeless, orphan electrons. When a slab of a dielectric material is inserted between the plates of a capacitor, the capacitance increases. Why does this happen? Because when you put the dielectric material in the electric field produced by the charged conducting plates of the capacitor, positive and negative charges appear on the surfaces of the dielectric. The face close to the positively charged plate acquires negative charge and the one close to the negatively charged plate acquires positive charge (Figure 8.1). This reduces the electric field in the space occupied by the slab, resulting in increased capacitance. But the dielectric material does not have free electrons which can move and redistribute on the surface. Then from where does the so-called "induced charge" appear on its faces? What happens at the atomic level when a dielectric material is placed in an electric field? And what kind of electric field is produced by the dielectric when charges appear on it? We shall discuss the electrical behaviour of dielectric materials in this chapter.

8.1 Polar and Nonpolar Molecules

Materials are made of molecules. As you know, a vast majority of molecules contain more than one atom, whereas those of the noble gas elements have only one atom each. Let us call this entity a molecule, even if it has just one atom. In each molecule, positive and negative charges have a distribution. In a water molecule the positive charges are lumped at three places—$+e$, $+e$

and $+8e$ at the three nuclei, and electron clouds have particular distributions. The electron clouds are shifted towards the oxygen side, making this end of the molecule negatively charged. The hydrogen end of the molecule has positive charge. The total charge of the water molecule is zero, but its electric dipole moment is not. In fact, water is a very good solvent because of the large dipole moment of its molecules.

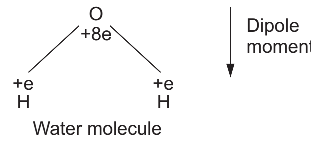


Fig. 8.2

Then there are substances like carbon tetrachloride and carbon dioxide in which the molecules have no dipole moment. The electron cloud distribution is symmetrical, giving zero dipole moment.

Molecules with a nonzero electric dipole moment are called *polar molecules* and those with zero dipole moment are said to be *nonpolar*. Dielectric materials made up of polar molecules are called *polar dielectrics* and those made up of nonpolar molecules are called *nonpolar dielectrics*.

Even if you take a volume element $d\tau$ having several thousand molecules in a polar dielectric, the dipole moment of the molecules in $d\tau$ taken together may be zero. Such is the case with most materials not placed in an electric field or not treated specially. This is because molecules are randomly oriented in a material and the vector sum of the individual dipole moments becomes almost zero in any volume containing a large number of them.

8.2 Polarization *P* in a Dielectric Material

In general, the dipole moment of all the molecules taken together in any small volume $d\tau$ of a dielectric material is zero. But if the material is placed in an electric field, a dipole moment appears in each such $d\tau$. The mechanism of the appearance of dipole moment is different for polar and nonpolar dielectrics. For polar dielectrics, each molecule has a permanent dipole moment. In a given volume $d\tau$, there are thousands of molecules and they are randomly oriented in the absence of the field. But why are they randomly oriented? The main causes are thermal energy and interactions with the neighbouring molecules.

When an electric field is applied, it exerts a torque

$$\tau = \boldsymbol{p} \times \boldsymbol{E}$$

on each dipole, and this torque tries to orient the molecule to bring the dipole moment along the direction of the field. The thermal interactions are still there and hence the alignment is not perfect, at least for small fields. However, the randomization is also not perfect. As a result, if you add the dipole moments of all the molecules in a small volume $d\tau$, you will get some dipole moment $d\boldsymbol{p}$. The quantity

$$\boldsymbol{P} = \frac{d\boldsymbol{p}}{d\tau} \tag{8.1}$$

is called the polarization at the location of the volume element $d\tau$.

If you take the volume element $d\tau$ at different places in the dielectric, the dipole moment $\boldsymbol{dp}$ may have different magnitudes and different directions. So, the polarization $\boldsymbol{P}$ is a function of space point and you can write it as $\boldsymbol{P}(\boldsymbol{r})$.

What happens if you put a nonpolar dielectric in an electric field? Initially, the charge distribution in each molecule is such that its dipole moment is zero. But because of the applied electric field, the charge distribution changes. The negative-charge clouds shift to some extent (they still remain bound to the molecules) opposite to the direction of the field and this causes dipole moments in the molecules in the direction of the field. We call them *induced dipole moments*. So if you place a nonpolar dielectric material in an external field, dipole moments will be induced in its molecules. You can look at the total dipole moment in a small volume $d\tau$ and define polarization at the location of the volume using Equation 8.1.

Electron shift also occurs in polar dielectrics, together with molecular rotation when they are placed in an electric field.

The actual mechanism of appearance of polarization is much more complicated than described here in terms of electron shift and rotation.

The two kinds of dielectrics we have talked about may give you the impression that without an applied electric field, the polarization $\boldsymbol{P}$ in a material is zero and with an electric field, the polarization is along the direction of the applied field. This is true for a class of materials called *isotropic linear dielectrics*, or simply linear dielectrics. In fact, for such dielectrics, the polarization is proportional to the electric field. We also have *anisotropic materials,* in which the polarization $\boldsymbol{P}$ may be in a direction different from the applied electric field. We can also suitably synthesize and treat materials so that a nonzero polarization $\boldsymbol{P}$ can exist in them without any applied field. This means that the molecules have dipole moments already aligned in a direction so that in any volume $d\tau$, you have a substantial dipole moment, even without any applied field. Call this *permanent polarization* or *frozen-in polarization*. Materials that can have frozen-in polarization are called *ferroelectric materials*.

However, here we will largely talk about linear isotropic dielectrics (or linear dielectrics), in which the polarization is in the direction of the electric field existing there and the magnitude of the polarization is proportional to the applied electric field.

If you have studied about polarization of light in school, please do not confuse it with polarization in dielectrics, which we are talking about in this chapter. Apart from having similar names, they do not have much in common.

8.3 Bound and Free Charges

Consider a rectangular slab of a linear isotropic dielectric material, placed in the space between the plates of a parallel-plate capacitor, as shown in Figure 8.3.

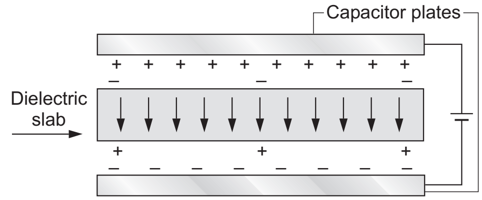


Fig. 8.3

When the capacitor is uncharged, there is no electric field and the dipole moment in any volume $d\tau$ of the dielectric slab is also zero. Remember, our $d\tau$ should contain at least a few thousand molecules. If you take $d\tau$ anywhere in the material, wherever you like, the charge density is zero. Now charge the capacitor. The charge on the capacitor creates an electric field and the dielectric in this field becomes polarized. The direction of the polarization is the same as that of the electric field. It is downwards in Figure 8.3, where the upper plate of the capacitor is given a positive charge and the lower plate, a negative charge. The dipole moment has appeared in any $d\tau$ you take, but do you also have a nonzero charge density in the dielectric? The whole slab is of course neutral. By connecting the capacitor plates to a battery, you are not putting any charge on the dielectric slab and hence the total charge on it must be zero. But there could be positive charge densities at some places and negative charge densities at some other places. In fact, a negative surface charge density appears on the upper surface of the slab and an equal positive surface charge density appears on the lower surface. The charge density in the interior of the slab is still zero.

To understand how charge densities can appear when a material gets polarized, let us use a model. Models are not actual descriptions of situations but represent real situations for the purpose for which they are formed, and give correct results.

Assume that the total positive charge in a given molecule is q, distributed in a volume τ_0 with a uniform charge density. Also assume that the total negative charge $(-q)$ in it is distributed in another volume τ_0 with a uniform charge density. Positive charge comes from the nuclei and negative charge from the electrons. You can talk about a uniform charge density because you are not interested in fluctuations at angstrom-like length variation. That is why any small volume element to be considered should contain at least a few thousand molecules. Initially, when there is no polarization in the material, the two distributions are exactly superimposed on each other in each molecule, and the net charge density as well as the dipole moment in any given volume is zero.

The dielectric becomes polarized when the negative charge distribution is shifted relative to the positive charge distribution by a small distance in the molecules. Denote the shift by s, which is the same everywhere if the polarization is uniform, and is different at different places if the polarization is nonuniform. The dipole moment developed in the molecule is then qs, opposite to the shift. Let the unit vector in the direction of the shift of the negative charge be denoted by $\hat{\boldsymbol{e}}$. The dipole moment of the molecule is then $-qs\hat{\boldsymbol{e}}$.

Consider a small volume $d\tau$ in the dielectric material. If there are N molecules per unit volume, the dipole moment generated in this volume will be

$$\boldsymbol{dP} = (Nd\tau)(-qs\hat{\boldsymbol{e}}).$$

So the polarization created will be $\boldsymbol{P} = \dfrac{(d\boldsymbol{p})}{(d\tau)} = -Nqs\hat{e}$. The magnitude of the polarization is

$$P = Nqs. \tag{8.2}$$

This model represents quite closely the actual process of polarization of nonpolar dielectrics, where the centre of the electron cloud in a molecule coincides with that of the positive charge to start with. As the electric field is applied, the electron cloud shifts a small distance relative to the positive charge and a dipole moment is created. As far as the effects of polarization are concerned, this model works equally well for polar dielectrics, where the dipole moment appears due to the alignment of molecules already having a dipole moment.

Now consider the dielectric slab shown in Figure 8.3, which is polarized in the downward direction. This slab alone has been shown in Figure 8.4. As the electric field created by the capacitor plates is uniform in the space between them, the polarization $\boldsymbol{P}$ in the dielectric slab is also uniform and hence the displacement s will be the same everywhere. Consider a small, flat, disk-type volume $d\tau$ in the interior of the slab (Figure 8.4). Let the faces of this volume be parallel to those of the slab. During the process of polarization, the negative charge of each molecule shifts in the direction opposite to the polarization (our model) by a distance s. In Figure 8.4, the electric field is downwards, and so the negative charges shift in the upward direction. During the shift, some negative charge enters the volume $d\tau$ from below and an equal negative charge goes out of it from above. Thus the net charge in the volume $d\tau$ remains zero. There will be no net charge density in the interior of the slab.

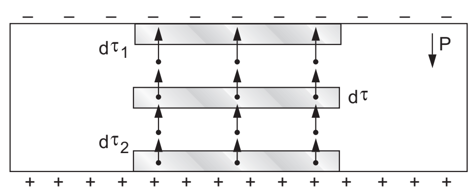


Fig. 8.4

Now consider a similar disk-type volume $d\tau_1$ on the upper surface, as shown in Figure 8.4. Each negative charge of the slab shifts a little in the upward direction. Thus, some negative charge will enter $d\tau_1$ from below but no negative charge will leave it. This is because there is no material above $d\tau_1$ where the negative charge can go. Thus, the volume $d\tau_1$ will get a net negative charge. As the interior of the dielectric slab does not have a net charge, the negative charge appears at the upper surface of $d\tau_1$. In fact, the whole of the upper surface of the slab becomes negatively charged.

Now consider a flat, disk-type volume $d\tau_2$ in the dielectric slab on the lower surface, also shown in Figure 8.4. Negative charges will leave $d\tau_2$ from above but nothing will enter it from below. This is because there is no material below $d\tau_2$ from where negative charge can enter $d\tau_2$. So, positive charge will appear on the lower surface of the slab. Thus, polarization has caused a surface charge distribution on the slab. The total charge on the slab is still zero.

What kind of charge is this? It has appeared due to the small shift in the molecular charge distribution or due to the alignment of molecules in a particular direction. Each electron is still bound to its parent molecule. The charge appearing in a dielectric material due to polarization is therefore called *bound charge*. The charge appearing on the surfaces has not come from faraway places. It exists only because of the molecules at the surface, in each of which the negative charge cloud is slightly shifted with respect to the positive charge in it. You can also put extra electrons on the dielectric by touching it with a negatively charged rod, for instance. These electrons will not be bound to any molecule and will cause a *free charge*.

Note the small difference between the terms 'free electrons in a conductor' and 'free charge in a dielectric'. Any charge appearing due to electrons not bound to any molecule constitutes free charge. Free electrons also cause free charge, but there are other ways too of obtaining such charge. You can rub a dielectric with a suitable material and this action can transfer electrons to the dielectric. These electrons are not bound to the molecules and so cause a free charge although they cannot move throughout the material as free electrons do in a metal.

In our example of a linear isotropic slab in a uniform electric field, bound charge appears only at the surface. So there is no bound volume charge density in the material, but there is a bound surface charge density. In general, you can also have a bound volume charge density in a polarized dielectric.

8.4 Relation between Bound Charge Density and Polarization

Bound volume charge density

Suppose you have a dielectric block with polarization $\boldsymbol{P}$, which may not be uniform. Assume the molecular charge distribution in the dielectric to be a superposition of a positive charge density and a negative charge density. When there is no polarization, these coincide with each other, giving zero net charge and zero dipole moment in any volume. Polarization results when the negative charge distribution shifts by some distance s relative to the positive charge distribution in each molecule.

Imagine a closed surface S inside the dielectric block, enclosing a volume τ (Figure 8.5). Take a small surface area da on S. Let us calculate how much charge has gone out of or into S through this surface area element da during the polarization. Let the polarization at this location be $\boldsymbol{P}$, making an angle θ with the outward normal $\hat{n}$ to da.

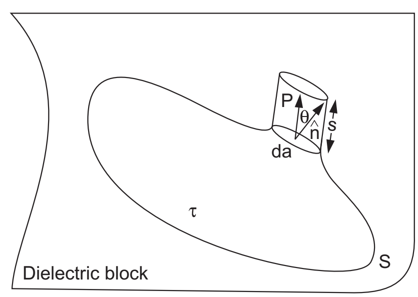


Fig. 8.5

Mentally construct a cylinder by dragging the area da through a distance s in the direction of the polarization, as suggested in Figure 8.5. The volume of this cylinder is

$$d\tau = (da)(s)(\cos\theta).$$

When the dielectric got polarized, the negative charge moved a distance s in the direction opposite to that of the polarization $\boldsymbol{P}$. Suppose there are N molecules per unit volume and each molecule has charge $+q$ and $-q$. All the negative charge $-Nqd\tau$ contained in this cylinder must have crossed da to move into S. The charge getting into S through da is, therefore,

$$\begin{aligned} -Nqd\tau &= -Nq(da)(s)\cos\theta \\ &= -Pda\cos\theta = -\boldsymbol{P}\cdot\boldsymbol{da}, \end{aligned}$$

where $\boldsymbol{da}$ is the area vector corresponding to the area da taken. The total charge going into S during the polarization is therefore

$$Q = -\oint_s \boldsymbol{P}\cdot\boldsymbol{da}.$$

At places where $\boldsymbol{P}$ makes an acute angle with $\boldsymbol{da}$, the quantity $\boldsymbol{P}\cdot\boldsymbol{da}$ is positive and at places where $\boldsymbol{P}$ makes an obtuse angle with $\boldsymbol{da}$, it is negative. The integration $\oint \boldsymbol{P}\cdot\boldsymbol{da}$ may be either zero or nonzero. If it is nonzero, it tells you that a net charge Q has appeared in the volume τ. This means there is some bound volume charge density ρ_b (the suffix b reminds us that we are dealing with bound charge density) in τ so that

$$Q = \int_\tau \rho_b d\tau.$$

Therefore, $$\oint_s \boldsymbol{P}\cdot\boldsymbol{da} = \int_\tau \rho_b d\tau.$$

Using the Gauss divergence theorem,

$$\oint_s \boldsymbol{P}\cdot\boldsymbol{da} = \int_\tau (\nabla\cdot\boldsymbol{P})d\tau.$$

So, $$-\int_\tau (\nabla\cdot P)d\tau = \int_\tau \rho_b d\tau$$

or $$\int_\tau (\rho_b + \nabla\cdot\boldsymbol{P})d\tau = 0.$$

Since this is true for any volume τ inside the dielectric, the integrand itself should be zero everywhere. So

$$\rho_b = -\nabla\cdot\boldsymbol{P}. \tag{8.3}$$

As you can see, if the polarization $\boldsymbol{P}$ is uniform, there is no bound volume charge density inside the dielectric as the divergence of $\boldsymbol{P}$ will be zero. If $\boldsymbol{P}$ is nonuniform, bound volume charge density appears if $\nabla \cdot \boldsymbol{P}$ is nonzero.

Bound surface charge density

Now consider the upper surface of the dielectric block in the situation shown in Figure 8.5 and take a small area da on it. Construct a cylinder of volume $d\tau$ by dragging da parallel to the direction of polarization by a distance s into the material. The flat areas are da, and the slant length is s, parallel to the direction of polarization. During the polarization, each negative charge tries to move a distance s in the direction opposite $\boldsymbol{P}$, and hence a charge $-Nqd\tau$ comes out of the cylinder and goes into the interior of the dielectric. This means the volume $d\tau$ will have a positive charge $Nqd\tau$ in it. As s is small, this charge will appear as a surface charge on da. Thus a surface charge density σ_b appears on the surface of the dielectric due to polarization, given by

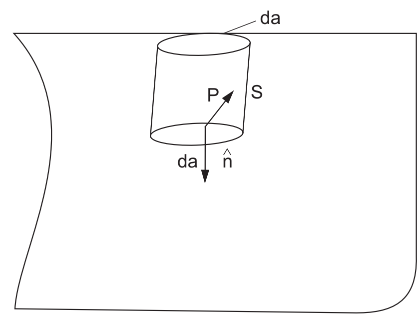


Fig. 8.6

$$\sigma_b = \frac{(dq)}{(da)} = \frac{Nqd\tau}{da} = \frac{Nqsda\cos\theta}{da} = P\cos\theta$$

or

$$\sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{n}}, \tag{8.4}$$

where $\hat{\boldsymbol{n}}$ is the unit vector along $\boldsymbol{da}$, that is, in the direction of the outward normal to the surface. The subscript b stands for 'bound'.

Thus, a polarized dielectric has a bound surface charge. If the polarization is nonuniform and its divergence is nonzero, bound charges appear in the interior of the volume also. Though these charges are still bound to the molecules, they produce an electric field and a potential in the same way as any other charge distribution.

Remember that the concept of the shifting of the negative charge by a distance s is only a model and does not represent the real process of polarization. But this model can be used to obtain correct relations.

This exercise should convince you that because of the stretching/rotation of the molecules, the dielectric acquires a surface charge and possibly a volume charge. These charge densities are related to the polarization $\boldsymbol{P}$ as given by Equations 8.3 and 8.4. These equations will be very helpful to you while solving problems involving dielectrics.

EXAMPLE 8.1 **A rectangular dielectric slab is uniformly polarized with polarization vector *P* perpendicular to the larger faces as shown. What is the bound charge density?**

Solution Look at Figure 8.7. The solid arrows show the direction of the polarization $\boldsymbol{P}$.

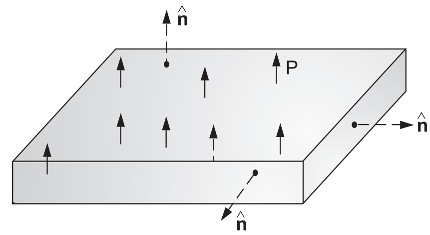


Fig. 8.7

At the top surface, $\boldsymbol{P}$ and the outward normal $\hat{\boldsymbol{n}}$ are in the same direction. So the surface charge density is $\sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{n}} = P$.

At the bottom surface, $\boldsymbol{P}$ and the outward normal $\hat{\boldsymbol{n}}$ are in opposite directions. So $\sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{n}} = -P$.

On the side surfaces, $\boldsymbol{P}$ and the outward normal $\hat{\boldsymbol{n}}$ are perpendicular to each other. So σ_b is zero.

In the interior, $\rho_b = -\nabla \cdot \boldsymbol{P} = 0$.

Hence charges appear only at the top and bottom surfaces with surface charge densities $+P$ and $-P$ respectively.

8.5 The Electric Field Due to a Uniformly Polarized Sphere

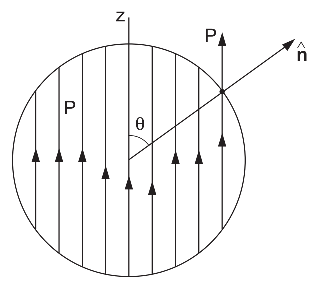


Fig. 8.8

Suppose a sphere made of a dielectric material is uniformly polarized, the polarization being $\boldsymbol{P}$ everywhere (Figure 8.8). The uniformly drawn parallel lines represent the polarization field. Because of this, some charge will appear on the surface of the sphere. The surface charge density at a point on the surface will be

$$\sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{n}} = P \cos \theta,$$

where θ is the angle between the radial direction and the polarization vector. There will be no volume charge density as $\nabla \cdot \boldsymbol{P} = 0$ inside the dielectric. The surface on the upper hemisphere will become positively charged (as $\theta < 90°$) and that of the lower hemisphere will become negatively charged (as $\theta > 90°$). How do you get the electric field due to this charge distribution $\sigma_b = P \cos \theta$?

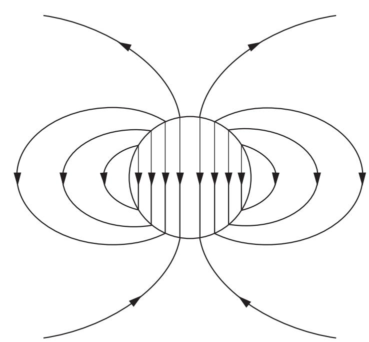


Fig. 8.9

The simplest way is to go back to Chapter 4, where the field due to a surface charge distribution $\sigma = \sigma_0 \cos \theta$ on the spherical surface $r = R$ was derived. Here P takes the place of σ_0. Thus, throughout the inside of the sphere, the field is uniform and given by

$$\boldsymbol{E} = -\frac{\boldsymbol{P}}{3\varepsilon_0}.$$

For points outside, the sphere is like a dipole of dipole moment $\boldsymbol{p} = \frac{4}{3}\pi R^3 \boldsymbol{P}$. And the electric field is

$$\boldsymbol{E} = \frac{R^3 P}{3\varepsilon_0 r^3}(2\cos\theta \boldsymbol{r} + \sin\theta \theta),$$

where the symbols have standard meanings. The electric field lines due to a uniformly polarized sphere are sketched in Figure 8.9.

8.6 Field due to Long, Cylindrical Dielectric Polarized Uniformly and Transversely

Suppose you have a long, cylindrical dielectric of radius R with its axis along the z-axis, polarized uniformly and transversely with uniform polarization $P\hat{\boldsymbol{i}}$. The bound surface charge density is

$$\sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{n}} = P\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{s}} = P\cos\phi.$$

Consider two long cylinders, each of radius R, one with a uniform volume charge density $-\rho_0$ and the other, $+\rho_0$. Let the axis of the first be along the z-direction and that of the other be displaced by a distance d along the x-direction. Let d be very small and ρ_0 satisfy

$$\rho_0 d = P. \tag{i}$$

Figure 8.10 shows the cross sections of the two cylinders in the x–y plane. In the overlap region, the net charge density is zero. This is essentially the full cylindrical volume as d is very small. The layer that is charged is like the surface of the cylinder. The charge density on this surface is

$$\sigma = \rho_0 d\cos\phi = P\cos\phi \quad \text{[using (i)]}.$$

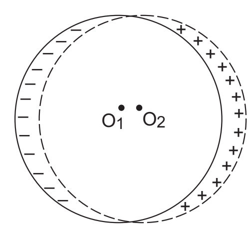


Fig. 8.10

Thus the polarized cylinder is equivalent to the combination of these two cylinders.

As the cylinder is long, nothing depends on z. The electric field or potential at (s, ϕ, z) is the same as that at $(s, \phi, 0)$. Thus you need to calculate the fields in the x–y plane only.

$$s < R.$$

Figure 8.11 shows a point A in the cylinder at a distance s from the axis. The electric field due to a large, uniformly charged cylinder of radius R is

$$\boldsymbol{E} = \frac{\rho \boldsymbol{s}}{2\varepsilon_0} \text{ for } s < R, \tag{ii}$$

where ρ is the volume charge density. Apply equation (ii) to the two cylinders in question to get the field at A (Figure 8.11).

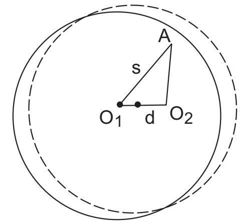


Fig. 8.11

$$\boldsymbol{E} = -\frac{\rho \boldsymbol{O_1 A}}{2\varepsilon_0} + \frac{\rho \boldsymbol{O_2 A}}{2\varepsilon_0}$$

$$= \frac{\rho}{2\varepsilon_0}(\boldsymbol{O_2 A} - \boldsymbol{O_1 A}) = \frac{\rho}{2\varepsilon_0}\boldsymbol{O_2 O_1} = -\frac{\rho \boldsymbol{d}}{2\varepsilon_0} = -\frac{\boldsymbol{P}}{2\varepsilon_0}.$$

$s > R$.

To get the fields outside, first calculate the potential. The potential due to a line charge with linear charge density λ is $V = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{s}{s_0}$, where s_0 is a fixed distance. A long, uniformly charged cylinder is equivalent to a long straight wire with $\lambda = \rho\pi R^2$. From Figure 8.12, the potential at a point A outside the cylinder is

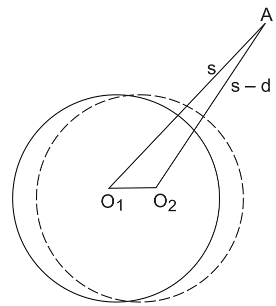


Fig. 8.12

$$V = \frac{\rho\pi R^2}{2\pi\varepsilon_0}\ln\frac{s}{s_0} - \frac{\rho\pi R^2}{2\pi\varepsilon_0}\ln\frac{|\boldsymbol{s}-\boldsymbol{d}|}{s_0}$$

$$= \frac{\rho R^2}{2\varepsilon_0}\ln\frac{s}{|\boldsymbol{s}-\boldsymbol{d}|}$$

$$= \frac{\rho R^2}{2\varepsilon_0}\ln\left[\left|1 - \frac{2d\cos\phi}{s} + \frac{d^2}{s^2}\right|\right]^{-\frac{1}{2}}.$$

Expanding up to the order of d/s,

$$V \approx \frac{\rho R^2}{2\varepsilon_0}\ln\left[1 + \frac{d\cos\phi}{s}\right] \approx \frac{\rho R^2}{2\varepsilon_0}\frac{d\cos\phi}{s}.$$

The electric field is $\boldsymbol{E} = -\nabla V = \frac{\rho R^2 d}{2\varepsilon_0 s^2}[\cos\phi\hat{s} + \sin\phi\hat{\phi}]$

$$= \frac{PR^2}{2\varepsilon_0 s^2}[\cos\phi\hat{s} + \sin\phi\hat{\phi}].$$

8.7 Bound Surface Charge on the Surface of a Cavity

If there is a cavity in a dielectric material, you have a surface of the dielectric where the cavity starts. If the dielectric is polarized, you will have bound surface charge on this surface too. The expression will be given by the same equation, $\sigma_b = \boldsymbol{P}\cdot\hat{\boldsymbol{n}}$. The normal $\hat{\boldsymbol{n}}$ will be drawn towards the cavity.

EXAMPLE 8.2 **A cylindrical block of dielectric material, having radius R and length L, is kept in an electric field. The axis of the block is taken as the z-axis. The block has a cavity, again cylindrical, with the axis parallel to the z-axis. Due to the electric field, a uniform polarization $\boldsymbol{P} = P_0\hat{\boldsymbol{k}}$ appears in the whole of the material. Find the bound charges induced.**

Solution The situation is shown in Figure 8.13. As the polarization is uniform in the material, there is no bound volume charge density,

$$\rho_b = -\nabla \cdot \boldsymbol{P} = 0.$$

Thus, charges will be induced only on the surfaces. This block has six surfaces, three towards the outside and three towards the cavity. These are the top surface, bottom surface, outer curved surface, top surface of the cavity, bottom surface of the cavity and the curved surface of the cavity. Figure 8.13 shows the normal $\hat{\boldsymbol{n}}$ drawn at each of the six surfaces. Note that the normal in each case is drawn away from the material.

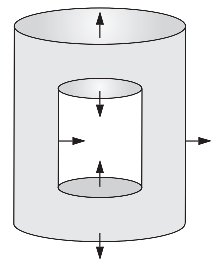


Fig. 8.13

Top surface The normal is $\hat{\boldsymbol{n}} = \hat{\boldsymbol{k}}$. The surface charge density is

$$\sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{n}} = P_0\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}} = P_0.$$

Bottom surface The normal is $\hat{\boldsymbol{n}} = -\hat{\boldsymbol{k}}$. The surface charge density is

$$\sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{n}} = P_0\hat{\boldsymbol{k}} \cdot (-\hat{\boldsymbol{k}}) = -P_0.$$

Outer curved surface The normal is $\hat{\boldsymbol{n}} = \hat{\boldsymbol{s}}$. The surface charge density is

$$\sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{n}} = P_0\hat{\boldsymbol{k}} \cdot (\hat{\boldsymbol{s}}) = 0.$$

Top surface of cavity The normal is $\hat{\boldsymbol{n}} = -\hat{\boldsymbol{k}}$. The surface charge density is

$$\sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{n}} = P_0\hat{\boldsymbol{k}} \cdot (-\hat{\boldsymbol{k}}) = -P_0.$$

Bottom surface of cavity The normal is $\hat{\boldsymbol{n}} = \hat{\boldsymbol{k}}$. The surface charge density is

$$\sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{n}} = P_0\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}} = P_0.$$

Curved surface of cavity The normal is $\hat{\boldsymbol{n}} = -\hat{\boldsymbol{s}}$. The surface charge density is

$$\sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{n}} = P_0\hat{\boldsymbol{k}} \cdot (-\hat{\boldsymbol{s}}) = 0.$$

EXAMPLE 8.3 **A disc-shaped cavity is carved out in a large block of dielectric. The thickness of the cavity is small compared to its width. The dielectric has a uniform polarization $\boldsymbol{P}$ in the direction parallel to the axis of the cavity. Find the electric field in the cavity due to polarization.**

Solution The situation is shown in Figure 8.14. Surface charge density will appear on the flat surfaces of the cavity. On the top surface, it will be $-P$ and on the bottom surface, $+P$. As the height of the disk is small compared to the width, the electric field in the cavity is almost uniform, except for points close to the edges. From each of the surfaces, the electric field will be $\frac{P}{2\varepsilon_0}$ in the upward direction, that is, in the direction of polarization in the material. Thus, the field in the cavity is $\boldsymbol{E}_{\text{in}} = \frac{\boldsymbol{P}}{\varepsilon_0}$.

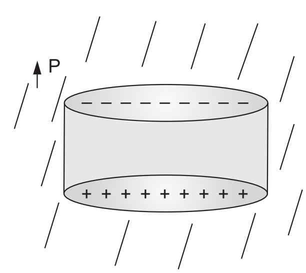


Fig. 8.14

8.8 Electric Susceptibility

Linear dielectrics form a class of dielectric materials in which the polarization $\boldsymbol{P}$ is proportional to the resultant electric field $\boldsymbol{E}$ existing at that place. The polarization is in the direction of the resultant electric field and the proportionality constant is independent of direction. For a linear dielectric, we write

$$\boldsymbol{P} = \varepsilon_0 \chi \boldsymbol{E}. \tag{8.5}$$

The quantity χ is a constant for the given material and is independent of the magnitude and direction of $\boldsymbol{E}$, as long as $\boldsymbol{E}$ is not very large. χ is called the *electric susceptibility* of the material. Remember that $\boldsymbol{E}$ is the resultant field at the point inside the dielectric where $\boldsymbol{P}$ is being calculated. This includes the field due to any external charges and also due to the bound charges appearing on the material itself due to polarization.

8.9 Molecular Polarizability and Clausius–Mossotti Equation

A linear dielectric material made up of nonpolar molecules gets polarized when its molecules acquire dipole moment due to the electric field acting on them. For a nonpolar molecule, the dipole moment $\boldsymbol{p}$ induced is proportional to the electric field $\boldsymbol{E}$ present at the location of the molecule,

$$\boldsymbol{p} = \alpha \boldsymbol{E}. \tag{8.6}$$

The proportionality constant α is called the *polarizability* of the molecule or molecular polarizability. If the dielectric is made up of atoms, you can similarly define atomic polarizability.

The polarizability of an atom or a molecule tells you how simple or difficult it is to induce a dipole moment in it. A large polarizability will mean that the electron clouds can be easily displaced and a small applied electric field can induce a large dipole moment in the molecule.

Now consider a linear nonpolar dielectric. Suppose it is placed in an electric field and it acquires a polarization $\boldsymbol{P}$. The net electric field in the dielectric (the applied field plus the field due to all the polarized molecules) is $\boldsymbol{E}$. The relation between $\boldsymbol{P}$ and $\boldsymbol{E}$ is

$$\boldsymbol{P} = \varepsilon_0 \chi \boldsymbol{E}, \tag{i}$$

where χ is electrical susceptibility. As the polarization $\boldsymbol{P}$ in the dielectric results from the dipole moments induced in the individual molecules, you would expect there to be a relation between molecular polarizability α and electrical susceptibility χ. This relation is given by the *Clausius–Mossotti equation* discussed below.

Suppose there are N molecules per unit volume in the dielectric and each molecule has a dipole moment $\boldsymbol{p}$. Then the dipole moment per unit volume will be

$$\boldsymbol{P} = n\boldsymbol{p}. \tag{ii}$$

Can you write $\boldsymbol{P} = \varepsilon_0 \chi \boldsymbol{E}$, $\boldsymbol{p} = \alpha \boldsymbol{E}$ and using (ii) get the relation between α and χ? No. In the relation $\boldsymbol{P} = \varepsilon_0 \chi \boldsymbol{E}$, the electric field $\boldsymbol{E}$ is the net field existing in the dielectric that includes contributions from all the molecular dipole moments. In the relation $\boldsymbol{p} = \alpha \boldsymbol{E}$, applicable to a molecule separately, $\boldsymbol{E}$ is the electric field acting on that molecule and does not include a contribution from that particular molecule itself. The charge clouds of a molecule get displaced due to the field of all charges other than that of the molecule itself. So the electric field in $\boldsymbol{p} = \alpha \boldsymbol{E}$ is different from that in $\boldsymbol{P} = \varepsilon_0 \chi \boldsymbol{E}$. Let us express the net electric field due to all the charges as $\boldsymbol{E}$ and the electric field due to all the charges other than the molecule under consideration as $\boldsymbol{E}'$. So $\boldsymbol{E}'$ = (net field $\boldsymbol{E}$ in the dielectric) – (field due to the one molecule under consideration). The equations are

$$\boldsymbol{p} = \varepsilon_0 \chi \boldsymbol{E}, \quad \boldsymbol{p} = \alpha \boldsymbol{E}' \text{ and } \boldsymbol{P} = N\boldsymbol{p}.$$

How do you relate $\boldsymbol{E}$ and $\boldsymbol{E}'$? Consider the molecule itself as a uniformly polarized sphere of some radius. As the dielectric has uniform polarization $\boldsymbol{P}$, this molecule can also be thought to be having polarization $\boldsymbol{P}$. The field inside the molecule due to the molecular dipole moment is then $-\dfrac{\boldsymbol{P}}{3\varepsilon_0}\cdot$

So,
$$\boldsymbol{E}' = \boldsymbol{E} - \left(-\frac{\boldsymbol{P}}{3\varepsilon_0}\right) = \boldsymbol{E} + \frac{\boldsymbol{P}}{3\varepsilon_0}$$

$$= \boldsymbol{E} + \frac{\varepsilon_0 \chi \boldsymbol{E}}{3\varepsilon_0} = \boldsymbol{E}\left(1 + \frac{\chi}{3}\right)\cdot$$

Thus,
$$\boldsymbol{P} = N\boldsymbol{p} = N\alpha \boldsymbol{E}' = N\alpha\left(1 + \frac{\chi}{3}\right)\boldsymbol{E}$$

or
$$\varepsilon_0 \chi \boldsymbol{E} = N\alpha\left(1 + \frac{\chi}{3}\right)\boldsymbol{E}$$

or
$$\alpha = \frac{\varepsilon_0 \chi}{N\left(1 + \frac{\chi}{3}\right)} = \frac{3\varepsilon_0}{N}\frac{\chi}{\chi + 3}\cdot$$

Writing
$$K = 1 + \chi,$$

$$\alpha = \frac{3\varepsilon_0}{N}\frac{K-1}{k+2}\cdot \tag{8.7}$$

This is the Clausius–Mossotti equation.

The constant K is called the dielectric constant. We will talk more about it in the next chapter.

8.10 Some Phenomena in Dielectrics

Actual dielectric materials show a variety of phenomena due to molecular interaction. Some of them are described below.

Dielectric relaxation

In general, the polarization $\boldsymbol{P}$ in a dielectric material depends on the electric field applied. You can apply or change the electric field very fast, but $\boldsymbol{P}$ cannot change that fast, especially in polar dielectrics. Due to forces from the neighbouring molecules, molecular rotation takes finite time. This time depends on the actual material and varies in a wide range. Similarly, if you switch off the electric field, it takes finite time for the polarization to reduce to zero; sometimes it can even retain some polarization after the field is removed. This phenomenon is called dielectric relaxation.

Dielectric breakdown

Dielectrics are essentially bad conductors of electricity. This is because they do not have a large number of free electrons. However, if the magnitude of the applied field is very large, molecular bonds can break and electrons may get detached from the molecules. The dielectric then may become conducting and lose its dielectric character. This phenomenon is called dielectric breakdown. Often the breakdown starts in a portion of the dielectric and then travels along a path. The minimum electric field at which dielectric breakdown occurs is called the dielectric strength of that material. It crucially depends on any kind of impurities in the material, surface roughness, etc. For dry air at atmospheric pressure, the dielectric strength is 3kV/mm.

Leakage current

An ideal dielectric should have zero conductivity. But in real materials, some conductivity is possible. This is because some electrons gain enough energy from thermal processes to break bonds. Such residual conductivity causes heat loss when used in capacitors or other devices.

Concepts at a Glance

1. Dielectrics are materials in which almost all electrons are bound to the atoms.
2. In polar dielectrics, molecules have permanent electric dipole moments due to their structure. In nonpolar dielectrics, molecules do not have permanent dipole moments.
3. In general, the bulk dipole moment in a given volume is zero. For polar dielectrics, this is due to the thermal randomization of molecular dipole directions.
4. In applied electric fields, or due to special procedures, a material acquires a dipole moment in each volume element.
5. The dipole moment per unit volume is called polarization $\boldsymbol{P}$. It is defined at each point of the dielectric.
6. Because of polarization, the bound electron clouds shift/reorient and this gives a charge density to the material, though the total charge remains zero.
7. $\rho_b = \nabla \cdot \boldsymbol{P}$ and $\sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{n}}$.
8. For a linear dielectric, $\boldsymbol{P} = \varepsilon_0 \chi \boldsymbol{E}$.

EXERCISES

Problems

1. Consider a cubical volume of $1\,\text{cm}^3$ filled with liquid helium (density $0.14\,\text{g/cm}^3$). The axes chosen are shown in Figure 8E.1. An electric field is applied in the z-direction, because of which the electrons in each atom shift by 0.1 nm opposite the field. Find (a) the dipole moment induced in each atom, (b) the polarization vector and (c) the bound charge induced on the surface.

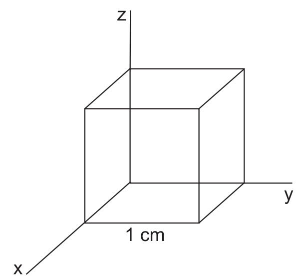


Fig. 8E.1

[**Ans.** (b) $(67.2\,\text{C/m}^2)\hat{\boldsymbol{k}}$ (c) 6.72 mC on the top surface and –6.72 mC on the bottom surface]

2. A thin slab is uniformly polarized in the direction perpendicular to the faces. The magnitude of the polarization is P. Find the electric field just outside and just inside the slab due to the polarization.

3. The polarization in a material varies in the $\boldsymbol{x}$-direction and is given by $\boldsymbol{P} = P_a e^{-x^2/a^2}\hat{\boldsymbol{z}}$, where a is a constant. What is the charge density that appears in the material? [**Ans.** $\dfrac{2xP_0 e^{-x^2/a^2}}{a^2}$]

4. A dielectric cube is placed with its centre at the origin and edges parallel to the coordinate axes. It has a polarization $\boldsymbol{P} = k\boldsymbol{r}$, where k is a constant. Find all the bound charge densities. What is the total bound charge? [**Ans.** $\rho_b = -3k,\ |\sigma_b| = \dfrac{ka}{2}$ on each face]

5. The polarization in a dielectric cube of edge length a placed with its centre at the origin is given by $\boldsymbol{P} = \lambda_0\hat{\boldsymbol{r}}$. Find (a) the bound volume charge density and (b) surface charge density at the point $(a/2, 0, a/4)$. [**Ans.** (a) $\rho_b = \dfrac{2\lambda_0}{\sqrt{5}}$]

6. A small spherical cavity is carved out from a large block of a dielectric material. A uniform polarization $\boldsymbol{P}$ exists in the material. Find the electric field inside the cavity. [**Ans.** $\dfrac{\boldsymbol{P}}{3\varepsilon_0}$]

7. A long cylinder of radius R is kept with its axis along the z-axis. It is uniformly polarized with polarization $\boldsymbol{P} = P_0\hat{\boldsymbol{i}}$. Show that the electric field outside the cylinder due to the bound charges can be written as $\dfrac{R^2}{2\varepsilon_0 s^2}[2(\boldsymbol{P}\cdot\hat{\boldsymbol{s}}) - \boldsymbol{P}]$.

8. A long, thin cylindrical cavity is carved out from a big dielectric material with a uniform polarization $\boldsymbol{P}$ parallel to the length of the cavity. The radius of the cavity is r and the length is L. Find the electric field at the centre of the cavity. [**Ans.** $\dfrac{Pr^2}{\varepsilon_0 L^2}$]

9. A short cylinder of radius R and length L has a uniform polarization P parallel to its axis. Find the electric field at the centre. **[Ans.** $\frac{PL}{2\varepsilon_0}\left[\frac{1}{L}-\frac{1}{\sqrt{L^2+4R^2}}\right]$]

10. A cylinder made of a linear dielectric with susceptibility $\chi = 1.0$ has a radius 5.0 mm and height 2.0 mm. It is uniformly polarized with the polarization vector 2.0 C/m^2 parallel to the axis. Find the electric field at a point on the axis of the cylinder at a distance 2.0 m from the cylinder.

11. The polarization field in a region is given by $\boldsymbol{P} = 0$ for $r < a$ and $\boldsymbol{P} = \frac{ke^{-r/a}}{r^2}\hat{\boldsymbol{r}}$ for $r > a$ in spherical polar coordinates. Find the bound charge contained in the region $r < 2a$. **[Ans.** $\frac{k}{2\pi}\left(\frac{1}{e}-\frac{1}{e^2}\right)$]

12. A dielectric sphere of susceptibility χ is placed in an otherwise uniform electric field $\boldsymbol{E}$. Assume that the polarization produced in the sphere is uniform. Find the electric field inside the sphere. Justify the assumption from your result. **[Ans.** $\frac{3\varepsilon_0\chi\boldsymbol{E}}{3+\chi}$]

❑

9 Displacement Field: Linear Dielectrics

9.1 Definition of Displacement Field

Given a charge distribution, you can, in principle, calculate the electric field $\boldsymbol{E}$ at any point. If the point is inside a dielectric material, you also have a polarization $\boldsymbol{P}$ (dipole moment per unit volume) at that point. You can then construct a vector quantity

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} \tag{9.1}$$

at each point. Even in empty space or in a conductor where there is no polarization, you can define $\boldsymbol{D}$. Here $\boldsymbol{P} = 0$ and so

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E}.$$

It turns out that this quantity $\boldsymbol{D}$ is intimately related to the free charge density. Also, it has a special significance when the fields vary with time. The vector $\boldsymbol{D}$ in Equation 9.1 is given a separate name, *displacement vector*. As $\boldsymbol{E}$ and $\boldsymbol{P}$ are both vector fields (they are functions of space points), so is $\boldsymbol{D}$. Hence it is also called displacement field.

Why displacement? Maybe because polarization itself is caused by the displacement of charged electronic clouds.

EXAMPLE 9.1 **A dielectric slab is placed in the space between the plates of a capacitor as shown in Figure 9.1. The charge densities on the plates are $+\sigma$ and $-\sigma$, and a uniform polarization $\boldsymbol{P} = P\hat{e}$ appears in the slab, where $\hat{e}$ is the unit vector in the direction perpendicular to the plates as shown. Find the displacement vector $\boldsymbol{D}$ at the points A_1, A_2 and A_3. Assume that the plates and the slab have large surfaces.**

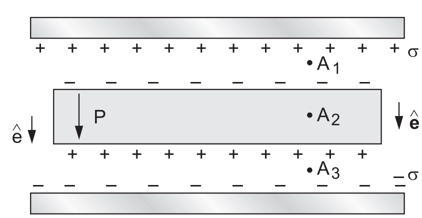


Fig. 9.1

Solution As the polarization in the slab is uniform, $\rho_b = -\nabla \cdot \boldsymbol{P} = 0$. Hence there is no bound charge in the volume of the slab.

On the top surface of the slab, a surface charge density

$$\sigma_{b_1} = \boldsymbol{P} \cdot \hat{\boldsymbol{n}} = (P\hat{\boldsymbol{e}}) \cdot (-\hat{\boldsymbol{e}}) = -P$$

will appear and on the bottom surface of the slab, a surface charge density

$$\sigma_{b_2} = \boldsymbol{P} \cdot \hat{\boldsymbol{n}} = (P\hat{\boldsymbol{e}}) \cdot (\hat{\boldsymbol{e}}) = P$$

will appear.

Point A_1

This point is between the upper plate of the capacitor and the slab. At this point, the

field due to the charge on the upper plate $= \dfrac{\sigma}{2\varepsilon_0}\hat{\boldsymbol{e}}$,

field due to the charge on the lower plate $= \dfrac{\sigma}{2\varepsilon_0}\hat{\boldsymbol{e}}$,

field due to the charge on the upper surface of the slab $= \dfrac{\sigma_{b_1}}{2\varepsilon_0}(-\hat{\boldsymbol{e}}) = \dfrac{+P}{2\varepsilon_0}\hat{\boldsymbol{e}}$,

field due to the charge on the lower surface of the slab $= \dfrac{\sigma_{b_2}}{2\varepsilon_0}(-\hat{\boldsymbol{e}}) = \dfrac{-P}{2\varepsilon_0}\hat{\boldsymbol{e}}$.

Adding all four contributions, the net field is $\boldsymbol{E} = \dfrac{\sigma}{\varepsilon_0}\hat{\boldsymbol{e}}$.

In the empty space, there is no material and hence no dipole moment. Thus $\boldsymbol{P} = 0$.

So, $\quad \boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} = \varepsilon_0 \left(\dfrac{\sigma}{\varepsilon_0}\right)\hat{\boldsymbol{e}} = \sigma\hat{\boldsymbol{e}}$.

Point A_2

Here, the

field due to the charge on the upper plate $= \dfrac{\sigma}{2\varepsilon_0}\hat{\boldsymbol{e}}$,

field due to the charge on the lower plate $= \dfrac{\sigma}{2\varepsilon_0}\hat{\boldsymbol{e}}$,

field due to the charge on the upper surface of the slab $= \dfrac{-P}{2\varepsilon_0}\hat{\boldsymbol{e}}$,

field due to the charge on the lower surface of the slab $= \dfrac{-P}{2\varepsilon_0}\hat{\boldsymbol{e}}$.

So, the net field is $\boldsymbol{E} = \left(\dfrac{\sigma}{\varepsilon_0} - \dfrac{P}{\varepsilon_0}\right)\hat{\boldsymbol{e}}$.

Thus $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} = (\sigma - P)\hat{\boldsymbol{e}} + \boldsymbol{P}\hat{\boldsymbol{e}} = \sigma\hat{\boldsymbol{e}}$.

Point A_3

As in the case of point A_1, at point A_3 also

$$\boldsymbol{D} = \sigma\hat{\boldsymbol{e}}.$$

Did you note the interesting behaviour of $\boldsymbol{D}$? Though the field $\boldsymbol{E}$ is different in the slab and outside it, the displacement $\boldsymbol{D}$ has the same value throughout.

9.2 Gauss's Law in Terms of *D*

Gauss's law is given by

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}, \tag{i}$$

where ρ is the net charge density at the point for which $\boldsymbol{E}$ and ρ have been written. It may come from free charges, bound charges or both, whichever are present at that point. Let the free charge density be ρ_f and the bound charge density be ρ_b. The net charge density is

$$\rho = \rho_b + \rho_f.$$

Gauss's law (i), therefore becomes

$$\nabla \cdot \boldsymbol{E} = \frac{\rho_b + \rho_f}{\varepsilon_0} = \frac{(-\nabla \cdot \boldsymbol{P}) + \rho_f}{\varepsilon_0}$$

or $$\varepsilon_0(\nabla \cdot \boldsymbol{E}) + (\nabla \cdot \boldsymbol{P}) = \rho_f$$

or $$\nabla \cdot (\varepsilon_0 \boldsymbol{E} + \boldsymbol{P}) = \rho_f \tag{9.2}$$

or $$\nabla \cdot \boldsymbol{D} = \rho_f.$$

Integrating over any closed surface enclosing a volume τ,

$$\int_\tau (\nabla \cdot \boldsymbol{D})\, d\tau = \int_\tau \rho_f\, d\tau.$$

Using the Gauss divergence theorem,

$$\int_\tau (\nabla \cdot \boldsymbol{D})\, d\tau = \oint_S \boldsymbol{D} \cdot \boldsymbol{da}.$$

Also $\int_\tau \rho_f\, d\tau$ is the total free charge Q_{free}, enclosed in the volume.

Thus, $$\oint D \cdot da = Q_{\text{free, encl.}}. \tag{9.3}$$

The flux of $\boldsymbol{D}$ through a closed surface is equal to the net "free charge" enclosed in the volume bounded by the closed surface.

Equations 9.2 and 9.3 give you Gauss's law in terms of free charges only. Even if bound charges are present, you ignore them while writing these equations. If symmetry permits, you can get $\boldsymbol{D}$ at a given point in a given problem using these equations. Please do not think that the equation $\nabla \cdot \boldsymbol{D} = \rho_f$ is valid only when a dielectric is present or that $\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$ is valid only in empty space. The two forms are equally valid in "all" situations. If you want to write the total charge density, use $\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$. If you wish to write only the free charge density, use $\nabla \cdot \boldsymbol{D} = \rho_f$.

EXAMPLE 9.2 **Consider the situation given in Example 9.1. In Figure 9.2, a Gaussian surface is shown by dotted lines. Using Gauss's law, find the displacement vector D at point A_2.**

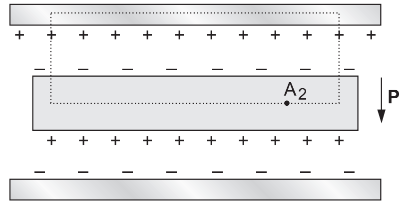


Fig. 9.2

Solution We will use Equation 9.3, that is, $\int_S \boldsymbol{D} \cdot d\boldsymbol{a} = Q_{\text{free, encl.}}$. First, look at the part of the Gaussian surface inside the upper plate of the capacitor. Inside the conducting plate, the field $\boldsymbol{E}$ is zero. Also, inside a conductor, you don't have any appreciable dipole moment as the outer electrons are free to move and separate out from the core, and the deformation of the core is quite small. Thus we take $\boldsymbol{P} = 0$ inside the conductor. So $\boldsymbol{D}$ is zero. Therefore, $\int \boldsymbol{D} \cdot d\boldsymbol{a}$ on this surface is zero. At the four surfaces perpendicular to the plates, $\int \boldsymbol{D} \cdot d\boldsymbol{a} = 0$ as $\boldsymbol{D}$ is perpendicular to $d\boldsymbol{a}$. So, there is a contribution only from the surface that passes through A_2. Let the area of this surface be A and the magnitude of the displacement be $\boldsymbol{D}$. The displacement vector here will be downwards as both $\boldsymbol{E}$ and $\boldsymbol{P}$ are downwards. Thus,

$$\boldsymbol{D} \cdot d\boldsymbol{a} = DA.$$

The free charge enclosed by the Gaussian surface is σA, where σ is the surface charge density on the upper plate of the capacitor. So, using Gauss's law,

$$\boldsymbol{D} \cdot d\boldsymbol{a} = Q_{\text{free, encl.}}$$

or $$DA = \sigma A$$

or $$D = \sigma.$$

Thus the displacement vector at A_2 is as obtained in Example 9.1.

EXAMPLE 9.3 **A point charge q is embedded at the centre of a sphere of radius R, made of a linear dielectric material of electric susceptibility χ_e. Find (a) the displacement vector, (b) the electric field, (c) the polarization vector and (d) the bound charge density everywhere.**

Solution (a) The only free charge is at the centre of the sphere and that is q.

From Gauss's law,

$$\int \boldsymbol{D} \cdot d\boldsymbol{a} = q.$$

From spherical symmetry,

$$D \times 4\pi r^2 = q$$

or $$D = \frac{q}{4\pi r^2}.$$

Thus $\quad \boldsymbol{D} = \dfrac{q}{4\pi r^2}\hat{\boldsymbol{r}}.$

This is true for all r.

(b) $\quad \boldsymbol{D} = \varepsilon_0(1+\chi_e)\boldsymbol{E}$

or $\quad \boldsymbol{E} = \dfrac{\boldsymbol{D}}{\varepsilon_0(1+\chi_e)}.$

For $r < R$,

$$\boldsymbol{E} = \frac{1}{\varepsilon_0(1+\chi_e)}\cdot\frac{q}{4\pi r^2}\hat{\boldsymbol{r}} = \frac{q}{4\pi\varepsilon_0(1+\chi_0)r^2}\hat{\boldsymbol{r}}.$$

For $r > R$,

$$\chi = 0.$$

So, $\quad \boldsymbol{E} = \dfrac{\boldsymbol{D}}{\varepsilon_0} = \dfrac{q}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}}.$

(c) $\boldsymbol{P} = \varepsilon_0\chi_e\boldsymbol{E}.$

For $r < R$,

$$\boldsymbol{P} = \varepsilon_0\chi_e\boldsymbol{E} = \frac{\chi_e q}{4\pi(1+\chi_e)r^2}\hat{\boldsymbol{r}}.$$

For $r > R$,

$$\boldsymbol{P} = 0.$$

(d) The bound surface charge density

$$\sigma_b = \boldsymbol{P}\cdot\hat{\boldsymbol{n}} = \frac{\chi_e q}{4\pi(1+\chi_e)R^2}.$$

The bound volume charge density

$$\rho_b = -\nabla\cdot\boldsymbol{P} = -\frac{\chi_e\varepsilon_0}{1+\chi_e}\nabla\cdot\left(\frac{q}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}}\right)$$

$$= -\frac{\chi_e\varepsilon_0}{1+\chi_e}\frac{q\delta^3(\boldsymbol{r})}{\varepsilon_0} = -\frac{\chi_e q}{1+\chi_e}\delta^3(\boldsymbol{r}).$$

Can there be an equation like Coulomb's law for *D*?

The bound charge does not occur in Gauss's law in terms of **D**. Even if there are bound charges, you ignore them while writing the right-hand side of Gauss's law $\nabla\cdot\boldsymbol{D} = \rho_f$ or $\int \boldsymbol{D}\cdot d\boldsymbol{a} = Q_{\text{free, encl.}}$. The structures of the two equations

$$\nabla\cdot\boldsymbol{D} = \rho_f$$

and $$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$

are identical and you may be tempted to think that $\boldsymbol{D}$ can be obtained from the free charge density ρ_f in the same way as $\boldsymbol{E}$ is obtained from the net charge density ρ. In particular, as $\boldsymbol{E}$ can be obtained from ρ by using Coulomb's law,

$$\boldsymbol{E} = \int \frac{\rho(\boldsymbol{r}-\boldsymbol{r}')}{4\pi\varepsilon_0\ |\boldsymbol{r}-\boldsymbol{r}'|^3} d\tau,$$

can you write a similar equation for $\boldsymbol{D}$,

$$\boldsymbol{D} = \int \frac{\rho_f(\boldsymbol{r}-\boldsymbol{r}')}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|^3} d\tau, \tag{i}$$

ignoring any bound charge? Consider the following example.

Suppose there is a dielectric rod AB and a free charge q is placed at position $\boldsymbol{r}'$ near the rod as shown in Figure 9.3. The charge produces an electric field, because of which the rod AB gets polarized. Negative charges appear on the rod close to q and positive charges appear on it farther away. The electric field at a point C with position vector $\boldsymbol{r}$ is

$$\boldsymbol{E} = \boldsymbol{E}_q + \boldsymbol{E}_p,$$

where $\boldsymbol{E}_q$ is the field at C due to the charge q and $\boldsymbol{E}_p$ is the field there due to the charges appearing on the rod because of polarization. The displacement vector at C is

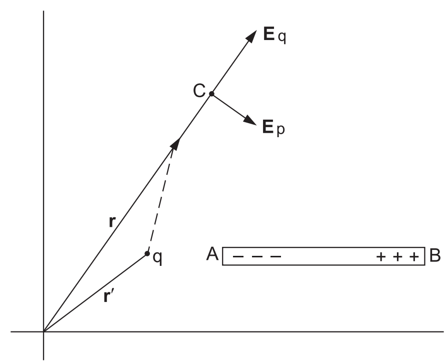


Fig. 9.3

$$\begin{aligned}\boldsymbol{D} &= \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} \\ &= \varepsilon_0(\boldsymbol{E}_q + \boldsymbol{E}_p) \qquad (\boldsymbol{P} \text{ at C is zero as there is no material here}) \\ &= \frac{q(\boldsymbol{r}-\boldsymbol{r}')}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|^3} + \varepsilon_0 \boldsymbol{E}_p. \end{aligned} \tag{ii}$$

If you ignore the bound charge and use (i) to calculate $\boldsymbol{D}$, you will get only the first term in (ii). Thus (i) is incorrect and you cannot obtain $\boldsymbol{D}$ from free charges using an expression similar to that for Coulomb's law.

But why is it incorrect? What is wrong with the argument that was used to write (i)? Indeed $\nabla \cdot \boldsymbol{D} = \rho_f$ and $\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$ have the same mathematical structure. Replace $\frac{\rho}{\varepsilon_0}$ by ρ_f and $\boldsymbol{E}$ by $\boldsymbol{D}$, and you get Gauss's law. Why does a similar process not give Coulomb's law for $\boldsymbol{D}$?

Though the divergence of $\boldsymbol{E}$ and that of $\boldsymbol{D}$ have a similar mathematical structure, their curls do not have any similarity. While $\nabla \times \boldsymbol{E} = 0$ for electrostatic fields, $\nabla \times \boldsymbol{D}$ need not be zero. That is why $\boldsymbol{E}$ and $\boldsymbol{D}$ cannot always be treated in the same manner.

So, if you need $\boldsymbol{D}$, use only Gauss's law for $\boldsymbol{D}$ and do not try a Coulomb-like equation. If symmetry permits you to evaluate $\int \boldsymbol{D} \cdot d\boldsymbol{a}$, get $\boldsymbol{D}$ from there. If not, find $\boldsymbol{E}$ and $\boldsymbol{P}$ separately and use $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$ to obtain $\boldsymbol{D}$.

9.3 Boundary Conditions on *D*

Consider a surface S separating two materials of different kinds. The materials may be two kinds of dielectrics, or one may be a dielectric and the other empty space, or one a metal and the other a dielectric, or there may be some other combinations. Call one side of S Side 1 and the other side, Side 2 (Figure 9.4). From any point A on the surface, draw a normal $\hat{n}$ towards Side 1.

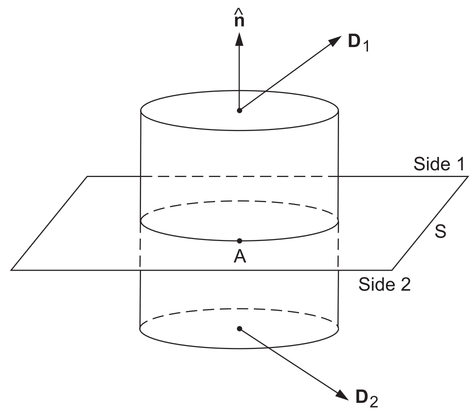


Fig. 9.4

Let $\boldsymbol{D}_1$ be the displacement vector at a point very close to point A, on Side 1, and $\boldsymbol{D}_2$ be the displacement vector at a point very close to point A, on Side 2. The components of $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ along $\hat{\boldsymbol{n}}$ are written as D_{1n} and D_{2n}.

Draw a Gaussian surface in the shape of a pillbox, as shown in the figure. The two flat surfaces are parallel to the surface S, one on Side 1 the other on Side 2. The area of each of these surfaces is Δa and the thickness of the box is vanishingly small. Thus $\int \boldsymbol{D} \cdot d\boldsymbol{a}$ over the curved surface of the pillbox tends to zero. So,

$$\begin{aligned} \boldsymbol{D} \cdot d\boldsymbol{a} &= \boldsymbol{D}_1 \cdot \Delta a\hat{\boldsymbol{n}} - \boldsymbol{D}_2 \cdot \Delta a\hat{\boldsymbol{n}} \\ &= (D_{1n} - D_{2n})\Delta a. \end{aligned}$$

If the free surface charge density at point A is σ_f, the free charge enclosed in the pillbox is $\sigma_f \Delta a$. According to Gauss's law,

$$\int \boldsymbol{D} \cdot d\boldsymbol{a} = q_{\text{free, encl.}}$$

or $$(D_{1n} - D_{2n})\Delta a = \sigma_f \Delta a$$

or $$D_{1n} - D_{2n} = \sigma_f. \tag{9.4}$$

This is almost a replica of the derivation of the boundary condition for the normal component of $\boldsymbol{E}$, that is, $E_{1n} - E_{2n} = \frac{\sigma}{\varepsilon_0}$, derived earlier. Never mind, it helps you revise the procedure.

No attempt will be made to derive the boundary condition for the tangential component as $\nabla \times \boldsymbol{D} \neq 0$ and hence the variation in this component is not related to the free or total charge in any simple manner.

EXAMPLE 9.4 **The displacement vector D_1 at a point close to a surface has a magnitude of 5 units and D_2, also close to the surface but on the other side, has a magnitude of 6 units, as shown in Figure 9.5. If the angle made by D_1 with the upward normal to the surface is 30°, what will be the angle made by D_2 with this normal? The surface has only bound charges.**

Solution As the surface does not have any free charge on it,

$$D_{1n} - D_{2n} = 0$$

or $$D_1 \cos 30^\circ - D_2 \cos\theta = 0,$$

where θ is the angle made by $\boldsymbol{D}_2$ with the upward normal.

Thus $$5 \times \frac{\sqrt{3}}{2} = 6\cos\theta$$

or $$\cos\theta = \frac{5\sqrt{3}}{12}$$

or $$\theta = \cos^{-1}\left(\frac{5\sqrt{3}}{12}\right).$$

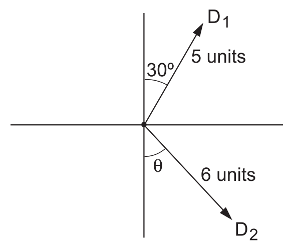


Fig. 9.5

9.4 Displacement Vector in a Linear Dielectric: Dielectric Constant

The polarization and the next electric field at a point in a linear dielectric are related to each other as

$$\boldsymbol{P} = \varepsilon_0 \chi E. \tag{9.5}$$

The displacement vector $\boldsymbol{D}$ in a linear dielectric with susceptibility χ is

$$\begin{aligned} \boldsymbol{D} &= \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} = \varepsilon_0 \boldsymbol{E} + \varepsilon_0 \chi \boldsymbol{E} \\ &= \varepsilon_0 (1+\chi)\boldsymbol{E} = \varepsilon_0 K\boldsymbol{E}. \end{aligned} \tag{9.6}$$

The constant $K = 1 + \chi$ is called the *dielectric constant* of the material. It is also called *relative permittivity* and when this name is used, it is represented by the symbol ε_r. Table 9.1 gives the dielectric constants of some linear dielectrics.

Table 9.1

Material	Dielectric constant
Air	1.00054
Diamond	5.7
Salt	5.9
Water	80.1

The dielectric constants listed here are for constant electric fields, as is the case in electrostatic situations. If you somehow apply an electric field that changes rapidly with time, the dielectric constant may be very different.

The dielectric constant of water is very large, which is a gift of nature. This helps in microwave cooking.

The constant

$$\varepsilon = \varepsilon_0 K \tag{9.7}$$

is called the permittivity of the material. Thus

$$\boldsymbol{D} = \varepsilon \boldsymbol{E} \tag{9.8}$$

for linear dielectrics.

EXAMPLE 9.5 **Show that bound charges appear on the surface of a linear dielectric unless free charges are present in its volume.**

Solution The bound volume charge density is

$$\begin{aligned}\rho_b &= -\nabla \cdot \boldsymbol{P}\\ &= -\nabla \cdot \varepsilon_0 \chi \boldsymbol{E} = -\varepsilon_0 \chi \left(\frac{\rho}{\varepsilon_0}\right)\\ &= -\chi(\rho_b + \rho_f)\end{aligned}$$

or $$\rho_b = -\frac{x\rho_f}{1+x}.$$

Thus if $\rho_f = 0$, ρ_b is also zero.

EXAMPLE 9.6 **A point charge *q* is placed at a distance *d* from a large, plane, linear dielectric slab of dielectric constant *K*. Calculate the induced surface charge density, close to the foot of the perpendicular from *q* to the slab.**

Solution Suppose the surface charge density at the foot of the perpendicular is σ_b. For a linear dielectric, there is no bound volume charge density unless there is a free charge density in it. The field just inside the dielectric below the foot of the perpendicular is

$$E = \frac{q}{4\pi\varepsilon_0 d^2} + \frac{\sigma_b}{2\varepsilon_0}$$

in the downward direction. The polarization at this point is

$$P = \varepsilon_0 \chi E = \varepsilon_0 (K-1) E$$

in the downward direction.

But $\sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{n}}$. Here $\hat{\boldsymbol{n}}$ is the unit vector in the upward direction. Hence,

$$\sigma_b = -\varepsilon_0 (K-1) \left[\frac{q}{4\pi\varepsilon_0 d^2} + \frac{\sigma_b}{2\varepsilon_0} \right]$$

or $$\sigma_b \left[1 + \frac{\varepsilon_0 (K-1)}{2\varepsilon_0} \right] = -\varepsilon_0 (K-1) \frac{q}{4\pi\varepsilon_0 d^2}$$

or $$\sigma_b = -\frac{1}{2\pi} \cdot \frac{K-1}{K+1} \cdot \frac{q}{d^2}.$$

9.5 Capacitor with a Dielectric Slab

Consider a parallel-plate capacitor with large plate areas. A dielectric slab is inserted in the space between the plates. Also suppose the dielectric is linear and its dielectric constant is K. The plates have charges $+Q$ and $-Q$, and charge densities σ and $-\sigma$.

Field in empty space between capacitor plates and dielectric slab

Let P_1 be a point between the upper plate of the capacitor and the slab (Figure 9.6). Draw a rectangular closed surface passing through P_1 as shown in the figure. Two of the six surfaces are parallel to the capacitor plates and four are perpendicular. One of the surfaces parallel to the capacitor plates passes through the capacitor plate and the other through P_1. The electric field is zero inside the capacitor plate and is perpendicular outside it. So only the face passing through P_1 will give nonzero flux of electric field. If A be the area of this face, the flux is

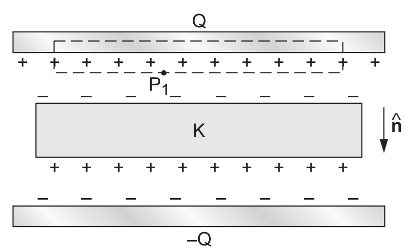


Fig. 9.6

$$\boldsymbol{E} \cdot \boldsymbol{da} = EA.$$

The total charge enclosed by this Gaussian surface is σA. Thus, by Gauss's law,

$$EA = \frac{\sigma A}{\varepsilon_0}$$

or $$E = \frac{\sigma}{\varepsilon_0}$$

or $$\mathrm{E} = \frac{\sigma}{\varepsilon_0}\hat{\boldsymbol{n}}, \tag{9.9}$$

where $\hat{\boldsymbol{n}}$ is the unit vector as shown in the figure. We had obtained this result in Example 9.1 too.

But this is also the expression for the electric field in a parallel-plate capacitor even without a dielectric slab. So, does the slab have no effect on the electric field in the empty space inside the capacitor? Yes and no. If you keep the charge density $\pm\sigma$ the same, inserting a dielectric slab will not affect the field, which will remain $(\sigma/\varepsilon_0)\hat{\boldsymbol{n}}$. This will be the case with an isolated charged capacitor. But if you connect a battery to the capacitor plates and then insert or remove the dielectric slab, the battery will supply charges to change σ and then the electric field.

You can work out the electric field in the empty space between the lower plate of the capacitor and the dielectric slab in a similar manner. This is also given by Equation 9.9.

Electric field inside the dielectric

The displacement vector is $\boldsymbol{D} = \sigma\hat{\boldsymbol{n}}$ everywhere inside the dielectric slab as well as in the space between the slab and the capacitor plates. This has been shown earlier. If the electric field inside the slab is $\boldsymbol{E}'$,

$$\boldsymbol{D} = \varepsilon_0 K \boldsymbol{E}'$$

or $$\boldsymbol{E}' = \frac{\boldsymbol{D}}{\varepsilon_0 K} = \frac{\sigma}{\varepsilon_0 K}\hat{\boldsymbol{n}}. \tag{9.10}$$

In the empty space between the slab and the plates, the electric field is $\boldsymbol{E} = \frac{\sigma}{\varepsilon_0}\hat{\boldsymbol{n}}$.

So, $$\boldsymbol{E}' = \frac{\boldsymbol{E}}{K}\cdot \tag{9.11}$$

Charge induced on the surface of the slab

Inside the dielectric slab,

$$\boldsymbol{D} = \sigma\hat{\boldsymbol{n}}$$

and $$\boldsymbol{E} = \frac{\sigma}{\varepsilon_0 K}\hat{\boldsymbol{n}}.$$

So $$\boldsymbol{P} = \boldsymbol{D} - \varepsilon_0\boldsymbol{E} = \sigma\hat{\boldsymbol{n}} - \frac{\sigma}{K}\hat{\boldsymbol{n}} = \sigma\left(1 - \frac{1}{K}\right)\hat{\boldsymbol{n}}.$$

The bound charge density on the upper surface of this slab is

$$\sigma_b = \boldsymbol{P}\cdot(-\hat{\boldsymbol{n}}) = -\sigma\left(1 - \frac{1}{K}\right)$$

and that on the lower surface is

$$\sigma_b = \boldsymbol{P}\cdot\hat{\boldsymbol{n}} = \sigma\left(1 - \frac{1}{K}\right)\cdot$$

Remember that in the equation $\sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{n}}$ which helps us calculate bound surface charge density on a material, the unit vector is taken in the direction of the normal going away from the material. That is why we used $(-\hat{\boldsymbol{n}})$ for the upper surface and $\hat{\boldsymbol{n}}$ for the lower surface with the direction of $\hat{\boldsymbol{n}}$ as taken in Figure 9.6.

The magnitude of the induced charge density is less than the charge density of the capacitor plates by a factor of $\left(1 - \frac{1}{K}\right)\cdot$

Change in capacitance due to the dielectric

Suppose the space between the plates of a capacitor is completely filled with a linear dielectric having dielectric constant K. The charges on the capacitor plates are $+Q$ and $-Q$. These are free charges. Let the plate area, assumed to be large, of each plate be A. Let the separation between the plates be d. The electric field in the dielectric is

$$\boldsymbol{E} = \frac{\sigma}{\varepsilon_0 K}\hat{\boldsymbol{n}} = \frac{Q}{A\varepsilon_0 K}\hat{\boldsymbol{n}}.$$

The potential difference between the plates is

$$V = Ed = \frac{Qd}{A\varepsilon_0 K}\cdot$$

Thus the capacitance is

$$C = \frac{Q}{V} = \frac{K\varepsilon_0 A}{d}\cdot$$

If there is no dielectric between the plates, the capacitance is $\frac{\varepsilon_0 A}{d}\cdot$ Thus by filling up the dielectric, the capacitance is increased by a factor K. This is true for capacitors of any shape.

Capacitors commonly used in circuits have dielectric material filled between metallic plates. Apart from increasing the capacitance, this also gives a stable structure to the capacitor. Otherwise, keeping the metallic plates at a very small separation would be difficult.

Electrostatic energy of a dielectric-filled capacitor

Look at Figure 9.7. The space between the capacitor plates is filled with a dielectric of dielectric constant K. Suppose the capacitor is uncharged to start with and charges in steps of dq are taken out from the lower plate and shifted to the upper plate. This way, a total charge Q is put on the upper plate. The charge on the lower plate then becomes $-Q$.

Fig. 9.7

Look at the step when the charges on the plates are q and $-q$. The potential difference between the plates at this time is

$$V = Ed = \frac{\sigma}{\varepsilon_0 K} d = \frac{qd}{A\varepsilon_0 K}\cdot$$

The work done in shifting the charge dq at this stage from the lower plate to the upper plate is

$$dW = (dq)\,V = \frac{qd}{A\varepsilon_0 K} dq.$$

The total work done in bringing charge Q from the lower plate to the upper plate is, therefore,

$$W = \int_0^Q \frac{qd}{A\varepsilon_0 K} dq = \frac{Q^2 d}{2\varepsilon_0 AK}\cdot$$

This work done equals the energy stored in the capacitor at the end. Thus the energy stored is $U = \frac{Q^2 d}{2\varepsilon_0 AK}\cdot$ You can express the energy in terms of the electric field. The electric field in the space between the capacitor plates (filled with the dielectric) is

$$E = \frac{Q}{\varepsilon_0 AK}\cdot$$

or $$Q = \varepsilon_0 EAK.$$

Thus $$U = \frac{1}{2}\varepsilon_0 KE^2 Ad.$$

The energy per unit volume is

$$\frac{U}{Ad} = \frac{1}{2}\varepsilon_0 KE^2. \tag{9.12}$$

In empty space, the energy per unit volume of the electric field is $\frac{1}{2}\varepsilon_0 E^2$. So in the presence of a dielectric, the energy density is increased by the factor K.

The electrostatic energy corresponding to a charge distribution is indeed given by $\int \frac{1}{2}\varepsilon_0 E^2 d\tau$ over all space. Here the energy of the capacitor is more than this. This is because it also includes the energy of the molecular dipoles, which have been oriented along a particular direction.

9.6 Force on Dielectric Partially Inserted in Capacitor

In many simple applications using parallel-plate capacitors, we assume large plate areas and take the electric field to be perpendicular to the plates in the space between the plates. This is a good approximation as long as you are well within the capacitor. Near the edges of the plates, however, the electric field has to change direction and it only gradually falls to negligible values as you go outside. This is known as fringing of the field, which is schematically shown in Figure 9.8. Remember that $\nabla \times \boldsymbol{E} = 0$ and hence the tangential component has to be continuous across any surface. An abrupt change in the field from a finite value to zero as you move from inside to outside, does not satisfy the curl equation $\nabla \times \boldsymbol{E} = 0$. One appreciable effect of fringing could be that a dielectric slab partially inserted in a capacitor experiences a force pulling it inwards.

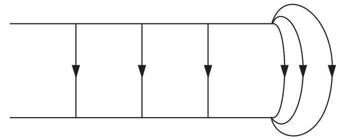


Fig. 9.8

EXAMPLE 9.7 **Figure 9.9 shows a parallel-plate capacitor with plate of width b and length l. The separation between the plates is d. The plates are rigidly clamped and connected to a battery of emf V. A dielectric slab of thickness d and dielectric constant K is slowly inserted between the plates.**

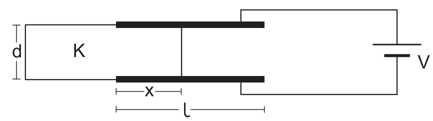


Fig. 9.9

(a) Calculate the energy of the system when a length x of the slab is introduced into the capacitor.

(b) What force should be applied on the slab to ensure that it goes slowly into the capacitor? Neglect any effect of friction or gravity.

Solution (a) Consider the part of the capacitor with the dielectric slab. The plate area of this part is bx. Its capacitance is

$$C_1 = \frac{K\varepsilon_0 bx}{d}.$$

Similarly, the capacitance of the part without the dielectric is

$$C_2 = \frac{\varepsilon_0 b(l-x)}{d}.$$

These two parts are connected in parallel. The capacitance of the system is, therefore,

$$C = C_1 + C_2 = \frac{\varepsilon_0 b}{d}[l + x(K-1)]. \qquad \text{(i)}$$

The energy of the capacitor is

$$U = \frac{1}{2}CV^2 = \frac{\varepsilon_0 bV^2}{2d}[l + x(K-1)].$$

(b) Suppose the electric field attracts the dielectric slab with a force F. An external force of equal magnitude F should be applied in the opposite direction so that the plate moves slowly (no acceleration).

Consider the part of the motion in which the dielectric moves a distance dx further inside the capacitor. The capacitance increases to $C + dC$. As the potential difference remains constant at V, the battery has to supply a further charge

$$dQ = (dC)\,V$$

to the capacitor. The work done by the battery is, therefore,

$$dW_b = VdQ = (dC)\,V^2.$$

The work done by the external force F during the displacement is

$$dW_e = (-Fdx).$$

The total work done on the capacitor is

$$dW_b + dW_e = (dC)\,V^2 - Fdx.$$

This should be equal to the increase dU in the stored energy. Thus,

$$\frac{1}{2}(dC)\,V^2 = (dC)\,V^2 - Fdx$$

or $$F = \frac{1}{2}V^2\frac{dC}{dx}.$$

Using equation (i),

$$F = \frac{\varepsilon_0 bV^2(K-1)}{2d}.$$

Thus, the electric field attracts the dielectric into the capacitor with a force $\dfrac{\varepsilon_0 bV^2(K-1)}{2d}$ and an equal force should be applied in the opposite direction to insert the dielectric slab slowly into the capacitor.

9.7 Ferroelectric Materials

We have talked about linear dielectrics, in which the polarization $\boldsymbol{P}$ is proportional to the net field $\boldsymbol{E}$ inside the material. This field consists of an applied electric field and also the field due to polarized material. In order to have a polarization $\boldsymbol{P}$ in a linear dielectric, you must apply an external field.

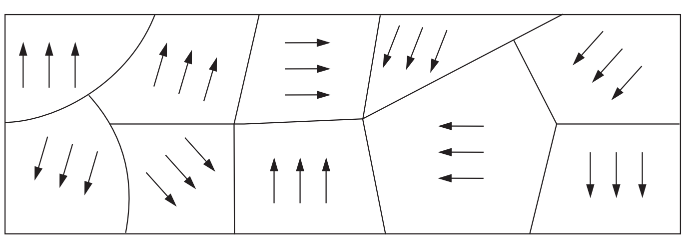


Fig. 9.10

As stated in the previous chapter, not all dielectric materials are linear. There is an important class of materials which can exhibit polarization even if there is no applied electric field. Polarization can be 'frozen' in such materials. Barium titanate ($BaTiO_3$) is a material representative of this class. It can be made in powder form or in thin-film form in a particular manner. Below 130 °C, barium, titanium and oxygen atoms are arranged in a crystalline tetragonal structure. There are regions in which thousands of $BaTiO_3$ molecules, arranged in a crystalline, tetragonal structure, are so aligned that their electric dipole moments point in the same direction. So you get electric domains with large electric dipole moments (Figure 9.10). Domain size varies with process of preparation but could typically be of the order of $1\,\mu m\,(10^{-6}$ m) or so. In any sizeable material, say 1 mg (milligram), there are a large number of such domains but these are randomly oriented and hence the observable polarization is still zero. But a small electric field can orient the domains and can, therefore, give rise to a very large polarization. The susceptibility χ or dielectric constant K becomes of the order of thousands in such materials. Once you have aligned the domains, you can remove the applied electric field and the alignment will not be destroyed totally. So you get a polarization with zero applied field. Such a material is called a *ferroelectric material*.

The polarization in a ferroelectric material depends on the history of application of electric field. Suppose you put a ferroelectric slab in an electric field in the "upward" direction. The domains will get aligned in the upward direction [Figure 9.11(a)]. Now remove the applied field. The domains will remain aligned to some extent and the polarization will be in the upward direction [Figure 9.11(b)]. Now apply an electric field in the "downward" direction. The domains get aligned in the downward direction [Figure 9.11(c)]. Remove the applied field and the slab remains polarized in the downward direction [Figure 9.11(d)].

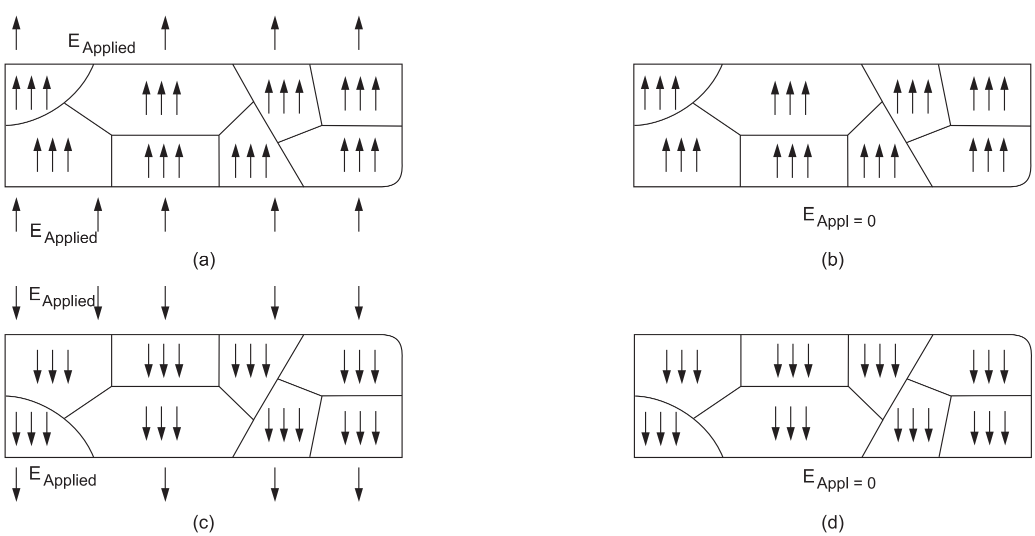


Fig. 9.11

So with zero applied field, the polarization can be in the upward or downward direction, depending on the previous application of the field. Such properties make ferroelectric materials a suitable class for computer memory applications. You can arrange for an array of ferroelectric particles and also for the application of electric fields in the upward and downward directions at specific locations of the particles. Call polarization in one direction 0 and that in the opposite direction 1. So by applying electric fields in proper directions, you can write any information in terms of 0 and 1 in this array of particles. The memory would be nonvolatile, because even if the power is switched off and the fields go to zero, the polarizations in the particles will retain their directions.

Concepts at a Glance

1. $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$
2. $\nabla \cdot \boldsymbol{D} = P_{\text{free}}, \oint \boldsymbol{da} = q_{\text{free, enclosed}}$
3. $D_{1n} - D_{2n} = \sigma_f$
4. For a linear dielectric, $\boldsymbol{D} = K\varepsilon_0 \boldsymbol{E}$, $\boldsymbol{P} = \varepsilon_0 \chi \boldsymbol{E}$, $\chi = K - 1$.
5. The dielectric constant of water for a constant electric field is very high, 80.
6. The electrostatic energy density in a dielectric (including the dipole–dipole interactions) is $u = \frac{1}{2} K\varepsilon_0 E^2$.
7. In ferroelectric materials like $BaTiO_3$, electric dipoles make domains and there can be polarization even without an applied electric field.

EXERCISES

Based on Concepts

1. We define potential corresponding to the electrostatic field E. Can we define potential due to the displacement field D? Justify your answer.
2. In vacuum, **D** and **E** must be parallel. True or false?
3. What is $\nabla \times (\boldsymbol{D} - \boldsymbol{P})$ in an electrostatic situation?
4. In a linear dielectric material, the magnitude of the displacement vector $\boldsymbol{D}$ at any point is always larger than that of the polarization vector. True or false?
5. Which of the fields ***D***, ***E***, ***P*** must necessarily change as you come out of a linear dielectric placed in an electric field?

6. The space inside a thin, spherical, conducting shell of radius R is filled with a dielectric material of dielectric constant K. The shell is given a charge Q. What is the field inside the shell? What is the field outside the conducting shell?

Problems

1. The space between the plates of a parallel-plate capacitor is completely filled with two slabs of linear dielectric materials (Figure 9E.1). Each slab has thickness a so the total distance between the plates is $2a$. Slab 1 has a dielectric constant 2 and Slab 2 has a dielectric constant 1.5. The free charge density on the top plate (on the inner side of Slab 1) is σ and that on the bottom plate is $-\sigma$.

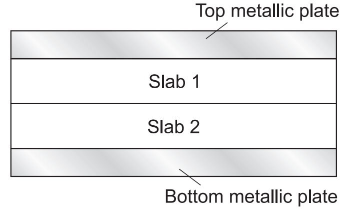


Fig. 9E.1

   (a) Find $\boldsymbol{D}$, $\boldsymbol{P}$ and $\boldsymbol{E}$ in each slab.
   (b) Find the potential difference between the plates.
   (c) Find the location and amount of all the bound charges.
   (d) From the charges, calculate the field $\boldsymbol{E}$ in each slab and see that you get back the values found in part (a). [**Ans.** (a) $\boldsymbol{D} = \sigma\hat{\boldsymbol{n}}$ everywhere; $\boldsymbol{E} = -\dfrac{\sigma}{2\varepsilon_0}\hat{\boldsymbol{n}}$ in Slab 1 and $\dfrac{2\sigma}{3\varepsilon_0}\hat{\boldsymbol{n}}$ in Slab 2; $\boldsymbol{P} = \dfrac{\sigma}{2}\hat{\boldsymbol{n}}$ in Slab 1 and $\dfrac{\sigma}{3}\hat{\boldsymbol{n}}$ in Slab 2, $\hat{\boldsymbol{n}}$ is the unit vector in the downward direction.]

2. A parallel-plate capacitor is made from plates of area A separated by a distance d. Half the space between the plates is filled with a dielectric material of dielectric constant K as shown in Figure 9E.2. If the capacitor is maintained at a potential difference V, find

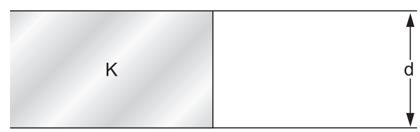


Fig. 9E.2

   (a) $\boldsymbol{D}$ and $\boldsymbol{E}$ in each half of the capacitor,
   (b) the bound charge density appearing on the dielectric, and
   (c) the electrostatic energy.

[**Ans.** (a) $\boldsymbol{D} = \left(\dfrac{\varepsilon_0 KV}{d}\right)\hat{\boldsymbol{n}}$ in the dielectric; $\varepsilon_0 \dfrac{V}{d}\hat{\boldsymbol{n}}$ in the other half; (b) $\dfrac{\varepsilon_0 V}{d}(K-1)$; (c) $\dfrac{\varepsilon_0 V^2 A}{4d}(K-1)$]

3. Consider the situation shown in Figure 9E.3. All the surfaces of the metal plates and the dielectric slabs are large. $+\sigma$ and $-\sigma$ are the surface charge densities on the plates. Show separately plots of D, E and P as functions of z.

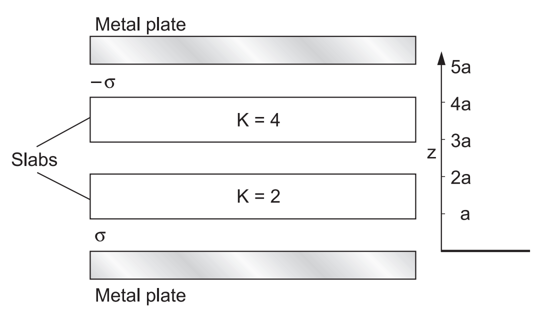


Fig. 9E.3

4. A water molecule has a dipole moment of 6.0×10^{-30} C m. Suppose the molecules have their dipoles aligned parallel to each other in a drop (radius 0.2 mm) of water. Find the maximum electric field just outside the surface of the drop. [**Ans.** 1.5×10^{10} V/m]

5. A positive point charge q is placed at the centre of a spherical shell made of a material with dielectric constant 1.5 and inner and outer radii a and b. Find the fields $\boldsymbol{D}$, $\boldsymbol{P}$ and $\boldsymbol{E}$ everywhere.

[**Ans.** $D = D_0 = \dfrac{q}{4\pi r^2}$ everywhere, $E = \dfrac{D_0}{\varepsilon_0}$ for $r < a$ and $r > b$, $\dfrac{D_0}{\varepsilon_0 K}$ for $a < r < b$, $P = \dfrac{D_0(K-1)}{K}$ for $a < r < b$, 0 otherwise]

6. A long conducting cylinder of radius R carries a uniform surface charge density σ. It is surrounded by a coaxial dielectric shell of inner and outer radii a and b, the dielectric constant of the shell being K. Find (a) $\boldsymbol{D}$ in the dielectric, (b) the electric field everywhere, and (c) the bound charge densities appearing on the shell. [**Ans.** (c) $-\dfrac{(K-1)R\sigma}{Ka}$ on the inner surface and $\dfrac{(K-1)R\sigma}{Kb}$ on the outer surface]

7. A long line charge with linear charge density λ is embedded along the axis of a long, solid, cylindrical dielectric of dielectric constant K. Find the $\boldsymbol{D}$ and $\boldsymbol{P}$ vectors outside and inside the dielectric.

[**Ans.** $\boldsymbol{P}$ inside the dielectric $= \dfrac{\lambda}{2\pi r}\left(1 - \dfrac{1}{K}\right)\hat{s}$]

8. A coaxial, cylindrical capacitor has inner and outer radii a and $4a$. The space between the surfaces is filled with two coaxial, cylindrical layers of dielectric materials. One of the layers has a dielectric constant of 1.5 and occupies the space $a < s < 2a$. The other layer, having a dielectric constant of 3.0, occupies the space $2a < s < 4a$. Find the capacitance of the capacitor per unit length. [**Ans.** $\dfrac{2\pi\varepsilon_0}{\ln 2}$]

9. An infinite circular cylinder of dielectric constant K is placed in a vacuum that has a transverse uniform electric field $\boldsymbol{E}_0$. Assuming the polarization to be uniform, find the field inside the dielectric. [**Ans.** $\dfrac{2E_0}{K+1}$]

10. A spherical conductor of radius a, carries a charge Q. It is surrounded by a linear dielectric material of susceptibility χ_e, up to a radius b. Find the energy of this configuration. [**Ans.** $\dfrac{Q^2[b + a(K^2 - 1)]}{8\pi\varepsilon_0 K^2 ab}$]

11. A cylinder of length $2L$ and radius L is placed with its centre at the origin and the axis along the z-axis. It is uniformly polarized with polarization $P_0\hat{\boldsymbol{k}}$. Let ε be a very small length. (a) Sketch the $\boldsymbol{D}$ field lines. (b) Find the value of $\boldsymbol{D}$ at $(0, 0, L + \varepsilon)$. (c) What is the value of $\boldsymbol{D}$ at $(0, 0, L - \varepsilon)$? (d) Find the value of $\boldsymbol{D}$ at the origin. [**Ans.** (b) $\dfrac{P}{\sqrt{5}}\hat{\boldsymbol{k}}$ (c) $\dfrac{P}{\sqrt{5}}\hat{\boldsymbol{k}}$ (d) $\dfrac{P}{\sqrt{2}}\hat{\boldsymbol{k}}$]

12. A long, thin rod of dielectric constant K is placed in an otherwise uniform external electric field $\boldsymbol{E}_0$ along the axis of the rod. Find the $\boldsymbol{D}$ vector inside the rod, away from the ends. [**Ans.** $\varepsilon_0 K\boldsymbol{E}_0$]

13. A thin disk of radius $\boldsymbol{R}$, thickness $d\,(<< R)$ and dielectric constant K is placed in an otherwise uniform electric field $\boldsymbol{E}_0$ along the axis of the disk. Find the $\boldsymbol{D}$ vector inside the disk, away from the edges.

[**Ans.** $\varepsilon_0\boldsymbol{E}_0$]

14. A metallic sphere of radius a is placed concentrically within a spherical, metallic shell of inner and outer radii b and c respectively. The space between the two is filled with a dielectric material of dielectric constant K. The inner sphere is given a charge q. Find $\boldsymbol{E}$, $\boldsymbol{P}$ and $\boldsymbol{D}$ fields for each of the regions $r < a$, $a < r < b$, $b < r < c$ and $r > c$. [**Hint:** $\boldsymbol{D}$ is the same for all the regions where $r > a$]

15. A dielectric sphere of radius R and dielectric constant K has a concentric cavity of radius a . A particle of charge q is placed at the centre of the cavity. Find all charge densities appearing in the sphere.

[**Ans.** $-\dfrac{q(K-1)}{4\pi Ka^2}$ at the inner surface]

16. A large piece of dielectric of irregular shape is placed in an external electric field. The resultant electric field inside the dielectric is $\boldsymbol{E}_0$ everywhere and the polarization is $\boldsymbol{P}$, so that the displacement is $\boldsymbol{D}_0 = \varepsilon_0 E_0 + \boldsymbol{P}$. Assume that the polarization remains uniform even when cavities are carved out in the dielectric. A small, spherical cavity is carved out of the material. Find the field in the cavity in terms of $\boldsymbol{E}_0$ and $\boldsymbol{P}$. Also find the displacement in the cavity in terms of $\boldsymbol{D}_0$ and $\boldsymbol{P}$. [**Ans.** $\boldsymbol{E}_0 - \dfrac{\boldsymbol{P}}{3\varepsilon_0}$, $\boldsymbol{D}_0 - \dfrac{2}{3}\boldsymbol{P}$]

17. A tank is filled with distilled water, which has a dielectric constant of 80. A nearly uniform electric field is set up in the air above the water. Taking the x–y plane on the water surface, and the z-axis pointing vertically upwards, the field above the water is $E = (100\hat{\boldsymbol{i}} - 100\hat{\boldsymbol{k}})$ V/m. Find the polarization of the water just below the surface. What is the bound charge density on the water surface?

[**Ans.** $\sigma_b = \varepsilon_0(99 \text{ V/m})$]

18. A cylindrical capacitor of inner and outer radii a and b is kept vertically in a liquid of dielectric constant K and density ρ. The height of the capacitor above the liquid surface is H. The capacitor plates are connected to a battery which maintains a potential difference between them at V. (a) Suppose, at a certain instant, the liquid is up to a height h in the capacitor. Find the electrostatic field energy stored in the capacitor. (b) Suppose the height is further increased by dh. Show that the work done by the battery is twice the increase in electrical energy. (c) Using the principle of conservation of energy, show that the electrical force on the liquid is given by $F = \dfrac{dV}{dh}$ and find its value. (d) Work out the value of h at equilibrium. [**Ans.** (a) $\dfrac{\pi\varepsilon_0 V^2}{\ln(b/a)}[(K-1)h + H]$ (c) $\dfrac{\pi\varepsilon_0 V^2(K-1)}{\ln(b/a)}$ (d) $\dfrac{\varepsilon_0(K-1)V^2}{\rho g(b^2-a^2)\ln(b/a)}$]

19. Show that the electric field lines "refract" at the interface between two dielectrics obeying $\dfrac{\tan\theta_2}{\tan\theta_1} = \dfrac{K_2}{K_1}$, where θ denotes the angle of the electric field with the normal and K denotes the dielectric constant.

20. Suppose the region $z < 0$ is filled with a dielectric of dielectric constant 4 and the region $z > 0$ is filled with a dielectric of dielectric constant 3. Both regions have uniform electric fields. The field in $z > 0$ is given by $\boldsymbol{E}_1 = (5\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}})$ V/mm. Find
    (a) the electric field in the region $z < 0$,
    (b) the angles made by the fields in the two regions with the interface,
    (c) the energy density in both regions,
    (d) the energy within a cube of side 2 cm centred on (3 cm, 4 cm, –5 cm), and
    (e) the surface charge density appearing at the interface.

[**Ans.** (a) $\left(\dfrac{20}{3}\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}\right)$ V/mm, (d) $\left(-\dfrac{5}{3}\text{ V/mm}\right)\varepsilon_0$]

21. Consider a parallel-plate capacitor with charge Q, detached from the voltage source. The plates have length l and width b, and are separated by a distance d. A dielectric slab of thickness d and width b is partially inserted in the capacitor with a length x inside the capacitor. Find the force exerted by the capacitor on the slab. [**Ans.** $\dfrac{Q^2 d}{2\varepsilon_0 b}\dfrac{K-1}{[l + x(K-1)]^2}$]

❑

10 Electric Currents

10.1 Introduction

An electric bulb glows when there is a current in the filament, a ceiling fan rotates when there is a current in its coils, your body parts receive signals from the brain when there is a current in the neurons, and so on. What is electric current?

As you know, an electric current results from the flow of charges. We generally talk about electric current through a surface. If charges cross a surface from one side to the other, we say there is a current through the surface. The direction of the current is the same as that of the flow of the charges if the charges are positive [Figure 10.1(a)], and is opposite to the direction of flow if they are negative [Figure 10.1(b)]. If both positive and negative charges cross the surface simultaneously, you have to add contributions from each, carefully looking at the corresponding directions [Figure 10.1(c)].

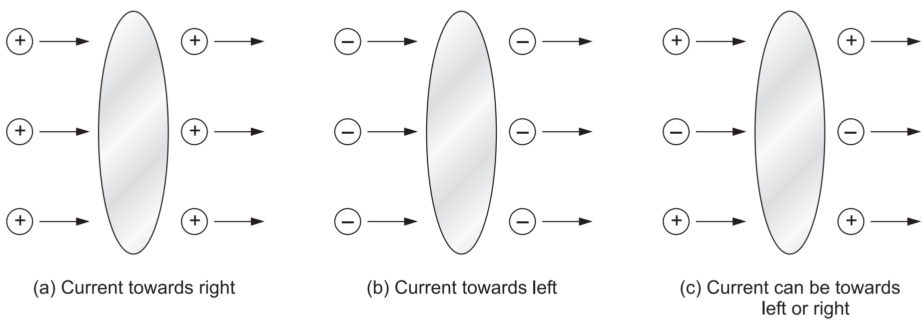


Fig. 10.1

If a net charge Δq flows through a surface in time Δt, the average current through this surface during this time is

$$i_{\text{av}} = \frac{\Delta q}{\Delta t} \cdot$$

If Δt is very small, the average current tends to the instantaneous current at that time.

$$i = \frac{dq}{dt} \cdot \qquad (10.1)$$

This may be seen as a charge dq crossing the surface in a small time interval dt, where dq is a small amount of charge (still much more than the charge on an electron). The SI unit of electric current is ampere, the symbol of which is A.

10.2 Examples of Electric Current

Electric current is produced whenever you have charges in motion. Here are some situations where we deliberately put charges in motion to produce current.

A beam of charged particles

In the good old days, the most important component of the TV set was its picture tube. Electrons were ejected from the cathode at one end of this tube. These electrons travelled as a beam and fell on the screen at a particular point where a reaction caused brightness. Along the path from the cathode to the screen, forces were exerted on the electrons that deviated the beam in such a way that the beam reached the desired location on the screen at the desired time. These forces were controlled by the TV signal received by the circuit in the TV set and hence you got the picture according to the way these signals were sent.

The electron beam going from the cathode to the screen constituted an electric current in the opposite direction. The modern generation of TVs like LED and plasma TVs use a very different technology.

In an ion-beam accelerator, ions are produced at one place and are moved in an evacuated tube. Forces are applied during this motion to accelerate them to high kinetic energies. These accelerated ions are made to fall on materials where different kinds of experiments are performed or fabrications are done. The motion of charged ions in space constitutes the current. Quite often, the current due to the motion of charged particles in empty space is called convection current.

Current in an electrolyte

When you dissolve common salt in water, first the crystalline structure breaks and then the molecules too break up into the ions Na^+ and Cl^-. These ions are surrounded by water molecules. Similarly, if you put some sulphuric acid in water, you get H^+ and SO_4^- ions in the solution. In a torch cell, there are chemicals in the form of a paste in which positive and negative ions exist. A material that decomposes to give positive and negative ions is called an electrolyte.

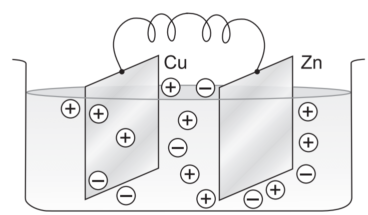


Fig. 10.2

Quite often, ions move in a solution or a paste in a systematic manner to produce electric current. Take a dilute acidic solution in a vessel, put a copper plate and a zinc plate in it, and connect the plates by a copper wire (Figure 10.2). Current starts flowing in the solution (as well as in the wire and the plates). Positive ions move towards the copper plate and negative ions towards the zinc plate. Each of these motions results in a current in the solution, between the two plates, from the zinc plate to the copper plate.

Current in a metal

This is the most common mode of electric current, which all of us are familiar with. As you know, a metal contains a large number of free electrons, which are not firmly attached to any of the atoms. If an electric field is applied, these electrons easily move in the direction opposite to the field. If you put a metallic block in an electric field, some of the electrons are accumulated on the surface in such a way that the electric field inside the block becomes zero everywhere. This is what we had discussed in the previous chapters. But suppose you apply an electric field in the metallic block and at the same time put up a mechanism by which the accumulated electrons are continuously removed from the block and fresh electrons are injected at those places from where the free electrons had been removed in the first place. This field will remain there. The free electrons will move due to it but this will not cause a surface charge (because of the special mechanism of removing and injecting charges) to nullify the field inside. In such cases, you have an electric field in the metallic block and electric current continues, for any length of time.

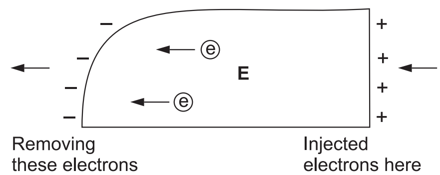


Fig. 10.3

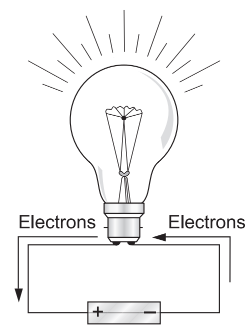


Fig. 10.4

Is it very difficult to design a mechanism to remove and inject electrons to keep the electric field inside a metallic block? Not really. When you connect a torch cell to a torch bulb, you do precisely this. The cell serves both purposes—to create an electric field in the filament (a conductor) of the bulb, and to maintain this field by removing the electrons from the filament at one end and injecting them into it at the other end.

10.3 Linear, Surface and Volume Current

(a) Linear current

If charges are initially arranged on a line and are made to move along it, the resulting current is called linear current or filamentary current. The free electrons in a wire, for example, are all situated in it. If you neglect its thickness, the wire is like a line. When the electrons move along this line (that is, along the wire), there is a linear current. Take a point P on the line and see how much charge crosses it per unit time (Figure 10.5).

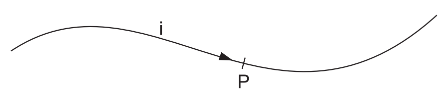


Fig. 10.5

The current is

$$I = \frac{dq}{dt}.$$

If $\boldsymbol{dl}$ is a small element along the line, in the direction of the current, $I\boldsymbol{dl}$ is called the *current element* at $\boldsymbol{dl}$.

Suppose charges are situated along a line and the linear charge density at a point P is λ. Also suppose the speed of the charges at P is v (Figure10.6). What will be the current at P? How much charge will cross P in a given time dt? Consider a point A, a distance vdt behind P. Whatever charge is contained in the length AP at a time t will cross through P in the next time interval dt. This is because each charge moves a distance vdt in time dt. The charge contained in AP at any given time is

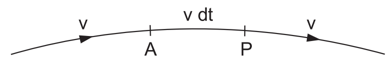


Fig. 10.6

$$dq = \lambda v dt.$$

Hence the current is

$$I = \frac{dq}{dt}$$

or $$I = \lambda v. \tag{10.2}$$

The direction of the current is the same as that of the velocity of the charges if λ is positive and opposite to it if λ is negative.

(b) Surface current

Current need not be confined to a line—it may be spread over a surface. Consider a thin metallic sheet. There are free electrons everywhere in it. Neglect the thickness of the sheet and treat it as just a surface. You can then say that the charges reside on this surface. If these charges start moving systematically on the surface, there is an electric current throughout the surface. Such a current is called *surface current*.

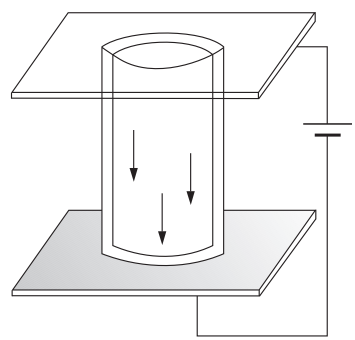


Fig. 10.7

To take another example, suppose you have a long, thin metallic tube and create a potential difference between the ends (Figure 10.7). Current will flow everywhere on its surface in a direction parallel to the axis of the tube. This is a surface current.

Surface current is described in terms of a vector quantity called *surface current density*, defined at each point of the surface. The usual notation for surface current density is $\boldsymbol{K}$. How does one define the magnitude and direction of $\boldsymbol{K}$?

Suppose P is a point on a surface on which there is a surface current (Figure 10.8). Suppose positive charges move on this surface, constituting the current. The direction of $\boldsymbol{K}$ at P is the same as that of the motion of the charges at this point. If the charges are negative, the direction of $\boldsymbol{K}$ is opposite.

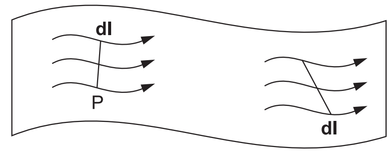


Fig. 10.8

To get the magnitude of $\boldsymbol{K}$, draw a small line dl at P on the surface, perpendicular to $\boldsymbol{K}$. Because of the current, charges will cross this line and will go from one side of dl to the other. The charges crossing dl per unit time make up the current dI through this length. The quantity

$$\boldsymbol{K} = \frac{dI}{dl}$$

is the magnitude of $\boldsymbol{K}$ at P. If you take a dl which is not perpendicular to $\boldsymbol{K}$ (Figure 10.8), the relation between $\boldsymbol{K}$ and dI will be

$$dI = |\boldsymbol{K} \times \boldsymbol{dl}|.$$

Take a small area da on the surface carrying the surface current. Let the surface current density at the location of this small area be $\boldsymbol{K}$. Then $\boldsymbol{K}\,da$ is called the *current element* in this case.

Suppose a surface carries a surface current caused by the motion of positive charges on it. Also suppose, at a point P on the surface, the surface charge density is σ and the charges are moving with a velocity v. What is the value of the surface current density here?

Draw a small line AB of length dl through P, perpendicular to the direction of motion of the charges (Figure 10.9). Draw a parallel line CD behind AB at a distance vdt. Each charge in this location moves a distance vdt in time dt. So all the charge contained in the rectangle ABCD at time t will cross AB in the next interval dt. But the charge contained in ABCD is $\sigma(dl)(vdt)$.

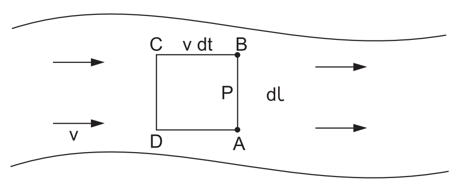


Fig. 10.9

So the current through dl is

$$dI = \sigma v dl$$

and the current density is

$$K = \frac{dI}{dl} = \sigma v.$$

As the direction of the current density is the same as that of the velocity (assuming σ to be positive),

$$\boldsymbol{K} = \sigma \boldsymbol{v}. \tag{10.3}$$

Check that this relation is valid even if σ is negative. The equation then tells us that $\boldsymbol{K}$ is opposite to $\boldsymbol{v}$.

(c) Volume current

Now suppose charges are not confined to a line or to a surface but are distributed in a volume. If these charges start moving in a systematic manner, there will be current in the volume. Such a current is called *volume current*.

To describe volume current quantitatively, we have to first define volume current density $\boldsymbol{J}$ at each point within the volume. Assuming the moving charges to be positive, the direction of $\boldsymbol{J}$ at a point is the same as the direction of motion of the charges there. If the moving charges are negative, the direction of $\boldsymbol{J}$ is opposite to that of the motion of the charges. To get the magnitude of the volume current density at a point P, draw an area da through this point, perpendicular to the direction of the current. If the current through this area is dI, the magnitude of the volume current density is

$$J = \frac{dI}{da}$$

or $$dI = J\,da.$$

If you take an area which is not perpendicular to the current, the current through the area is given by $dI = J da \cos\theta$, where θ is the angle between the current and the normal to the area.

Thus $$DI = \boldsymbol{J} \cdot \boldsymbol{da}.$$

In fact, all currents are volume currents. The charge has to reside on a material and all materials have a volume. Even if you think of a perfectly linear or surface current, it can be written in terms of volume current density utilizing Dirac delta functions. For example, a linear current I along the x-axis may be expressed as

$$\boldsymbol{J} = I\delta(y)\,\delta(z)\,\hat{\boldsymbol{i}}$$

and a surface current $K\hat{\boldsymbol{i}}$ on the x–y plane as

$$\boldsymbol{J} = K\delta(z)\,\hat{\boldsymbol{i}}.$$

Suppose the charges are contained in a volume with volume charge density ρ and move with velocity $\boldsymbol{v}$. Consider an area ΔA at a particular point P, perpendicular to the direction of motion of the charges (Figure 10.10). Take another area ΔA, a distance vdt behind the first area. Join the corresponding points to make a cylinder of volume $(\Delta A)\,vdt$.

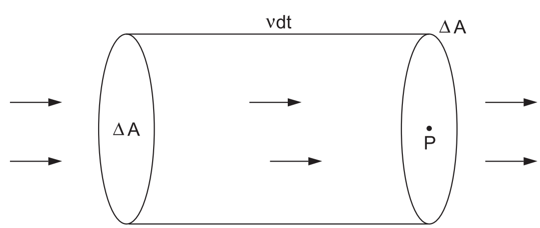


Fig. 10.10

Each charge moves a distance vdt in time dt. So all the charge contained in this cylinder at time t will cross the area ΔA in the next interval of time dt. This charge is $\rho(\Delta A)vdt$. So the current through ΔA is

$$I = \rho v \,\Delta A$$

and the volume current density is

$$J = \frac{\Delta I}{\Delta A} = \rho v.$$

Taking the direction into account,

$$\boldsymbol{J} = \rho \boldsymbol{v}. \tag{10.4}$$

EXAMPLE 10.1 **A straight, cylindrical wire of radius R is kept parallel to the z-axis and carries a current I along its length. Assuming that the current is distributed uniformly over the cross section of the wire, find the current density.**

Solution The current is along the z-direction everywhere in the wire. The cross-sectional area of the wire is πR^2 and the current I crosses it perpendicularly. Also, it is uniformly distributed over the cross section. So the current density is

$$\boldsymbol{J} = \frac{I}{\pi R^2}\hat{\boldsymbol{k}}$$

for $s < R$, where s is the distance from the central axis. For $s > R$, you are outside the wire and there is no current. So $\boldsymbol{J} = 0$.

EXAMPLE 10.2 **A uniformly charged sphere of radius R and volume charge density ρ rotates about one of its diameters with a uniform angular velocity ω. The rotation axis is along the z-axis. Find the current density everywhere.**

Solution Take the centre of the sphere as the origin and use spherical polar coordinates. At a point (r, θ, ϕ) inside the sphere, the velocity is

$$\boldsymbol{v} = \omega R \sin\theta\hat{\phi}.$$

So the current density is

$$\boldsymbol{J} = \rho\boldsymbol{v} = \rho\omega R \sin\theta\hat{\phi}.$$

For points outside the sphere, the current density is zero.

EXAMPLE 10.3 **A capacitor disk of radius R carries charge Q uniformly distributed on its surface. The disk is made to rotate about its axis at an angular velocity ω. What is the current density that results due to this rotation?**

Solution There is a surface current density everywhere on the surface of the disk. Taking the z-axis along the axis of the disk, $\boldsymbol{v} = \omega s\hat{\phi}$ and so the surface current density $\boldsymbol{K} = \sigma\boldsymbol{v} = \sigma\omega s\hat{\phi}$.

10.4 Equation of Continuity

Consider a volume V which contains a charge distribution given by the charge density $\rho(\boldsymbol{r}, t)$. The charges move with velocities which may be different at different places at any given time t. The charge density may depend on position as well as on time, and hence we represent it as $\rho(\boldsymbol{r}, t)$. The net charge at time t in the volume is

$$Q(t) = \int_{\text{volume}} \rho(\boldsymbol{r}, t)\, d\tau.$$

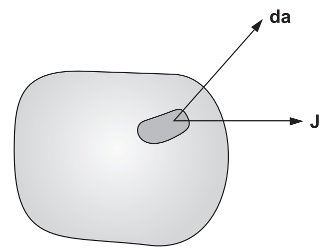


Fig. 10.11

Consider an area element da on the surface bounding the volume considered (Figure 10.11). The corresponding area vector $\boldsymbol{da}$ points along the outward normal to the surface. If the current density here is $\boldsymbol{J}(\boldsymbol{r}, t)$, the current through da is

$$dI = \boldsymbol{J}(\boldsymbol{r}, t) \cdot \boldsymbol{da}$$

meaning that, in the next time interval dt, a charge

$$(dI)\, dt = \boldsymbol{J} \cdot \boldsymbol{da}\, dt$$

crosses da and moves out of the volume through the surface element. The total charge going out of the volume in time dt is, then,

$$\int_{\text{surface}} \boldsymbol{J} \cdot \boldsymbol{da}\, dt.$$

The total charge Q in the volume is reduced by this quantity, which we may express as $-dQ$. So

$$-dQ = \int_{\text{surface}} \boldsymbol{J} \cdot \boldsymbol{da}\, dt$$

or $$-\frac{dQ}{dt} = \int_{\text{surface}} \boldsymbol{J} \cdot \boldsymbol{da} = \int_{\text{volume}} (\nabla \cdot \vec{J})\, d\tau \quad \text{(from the Gauss divergence theorem).} \tag{i}$$

But $$Q = \int_{\text{volume}} \rho\, d\tau.$$

So $$\frac{dQ}{dt} = \int_{\text{volume}} \frac{\partial \rho}{\partial t}\, d\tau.$$

We have used $\frac{d}{dt}$ on the left-hand side, but $\frac{\partial}{\partial t}$ on the right side. Why? Because Q is the total charge in the volume. Variation with $\boldsymbol{r}$ is meaningless. Thus, there is no need for a partial derivative. But ρ is written for $\rho(\boldsymbol{r}, t)$. Hence, you should treat $\boldsymbol{r}$ as a constant while differentiating with respect to time.

Thus, from (i),

$$-\int_{\text{volume}} \frac{\partial \rho}{\partial t} d\tau = \int_{\text{volume}} (\nabla \cdot \boldsymbol{J})\, dr$$

or

$$\int_{\text{volume}} \left(\nabla \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial \tau}\right) d\tau = 0. \tag{ii}$$

As equation (ii) is true for any volume, the integrand must itself be zero for all points separately. Thus

$$\nabla \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0. \tag{10.5}$$

This is a point-by-point relation. Both $\boldsymbol{J}$ and ρ may be functions of $\boldsymbol{r}$ and t. The divergence of $\boldsymbol{J}$ evaluated at any point $\boldsymbol{r}$ and the time derivative of the charge density ρ at the same point $\boldsymbol{r}$ are related to each other by Equation 10.5. This equation is called the *equation of continuity* for charges and comes from the fact that charge is locally conserved and can neither be created nor destroyed. Any charge going out of any small volume results in a decrease in the total charge in that volume.

If $\boldsymbol{J}$ and ρ depend only on $\boldsymbol{r}$ and not on t, $\nabla \cdot \boldsymbol{J} = 0$.

$\boldsymbol{J} \cdot \boldsymbol{da}$ may be positive or negative. If you consider a current-carrying wire and a volume between two cross sections, current enters at one cross section and leaves at the other (Figure 10.12). At the first cross section, $\boldsymbol{J} \cdot \boldsymbol{da}$ is negative, and at the second, it is positive. Remember, $\boldsymbol{da}$ is taken along the outward normal. If the total $\oint \boldsymbol{J} \cdot \boldsymbol{da} = 0$, there is no change in the net charge between the sections.

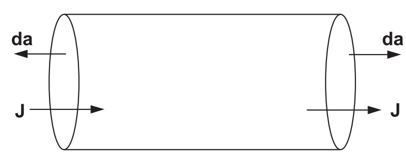


Fig. 10.12

10.5 Current in a Metallic Conductor

A metallic conductor has a large number of free electrons, which can move throughout its body. It also has positive ions, whose positions are almost fixed. The free electrons change their directions of motion and speeds every now and then. As a result, there is no net transfer of charge from one side of any cross section of the conductor to the other.

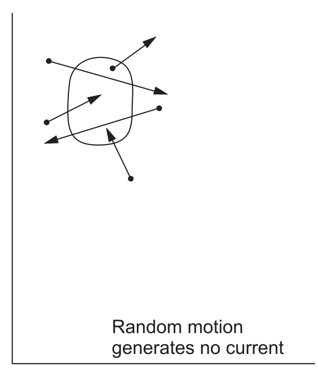


Fig. 10.13

Why do free electrons change their directions so frequently? If your guess is that they collide with the much heavier positive ions within the material and hence change direction, you are only partially correct. If the positive ions are all arranged in a regular fashion according to the crystal structure, the electrons can go much longer distances without suffering any collisions. This is essentially due to the wave nature of electrons. But in an actual metallic conductor, this is not the case. In the first place, the positive ions are not fixed. At any given temperature, they keep vibrating about their mean positions. Then, at some lattice positions, there are no ions, creating what we call vacancies. There may be other kinds of defects too in the lattice. All these imperfections in the crystal lead to the scattering of electrons. You can treat these scattering events as collisions. Whatever be the reasons of scattering, the fact remains that the free electrons keep changing their directions randomly and hence no net current appears unless you apply an electric field.

If you establish an electric field and maintain it (by arranging to remove any accumulated electrons and to inject fresh electrons at some other places), the field exerts a force on the free electrons in a particular direction (opposite to the field). How do the free electrons respond to the electric field? This is a complex phenomenon. You can think of a simple picture which is called Drude's model. There is still a random change in direction and speed when the electrons meet imperfections but between two such events, the field gives them a systematic displacement in one direction. So the electrons already moving in the direction of the force are accelerated by the field whereas those moving in the opposite direction are slowed down a bit. This imbalance causes a net transfer of charge across certain cross sections, resulting in an electric current.

Drift velocity

The motion of a free electron in a metal in the presence of an electric field can be thought of as a superposition of two kinds of motion—(a) a truly random motion in which the speed and direction change frequently and at random, and (b) a slow motion along a fixed direction opposite to that of the electric field. The second type of movement causes current, and is characterized by an average velocity v_d, called *drift velocity*. In time t, which is large compared to the average time between successive collisions, the electron drifts by a distance $x = v_d t$ in the direction opposite to the electric field (Figure 10.14). For the purpose of studying electric current in metals, you can assume that all free electrons move with this constant velocity v_d. If there are n free electrons per unit volume, the volume charge density corresponding to the free electrons is

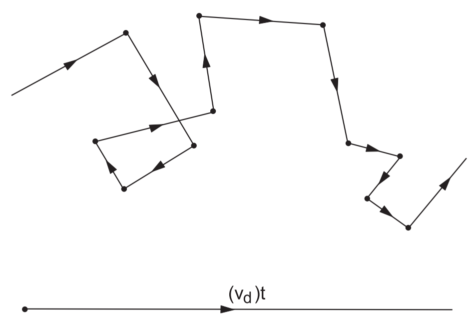


Fig. 10.14

$$\rho = -ne$$

and the volume current density is

$$\boldsymbol{J} = \rho \boldsymbol{v} = -ne\boldsymbol{v}_d. \tag{10.6}$$

The value of the drift velocity turns out to be proportional to the electric field $\boldsymbol{E}$ and the average time between successive collisions. A much simplified calculation can be made to show this. First assume that an electron suffers collisions at regular time intervals τ. Then the average collision time (average time between successive collisions) is also τ. Also assume that when an electron suffers a collision, it loses any systematic velocity that it might have gained from the electric field and moves in a random direction with random speed.

So what is the drift velocity? Just after the collision, the systematic component of the velocity is zero. As the force eE accelerates the electron for time τ, the systematic velocity increases linearly to $\frac{eE}{m}\tau$, when it collides again and its systematic velocity becomes zero. The average systematic velocity is therefore

$$v_d = \frac{1}{2}\frac{eE}{m}\tau$$

or

$$\boldsymbol{v}_d = -\frac{1}{2}\frac{e\tau}{m}\boldsymbol{E}. \tag{10.7}$$

This equation has been derived by making the simplest model of collisions and drift that could be thought of. It was assumed that each electron suffers a collision after a fixed time interval τ. This is certainly not true. An electron suffers a collision when it meets an imperfection. An electron which has just met an imperfection and started with zero systematic velocity may meet another imperfection very soon or after a much longer time. If you refine your model, allowing random variation in collision time, the constant factor (1/2) in Equation 10.7 will be modified. However, we are not interested in getting an exact relation between $\boldsymbol{v}_d$ and $\boldsymbol{E}$ [to obtain such a relation you need a more sophisticated description of free electrons, and not consider them as tiny particles but as continuous waves $\psi(\boldsymbol{r})$]. The message from Equation 10.7 is that the drift velocity $\boldsymbol{v}_d$ is proportional to the electric field $\boldsymbol{E}$ existing in the metal and also to the average collision time τ. So let us write the relation between $\boldsymbol{v}_d$ and $\boldsymbol{E}$ as

$$\boldsymbol{v}_d = -\frac{ke\tau}{m}\boldsymbol{E}, \tag{10.8}$$

where k is a dimensionless constant.

Ohm's Law

For current in a metallic conductor,

$$\boldsymbol{J} = -ne\boldsymbol{v}_d = \frac{kne^2\tau}{m}\boldsymbol{E}$$

or

$$\boldsymbol{J} = \sigma\boldsymbol{E}, \tag{10.9}$$

where

$$\sigma = \frac{kne^2\tau}{m}.$$

As k, n and τ are constants for a given metallic conductor in the given conditions of temperature, pressure, etc., so is σ. The constant σ is called the *electrical conductivity* of the material. Equation 10.9, which says that the current density in a conductor is proportional to the electric field there and the proportionality constant is the electrical conductivity of that conductor, is called Ohm's law.

The reciprocal of electrical conductivity is called *electrical resistivity* and is generally represented by the letter ρ.

$$\rho = \frac{1}{\sigma}\,. \tag{10.10}$$

Remember that we also commonly use the symbol ρ for volume charge density and σ for surface charge density. In other words, the same symbols denote different quantities.

The SI unit of current density $\boldsymbol{J}$ is A/m^2 and that for the electric field $\boldsymbol{E}$ is V/m. Thus, from Equation 10.9, σ has the SI unit $\frac{A/m^2}{V/m} = \frac{A}{Vm}$ or ampere/volt metre. This unit is also called siemens/m and is written as S/m.

Resistivity is simply the reciprocal of conductivity. So its SI unit is Vm/A or Ω m where Ω stands for ohm, i.e., volt/ampere.

Table 10.1 gives the values of resistivity and conductivity for some materials.

Table 10.1

Material	$\rho\,(\Omega\,\text{m})$	$\sigma = 1/\rho$
Silver	1.47×10^{-8}	6.8×10^{7}
Copper	1.72×10^{-8}	5.8×10^{7}
Gold	2.35×10^{-8}	4.25×10^{7}
Aluminium	2.63×10^{-8}	3.8×10^{7}
Tungsten	5.51×10^{-8}	1.81×10^{7}
Nickel	86.84×10^{-8}	0.11×10^{7}
Iron	9.71×10^{-8}	1.02×10^{7}
Magnesium	44×10^{-8}	0.22×10^{7}
Mercury	96×10^{-8}	0.1×10^{7}
Nichrome	100×10^{-8}	0.1×10^{7}
Silicon	640	1.56×10^{-3}
Germanium	0.46	2.17
Fused quartz	7.5×10^{17}	1.33×10^{-16}

In your schooldays, you must have learnt Ohm's law in a very different form. The current I flowing through a conductor is proportional to the potential difference applied between its ends.

$$I \propto V$$

or
$$I = \frac{V}{R}, \tag{10.11}$$

where R is a constant for the given piece of the conductor at a given temperature and is called its resistance. Are equations 10.9 and 10.11 equivalent? Yes, they are. Equation 10.9 is more microscopic as it talks about field $\boldsymbol{E}$ and current density $\boldsymbol{J}$ at each point $\boldsymbol{r}$ in the conductor. Equation 10.11 gives an overall relation between the current and the potential difference across a larger length of the conductor. Let us see how Equation 10.11 can be derived from the microscopic relation given by Equation 10.9, for a cylindrical metallic wire.

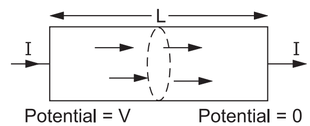


Fig. 10.15

Let the wire have a length L and area of cross section A. The ends are "maintained" at potentials zero and V (Figure 10.15). This results in a uniform electric field along the length of the wire. Taking this direction as the x-direction, the field is

$$\boldsymbol{E} = \frac{V}{L}\hat{\boldsymbol{i}}$$

everywhere in the wire. There will be some end effects at the surface of the wire but let us neglect those. The current density inside the wire is, then,

$$\boldsymbol{J} = \sigma \boldsymbol{E} = \sigma \frac{V}{L}\hat{\boldsymbol{i}}.$$

The current through any cross section of the wire is

$$I = \int_{\text{cross section}} \boldsymbol{J} \cdot d\boldsymbol{a} = \int \frac{\sigma V}{L} da = \frac{\sigma V}{L} A$$

or
$$I = \frac{V}{(L/\sigma A)} = \frac{V}{\rho L/A} \cdot$$

Writing
$$\rho \frac{L}{A} = R, \tag{10.12}$$

$$I = \frac{V}{R} \cdot \tag{10.13}$$

The quantity R is a constant for a given conductor in the given conditions. So you get your familiar form of Ohm's law.

Ohm's law is a material property and not a fundamental law of nature. Semiconductors, for example, do not follow Ohm's law and the current changes nonlinearly with electric fields $\boldsymbol{E}$. We call them *nonohmic materials*. If you apply a very high field, Ohm's law may not work for so-called ohmic materials either. But for reasonable fields commonly employed, a large number of materials including metals obey Ohm's law. You all know that the SI unit of resistance R is called ohm, with the symbol Ω.

A particular piece of material especially made to have a particular resistance is called a *resistor*.

Limitation of Drude's model

The description given here for the mechanism of conduction in metals (essentially Drude's model) is simplistic and unrealistic. For example, the conductivity derived from Equation 10.9 for a given metal differs quite significantly from the measured value. If the mean free path between the successive scattering events is λ, the collision time is $\tau = \frac{\lambda}{v_{th}}$, where v_{th} is the mean thermal speed of the electrons. Thus, the average collision time is inversely proportional to the mean thermal speed. From Equation (10.9), one may expect that

$$\sigma \propto \frac{1}{v_{th}}$$

or $$\rho \propto v_{th}$$

or $$\rho \propto \sqrt{T},$$

as the mean thermal speed is proportional to $\sqrt{T}$. But the resistivities of conductors are known to vary linearly with temperature.

Conduction in metals is a quantum-mechanical phenomenon. One must take into account that electrons can show wave character and that a wave can pass much larger distances through a periodic array of atoms than a particle, depending on the wavelength and the periodicity of the material structure. The fact that electrons obey the Pauli exclusion principle also plays a crucial role in conduction. An important factor in determining resistivity is the vibrational amplitude of the ions in the lattice. The larger the amplitudes of vibration, the larger is the scattering and the larger the resistivity.

The advancement of nanotechnology has equipped us to work with metallic structures of dimensions of the order of, say, tens of nanometres. You can easily construct a wire with such small diameters. When a potential difference is applied across such a wire and is varied, the current is not linear as suggested by Equation 10.11. In the lateral direction, the electron collision time is shortened as it meets the surface quickly. There are other effects too at such small lengths and Drude's model cannot be applied.

Charge density in current-carrying conductors

You have learnt that there is no electric field in a metal in an electrostatic situation. In fact, we used this as the main argument to say that the charge density is zero in such an object. Gauss's law is $\nabla \cdot \boldsymbol{E} = \rho/\varepsilon_0$ and if $\boldsymbol{E} = 0$ in a metal, ρ has to be zero.

But you do have an electric field in a current-carrying conductor. Can you have a nonzero charge density inside the conductor?

The equation of continuity is

$$\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t}.$$

This comes from the principle of conservation of charge and must be obeyed religiously. Consider a situation where the current density at any given position does not vary with time for the limited period of interest. Such a current is called a *steady current*. Steady currents are produced by charges or fields that do not change with time. In these situations, charge density at any point also does not change with time and so $\partial\rho/\partial t$ is zero. Thus, for steady currents,

$$\nabla \cdot \boldsymbol{J} = 0$$

or

$$\nabla \cdot (\sigma \boldsymbol{E}) = 0.$$

If the whole conductor is made of the same material, σ is the same everywhere and can be taken out of the divergence operation. So,

$$\sigma(\nabla \cdot \boldsymbol{E}) = 0$$

or

$$\nabla \cdot \boldsymbol{E} = 0, \text{ giving } \rho = 0 \text{ (using Gauss's law).}$$

So there is no charge density inside the conductor even if a steady current passes through it and an electric field exists in it. However, if there are different kinds of conductors, having different conductivities, joined together, there will be charge accumulated at the junctions.

EXAMPLE 10.4 **Two cylindrical wires of the same radius *a* are joined to form a composite cylindrical wire of the same radius. The electrical conductivity is σ_1 for one wire and σ_2 for the other. A current *I* passes through the wires as shown in Figure 10.16. Find the charge accumulated at the junction.**

Solution The current densities in the two wires are the same,

$$J = \frac{I}{\pi a^2}$$

in the direction of the current. Let us call this the x-direction. The electric fields in the two wires are

$$\boldsymbol{E}_1 = \frac{\boldsymbol{J}}{\sigma_1} = \frac{I}{\sigma_1 \pi a^2}\hat{\boldsymbol{i}}$$

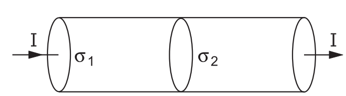


Fig. 10.16

and

$$\boldsymbol{E}_2 = \frac{\boldsymbol{J}}{\sigma_2} = \frac{I}{\sigma_2 \pi a^2}\hat{\boldsymbol{i}}.$$

Apply the boundary condition $E_{1n} - E_{2n} = \sigma/\varepsilon_0$ at the junction. Calling the side of the first wire Side 1 and that of the second wire Side 2, the normal $\hat{\boldsymbol{n}}$ is the same as $(-\hat{\boldsymbol{i}})$. Thus, $E_{1n} = -E_1$ and $E_{2n} = -E_2$, and so

$$\frac{I}{\sigma_2 \pi a^2} - \frac{I}{\sigma_1 \pi a^2} = \frac{\sigma}{\varepsilon_0}.$$

Distinguish between the different uses of the letter sigma—σ is the surface charge density at the junction surface, whereas σ_1 and σ_2 are conductivities. The charge accumulated on the junction is, therefore,

$$q = \sigma \pi a^2 = \varepsilon_0 I\left(\frac{1}{\sigma_2} - \frac{1}{\sigma_1}\right).$$

Charge appearing on the surface of a current-carrying wire

For a steady current in a conductor, there is no volume charge density. But is there a surface charge density on the surface of the conductor? Consider a long wire connected to a battery. The current I is along the wire and the electric field $\boldsymbol{E}$ also follows the same direction $(\boldsymbol{J} = \sigma \boldsymbol{E})$. What produces this electric field? Electric field can be produced either by charges or by changing magnetic fields. In this case, there is no question of a changing magnetic field. So where are the charges producing this field?

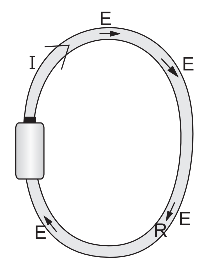


Fig. 10.17

You may guess that charges are accumulated at the terminals of the battery and this might produce the required electric field. This is a natural guess but, unfortunately, incorrect. Any charge distribution confined to the battery will not produce a field $\boldsymbol{E}$ that is confined mainly to the wire, has the same magnitude everywhere in the wire although the distance from the battery varies widely, and changes its direction with the turns in the wire. The only other place where charges can reside is the surface of the conductor. Charges do appear on the surface and the charge density is different at different places on the wire to ensure the right kind of electric field in the wire.

10.6 EMF

To maintain a current in a metallic wire, you have to continuously inject electrons at one end and remove them from the other, keeping a nonzero electric field in the wire. To do this, you need a special device. For example, you can connect a torch cell to the ends of the wire to maintain the field. What goes on inside the cell? Typically, you have some chemicals in the cell and two electrodes. Due to chemical reactions and the particular electronic structure of the electrodes, the positively charged ions are pushed towards one electrode (positive terminal) and the negatively charged ions, towards the other electrode (negative terminal). The situation is schematically shown in Figure 10.18(a).

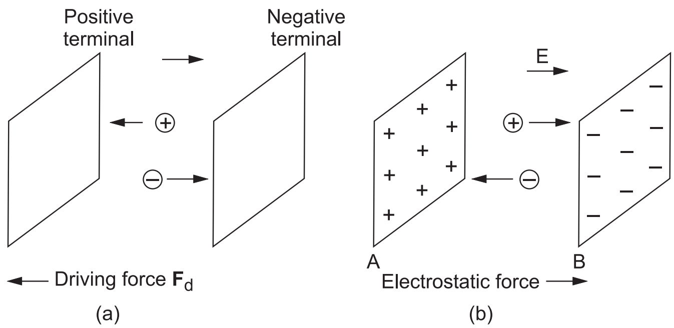


Fig. 10.18

The force on a charge q for this push comes from the local chemistry of the chemicals and the electrodes, and cannot be given by equations like $\boldsymbol{F} = q\boldsymbol{E}$ where $\boldsymbol{E}$ may be an overall electric field existing between the electrodes. Let us call this force the driving force and denote it by the symbol $\boldsymbol{F}_d$.

Indeed, when opposite charges accumulate at the electrodes, they create an overall electric field between the electrodes. This electric field $\boldsymbol{E}$ also exerts forces on the ions in the cell. The force on a given ion due to this field is opposite to the driving force [Figure 10.18(b)]. At equilibrium, an amount of charge accumulates on the electrodes which ensures that

$$\boldsymbol{F}_d = -q\boldsymbol{E}.$$

In this case, the overall electric field exerts forces on the ions equal and opposite to the driving force and no further accumulation takes place. The potential difference between the electrodes A and B is

$$V_{\mathrm{A}} - V_{\mathrm{B}} = -\int_{\mathrm{B}}^{\mathrm{A}} \boldsymbol{E} \cdot d\boldsymbol{l}.$$

On the other hand,

$$\varepsilon = \frac{1}{q}\int_{\mathrm{B}}^{\mathrm{A}} \boldsymbol{F}_d \cdot d\boldsymbol{l} \tag{10.14}$$

is called the emf of the battery. You may wonder that, if $\boldsymbol{F}_d = -q\boldsymbol{E}$, what is the difference between the two integrals. Indeed their values are the same in the situation we are discussing, but they represent two different physical quantities resulting from totally different mechanisms. While $\boldsymbol{F}_d$ appears due to the chemical reactions, $\boldsymbol{E}$ comes from the electrostatic field created in the electrolyte. While $\boldsymbol{F}_d$ tries to cause charge separation, $\boldsymbol{E}$ attempts to reduce it.

When you connect the two terminals of a torch cell by a metallic wire, a current starts in the wire (Figure 10.19). Because of the charges accumulated at the electrodes, electric fields are set up in the wire, some charges appear on the surface of the wire and finally, in the steady state, you have a steady electric field E of constant magnitude appearing in the entire wire. The wire draws electrons from the negative terminal and deposits them at the positive terminal. This would have decreased the concentration of the charge at the electrodes but the driving force brings these electrons back to the negative electrode to ensure equilibrium in the cell. In some cases, the concentration on the electrodes, when a continuous current is drawn from the cell, is less than that when no current is drawn. Then the electrostatic field $\boldsymbol{E}$ in the cell is reduced from its original value (no-current situation). Accordingly, the potential difference

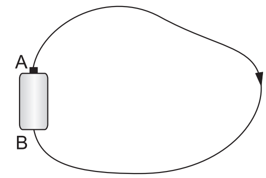


Fig. 10.19

$$V = \int \boldsymbol{E} \cdot \boldsymbol{dl}$$

is less than the emf

$$\varepsilon = \frac{1}{q}\int \boldsymbol{F}_d \cdot \boldsymbol{dl}.$$

If the emf and the potential difference between the terminals of a battery remain the same even when a current is drawn from it, we call it an *ideal cell* or an *ideal battery*.

In the example taken, the driving force was confined to one portion of the circuit, that is, in the cell. In some cases, the driving force may be generated throughout the circuit. We shall discuss induced emf in one of the later chapters, in a situation where the driving force will be of this type. You can define the emf of the whole circuit as

$$\varepsilon = \frac{1}{q}\oint \boldsymbol{F}_d \cdot \boldsymbol{dl}, \tag{10.15}$$

where $\boldsymbol{F}_d$ is the driving force on the charge q. This integration is to be taken at the same instant of time. In different segments of the circuit, you have to calculate $\boldsymbol{F}_d \cdot \boldsymbol{dl}$ and add the values to get the integral and then divide by q. The whole purpose of this discussion is to emphasize that there are two different kinds of force on the charged particles in a circuit. One is due to an overall electric field produced due to charge accumulation at the surface of the conductor or the electrodes of a cell, or anywhere else. Another is the driving force, which has its origin somewhere else, like a chemical reaction or some other mechanism. You should be able to distinguish between the two. The emf is related to the driving force, and the potential difference, to the electrostatic force. Indeed both have the same unit, the volt.

We have been using the abbreviation emf for quite some time without telling you the full form, intentionally. The full form of emf is *electromotive force*. However, it is not really a force as you can now realize. It is related to the driving force through Equation 10.14 or 10.15. Remembering it as electromotive force can create some confusion at some stage. So better remember it as emf only.

10.7 Electric Circuits

With batteries (a chemical cell or combination of cells) and resistors, you can set up a variety of circuits. You all know about series and parallel combinations of resistors, Kirchhoff's laws and methods to analyze circuits. A battery has a definite positive terminal and a definite negative terminal. The potential of the positive terminal is higher than that of the negative terminal. Thus a battery provides current only in one direction. It is, therefore, called a direct current source or dc source of emf. You can also have an alternating current source (ac source), in which the direction of current alternates.

10.8 A Leaky Capacitor

If you connect a capacitor of capacitance C to a battery, charges flow from the battery terminals, putting up charge $+q$ on one plate and $-q$ on the other (Figure 10.20). The total charge q is such that the potential difference $V = q/C$ between the plates becomes equal to the emf ε of the battery. This means $q = \varepsilon C$. Once this is achieved, no further movement of charge takes place. The charge moving from the battery and piling up on the capacitor plates takes time. This is called charging of the capacitor. The charge on the capacitor (positive plate) as a function of time is given as

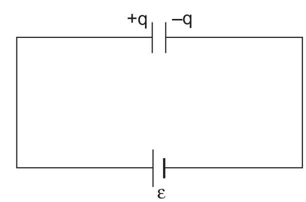


Fig. 10.20

$$q = \varepsilon C(1 - e^{-t/RC}),$$

where R is the resistance in the circuit including that of the connecting wires, and $t = 0$ is the time when the connections are made. The product RC has dimensions of time and is called the *time constant* of the circuit. In one time constant, about 63% of the maximum charge reaches the capacitor.

If you already have a charged capacitor and connect the plates by a conducting path, charges flow from one plate to the other till both the plates becomes neutral. This is called discharging of the capacitor. If the plates are connected at time $t = 0$ when the charge is q_0, the charge at time t is given by

$$q = q_0 e^{-t/RC}.$$

In one time constant, 37% of the charge goes away.

In general, a capacitor is made with a dielectric material filled in the space between its plates. This increases the capacitance on the one hand and provides mechanical strength and stability to the capacitor on the other. A perfect dielectric would not have any conductivity. But real dielectrics might have some conductivity. Figure 10.21 suggests a situation of this type. The capacitor plates are shown as A and B with a dielectric filling up space. The dielectric has permittivity ε and conductivity σ.

What happens if you give some charge to the capacitor and leave it? The capacitor will get discharged in due course as the charge will go from one plate to the other through the dielectric. Thus it is a leaky capacitor. What is the time constant for this capacitor? It is still given by *RC* where *R* is the resistance and *C*, the capacitance. Because of electric conduction, charges will flow from one conductor to the other through the dielectric material and hence will give rise to a current *I*. Suppose the potential difference between the two plates at a particular time *t* is V_0.

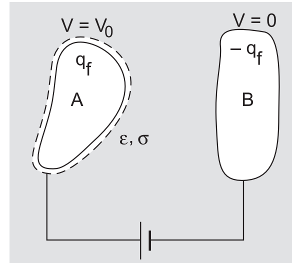


Fig. 10.21

Consider a surface just outside the conductor A, enclosing it. In Figure 10.21, this is shown by dotted lines. The free charge on the conductor A at time *t* is

$$q_f = \oint \boldsymbol{D} \cdot d\boldsymbol{a} = \varepsilon \oint \boldsymbol{E} \cdot d\boldsymbol{a}.$$

So the capacitance is $C = \dfrac{q_f}{V} = \dfrac{\varepsilon \oint \boldsymbol{E} \cdot d\boldsymbol{a}}{V_0}\cdot$

As time passes, charges flow between the plates, and q_f decreases. This causes a current through the dielectric. The current *I* at time *t* also goes through the closed surface around A, and hence is given by

$$I = \int \boldsymbol{J} \cdot d\boldsymbol{a} = \sigma \oint \boldsymbol{E} \cdot d\boldsymbol{a}.$$

So the resistance is $R = \dfrac{V}{I} = \dfrac{V_0}{\sigma \oint \boldsymbol{E} \cdot d\boldsymbol{a}}\cdot$

The product $RC = \dfrac{V_0}{\sigma \oint \boldsymbol{E} \cdot d\boldsymbol{a}} \cdot \dfrac{\varepsilon \oint \boldsymbol{E} \cdot d\boldsymbol{a}}{V_0}$

or $$RC = \frac{\varepsilon}{\sigma}\cdot \tag{10.16}$$

EXAMPLE 10.5 **The space between the plates of a parallel-plate capacitor is filled with a material of dielectric constant 5.0 and conductivity 4.0×10^{-5} S/m (Figure 10.22). Assume the conductivity of the plates to be very high. The plate area is 100 cm^2 and the separation between the plates is 1.0 mm. A battery of 6.0 V is connected to the capacitor. (a) Find the charge on the capacitor in the steady**

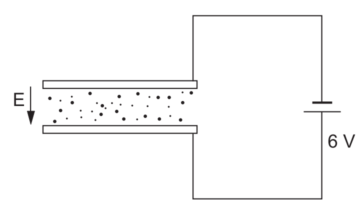


Fig. 10.22

state. (b) Work out the current through the capacitor in the steady state. (c) The battery is now disconnected. In how much time will the charge on the capacitor reduce to half its original value?

Solution (a) The electric field in the space between the plates is

$$E = \frac{6.0\text{ V}}{10^{-3}\text{ m}} = 6000\text{ V/m}.$$

Using the boundary condition $D_{1n} - D_{2n} = \sigma_f$, the free charge density on a plate is

$$\sigma_f = D = \varepsilon E = 5 \times 8.85 \times 10^{-12} \times 6000\text{ C/m}^2$$
$$= 2.655 \times 10^{-7}\text{ C/m}^2.$$

So, $$Q = \sigma_f A = 100 \times 10^{-4} \times 2.655 \times 10^{-7}\text{ C}$$
$$\approx 2.7 \times 10^{-9}\text{ C}.$$

(b) The current density is

$$J = \sigma E = 4.0 \times 10^{-5} \times 6000\text{ A/m}^2$$
$$= 24 \times 10^{-2}\text{ A/m}^2.$$

Current $$= J \cdot A = 24 \times 10^{-2} \times 100 \times 10^{-4} = 2.4 \times 10^{-3}\text{ A}.$$

(d) After the battery is disconnected, the capacitor will discharge through the material in it with a time constant

$$RC = \varepsilon/\sigma = \frac{5 \times 8.85 \times 10^{-12}}{4.0 \times 10^{-5}} = 1.1 \times 10^{-6}\text{ s}.$$

The charge at time t is

$$q = q_0 e^{-t/RC}.$$

For $$q = q_0/2,$$

$$\frac{q_0}{2} = q_0 e^{-t/RC}$$

or $$t = RC \ln 2 = (1.1 \times 10^{-6}) \ln 2\text{ s} \approx 7.6 \times 10^{-7}\text{ s}.$$

Figure 10.21 shows the dielectric medium to be on all sides of the capacitor plates. In the present example, it is only on one side. Make sure that the derivation of $RC = \varepsilon/\sigma$ is applicable to the present case too.

10.9 The Child–Langmuir Equation

Suppose you have two conducting plates placed at a separation of d, as in a parallel-plate capacitor (Figure 10.23). Consider the plates to be large and perpendicular to the x-axis, say at $x = 0$ and $x = d$. Suppose the plate at $x = 0$ is heated and maintained at a temperature so that electrons emerge from this plate at a certain rate. This process is called thermionic emission.

The electrons will be attracted towards the plate at $x = d$ maintained at a higher potential V_0. Soon a steady state will appear where, as a function of x, the potential V_0, the charge density ρ and the velocity v of the electrons will assume values independent of time. The charge built up due to the presence of electrons between the plates is called space charge. As the space charge density is constant over time, the current I in the region because of the movement of electrons will also be constant. The Child–Langmuir equation is relation between I and V_0.

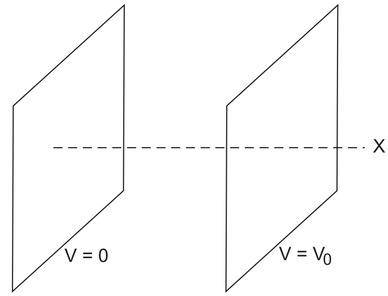


Fig. 10.23

Suppose the charge density at x is $\rho(x)$ and the potential here is $V(x)$. The speed v of an electron here will be $v = \sqrt{\dfrac{2K}{m}} = \sqrt{\dfrac{2eV}{m}}$.

You can write

$$J = \rho v$$

or $$\frac{I}{A} = \rho v$$

or $$\rho(x) = \frac{I}{Av} = \frac{I\sqrt{m}}{A\sqrt{2eV}}.$$

So, from Poisson's equation,

$$\frac{d^2 V}{dx^2} = -\frac{I\sqrt{m}}{\varepsilon_0 A\sqrt{2e}}\frac{1}{\sqrt{V}}.$$

Put $$\frac{dV}{dx} = V'.$$

Then $$\frac{d}{dx}(V') = -\frac{I\sqrt{m}}{\varepsilon_0 A\sqrt{2e}}\frac{1}{\sqrt{V}}$$

or $$\frac{dV'}{dV}\cdot\frac{dV}{dx} = -\frac{I\sqrt{m}}{\varepsilon_0 A\sqrt{2e}}\frac{1}{\sqrt{V}}$$

or $$V'\frac{dV'}{dV} = -\frac{I\sqrt{m}}{\varepsilon_0 A\sqrt{2e}}\frac{1}{\sqrt{V}}$$

or $$V'dV' = -\frac{I\sqrt{m}}{\varepsilon_0 A\sqrt{e}}\frac{dV}{\sqrt{2V}}$$

or $$\frac{V'^2}{2} = +\frac{I\sqrt{m}}{\varepsilon_0 A\sqrt{e}}\sqrt{V} + C_1$$

or $$\frac{1}{2}\left(\frac{dV}{dx}\right)^2 = \frac{I}{\varepsilon_0 A}\sqrt{\frac{m}{e}}\sqrt{V} + C_1.$$

At the plate, $x = 0$. The electrons come out with very little speed. Thus, the current is negligible and so is the electric field. If we ignore any surface charge, the boundary condition on the electric field tells us that the field just outside is also negligible.

This gives $C_1 = 0$.

Thus $$\left(\frac{dV}{dx}\right)^2 = \frac{2I}{\varepsilon_0 A}\sqrt{\frac{m}{e}}\sqrt{V}$$

or $$\frac{dV}{dx} = \left[\frac{2I}{\varepsilon_0 A}\sqrt{\frac{m}{e}}\right]^{1/2} V^{1/4}$$

or $$V^{-1/4}dV = \left[\frac{2I}{\varepsilon_0 A}\sqrt{\frac{m}{e}}\right]^{1/2} dx$$

or $$\frac{4}{3}V^{3/4} = \left[\frac{2I}{\varepsilon_0 A}\sqrt{\frac{m}{e}}\right]^{1/2} x + C_2$$

At $x = 0,\ V = 0$.

Then $$C_2 = 0.$$

So, $$V^{3/2} = \frac{9}{16}\cdot\frac{2I}{\varepsilon_0 A}\sqrt{\frac{m}{e}}\,x^2.$$

At $x = d$, $V = V_0$. Putting this in the above equation,

$$I = \frac{8A\varepsilon_0}{9d^2}\sqrt{\frac{e}{m}}\,V_0^{3/2}.$$

This is called the Child–Langmuir equation.

Concepts at a Glance

1. For linear current, $I = \lambda v$. For surface current, $\boldsymbol{K} = \sigma \boldsymbol{v}$. And for volume current, $\boldsymbol{J} = \rho \boldsymbol{v}$.
2. The equation of continuity for conservation of charge is $\nabla \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0$.
3. If $\nabla \cdot \boldsymbol{J} = 0$, the current is known as steady current.
4. The resistivity of a metal results from the frequent scattering of free electrons from imperfections in the crystalline structure of the metallic object.
5. $\boldsymbol{J} = \sigma \boldsymbol{E}$ is called Ohm's law. It is valid for a class of materials under specific situations.
6. emf $\varepsilon = \frac{1}{q}\int \boldsymbol{F}_d \cdot d\boldsymbol{l}$.

EXERCISES

Based on Concepts

1. A charged particle moves with a constant velocity v. Is this a steady current?
2. Which of the following are examples of steady current? (a) A ring of radius R having a uniform linear charge density λ, rotating about its axis with a constant angular velocity ω. (b) A ring of radius R with a uniform linear charge density λ, rotating about one of its diameters with a constant angular velocity ω. (c) A ring of radius R having a nonuniform linear charge density $\lambda(\varphi)$, rotating about its axis with a constant angular velocity ω.
3. Two long, straight threads carry charges $+\lambda$ and $-\lambda$ per unit length. The threads are kept parallel to each other at a small separation. The positively charged thread is set moving along its length with a velocity v. The other thread is also set moving but with a velocity $-v$. Is the net current through a plane cutting the two threads perpendicularly zero, or along v or along $-v$?
4. A metallic wire contains lots of free electrons and positive ions. If the wire is moved, does a current flow in space?
5. When you press buttons on the TV remote control, does an electric current pass from the remote to the TV set?
6. A metallic sheet of thickness t has a current density $\boldsymbol{J}$. If you neglect the thickness and treat the sheet as a surface, the current has to be considered as a surface current. What will be the corresponding surface current density?
7. A current flows in a straight cylindrical wire so that an electric field $\boldsymbol{E}$ is maintained in the wire. Will there be an electric field parallel to the wire, just outside it? (Consider $\nabla \cdot \boldsymbol{E}$.)
8. For a capacitor embedded in a large dielectric, the product RC was derived to be ε/σ. Justify qualitatively why RC does not depend on the separation between the plates.

Problems

1. An isolated radioactive source Na^{22} is kept at the origin and emits positrons isotropically at a rate of 10^7 particles per second. Consider a spherical surface with the centre at the origin and radius 10 cm. Find the current density through the surface. [**Ans.** 0.13 A/m^2]
2. A long, thin cylindrical pipe of radius R carries a current I going along its length. The current is uniformly distributed along the surface of the pipe. Take the z-axis along the axis of the pipe in the direction of the current. Write the expression for the surface current. [**Ans.** $\frac{I}{2\pi r}\hat{\boldsymbol{k}}$]
3. A cylindrical wire of radius R_1 carries a current I distributed uniformly over its cross section. The same current returns along the surface of a thin, cylindrical, coaxial shell of a larger radius R_2. Taking an appropriate coordinate system, write expressions for volume current density in the wire and surface current density on the surface the wire.
4. A sphere of radius R carries a charge Q uniformly distributed on its surface. If the sphere rotates with on angular velocity ω about the z-axis, which is taken along one of its diameters, what is the current density everywhere? [**Ans.** $\boldsymbol{K} = \frac{Q\omega \sin\theta}{4\pi r}\hat{\phi}$ on the surface]

5. Figure 10E.1 shows a thin, spherical, metallic surface of radius R. Two diametrically opposite points on this sphere are connected to a circuit carrying current I. Using the spherical coordinate system with the z-axis along the diameter passing through these points, give the expression of the current density on the surface. [**Ans.** $\frac{I}{2\pi R \sin\theta}\hat{\theta}$]

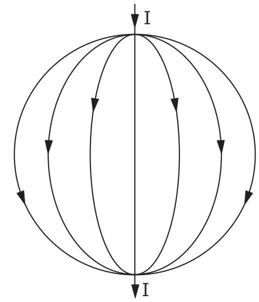


Fig. 10E.1

6. A circular disc of radius a carries a uniform surface charge distribution with total charge Q. It rotates about its axis with a constant angular velocity ω. Find the current through any radial line drawn from the centre. [**Ans.** $Q\omega/2\pi$]

7. Two concentric, spherical, metallic shells of radius a and $b (a < b)$ are separated by a weakly conducting material of conductivity σ. (a) If they are maintained at a potential difference V, what is the current flowing from one shell to the other? (b) What is the resistance between the shells?
[**Ans.** (a) $\frac{4\pi ab\sigma V}{b-a}$ (b) $R = \frac{b-a}{4\pi ab\sigma}$]

8. Pure iron has a resistivity of $10 \times 10^{-6}\ \Omega$ cm at 20 °C. The resistivity of copper at this temperature is $1.77 \times 10^{-6}\ \Omega$ cm. Consider two different composite conductors, each 1 m long and having a square cross section 8 cm on a side. Conductor A is made by joining a square bar of iron 50 cm long and a similar bar of copper end to end. Conductor B consists of 1-m rods of copper and iron clamped side by side.
   (a) What is the resistance in ohms of each conductor between the ends?
   (b) If a steady current flows in conductor A, in which material will the power dissipation be greater?
   (c) Answer the same question for conductor B.

9. Current I flows through a cylindrical bar of cross-sectional area A. The material of the bar has resistivity ρ and permittivity ε. Find the polarization $\boldsymbol{P}$ developed in the bar. [**Ans.** $\frac{(\varepsilon - \varepsilon_0)I\rho}{A}$]

10. A conducting wire of uniform cross section A, length L and resistivity ρ is connected to an ideal battery of emf ε, as shown in Figure 10E.2. Find
   (a) the current density in the wire,
   (b) the magnitude of the electric field in the wire,
   (c) $\oint \boldsymbol{E} \cdot d\boldsymbol{l}$ on the path ABCDA, and
   (d) $\int \boldsymbol{E} \cdot d\boldsymbol{l}$ on the path BC. [**Ans.** (c) zero, (d) $\varepsilon/3$]

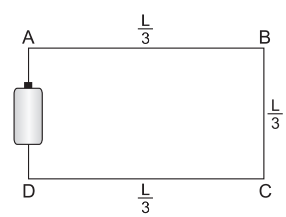


Fig. 10E.2

11. Find the current that would flow between two metal spheres, each of radius a, immersed deep under the sea and held quite far apart. The potential difference between the spheres is V. The conductivity of sea water is σ. [**Ans.** $2\pi\sigma Va$]

12. AB and CD are two wires taken from the same pool. The resistance of the wires per centimetre is $1\,\Omega$. The resistance of AB is $10\,\Omega$ and that of CD is $20\,\Omega$. These are connected in a circuit as shown in Figure 10E.3. The connecting wires are thick enough for you to neglect their resistances. Find the electric field in the wire AB and that in CD. What can you say about the electric field in the connecting wire? [**Ans.** 30 V/m and 15 V/m]

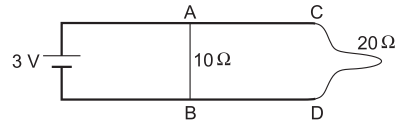


Fig. 10E.3

13. A conducting line is made as follows. A cylindrical tube of inner and outer radii a and b is made from a material of conductivity σ_1. Surrounding this, there is another tube of inner and outer radii b and c, made of material of conductivity σ_2. Assume that this line carries a current I and that the potential over any cross section is uniform. Find the current carried by each tube separately.

[**Ans.** $\dfrac{I(b^2-a^2)}{\sigma_1(b^2-a^2)+\sigma_2(c^2-b^2)}$ through the inner tube]

14. A metallic wire is in the shape of a semicircular arc with square cross section (Figure 10E.4). Using cylindrical coordinates, the region occupied by the wire is $R < s < R+a,\ 0 < \phi < \pi,\ 0 < z < a$. A current I flows through the wire. Use the fact that the curl of the electric field in the wire must be zero. Find the current density in the wire. [**Ans.** $\dfrac{I}{a\ln\dfrac{R+a}{R}}\cdot\dfrac{1}{s}\hat{\phi}$]

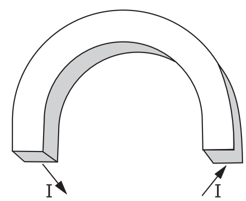


Fig. 10E.4

15. A cylindrical wire of cross-sectional area A, placed between $x = x_1$ and $x = x_2$, is made of a material with varying composition. As a result, the conductivity varies as $\sigma = \sigma_0 + kx$. A current I uniformly distributed over the cross section passes through the wire. Find the charge density appearing in the wire. [**Ans.** $\dfrac{\varepsilon_0 kI}{A(\sigma_0+kx)^2}$]

❑

11 Magnetic Fields

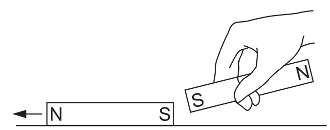


Fig. 11.1

Children enjoy watching magnets repel and attract each other. Keep one magnet on a table, another in your hand. Move your hand near the magnet on the table. The one on the table moves about. A magnet can lift iron nails, clips, coins, and so on. A button magnet sticks to the doors and sides of metallic almirahs and refrigerators. Moving a magnet near a compass needle makes the needle dance on its support.

Two magnets exert forces on each other even if not in contact. What kind of force is this? It is neither gravitational nor electrostatic. All portions of the magnets are uncharged and do not exert electrostatic force. The force between the magnets is magnetic force. As you have seen, the idea of an object exerting a force directly on another object at a distance is inconsistent with the theory of relativity. And this theory has established itself so strongly that there is no reason to disbelieve it at the present state of knowledge. This prompted us to describe the electric forces between two charged particles as a two-step process. A charge produces an electric field $\boldsymbol{E}$, which exerts a force on another charge placed in the field. In a similar fashion, a magnet produces a magnetic field, which in turn exerts a force on another magnet placed in the field. The symbol generally used to denote magnetic field is $\boldsymbol{B}$.

Magnets are not the only objects that give rise to a magnetic field and experience magnetic force. Place a compass needle close to a current-carrying wire—the needle gets deflected. This means the current in the wire also produces a magnetic field. If you put a charge at rest near a current-carrying wire, there is no force on the charge, which will remain stationary. The same is the case if you put a charge at rest near a magnet. The magnetic field due to the current or a magnet exerts no force on this charge. But if you project the charge with a velocity $\boldsymbol{v}$ parallel to the wire, it is deflected towards or away from the wire. The magnetic field exerts a force on a moving charged particle. If you project the charge in different directions, you will find that the deflection is also different in every case. The direction of magnetic force depends on the direction of the velocity.

The basic source for producing an electric field is charge. It turns out that the basic source for producing a magnetic field is electric current. A magnet produces a magnetic field because it consists of tiny atomic currents, aligned in a certain way. And these atomic currents result from the motion of the electrons that each atom has. There is also a quantity called the spin of an electron, which has an effect similar to that of a tiny current. So the two questions to be examined are: (a) given a current distribution, how do you calculate the magnetic field, and (b) how does a magnetic field exert force on different objects placed in it?

Question (b) will be taken up in this chapter and question (a) in the next. There is yet another source of magnetic field—a changing electric field. That will be discussed in a much later chapter. But before all that, we shall talk about an interesting relation between magnetism and relativity.

11.1 Magnetism is a Relativistic Phenomenon

The results of special relativity are quite surprising and appear to be unbelievable. One of these is length contraction. Suppose you look at a rod kept at rest with respect to you and find its length to be L. Then start moving with a velocity v parallel to the rod. In other words, the rod is moving with a velocity v along its length, with respect to you. If you measure the length of the same rod now, it will turn out to be smaller. The new length will be

$$L' = L\sqrt{1 - v^2/c^2}.$$

So the length of a bicycle on its stand is larger than the length of the same bicycle when your brother rides it. The length of a stationary train at a platform is larger than when it runs, each time as seen by an observer on the platform. A square plate moved parallel to one of its edges becomes rectangular because the length parallel to the motion contracts.

But have you ever observed such length contraction? No. That is why it is hard to believe it happens. However, relativistic phenomena do occur in the real world. The only thing preventing us from observing them is the magnitude of the speed v. This speed is very small compared to that of light c, in situations we commonly encounter. Bicycles move at speeds of, say, 10 km/h (≈ 3 m/s) and the maximum speed of the Rajdhani Express is about 150 km/h (≈ 42 m/s). For an aeroplane moving at 300 m/s (more than 1000 km/h), $v^2/c^2 = 10^{-12}$. So the change in length will be hardly 1 part in 10^{12}. No ordinary perception of sense will enable us to detect such small changes in length in objects moving at such speeds.

We do not detect these relativistic effects because they are so tiny. Why, then, should we believe that they are there? The theory of relativity is counter-intuitive, and takes a lot of logical rigour to understand.

However, magnetism is a fairly simple and observable relativistic effect. When you see a current-carrying wire deflecting a compass needle, you observe a relativistic effect. Let us see how.

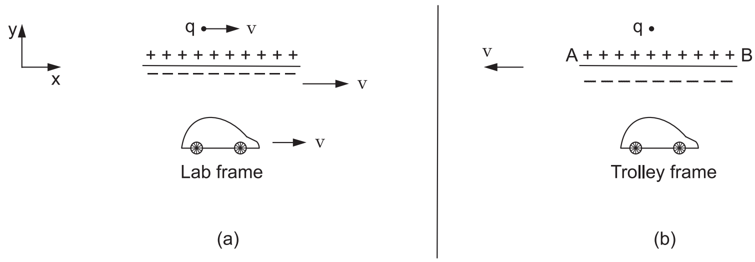


Fig. 11.2

Consider a long metallic wire that has positive ions and free electrons placed along the x-axis in the lab. The free electrons in the wire move with a velocity v along its length causing a current I. The positive ions in the wire are at rest. In Figure 11.2(a), the electrons move along the x-axis and the current is along the negative x-direction. Assume that the wire is electrically neutral, that is, the linear charge density for the positive ions has the same magnitude as that for the free electrons, and let the common magnitude be λ_0. Suppose a positive charge q is placed near the wire at a distance y and is moved parallel to the wire with the same speed v as that of the negative charges. A trolley also moves in the lab with a velocity v parallel to the wire.

Not considering lengths of the order of an angstrom, any small portion of the wire, as seen from the lab, is electrically neutral and so does not produce any electric field at the position of the charge q. There is no electric force on q.

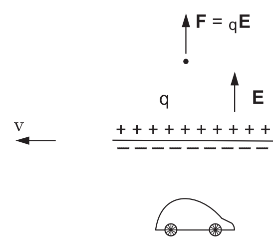


Fig. 11.3

Look at the situation from the trolley frame [Fig. 11.2(b)]. In this frame, the trolley, the charge q and the free electrons are at rest (remember we are talking about systematic velocity and not random velocity). The positive ions are moving with a velocity $-v$. Think of the negative-charge line. It is at rest in the trolley frame and moves with speed v in the lab frame. So the length AB between any two points A and B on the negative-charge line will be larger in the trolley frame than in the lab frame. This means the same charge is contained in a larger length as seen from the trolley frame and in a smaller length as seen from the lab frame. The magnitude of the linear charge density (charge/length) will therefore be smaller in the trolley frame and larger in the lab frame. If we call the linear charge density of the negative charges λ'_- in the trolley frame,

$$\lambda'_- = -\lambda_0\sqrt{1-\frac{v^2}{c^2}}\,. \tag{i}$$

Next consider the line of positive charges. This was at rest in the lab and is moving in the trolley frame. So the length between any two points on this line will be larger in the lab and smaller in the trolley frame. The linear charge density of the positive-charge line will therefore be larger in the trolley frame and smaller in the lab frame. If we call the linear charge density of the positive charges λ'_+ in the trolley frame,

$$\lambda'_+ = \frac{\lambda_0}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{ii}$$

Thus $\lambda'_+ + \lambda'_- \neq 0$ and the wire is electrically charged with linear charge density

$$\lambda' = \lambda'_+ + \lambda'_- = \lambda_0 \left[\frac{v^2/c^2}{\sqrt{1-\frac{v^2}{c^2}}} \right].$$

Accordingly, there will be an electric field $E = \dfrac{\lambda'}{2\pi\varepsilon_0 y}$ in the direction away from the wire in the trolley frame. Call it the y-direction (Figure 11.3).

Hence a force

$$\boldsymbol{F} = q\boldsymbol{E} = \frac{q\lambda'}{2\pi\varepsilon_0 y}\hat{\boldsymbol{j}} = q\lambda_0 \frac{v^2}{c^2\sqrt{1-\frac{v^2}{c^2}}}\frac{1}{2\pi\varepsilon_0 y}\hat{\boldsymbol{j}}$$

$$= qv\frac{\lambda_0 v}{2\pi\varepsilon_0 c^2 y}\frac{1}{\sqrt{1-v^2/c^2}}\hat{\boldsymbol{j}}$$

$$= qv\left(\frac{\mu_0 I}{2\pi y}\right)\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\hat{\boldsymbol{j}}$$

will act on the charge. The charge will thus move away from the wire.

Come back to the lab frame. If the perpendicular distance of the charge q from the wire is continuously increasing in the trolley frame, it should do so in the lab frame also. But there is no electric field in the lab frame. Then why does the charge go away from the wire? There obviously has to be another field that pushes the charge away from the wire.

The relativistic phenomenon of length contraction guarantees that electric current must produce a field other than electric field and this is nothing but what we define as magnetic field. We shall discuss more on how electric field, magnetic field and relativity are related, in a later chapter.

If you remember the equation for the magnetic field from your school physics, you can immediately recognize the quantity in parentheses in equation (i) as the magnitude of the magnetic field due to the current. The force in the trolley frame and that in the lab frame are also different and are related through the force-transformation equation.

11.2 Force on a Charged Particle Moving in a Magnetic Field

The expression for the force on a charged particle due to a magnetic field is slightly more complex than that due to an electric field. First, it is velocity-dependent—there is no magnetic force on a stationary charge even if a magnetic field exists. The next observation is that the magnetic force deflects the moving charged particle and does not increase or decrease its speed. The magnetic force is perpendicular to the velocity. It turns out that the magnetic force on a particle with a charge q, moving with a velocity $\boldsymbol{v}$ at a place where the magnetic field is $\boldsymbol{B}$, is given by

$$\boldsymbol{F}_{\text{mag}} = q\boldsymbol{v} \times \boldsymbol{B}. \tag{11.1}$$

In addition, if there is an electric field $\boldsymbol{E}$ also at the same place, an electric force

$$\boldsymbol{F}_{\text{e}} = q\boldsymbol{E}$$

acts on the particle. The sum of the electric and the magnetic force is then

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}). \tag{11.2}$$

This is also called *Lorentz force*. Carefully study Equation 11.1. The force is velocity-dependent. Are you familiar with any other force that is velocity-dependent? You must have studied at least one such force earlier.

How did physicists come up with Equation 11.1? Through sheer hard work. Many physicists made many observations in the laboratory, leading to this compact and beautiful expression for magnetic force. You can immediately learn a few things from this equation.

(a) The magnetic force is perpendicular to both $\boldsymbol{v}$ and $\boldsymbol{B}$. If you think in terms of field lines, magnetic force is perpendicular to the magnetic field lines, unlike electric force, which is along the electric field lines.

(b) $\boldsymbol{F} \cdot \boldsymbol{v} = 0$. So magnetic force does not do any work on a charged particle and hence does not change the kinetic energy of the particle. In other words, it does not change the speed of the charged particle. It can only change the direction of motion.

Unit of *B*

From the equation $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$, the SI unit of the field $\boldsymbol{B}$ is newton–second/(coulomb–metre), or N/(A m). There is a separate name for this unit, *tesla*, the symbol of which is T. The tesla is a large unit for most common situations. You know that the earth produces a magnetic field in its surroundings. That is why a compass needle, away from all magnets and magnetic materials, points in the north–south direction. The typical value of the earth's magnetic field near its surface is around 5×10^{-5} T or $50\,\mu$T. Bar magnets commonly used in school laboratories produce a magnetic field of the same order near their poles. Nowadays we have specially made Ni–Fe–B magnets or rare-earth magnets which give fields of the order of 0.5 T near their poles. These are termed very strong permanent magnets. Larger magnetic fields are created by passing large currents in certain coils. Large currents also mean that the coils get heated considerably and hence an arrangement for continuous cooling is an essential part of such systems. Magnetic fields of the order of 20 T are used by researchers to investigate several properties of materials. For such large fields, currents in superconducting coils are used.

Another unit called *gauss* is also in use. The relation between the gauss and the tesla is

$$1\text{ T} = 10\text{ kG} = 10^4\text{ G}.$$

11.3 Examples of Motion of Charged Particles in a Magnetic Field

Motion of a particle in a uniform magnetic field

Depending on the initial conditions, the motion of a charged particle in a uniform magnetic field will have different characters. The path the particle takes is governed by the force law $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$ and the initial conditions. Suppose a particle of mass m and charge q moves in a uniform magnetic field of magnitude B_0. Take the direction of the magnetic field as the z-axis. The force on the particle at time t is

$$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B} = q(v_x\,\hat{\boldsymbol{i}} + v_y\,\hat{\boldsymbol{j}} + v_z\,\hat{\boldsymbol{k}}) \times B_0\,\hat{\boldsymbol{k}}$$

$$= -qB_0 v_x\,\hat{\boldsymbol{j}} + qB_0 v_y\,\hat{\boldsymbol{i}}.$$

Thus $\quad F_x = qB_0 v_y,$

$$F_y = -qB_0 v_x$$

and $\quad F_z = 0.$

Then, $$\frac{dv_x}{dt} = \frac{qB_0}{m} v_y, \tag{i}$$

$$\frac{dv_y}{dt} = -\frac{qB_0}{m} v_x \tag{ii}$$

and $$\frac{dv_z}{dt} = 0. \tag{iii}$$

This last equation gives v_z = constant.

There is no acceleration along the z-direction. If the particle is given an initial velocity that has a z-component, this component will remain constant. If the z-component of the initial velocity is zero, it will remain zero, meaning that the motion will be confined to the x–y plane (or a plane parallel to it).

(a) Particle projected perpendicular to the field Suppose, at time $t = 0$, the particle is given a velocity in the x–y plane. You can as well take the direction of this initial velocity to be along the x-axis. Then, at $t = 0$, $v_x = v_0$ and $v_y = v_z = 0$. Also take the point where the particle is located at $t = 0$ as the origin. From equation (iii), v_z remains zero and the motion continues in the x–y plane.

Now look at equations (i) and (ii). Multiply equation (ii) by i (that is, $\sqrt{-1}$) and add to equation (i).

$$\frac{d}{dt}(v_x + iy) = \frac{qB_0}{m}(v_y - iv_x) = -\frac{iqB_0}{m}(v_x + iv_y).$$

Writing $v_x + iv_y = V$, this becomes

$$\frac{dV}{dt} = -i\frac{qB_0}{m}V$$

or $$\frac{dV}{V} = -i\frac{qB_0}{m}dt$$

or $$\frac{dV}{V} = -i\omega dt,$$

where $\omega = qB_0/m$.

This gives

$$V = V_0 e^{-i\omega t},$$

where V_0 is the value of V at time $t = 0$. At $t = 0$, $v_x = v_0$ and $v_y = 0$. Therefore, $V_0 = v_0$ and so,

$$V = v_0 e^{-i\omega t} \tag{iv}$$

or $$v_x + iv_y = v_0 \cos \omega t - iv_0 \sin \omega t$$

or $$v_x = v_0 \cos \omega t \tag{v}$$

and $$v_y = -v_0 \sin \omega t. \tag{vi}$$

You can also get these equations by differentiating equation (i) with respect to time and putting the value of $\frac{dv_y}{dt}$ from equation (ii). This gives

$$\frac{d^2 v_x}{dt^2} = -\left(\frac{qB_0}{m}\right)^2 v_x = -\omega^2 v_x. \tag{vii}$$

You must have encountered a similar equation while studying the simple harmonic motion of a particle. The acceleration $\frac{d^2 x}{dt^2}$ of such a particle is written as

$$\frac{d^2 x}{dt^2} = -\omega^2 x,$$

which gives $x = A\cos(\omega t + \phi)$. You can write the solution of equation (vii) as $v_x = A\cos(\omega t + \phi)$ and using the initial conditions, $v_x = v_0 \cos \omega t$, which reproduces equation (v). Similarly for v_y.

Write $v_x = \frac{dx}{dt}$ and $v_y = \frac{dy}{dt}$ and solve for x and y.

From equation (v),

$$x = \frac{v_0}{\omega} \sin \omega t + x_0.$$

As $x = 0$ at $t = 0$, x_0 becomes zero. So,

$$x = \frac{v_0}{\omega} \sin \omega t = \frac{mv_0}{qB_0} \sin\left(\frac{qB}{m} t\right). \tag{viii}$$

From equation (vi),

$$y = \frac{v_0}{\omega} \cos \omega t + y_0.$$

At $t = 0$, $y = 0$. So $y_0 = -\frac{v_0}{\omega}$, and therefore,

$$y + \frac{v_0}{\omega} = \frac{v_0}{\omega} \cos \omega t$$

or

$$y + \frac{mv_0}{qB} = \frac{mv_0}{qB} \cos \omega t. \tag{ix}$$

Equations (viii) and (ix) give the position of the particle as a function of time.

Square both sides of equations (viii) and (ix) and add to get

$$x^2 + \left(y + \frac{mv_0}{qB}\right)^2 = \left(\frac{mv_0}{qB}\right)^2. \tag{x}$$

This is the equation of the circle with the centre at $\left(0, -\frac{mv_0}{qB}\right)$ and radius equal to

$$r = \frac{mv_0}{qB}. \tag{11.3}$$

Thus the particle moves in a circular path of radius $\frac{mv_0}{qB}$ as shown in Figure 11.4. The dots in the small circles show that the magnetic field is perpendicular to the plane of the drawing, coming out of the page. From equations (v) and (vi),

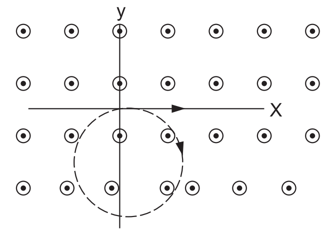


Fig. 11.4

$$v^2 = v_x^2 + v_y^2 = v_0^2,$$

showing that the speed of the particle on the circle remains the same. You already know that a magnetic field is incapable of changing the speed of a particle. Thus a particle projected perpendicular to a uniform magnetic field moves with a uniform speed along a circular path. The angular velocity of the motion is

$$\omega = \frac{qB}{m}, \tag{11.4}$$

giving the time period

$$T = \frac{2\pi}{\omega} = \frac{2\pi m}{qB}. \tag{11.5}$$

As you can see, it is independent of the speed v_0. Project the particle at any speed—it will complete the circle within the same time. The radius, of course, depends on the speed (Equation11.3).

(b) Particle projected in arbitrary direction If the initial velocity also has a z-component u_z, this component remains unchanged.

$$v_z = u_z$$

or $$z = u_z t.$$

The equations for the x- and y-direction remain the same as derived in part (a). So the particle rotates with angular velocity $\omega = qB/m$ in the x–y plane and at the same time drifts along the z-direction with uniform speed u_z. It thus describes a helical path (a path shaped like a spring) with pitch

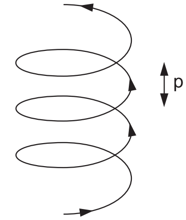


Fig. 11.5

$$p = u_z T = \frac{2\pi m u_z}{qB}.$$

<u>EXAMPLE 11.1</u> **A particle of charge +e and mass $4m_p$ moves with a kinetic energy 2 MeV (m_p = mass of proton). It has to be bent in an arc of radius 10 cm . Find the magnetic field required to do that.**

Solution You have $r = \dfrac{mv_0}{qB} = \dfrac{\sqrt{2mK}}{qB}$,

where $K = \dfrac{1}{2}mv^2$ is the kinetic energy of the particle. Thus,

$$B = \frac{\sqrt{2mK}}{qr} = \frac{\sqrt{2\times 4\times 1.67\times 10^{-27}\times 2\times 10^{6}\times 1.6\times 10^{-19}}}{1.6\times 10^{-19}\times 10^{-1}}\ \mathrm{T}$$

$$= 4.1\ \mathrm{T}.$$

The ability of a magnetic field to turn the direction of motion of charged particles has many useful applications. Let us talk about a few of these.

(i) The mass spectrometer The mass spectrometer is an instrument used to separate out particles according to their masses in a gas of ions. Even if the starting material is in solid form, it can be vaporized by heating and then ionized to get a fixed charge like +e, +2e, etc., on the ions. The ions of various masses constituting the ionized gas are accelerated by an electric field to gain some velocity and are then injected into a region of constant magnetic field. Components with different masses move in circular paths of different radii. Generally, after completing a semicircle, they are collected by a detector system. As the radii are different, particles of different masses collect at different places.

For example, natural iron has about 2% $^{57}\mathrm{Fe}$ while almost all the rest is $^{56}\mathrm{Fe}$. For certain scientific experiments, one needs $^{57}\mathrm{Fe}$ alone or iron with an increased percentage of this isotope. Mass spectrometers can help separate it out. While studying air and water pollution, one needs to know the composition of certain elements in trace amounts. Once again, the mass spectrometer can be utilized to separate out the species on the basis of mass and analyze the quantities.

(ii) e-beam evaporation For several studies, thin films of materials are deposited on a given substrate. One method of doing this is to place the material in a crucible and the substrate at a distance above the crucible (Figure 11.6). The whole system is in a chamber, which is evacuated to low pressures. The material in the crucible is heated to take it to the vapour phase. The vapours go up, get deposited on the substrate and make a thin film.

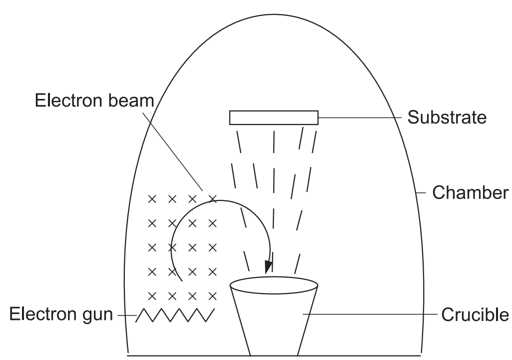


Fig. 11.6

A very effective way to heat the solid is to hit it with energetic electrons. A high voltage attracts the electrons to give them the required kinetic energy. A cathode is made to eject electrons by passing a current through a filament. The cathode and the filament are also placed in the chamber. The electrons start out as a beam (Figure 11.6) and then a magnetic field applied in the region bends the beam, making it fall into the crucible. In a few seconds, the solid melts and then vaporizes.

(iii) Accelerator beam lines Accelerated ions are widely used by scientists and engineers for a variety of purposes. The ions are made to hit a solid target and the modification made in the structure of the target can lead to interesting changes in its properties. In an accelerator lab, ions are accelerated and made to attain the desired kinetic energy in an accelerating tube. Using this ion beam, several types of experiments can be carried out, each of which needs different kinds of elaborate arrangements in the lab. The ions from the accelerating tube are made to go through a region in which a perpendicular magnetic field is applied. By applying magnetic fields of different strengths, the beam can be bent in different ways. A number of tubes are attached on the other side of the magnetic field at different angles. These tubes are called beam lines and are named according to the kind of experimental arrangement made at the end of the tube.

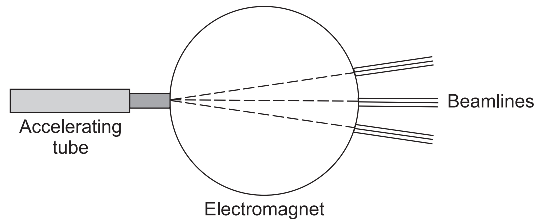


Fig. 11.7

When the beam is needed for a particular kind of experiment, an appropriate magnetic field is applied to turn the direction of the ions and direct the beam to the corresponding beam line.

The motion of a particle in a nonuniform magnetic field

(a) The motion of cosmic-ray particles in the earth's magnetic field A variety of particles arrive at the earth from all directions. Called cosmic-ray particles, these include high-energy protons, mesons and other particles. Those that arrive in the upper atmosphere from outside are called primary cosmic-ray particles. They interact with the molecules of the atmosphere and produce secondary cosmic-ray particles. All these cosmic-ray particles pass through the magnetic field of the earth before reaching the earth's surface. They are experimentally studied by sending balloons into the atmosphere.

The magnetic field due to the earth for points outside is like a dipole field as shown in Figure 11.8. The charged particles that move towards the geographical north pole of the earth (position a in Figure 11.8) or south pole (position b in Figure 11.8) move parallel to the magnetic field and hence magnetic force does not act on them. They reach the earth's surface at the polar regions vertically.

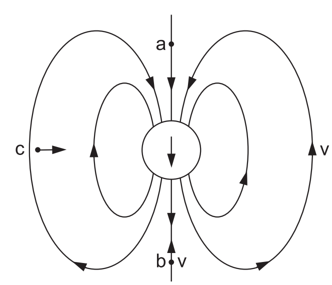


Fig. 11.8

On the other hand, particles that head towards an equatorial place (position c in Figure 11.8) are acted upon by a perpendicular magnetic field. The magnetic field at a place on the equator is horizontal and is towards the north. So the force $q\boldsymbol{v}\times\boldsymbol{B}$ on the positively charged particles is along the east and these particles are deflected toward the east. Many of the particles are deflected so much that they miss the earth's surface and go back. Thus the intensity of cosmic-ray particles reaching the earth's surface in the polar region is larger than the intensity of the particles reaching the equatorial region. This is called 'latitude effect'. Also, because of deflection towards the east, the particles with positive charge, which do manage to reach the surface of the earth, appear to come from the west. It has been found experimentally that cosmic-ray particles arrive at the equator preferentially from the west. This is called the 'east–west' effect. This provides evidence that cosmic-ray particles primarily have positive charge.

(b) Magnetic confinement Suppose there is a nonuniform magnetic field that may be represented by the magnetic field lines shown in Figure 11.9 (solid lines). The field is weak in the middle and strong near the ends. The field lines are more widely spaced in the middle than near the ends. Also, the direction of the field at points not on the x-axis keeps changing as you move parallel to the x-axis.

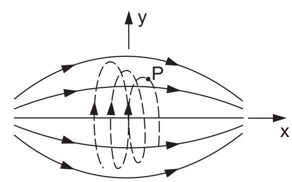


Fig. 11.9

Put a positively charged particle in the middle of the field and give it a velocity that is largely in the y–z plane but has a small component in the x-direction. Its path can be approximately described as helical—the particle moves in nearly circular paths in the y–z plane and moves ahead in the x-direction steadily. In the situation described in Figure 11.9, the path shown is clockwise as seen from the right end. That you can see from the force $q\boldsymbol{v}\times\boldsymbol{B}$. As the particle moves along the x-axis, the magnetic field becomes stronger and this reduces the radius of the helix. So it largely remains inside the region and does not go out from the sides. What about its motion in the x-direction? Consider the particle at point P in Figure 11.9. The field at the position of charge, that is P, has a positive x-component and a negative y-component. You can write this field as

$$\boldsymbol{B} = B_1\hat{\boldsymbol{i}} - B_2\hat{\boldsymbol{j}}$$

and the velocity as $\quad \boldsymbol{v} = v_1\hat{\boldsymbol{i}} - v_2\hat{\boldsymbol{k}}.$

The force on the particle is

$$\begin{aligned} q\boldsymbol{v}\times\boldsymbol{B} &= q[v_1\hat{\boldsymbol{i}} - v_2\hat{\boldsymbol{k}}]\times[B_1\hat{\boldsymbol{i}} - B_2\hat{\boldsymbol{j}}] \\ &= -qv_2B_1\hat{\boldsymbol{j}} - qv_1B_2\hat{\boldsymbol{k}} - qv_2B_2\hat{\boldsymbol{i}}. \end{aligned}$$

As you can see, the x-component of the force is negative. So, as the particle moves towards the x-axis, it is pushed back. If the starting x-component of velocity is not too high, it turns back towards the central region. And if it proceeds towards the left, where the field is strong, it is pushed back towards the central region.

So the converging field reflects the particle back into the main volume. A nonuniform magnetic field can, therefore, be used to confine charged particles within a volume not having a material boundary for quite some time. Confining charged particles within a volume without a material boundary is crucial for energy production from nuclear fusion. A lot of research is being done to design effective geometries of nonuniform magnetic fields so that the fusing nuclei at high temperatures ($\approx 10^7$ K) can be confined to a finite volume for a reasonably long time. At such high temperatures, one cannot have a material boundary to confine the particles.

Although the principle of magnetic confinement is well demonstrated by a magnetic field of the kind discussed above, this is not an efficient design to confine hot plasma for a long time. The most popular design is the Tokomak design, in which the magnetic field is primarily toroidal but superimposed by a transverse surrounding field.

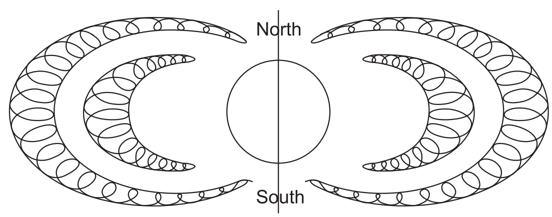


Fig. 11.10

(c) Van Allen radiation belts As you know, the earth's magnetic field is nonuniform. It is stronger near the polar regions and weaker near the equatorial region. Charged particles like protons or electrons in the atmosphere are often trapped by such a nonuniform field. Figure 11.10 shows such a region in cross section; in 3-D, it is a ring-type region surrounding the earth. The particle spirals and moves ahead in the region. But at some stage, the field becomes strong enough to push it back into the central region. With the speed achieved, it moves towards the other side and is again sent back by the stronger field. It keeps bouncing back and forth between the two near ends of the region. It takes only a few seconds for the particles to make an oscillation. Experimental evidence for such trapped particles was obtained by James A Van Allen and his collaborators in 1958 and hence such regions are called Van Allen belts. Researchers have found two such belts, one at an altitude of about 1000 km to 6000 km and the other at 13,000 km to 60,000 km. These are called the inner and outer Van Allen radiation belts.

11.4 Magnetic Force on a Current Element

Consider a linear electric current I. You can assume that there is a linear charge density λ along a particular curve and that these charges move with a velocity v along this curve. Then the current is given by

$$I = \lambda v.$$

Think of a small element $\boldsymbol{dl}$ along the current. The charge contained in this element at any time will be $dq = \lambda dl$. If a magnetic field $\boldsymbol{B}$ exists here due to all other currents, the magnetic force on this element is

$$\boldsymbol{dF} = (dq)\boldsymbol{v}\times\boldsymbol{B} = (\lambda dl)\boldsymbol{v}\times\boldsymbol{B}$$
$$= (\lambda v)\boldsymbol{dl}\times\boldsymbol{B} = I\boldsymbol{dl}\times\boldsymbol{B}. \tag{11.6}$$

This is the basic equation for the force on a linear current in a magnetic field. If you have to find the force on an extended length of the wire, you should integrate this expression under proper limits.

Suppose you have a current-carrying wire placed in a uniform magnetic field, as shown in Figure 11.11. The magnetic force on the part PRQ of the wire is

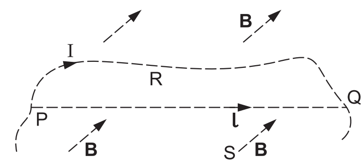


Fig. 11.11

$$\boldsymbol{F} = \int I\boldsymbol{dl}\times\boldsymbol{B} = I\left(\int \boldsymbol{dl}\right)\times\boldsymbol{B}$$
$$= I\boldsymbol{PQ}\times\boldsymbol{B} = I\boldsymbol{l}\times\boldsymbol{B}, \tag{11.7}$$

where $\boldsymbol{l}$ is the vector joining the ends of the portion of the wire on which the force is calculated. Whatever be the shape of the portion of the wire, the force depends only on the vector joining the ends of this portion.

If you have a surface current with surface current density $\boldsymbol{K}$, the force on an area da of this surface, when placed in a magnetic field $\boldsymbol{B}$, will be

$$\boldsymbol{dF} = \boldsymbol{K}da\times\boldsymbol{B}. \tag{11.8}$$

If the current is distributed in a volume with volume current density $\boldsymbol{J}$, and there is an external magnetic field $\boldsymbol{B}$ there, the force on a volume element $d\tau$ will be

$$\boldsymbol{dF} = \boldsymbol{J}d\tau\times\boldsymbol{B}. \tag{11.9}$$

11.5 Force on a Current Loop

Suppose an electric current I is established in a closed loop, and the loop is placed in a uniform magnetic field $\boldsymbol{B}$. Then the force on this loop due to the magnetic field

$$\int I\boldsymbol{dl}\times\boldsymbol{B} == I\left(\int \boldsymbol{dl}\right)\times\boldsymbol{B} = 0.$$

What happens if a current loop is placed in a nonuniform magnetic field? There is no general expression for the force on such a loop. But if the current loop is small in size, one can derive an expression. This will be done in the last section of the chapter.

11.6 Torque on a Current-Carrying Loop

The net force on a current loop placed in a uniform magnetic field is zero, but the net torque of the magnetic force need not be zero. We will first derive the expression for the torque on a rectangular loop placed in a uniform magnetic field and then, for a general plane loop.

Torque on a rectangular current loop

Suppose a rectangular loop of sides L and D carries a current I. Take the centre of the loop as the origin and lines parallel to the edges as the x- and y-axis (Figure 11.12). How do you get a current in the loop without putting a battery? The complete circuit is not shown here. Think of a circuit that can send a current in a rectangular loop. Let there be a uniform magnetic field $\boldsymbol{B} = B_1\hat{\boldsymbol{i}} + B_2\hat{\boldsymbol{j}} + B_3\hat{\boldsymbol{k}}$ in the region due to all the other currents. The magnetic forces on the four arms of the loop are

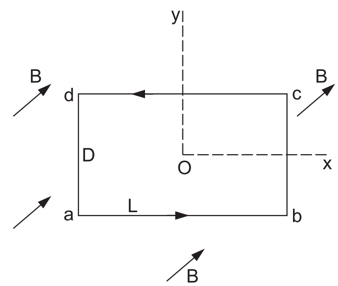


Fig. 11.12

ab: $\boldsymbol{F_1} = IL\,\hat{\boldsymbol{i}} \times (B_1\hat{\boldsymbol{i}} + B_2\hat{\boldsymbol{j}} + B_3\hat{\boldsymbol{k}}) = IL(B_2\hat{\boldsymbol{k}} - B_3\hat{\boldsymbol{j}})$

bc: $\boldsymbol{F_2} = ID\hat{\boldsymbol{j}} \times (B_1\hat{\boldsymbol{i}} + B_2\hat{\boldsymbol{j}} + B_3\hat{\boldsymbol{k}}) = ID(-B_1\hat{\boldsymbol{k}} + B_3\hat{\boldsymbol{i}})$

cd: $\boldsymbol{F_3} = -IL\hat{\boldsymbol{i}} \times (B_1\hat{\boldsymbol{i}} + B_2\hat{\boldsymbol{j}} + B_3\hat{\boldsymbol{k}}) = -IL(B_2\hat{\boldsymbol{k}} - B_3\hat{\boldsymbol{j}})$

da: $\boldsymbol{F_4} = -ID\hat{\boldsymbol{j}} \times (B_1\hat{\boldsymbol{i}} + B_2\hat{\boldsymbol{j}} + B_3\hat{\boldsymbol{k}}) = -ID(-B_1\hat{\boldsymbol{k}} + B_3\hat{\boldsymbol{i}})$.

You can assume that these forces act at the midpoints of the respective arms. You can do so because the force on any arm is uniformly distributed over the whole of its length. The points of application of the forces can therefore be taken as

midpoint of ab: $\boldsymbol{r_1} = -\dfrac{D}{2}\hat{\boldsymbol{j}}$,

midpoint of bc: $\boldsymbol{r_2} = \dfrac{L}{2}\hat{\boldsymbol{i}}$,

midpoint of cd: $\boldsymbol{r_3} = \dfrac{D}{2}\hat{\boldsymbol{j}}$, and

midpoint of da: $\boldsymbol{r_4} = -\dfrac{L}{2}\hat{\boldsymbol{i}}$.

The torques of the forces about the origin are

$$\tau_1 = \boldsymbol{r}_1 \times \boldsymbol{F}_1 = -\frac{D}{2}\hat{\boldsymbol{j}} \times IL(B_2\hat{\boldsymbol{k}} - B_3\hat{\boldsymbol{j}}) = -\frac{ILD}{2}B_2\hat{\boldsymbol{i}},$$

$$\tau_2 = \boldsymbol{r}_2 \times \boldsymbol{F}_2 = \frac{L}{2}\hat{\boldsymbol{i}} \times ID(-B_1\hat{\boldsymbol{k}} + B_3\hat{\boldsymbol{i}}) = \frac{ILD}{2}B_1\hat{\boldsymbol{j}},$$

$$\tau_3 = \boldsymbol{r}_3 \times \boldsymbol{F}_3 = \frac{D}{2}\hat{\boldsymbol{j}} \times (-IL)(B_2\hat{\boldsymbol{k}} - B_3\hat{\boldsymbol{j}}) = -\frac{ILD}{2}B_2\hat{\boldsymbol{i}},$$

and $$\tau_4 = \boldsymbol{r}_4 \times \boldsymbol{F}_4 = \left(-\frac{L}{2}\hat{\boldsymbol{i}}\right) \times (-ID)(-B_1\hat{\boldsymbol{k}} + B_3\hat{\boldsymbol{i}}) = \frac{ILD}{2}B_1\hat{\boldsymbol{j}}.$$

The net torque is

$$\tau = -ILDB_2\hat{\boldsymbol{i}} + ILDB_1\hat{\boldsymbol{j}} = IA(B_1\hat{\boldsymbol{j}} - B_2\hat{\boldsymbol{i}})$$
$$= IA\hat{\boldsymbol{k}} \times (B_1\hat{\boldsymbol{i}} + B_2\hat{\boldsymbol{j}} + B_3\hat{\boldsymbol{k}}) = IA\hat{\boldsymbol{k}} \times \boldsymbol{B},$$

where $A = LD$ is the area of the loop. Writing $A\hat{\boldsymbol{k}}$ as the area vector $\boldsymbol{A}$, the torque on the loop can be expressed, without reference to a specific choice of axes, as

$$\tau = I\boldsymbol{A} \times \boldsymbol{B}. \tag{11.10}$$

Remember, the direction of the area vector $\boldsymbol{A}$ is perpendicular to the area and can be obtained by curling the fingers of your right hand along the current. The direction of the area vector will be along the stretched thumb. Else, you can look at the loop from the perpendicular direction. If the current is anticlockwise, the area vector is towards you. If the current is clockwise, the area vector is away from you.

Torque on a general plane-current loop

Consider a plane-current loop in the x–y plane. The current in the loop is I. A uniform magnetic field $\boldsymbol{B}$ exists in space. Consider any small element $\boldsymbol{dl}$ on the loop at position $\boldsymbol{r}$ (Figure 11.13). The force on the element due to the magnetic field is

$$\boldsymbol{dF} = I\boldsymbol{dl} \times \boldsymbol{B}.$$

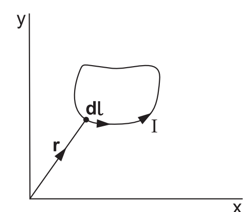


Fig. 11.13

The torque of this force about the origin is

$$\boldsymbol{d\tau} = \boldsymbol{r} \times \boldsymbol{dF} = I\boldsymbol{r} \times (\boldsymbol{dl} \times \boldsymbol{B}).$$

Using the identity

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C},$$

$$\boldsymbol{d\tau} = I[(\boldsymbol{r} \cdot \boldsymbol{B})\boldsymbol{dl} - (\boldsymbol{r} \cdot \boldsymbol{dl})\boldsymbol{B}].$$

The torque on the entire loop is

$$\tau = I\oint(\boldsymbol{r} \cdot \boldsymbol{B})\boldsymbol{dl} - I\left(\oint \boldsymbol{r} \cdot \boldsymbol{dl}\right)\boldsymbol{B}.$$

Now $$\oint \boldsymbol{r} \cdot \boldsymbol{dl} = \int (\nabla \times \boldsymbol{r}) \cdot \boldsymbol{da}$$ [using Stokes' theorem, $\oint \boldsymbol{A} \cdot \boldsymbol{dl} = \int (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{da}$]

$$= \nabla \times \boldsymbol{r} = 0.$$

Hence $$\oint \boldsymbol{r} \cdot \boldsymbol{dl} = 0.$$

So, $$\tau = I\oint(\boldsymbol{r} \cdot \boldsymbol{B})\boldsymbol{dl}. \tag{i}$$

To simplify $\oint(\boldsymbol{r}\cdot\boldsymbol{B})\boldsymbol{dl}$, use explicit Cartesian coordinates.

$$\oint(\boldsymbol{r}\cdot\boldsymbol{B})\boldsymbol{dl} = \oint[x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}})\cdot(B_1\hat{\boldsymbol{i}} + B_2\hat{\boldsymbol{j}} + B_3\hat{\boldsymbol{k}})][dx\hat{\boldsymbol{i}} + dy\hat{\boldsymbol{j}}]$$

$$= \oint(xB_1 + yB_2)[dx\hat{\boldsymbol{i}} + dy\hat{\boldsymbol{j}}]$$

$$= \hat{\boldsymbol{i}}B_1\oint x\,dx + \hat{\boldsymbol{j}}B_2\oint y\,dy + \hat{\boldsymbol{j}}B_1\oint x\,dy + \hat{\boldsymbol{i}}B_2\oint y\,dx. \tag{ii}$$

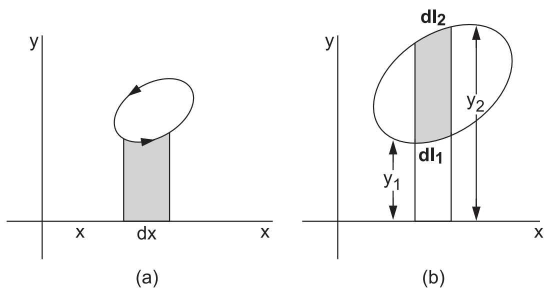


Fig. 11.14

For a closed loop, the integral $\oint x\,dx = \oint y\,dy = 0$. Now look at $\oint y\,dx$ with reference to Figure 11.14. The quantity $y\,dx$ represents the area under the small part $\boldsymbol{dl}$ of the curve [shown shaded in Figure 11.14(a)]. As it is a closed loop, you will traverse the point x twice—in one case dx will be positive and in the other case, negative. In Figure 11.14(b), such a pair is shown by $\boldsymbol{dl}_1$ and $\boldsymbol{dl}_2$. Convince yourself that $(y_1dx + y_2dx)$ is negative and its magnitude gives the area of the shaded strip. $\oint y\,dx$ is therefore equal to $(-A)$, where A is the area of the loop. Similarly, $\oint x\,dy = A$. From (ii),

$$\oint(\boldsymbol{r}\cdot\boldsymbol{B})\boldsymbol{dl} = \hat{\boldsymbol{j}}B_1A - \hat{\boldsymbol{i}}B_2A = A\hat{\boldsymbol{k}}\times(B_1\hat{\boldsymbol{i}} + B_2\hat{\boldsymbol{j}}) = \boldsymbol{A}\times\boldsymbol{B}.$$

Thus, from (i),

$$\tau = I\boldsymbol{A}\times\boldsymbol{B},$$

which is the same as Equation 11.10.

The quantity $I\boldsymbol{A}$ is called the *magnetic dipole moment* of the loop and is written as μ. Then the torque due to a uniform magnetic field $\boldsymbol{B}$ on a current loop is

$$\tau = \mu\times\boldsymbol{B} \tag{11.11}$$

The torque tries to line up the vector μ in the direction of the magnetic field. Do you remember the expression for the torque on an electric dipole due to a uniform electric field $\boldsymbol{E}$? It is

$$\tau = \boldsymbol{p}\times\boldsymbol{E},$$

where $\boldsymbol{p}$ is the electric dipole moment. Equation 11.11 has the same mathematical structure, justifying the name 'magnetic dipole moment' for a current-carrying loop. More about this in later chapters.

If the loop has more than one turn, its magnetic dipole moment will be NIA, where N is the number of turns. Often, one uses the term "magnetic moment" for magnetic dipole moment.

11.7 General Expression for Magnetic Dipole Moment

The magnetic dipole moment of a current loop is IA. Consider a plane loop carrying a current I (Figure 11.15). Let the area vector corresponding to the area of the loop be $\boldsymbol{A}$. Take the origin to be somewhere in the plane area bounded by the loop. Take an element $\boldsymbol{dl}$ on the loop. Join the ends to the origin. The area of this nearly triangular strip is $dA = \frac{1}{2}|\boldsymbol{r} \times \boldsymbol{dl}|$. Look at the vector $\frac{1}{2}\oint \boldsymbol{r} \times \boldsymbol{dl}$. The direction of each part $\frac{1}{2}\boldsymbol{r} \times \boldsymbol{dl}$ is the same as that of $\boldsymbol{A}$ and hence the direction of $\frac{1}{2}\oint \boldsymbol{r} \times \boldsymbol{dl}$ is the same as that of $\boldsymbol{A}$. Also, the magnitude of $\frac{1}{2}\oint \boldsymbol{r} \times \boldsymbol{dl}$ is the same as the total area of the plane surface bounded by the loop. Thus $\boldsymbol{A} = \frac{1}{2}\oint \boldsymbol{r} \times \boldsymbol{dl}$. The magnetic dipole moment of the current loop is, therefore,

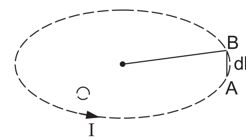


Fig. 11.15

$$\mu = \frac{1}{2} I \oint_{\text{loop}} \boldsymbol{r} \times \boldsymbol{dl}. \tag{11.12}$$

It turns out that this is the general expression for the magnetic dipole moment of a current loop. Even if the loop is not in a plane, you can use this expression. Taking the origin to be inside the loop is not necessary. By taking proper signs you can check that $\frac{1}{2}\oint \boldsymbol{r} \times \boldsymbol{dl}$ is still the area vector $\boldsymbol{A}$.

Equation 11.12 can be generalized for surface currents and volume currents too. Suppose you have a current distribution on a surface with surface current density $\boldsymbol{K}$. The magnetic dipole moment of this current distribution is obtained by replacing $I\boldsymbol{dl}$ by $\boldsymbol{K}da$. Thus

$$\mu = \frac{1}{2} \int_{\text{surface}} (\boldsymbol{r} \times \boldsymbol{K}) da. \tag{11.13}$$

Similarly, if you have a volume current distribution with volume current density $\boldsymbol{J}$, the magnetic dipole moment of this distribution is

$$\mu = \frac{1}{2} \int_{\text{volume}} (\boldsymbol{r} \times \boldsymbol{J}) d\tau. \tag{11.14}$$

11.8 The Pole Picture

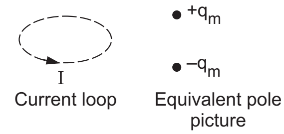


Fig. 11.16

The torque expression $\boldsymbol{p} \times \boldsymbol{E}$ for an electric dipole of dipole moment $\boldsymbol{p}$ was derived by taking the dipole to be a combination of two point charges $-q$ and $+q$ separated by a distance d such that $qd = p$. The expression for the torque on a current loop $\mu \times \boldsymbol{B}$ has a similar mathematical structure. One can, therefore, think of an imaginary model, equivalent to the current loop for the purpose of calculating torques, in terms of a combination of two magnetic poles (point sources) of strengths $-q_m$ and $+q_m$ separated by a distance d such that $q_m d = \mu$ (Figure 11.16). You can also call q_m magnetic charge. Like $\boldsymbol{F} = q\boldsymbol{E}$ for the force on an electric charge, one may assume $\boldsymbol{F} = q_m\boldsymbol{B}$ for the force on a magnetic pole. You can then derive the expression $\mu \times \boldsymbol{B}$ for the torque on the current loop in exactly the same way as used to derive the expression $\boldsymbol{p} \times \boldsymbol{E}$ for the torque on the electric dipole made of charges $-q$ and $+q$. The magnetic field due to a point magnetic pole q_m can be written as $\boldsymbol{B} = \frac{\mu_0}{4\pi} \frac{q_m(\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3}$, with the symbols having their usual meanings.

Customarily, the positive magnetic pole is called the north pole and the negative magnetic pole, the south pole—they are denoted by N and S.

The model in terms of magnetic poles is useful in many other situations too. For example, the force on a small current loop placed in an inhomogeneous magnetic field can be derived in exactly the same way as done for the force on an electric dipole in an inhomogeneous electric field (Chapter 4). It is given by

$$\boldsymbol{F} = (\mu \cdot \nabla)\boldsymbol{B} = \left[\mu_x \frac{\partial}{\partial x} + \mu_y \frac{\partial}{\partial y} + \mu_z \frac{\partial}{\partial z}\right]\boldsymbol{B}, \tag{11.13}$$

where μ is the dipole moment.

If the magnetic moment is in the z-direction,

$$\boldsymbol{F} = \mu_z \frac{\partial \boldsymbol{B}}{\partial z} \cdot \tag{11.14}$$

The permanent magnet is familiar to all of us. The ability to attract iron pins is much more at the ends of the magnet. Freely hanging magnets point towards the geographical north and south directions. The end pointing towards the north is called the north pole and the one pointing towards the south is called the south pole. Do they have a relation with the imaginary magnetic poles discussed here? Yes, but more on this topic later.

EXAMPLE 11.2 **The magnetic field lines due to a bar magnet of cylindrical shape are shown in Figure 11.17. A magnetic dipole of dipole moment *m* is placed on the pole face near the edge as shown. Will the dipole be drawn towards the axis or will it be pushed away from the axis?**

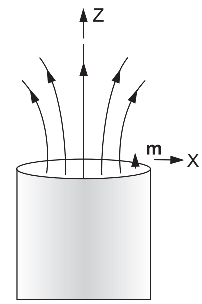


Fig. 11.17

Solution Taking the axes as shown in the figure,

$$\boldsymbol{m} = m\hat{\boldsymbol{k}}.$$

The force on the dipole is

$$\boldsymbol{F} = (\mu \cdot \nabla)\boldsymbol{B} = \mu_z \frac{\partial \boldsymbol{B}}{\partial z}\cdot$$

The x-component of this force is

$$F_x = \mu_z \frac{\partial B_x}{\partial z}\cdot \qquad \text{(i)}$$

Think of two points A and B at the same x (where the dipole is placed) but at slightly different values of z, as shown in a magnified way in Figure 11.18. The field line through the point B is less curved, the angle of the field with the vertical is smaller. Also, the magnitude of the field at B is smaller than that at A because of distance. Thus the x-component of the field at B is smaller than that at A, making $\frac{dB_x}{dz}$ negative. From (i), the x-component of the force F_x is negative and hence the dipole will be drawn towards the axis.

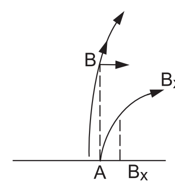


Fig. 11.18

11.9 The Energy of a Magnetic Dipole in a Magnetic Field

The expression for the torque on a magnetic dipole placed in a uniform magnetic field $\boldsymbol{B}$ is $\mu \times \boldsymbol{B}$. This means that if the dipole moment μ makes an angle θ with the magnetic field $\boldsymbol{B}$, the torque will be $\mu B \sin\theta$ to turn the dipole towards the field. As we did in the case of an electric dipole and electric field, we can talk about change in potential energy as the dipole is rotated in the field. Taking the potential energy to be zero when the dipole moment is perpendicular to the field, the general expression for potential energy will be

$$U = -\mu \cdot \boldsymbol{B}. \qquad (11.15)$$

11.10 Gyromagnetic Ratio

Think of a volume τ containing a material with mass density $d(\boldsymbol{r})$ and charge density $\rho(\boldsymbol{r})$. The parts of this distribution move with velocity $v(\boldsymbol{r})$, resulting in a current density

$$\boldsymbol{J}(\boldsymbol{r}) = \rho v(\boldsymbol{r}).$$

Because of the motion, there will be an angular momentum $\boldsymbol{L}$ and because of the current, there will be a magnetic dipole moment μ. If you can express the magnetic dipole moment as $\mu = \gamma \boldsymbol{L}$, the constant γ is called the gyromagnetic ratio for the system.

Suppose $\rho(\boldsymbol{r})/d(\boldsymbol{r}) = k$ is a constant. So, wherever there is mass, there is also a proportionate charge.

The magnetic dipole moment is

$$\begin{aligned}\mu &= \frac{1}{2}\int \boldsymbol{r}\times \boldsymbol{J}d\tau \\ &= \frac{1}{2}\int \boldsymbol{r}\times \rho \boldsymbol{v}d\tau \\ &= \frac{k}{2}\int \boldsymbol{r}\times d\boldsymbol{v}d\tau.\end{aligned}$$

The angular momentum is

$$\begin{aligned}\boldsymbol{L} &= \int \boldsymbol{r}\times \boldsymbol{dp} = \int \boldsymbol{r}\times (dm)\boldsymbol{v} \\ &= \int \boldsymbol{r}\times (dd\tau)\boldsymbol{v}.\end{aligned}$$

Thus $$\mu = \frac{k}{2}\boldsymbol{L}$$

or $$\gamma = \frac{k}{2}\cdot \tag{i}$$

The total charge in the volume is

$$q = \int \rho d\tau = k\int d d\tau = km.$$

Thus $$k = \frac{q}{m}$$

or $$\mu = \frac{q}{2m}\boldsymbol{L}. \tag{11.16}$$

....

EXAMPLE 11.2 **Consider a circular ring of radius R having a uniformly distributed charge q and mass m. It rotates about its axis at an angular speed ω. Find the (a) magnetic moment, (b) angular momentum about the axis, and (c) ratio of the magnetic moment to the angular momentum.**

Solution The time period of the motion is

$$T = \frac{2\pi}{\omega}\cdot$$

Consider any cross section of the ring. The charge q of the whole ring crosses this section in one time period. Hence the current is

$$I = \frac{q}{T} = \frac{q\omega}{2\pi}.$$

(a) The magnetic moment is

$$\mu = IA = \frac{q\omega}{2\pi} \cdot \pi R^2 = \frac{q\omega R^2}{2}.$$

(b) The angular momentum of the ring is

$$L = I\omega = mR^2\omega.$$

(c) The ratio

$$\frac{\mu}{L} = \frac{q\omega R^2}{2} \times \frac{1}{mR^2\omega} = \frac{q}{2m}.$$

Equation 11.16 is valid for all kinds of motion of particles or objects with charge and mass densities proportional to each other, even for atomic and subatomic particles. For example, electrons in their orbital motion obey this relation. However, the angular momentum due to the intrinsic spin of the electron (also for any other particle) and the corresponding magnetic dipole moment do not obey Equation 11.16. For electron spin, the relation is given by

$$\mu_s = -\frac{e}{m}\boldsymbol{s}, \tag{11.17}$$

meaning that γ is twice that corresponding to orbital motion.

11.11 The Hall Effect

Consider a conductor in the shape of a rectangular bar, carrying a uniformly distributed current I flowing along its length (Figure 11.19). Let us choose the x-axis to be along the length of the bar, and the y- and z-axes to be perpendicular to the side surfaces. Suppose a uniform magnetic field $\boldsymbol{B}$ exists in the y-direction, i.e., $\boldsymbol{B} = B\hat{\boldsymbol{j}}$. Consider a free electron moving with drift velocity $\boldsymbol{u} = -u\hat{\boldsymbol{i}}$. The magnetic field will exert a force

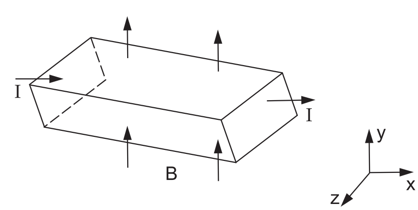


Fig. 11.19

$$\boldsymbol{F}_B = -e(-u\hat{\boldsymbol{i}}) \times (B\hat{\boldsymbol{j}}) = euB\hat{\boldsymbol{k}}$$

on it.

While moving along the bar, the free electrons will also drift along the z-direction and accumulate on the corresponding surface. A positive charge density will appear on the opposite surface. Such accumulation will create an electric field E_H along the z-direction. There will be a steady state when the charge accumulated on these side surfaces is so much that the magnetic force on the new charge carriers gets balanced by the electric force

$$eE_H = euB$$

or $$eE_H = \frac{J}{n}B$$

or $$E_H = \frac{JB}{ne},$$

where n is the number of free electrons per unit volume and J is the current density. If the width of the bar in the z-direction is d, a potential difference $E_H d$ is created between the side surfaces transverse to the current and the magnetic field. This is called the *Hall effect* and the potential difference referred to is called the *Hall voltage*.

The sign of the potential difference will be different if the charge carriers have positive charges. Thus, knowing the Hall voltage, one can know the sign of the charge carriers.

Many sensors used to measure the magnetic field in a region are made by utilizing the Hall effect. Current is passed through a probe of known parameters and by measuring the Hall voltage, B is calculated and displayed on the panel.

Concepts at a Glance

1. A magnetic field exerts a force on a moving charged particle, given by $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$.
2. The force exerted on a current element by a magnetic field is $\boldsymbol{dF} = i\boldsymbol{dl} \times \boldsymbol{B}$, $\boldsymbol{K} \times \boldsymbol{B}da$ or $\boldsymbol{J} \times \boldsymbol{B}d\tau$.
3. A magnetic field does not change the speed of a particle, but changes the direction of motion of the particle.
4. A non-uniform magnetic field can be used to confine charged particles within a given volume.
5. The earth's magnetic field has a pronounced effect on cosmic-ray particles. The number of these particles reaching the earth's surface per unit area is larger at the polar regions than at the equatorial region.
6. A current distribution (closed loop, closed surface or volume) has a dipole moment

   $$\mu = i\boldsymbol{A} \quad \text{or} \quad \frac{1}{2}\int I\boldsymbol{r} \times \boldsymbol{dl} \quad \text{or} \quad \frac{1}{2}\int \boldsymbol{r} \times \boldsymbol{K}da \quad \text{or} \quad \frac{1}{2}\int \boldsymbol{r} \times \boldsymbol{J}d\tau.$$
7. The force on a magnetic dipole is $\boldsymbol{F} = (\mu \cdot \nabla)\boldsymbol{B}$, which is zero for a uniform magnetic field.
8. A magnetic dipole placed in a magnetic field experiences a torque $\tau = \mu \times \boldsymbol{B}$.
9. The potential energy of a magnetic dipole in a magnetic field is $U = -\mu \cdot \boldsymbol{B}$.

EXERCISES

Based on Concepts

1. Is it possible for a charged particle to move with a uniform velocity (same speed, same direction) under the effect of a magnetic field and its weight? If it is, give the directions of the velocity and field for one such situation.
2. Give one evidence to support the statement that the particles in cosmic rays are primarily positively charged.
3. A charged particle moves along a circular path with constant speed under possible electric and magnetic fields. Can E be zero? If the answer is yes, what can be the direction of the B-field? Is it possible for B to be zero? If it is, what can be the direction of the electric field?
4. You are given a copper wire of length L, from which you have to make a circular loop. A current I_0 will be passed through it. You can make a one-turn loop or a two-turn loop as shown in Figure 11E.1. In which case is the magnetic dipole moment larger?

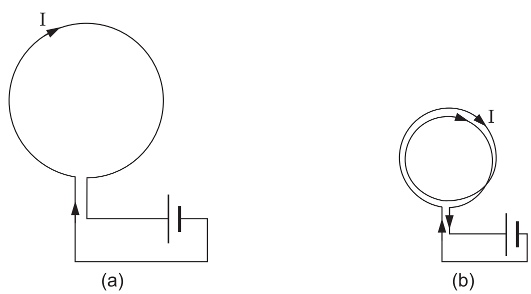


Fig. 11E.1

5. According to the equation $U = -\mu \cdot \boldsymbol{B}$, the energy is zero if the magnetic dipole moment is perpendicular to the magnetic field. Does this mean that the field B does not have any energy?
6. Consider a current I in a cylindrical wire with its axis along the z-axis and suppose the net charge density is zero. A charged particle is projected towards the wire with a velocity $v\hat{\boldsymbol{i}}$. As seen from the frame S′ of the charged particle, the charge density in the wire remains zero as there is no motion of the wire in the x-direction. As the charge is at rest in frame S′, it experiences no magnetic force. As $\rho = 0$, there is no electric field in S′ and hence there is no electric force. Hence the force on the charge is zero. The acceleration in S′ is therefore zero. But S′ moves in the lab frame S with a uniform velocity. Hence the acceleration of the particle in S is also zero and so the net force in S is zero! However, using $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$, the force is $qv\dfrac{\mu_0 I}{2\pi r}(-\hat{\boldsymbol{j}})$. Find the fallacy in this argument.

Problems

1. A particle of mass 30 μg with a charge 10 nC moves in a magnetic field $\boldsymbol{B} = (1.5\ \text{T})\hat{\boldsymbol{k}}$. At $t = 0$, it was at the origin and its velocity was $((5\sqrt{2}\ \text{cm/s})(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$. (a) Find its maximum distance from the x-axis during the motion. (b) Find its velocity at $t = (\pi/2)$ s. [**Ans.** (a) 20 cm, (b) $(10\hat{\boldsymbol{j}} + 5\sqrt{2}\hat{\boldsymbol{k}})$ cm/s]

2. A particular region in space has fields $\boldsymbol{E} = E_0 \hat{\boldsymbol{k}}$ and $\boldsymbol{B} = B_0 \hat{\boldsymbol{k}}$. A particle with charge-to-mass ratio λ starts from the origin at $t = 0$ with a velocity $v_0 \hat{\boldsymbol{i}}$. Find the position of the particle at time t.

[**Ans.** $x = y = 0$, $z = \dfrac{1}{2}\lambda E_0 t^2$]

3. An electron is emitted with negligible speed from the negative plate of a parallel-plate capacitor with large plates, charged to a potential difference V. The separation between the plates is d and a magnetic field B exists in the space in a direction parallel to the plates. Show that the electron will fail to strike the upper plate if $d > \left[\dfrac{2m_e V}{eB_0^2}\right]^{1/2}$.

4. Electrons emitted with negligible speed from an electron gun are accelerated through a potential difference V along the x-axis (Figure 11E.2). These electrons emerge from a narrow hole into a uniform magnetic field B directed along this axis. However, some of the electrons emerging from the hole make slightly divergent angles as shown in the figure. Show that these paraxial electrons are refocused on the x-axis at a distance $\sqrt{\dfrac{8\pi^2 mV}{eB^2}}$.

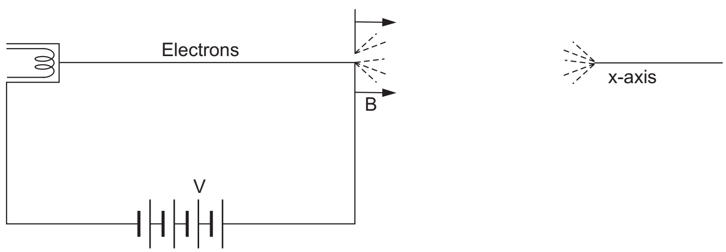


Fig. 11E.2

5. In 1897, J J Thomson found the e/m ratio for electrons as follows. An electron beam was accelerated through a potential difference and then passed through a region of electric and magnetic fields. The fields and the direction of the beam were perpendicular to each other. He then adjusted the electric field until the beam went straight through. (a) What was the speed of electrons, in terms of E and B? (b) Now he turned off the electric field and measured the radius of curvature due to B alone. What is the value of e/m in terms of E, B and R?

[**Ans.** (a) E/B, (b) $\dfrac{E}{RB^2}$]

6. A particle of mass m and charge q is released from the origin in a region where fields $\boldsymbol{E} = E_0 \hat{\boldsymbol{k}}$ and $\boldsymbol{B} = B_0 \hat{\boldsymbol{j}}$ exist.
   (a) Discuss qualitatively the nature of the path.
   (b) Find the speed of the particle as a function of its z-coordinate.
   (c) Write the components of the force in terms of v_x and v_z.
   (d) Show that $\dfrac{d^2 v_z}{dt^2} = -\left(\dfrac{qB_0}{m}\right)^2 v_z$.
   (e) Find the value of z for which the particle's direction of motion becomes perpendicular to the electric field.

[**Ans.** (b) $\sqrt{\dfrac{2qE_0 z}{m}}$, (e) $\dfrac{2mE_0}{qB_0^2}$]

7. A region has uniform electric and magnetic fields given by $\boldsymbol{E} = E_0\hat{\boldsymbol{i}}, \boldsymbol{B} = B_0\hat{\boldsymbol{j}}$. A particle of charge q and mass m starts from the origin with velocity $\boldsymbol{v}_0 = v_0\hat{\boldsymbol{j}}$. Find the position as a function of time and sketch the path. [**Ans.** $x = \dfrac{v_0}{\omega}\sin\omega t - v_0 t,\ y = v_0 t,\ z = \dfrac{v_0}{\omega}(1-\cos\omega t)$, where $v_0 = \dfrac{E_0}{B_0}$ and $\omega = \dfrac{qB}{m}$]

8. A particle of charge q and mass m has a velocity $u\hat{\boldsymbol{i}}$ at $t = 0$. The electric and magnetic fields in the region are $\boldsymbol{E} = E_0\hat{\boldsymbol{j}}$ and $\boldsymbol{B} = B_0\hat{\boldsymbol{k}}$. Find the velocity of the particle at time t. What would be the path if $u = E_0/B_0$? [**Ans.** $v_x = v_0 - (v_0 - u)\cos\omega t,\ v_y = (v_0 - u)\sin\omega t,\ v_z = 0$, where $v_0 = E_0/B_0$ and $\omega = \dfrac{qB}{m}$]

9. An aluminium ring of mass 20 g and radius 5 cm has a small cut in it. It is placed horizontally on a wooden platform which in turn is placed on the north pole of a strong magnet. The magnetic field at the site of the ring is 0.5 T, making an angle of 30° with the vertical. Find the current that should be passed through the ring so that it can be lifted due to the magnetic force. [**Ans.** ≈ 2.5 A]

10. A square coil of edge a with n turns carries a current i. It is kept on a smooth horizontal plate. A uniform magnetic field B exists in a direction parallel to an edge. The total mass of the coil is M. What is the minimum value of B at which the coil will start tipping over? [**Ans.** $\dfrac{Mg}{2nia}$]

11. A wire loop ABCDA carries a current I. As suggested in Figure 11.E3, ABC is a semicircle of radius R in the x–y plane and CDA is a semicircle of radius a in the y–z plane ($z > 0$ side). Calculate the magnetic dipole moment of the loop. [**Ans.** $\dfrac{I\pi}{2}(a^2\hat{\boldsymbol{i}} + R^2\hat{\boldsymbol{k}})$]

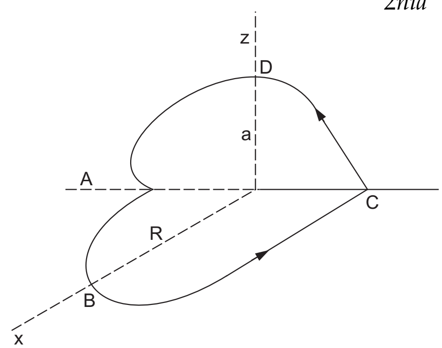


Fig. 11E.3

12. A circular disk of radius R carrying a uniform surface charge density σ rotates at an angular velocity ω about its axis. Find the magnetic dipole moment of this rotating disk. [**Ans.** $\dfrac{\pi\sigma\omega R^4}{4}$ along the axis]

13. A spherical shell of radius R has a charge q distributed uniformly on its surface. The shell rotates with an angular velocity ω about one of its diameters, which is taken as the z-axis. Find the magnetic dipole moment of this distribution. [**Ans.** $\dfrac{2}{3}qR^2\omega$]

14. A uniformly charged solid sphere with total charge q and mass m spins with an angular velocity ω about one of its diameters. Find (a) the magnetic moment of this sphere and (b) the angular momentum of the sphere. [**Ans.** (a) $\dfrac{q\omega R^2}{5}\hat{\boldsymbol{k}}$]

15. A circular current loop has an area 20 cm^2 and a current 10 mA in it. It is placed perpendicular to a magnetic field of 1.0 T. How much work does one have to do to rotate the loop by an angle of 180° about a diameter? [**Ans.** 40 μ J]

16. A current distribution produces a magnetic field $\boldsymbol{B} = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{\boldsymbol{r}} + \sin\theta)$ in spherical polar coordinates. A magnetic dipole of dipole moment m is placed at the point $(r = r_0, \theta = \theta_0, \phi = 0)$ in the direction $\hat{\theta}$. Find the work done to rotate it through an angle δ in the plane $\phi = 0$, towards the $\hat{\boldsymbol{r}}$ direction. [**Ans.** $\frac{\mu_0 m m'}{4\pi\varepsilon_0 r_0^3}(\sin\theta_0 - 2\cos\theta_0 \sin\delta - \sin\theta_0 \cos\delta)$]

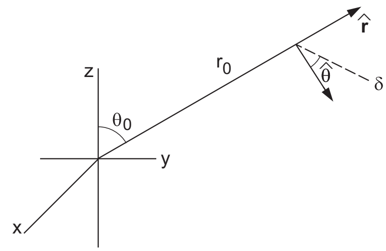


Fig. 11E.4

❑

12 Magnetostatics and the Biot–Savart Law

An electric current distribution (linear, surface or volume) produces a magnetic field in its surroundings. If the current densities do not change with time and charges are not piled up anywhere due to this current, the current is said to be steady. So a steady current is one that is constant in time, at least for a sufficiently long time. The current in a wire connected to a battery is a reasonably good example of steady current. There will be a steady current in a charged ring rotated about its axis with a constant angular velocity, but not in the same ring rotated about a diameter. A single charge q moving at a velocity v does not constitute a steady current even if v is constant. The magnetic field produced by a steady current is called a steady magnetic field, and the topic related to such fields is called *magnetostatics*.

So what are the basic rules for getting the magnetic field if the current distribution (steady) is given?

For an electrostatic field, we started by discussing Coulomb's law, which gives the electric field at $\boldsymbol{r}$ due a single point charge q placed at $\boldsymbol{r'}$. This law is expressed as

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q(\boldsymbol{r}-\boldsymbol{r'})}{|\boldsymbol{r}-\boldsymbol{r'}|^3}.$$

The field due to an extended charge distribution is obtained by invoking the principle of superposition and adding contributions from different parts of the distribution. Take a look at the *Biot–Savart law*, which gives the magnetic field due to a small current element. As done for the electric field, you can obtain the magnetic field due to a given current distribution by adding contributions from all such small elements in the distribution.

12.1 The Biot–Savart Law

The Biot–Savart law is generally stated for a linear current. Suppose a wire carries a current I and $\boldsymbol{dl'}$ is a small element of the wire, situated at a position $\boldsymbol{r'}$. The magnetic field due to this element at a point P situated at the position $\boldsymbol{r}$ is given by

$$\boldsymbol{dB} = \frac{\mu_0}{4\pi}\frac{I\boldsymbol{dl'}\times(\boldsymbol{r}-\boldsymbol{r'})}{|\boldsymbol{r}-\boldsymbol{r'}|^3}. \tag{12.1}$$

$(\boldsymbol{r}-\boldsymbol{r}')$ represents the vector from the current element to the field point P. The constant μ_0 has a value of $4\pi\times10^{-7}$ T m/A, and is called *permeability of free space* (or vacuum). It is a universal constant and its value is exact by definition. For an extended wire, you have to integrate the expression in Equation 12.1 under proper limits.

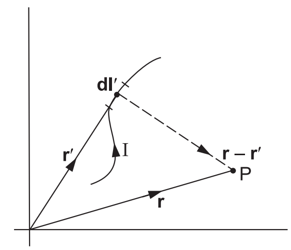


Fig. 12.1

In the case of a surface current and a volume current, the Biot–Savart law takes the following forms.

Surface current: $$\boldsymbol{dB}=\frac{\mu_0}{4\pi}\frac{\boldsymbol{K}\times(\boldsymbol{r}-\boldsymbol{r}')da'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \tag{12.2}$$

Volume current: $$\boldsymbol{dB}=\frac{\mu_0}{4\pi}\frac{\boldsymbol{J}\times(\boldsymbol{r}-\boldsymbol{r}')d\tau'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}. \tag{12.3}$$

This is the usual replacement $\boldsymbol{I}\boldsymbol{dl}'\to\boldsymbol{K}da'\to\boldsymbol{J}d\tau'$. The magnitude of the field in a particular direction, as seen from the current element, falls as the square of the distance from the element. However, it not only depends on the distance from the current element but on the direction as well. In the direction of the current element itself, the vector product $\boldsymbol{dl}'\times(\boldsymbol{r}-\boldsymbol{r}')$ becomes zero, telling us that there is no magnetic field produced along this direction. In the direction perpendicular to the current, the magnitude of the field is maximum. The direction of the magnetic field is perpendicular to that of the current as well as to the line joining the current element to the field point. It is important for you to remember this.

12.2 Magnetic Fields Due to Certain Current Distributions

The expressions describing the magnetic fields due to certain simple current distributions will now be derived. Perhaps you have studied many of these in your schooldays. You may still find something new in these derivations, like the use of the cylindrical coordinate system.

The magnetic field due to a straight segment of a wire

Suppose CD is a straight segment of a wire carrying a current I (Figure 12.2). Say you wish to calculate the magnetic field at a given point P due to the current in the segment CD. Indeed CD has to be a part of a larger circuit because current cannot simply vanish at D and cannot just start at C. But you are interested in the field at P only because of the current in the part CD of the circuit.

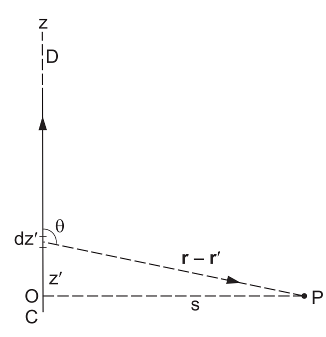


Fig. 12.2

Drop a perpendicular from P on the wire CD and take the foot of perpendicular O as the origin. Take CD as the z-axis and use the cylindrical coordinate system. The position vector of P is

$$\boldsymbol{r} = s\hat{\boldsymbol{s}}.$$

Now take an element $\boldsymbol{dl}$ on the wire at the position

$$\boldsymbol{r}' = z'\hat{\boldsymbol{k}}.$$

The element itself has length dz', so that

$$\boldsymbol{dl}' = dz'\hat{\boldsymbol{k}}.$$

Using the Biot–Savart law, the field at P due to the current element $\boldsymbol{dl}$ is

$$\boldsymbol{dB} = \frac{\mu_0}{4\pi}\frac{I\boldsymbol{dl}' \times (\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$$

$$= \frac{\mu_0 I}{4\pi}\frac{dz'\hat{\boldsymbol{k}} \times (s\hat{\boldsymbol{s}} - z'\hat{\boldsymbol{k}})}{(s^2+z'^2)^{3/2}} = \frac{\mu_0 Is}{4\pi}\frac{dz'}{(s^2+z'^2)^{3/2}}\hat{\phi}.$$

As the point P is given, the direction of $\hat{\phi}$ is fixed—into the page in Figure 12.2—for all elements of the wire.

Hence the net field at P is

$$\boldsymbol{B} = \frac{\mu_0 Is}{4\pi}\left[\int_C^D \frac{dz'}{(s^2+z'^2)^{3/2}}\right]\hat{\phi} \tag{i}$$

under proper limits. To evaluate the integral, put

$$z' = -s\cot\theta$$

so that $$dz' = s\,\mathrm{cosec}^2\theta\, d\theta.$$

Figure 12.2 shows that θ is the angle made by $(\boldsymbol{r}-\boldsymbol{r}')$ with the z-axis.

$$\int \frac{dz'}{(s^2+z'^2)^{3/2}} = \int \frac{s\,\mathrm{cosec}^2\theta}{s^3\mathrm{cosec}^3\theta}d\theta$$

$$= \frac{1}{s^2}(-\cos\theta). \tag{ii}$$

You can directly put limits on θ as you start at the end C and stop at D. Let θ_1 be the angle between CP and the z-axis, and θ_2 be the angle between DP and the z-axis (Figure 12.3). These are the limits on θ. Thus, using (ii),

$$\int_C^D \frac{dz'}{(s^2+z'^2)^{3/2}} = \frac{1}{s^2}(\cos\theta_1 - \cos\theta_2).$$

Using (i), $$\boldsymbol{B} = \frac{\mu_0 I}{4\pi s}(\cos\theta_1 - \cos\theta_2)\hat{\phi}. \tag{12.4}$$

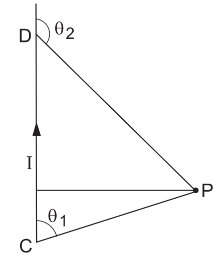


Fig. 12.3

If the wire extends to infinity on both sides, $\theta_1 = 0$ and $\theta_2 = \pi$ as you can check from Figure 12.3. The magnetic field due to such an infinitely long, straight wire carrying current I is, therefore,

$$\boldsymbol{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}. \tag{12.5}$$

What do the magnetic field lines look like for a long, straight current? The field is in the $\hat{\phi}$-direction and its magnitude depends only on the distance from the wire. The field lines are circular, with the wire as the axis. In other words, they are closed curves—they do not start from a point nor do they end at one. This is a very general result for all kinds of current distributions. Electric field lines start from positive charges and terminate at negative charges. These charges are the point-like sources of an electric field. A magnetic field has no point-like sources, and hence magnetic field lines do not start or end anywhere.

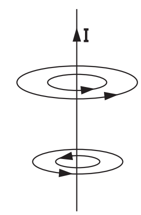


Fig. 12.4

<u>**EXAMPLE 12.1**</u> **In Figure 12.5, AB is part of a circuit carrying a current *I*. The point A is at $\left(-\frac{L}{2}, R, 0\right)$ and B is at $\left(\frac{L}{2}, R, 0\right)$. Find the magnetic field B at the point (0, 0, *z*) due to the current in the part AB.**

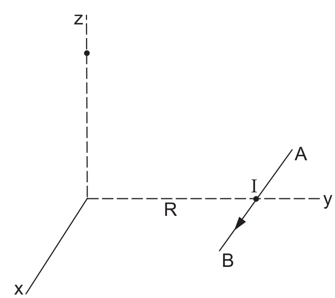


Fig. 12.5

Solution Take a small element $\boldsymbol{dl}$ on AB. Let it be at $(x', R, 0)$ and its length be dx'. Then,

$$\boldsymbol{dl'} = dx'\,\hat{\boldsymbol{i}},$$

$$\boldsymbol{r'} = R\hat{\boldsymbol{j}} + x'\,\hat{\boldsymbol{i}},$$

$$\boldsymbol{r} = -z\hat{\boldsymbol{k}}.$$

Thus, $$d\boldsymbol{B} = \frac{\mu_0 I}{4\pi}\frac{\boldsymbol{dl}\times(\boldsymbol{r}-\boldsymbol{r'})}{|\boldsymbol{r}-\boldsymbol{r'}|^3} = \frac{\mu_0 I dx'\,\hat{\boldsymbol{i}}\times(z\hat{\boldsymbol{k}} - x'\,\hat{\boldsymbol{i}} - R\hat{\boldsymbol{j}})}{4\pi(R^2+z^2+x'^2)^{3/2}}$$

$$= -\frac{\mu_0 I}{4\pi}(R\hat{\boldsymbol{k}} + z\hat{\boldsymbol{j}})\frac{dx'}{(R^2+z^2+x'^2)^{3/2}}.$$

Write $R^2 + z^2 = t^2$ and $x' = t\tan\theta$.

$$\int\frac{dx'}{(R^2+z^2+x'^2)^{3/2}} = \frac{1}{t^2}\sin\theta = \frac{1}{t^2}\frac{x'}{\sqrt{t^2+x'^2}} = \frac{1}{R^2+z^2}\frac{x'}{\sqrt{R^2+z^2+x'^2}}.$$

As you integrate over the wire AB, the value of x' changes gradually from $-L/2$ to $L/2$. Thus,

$$\boldsymbol{B} = -\frac{\mu_0 I}{4\pi}(R\hat{\boldsymbol{k}} + z\hat{\boldsymbol{j}})\int_{-L/2}^{L/2}\frac{dx'}{(R^2+z^2+x'^2)^{3/2}}$$

$$= -\frac{\mu_0 I}{4\pi}\frac{(R\hat{\boldsymbol{k}} + z\hat{\boldsymbol{j}})}{R^2+z^2}\left[\frac{x'}{\sqrt{R^2+z^2+x'^2}}\right]_{-L/2}^{L/2}$$

$$= -\frac{\mu_0 I}{4\pi} \frac{L}{\sqrt{R^2 + z^2 + \frac{L^2}{4}}} \frac{(R\hat{\boldsymbol{k}} + z\hat{\boldsymbol{j}})}{(R^2 + z^2)}.$$

The magnetic field due to a current-carrying circular loop at the centre

Suppose a circular loop of radius R in the x–y plane carries a current I, and you need to find the magnetic field at its centre. Take the origin to be at the centre and the z-axis to be along the axis of the loop. We will use the cylindrical coordinate system. Choose an element $\boldsymbol{dl'}$ by drawing radii making angles ϕ and $\phi + d\phi$ with the x-axis and picking up the portion of the loop in between. The element $\boldsymbol{dl}$ will have a length $Rd\phi$, and will be in the $\hat{\phi}$-direction. So,

$$\boldsymbol{dl'} = Rd\phi\hat{\phi}.$$

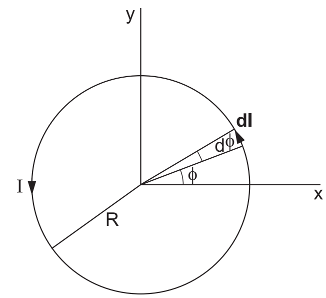


Fig. 12.6

We are not using ϕ' as there is no confusion regarding the source coordinates and field point coordinates.

We need the field at the origin. So,

$$\boldsymbol{r} = 0.$$

The position vector of the element $\boldsymbol{dl'}$ is

$$\boldsymbol{r'} = R\hat{\boldsymbol{s}}.$$

The field at the origin due to the element $\boldsymbol{dl'}$ is

$$\boldsymbol{dB} = \frac{\mu_0}{4\pi} \frac{I\boldsymbol{dl'} \times (\boldsymbol{r} - \boldsymbol{r'})}{|\boldsymbol{r} - \boldsymbol{r'}|^3} = \frac{\mu_0}{4\pi} \frac{IRd\phi\,\hat{\phi} \times (-R\hat{\boldsymbol{s}})}{R^3} = \frac{\mu_0 I}{4\pi R} d\phi\,\hat{\boldsymbol{k}}.$$

So, the field due to the entire loop is

$$\boldsymbol{B} = \left[\int_0^{2\pi} \frac{\mu_0 I}{4\pi R} d\phi\right]\hat{\boldsymbol{k}}$$

or
$$\boldsymbol{B} = \frac{\mu_0 I}{2R}\hat{\boldsymbol{k}}. \tag{12.6}$$

The field is along the axis of the loop.

See that when the current is along the $\hat{\boldsymbol{k}}$-direction (as in the case of a long, straight wire), the magnetic field is in the $\hat{\phi}$-direction, and when the current is in the $\hat{\phi}$-direction (circular loop), the magnetic field is along the $\hat{\boldsymbol{k}}$-direction (at the centre).

In both cases, you can get the direction of the field using the right-hand thumb rule. If the fingers of the right hand curl along the direction of the current, the field is along the stretched thumb. And if the current is along the stretched thumb, the field is along the curled fingers.

The magnetic field due to a current-carrying circular loop at a point on its axis

Suppose a circular loop of radius R carries a current I, and the magnetic field due to this current is needed at a point on the axis of the loop. The situation and the coordinate system are similar to those in Figure 12.6. The only change is in the field point, which is away from the origin on the z-axis, say at the position $\boldsymbol{r} = z\hat{\boldsymbol{k}}$. The modified situation is shown in Figure 12.7. The relevant vectors are

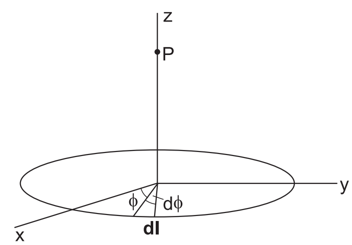


Fig. 12.7

$$\boldsymbol{dl'} = Rd\phi\hat{\phi},$$

$$\boldsymbol{r} = z\hat{\boldsymbol{k}}$$

and $$\boldsymbol{r'} = R\hat{\boldsymbol{s}}.$$

So,

$$\boldsymbol{dB} = \frac{\mu_0}{4\pi}\frac{I\boldsymbol{dl'}\times(\boldsymbol{r}-\boldsymbol{r'})}{|\boldsymbol{r}-\boldsymbol{r'}|^3} = \frac{\mu_0 I}{4\pi}\frac{Rd\phi\hat{\phi}\times(z\hat{\boldsymbol{k}}-R\hat{\boldsymbol{s}})}{(z^2+R^2)^{3/2}}$$

$$= \frac{\mu_0 IR}{4\pi}\frac{(z\hat{\boldsymbol{s}}+R\hat{\boldsymbol{k}})}{(z^2+R^2)^{3/2}}d\phi.$$

Thus,

$$\boldsymbol{B} = \frac{\mu_0 IR}{4\pi(z^2+R^2)^{3/2}}\left[z\int_0^{2\pi}\hat{\boldsymbol{s}}\,d\phi + R\int_0^{2\pi}\hat{\boldsymbol{k}}\,d\phi\right].$$

While the second integral within the square brackets is $2\pi\hat{\boldsymbol{k}}$, the first one is zero. Why is the first zero? Because, as ϕ changes from 0 to 2π, $\hat{\boldsymbol{s}}$ keeps changing its direction continuously and when the values of the terms constituting $\hat{\boldsymbol{s}}$ are added together, you get zero. If you do not understand this, write $\hat{\boldsymbol{s}} = \cos\phi\,\hat{\boldsymbol{i}} + \sin\phi\,\hat{\boldsymbol{j}}$ and integrate each term on the RHS separately. You will find that the integrals of both are zero.

So,

$$\boldsymbol{B} = \frac{\mu_0 IR}{4\pi(z^2+R^2)^{3/2}}\times 2\pi R\hat{\boldsymbol{k}}$$

$$= \frac{\mu_0 IR^2}{2(z^2+R^2)^{3/2}}\hat{\boldsymbol{k}}. \tag{12.7}$$

If you put $z = 0$ in this expression, you get $\boldsymbol{B} = \frac{\mu_0 I}{2R}\hat{\boldsymbol{k}}$, which is the same as Equation 12.6 for the field at the centre.

The field has been calculated at the points on the axis of the current loop. For points not on the axis, the integrations are difficult. One can use various approximation methods to evaluate the integrals numerically. It turns out that for all points in the plane of the circular current, the field is perpendicular to this plane. For points inside the circle, the field is in the $\hat{\boldsymbol{k}}$-direction and for points outside the circle, the field is in the $-\hat{\boldsymbol{k}}$-direction (the current is in the $\hat{\phi}$-direction). At other places, the field has $\hat{\boldsymbol{k}}$- and $\hat{\boldsymbol{s}}$-components. The field lines due to a circular current loop are qualitatively sketched in Figure 12.8.

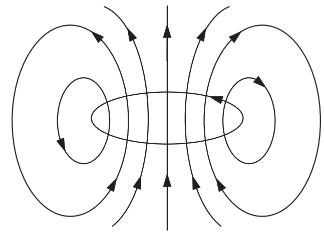


Fig. 12.8

The magnetic field due to a plane surface current

Suppose an infinite plane surface contains a uniform surface current $\boldsymbol{K_0}$. This means that, on the entire surface, charges move in one direction and that the current through a line dl perpendicular to the direction of the current is $K_0 dl$. The magnetic field is needed at a point P. Drop a perpendicular from P to the surface and take the foot of the perpendicular as the origin O. Take the line OP to be the z-axis, and the x-, y-axes to be in the plane itself. Also, take the x-axis to be along the direction of the current on this plane.

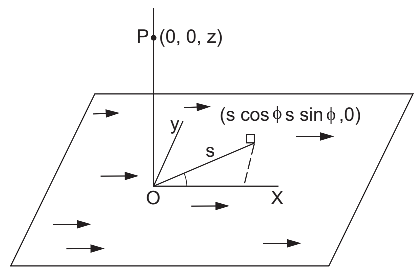


Fig. 12.9

Given this situation, the Cartesian coordinates of the point P are (0, 0, z) and the surface current density is $K_0\hat{\boldsymbol{i}}$. To take a current element, construct a small surface area $s\,ds\,d\phi$ as we do in the cylindrical coordinate system. The point A, where the element is constructed, has Cartesian coordinates ($s\cos\phi$, $s\sin\phi$, 0), and the element is made by changing s to $s+ds$ and ϕ to $\phi+d\phi$. The entire plane is covered when s goes from 0 to ∞ and ϕ goes from 0 to 2π.

The current element is $\boldsymbol{K}da' = K_0 s\,ds\,d\phi\hat{\boldsymbol{i}}$, the position vector of this element is $\boldsymbol{r}' = s\cos\phi\hat{\boldsymbol{i}} + s\sin\phi\hat{\boldsymbol{j}}$ and the position vector of the field point P is $\boldsymbol{r} = z\hat{\boldsymbol{k}}$.

Using the Biot–Savart law, the magnetic field at P due to the current element chosen is

$$\boldsymbol{dB} = \frac{\mu_0}{4\pi}\frac{\boldsymbol{K}da' \times (\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$$

$$= \frac{\mu_0 K_0}{4\pi}\frac{\hat{\boldsymbol{i}} \times (-s\cos\phi\,\hat{\boldsymbol{i}} - s\sin\phi\,\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}})s\,ds\,d\phi}{(s^2+z^2)^{3/2}}$$

$$= \frac{\mu_0 K_0}{4\pi}\frac{-s^2\sin\phi\,\hat{\boldsymbol{k}}\,ds\,d\phi - sz\hat{\boldsymbol{j}}ds\,d\phi}{(s^2+z^2)^{3/2}}.$$

The field due to the entire plane is

$$\boldsymbol{B} = \frac{\mu_0 K_0}{4\pi}\left[\int_{s=0}^{\infty}\int_{\phi=0}^{2\pi}\frac{-s^2 \sin\phi\, ds\, d\phi}{(s^2+z^2)^{3/2}}\hat{\boldsymbol{k}} + \int_{s=0}^{\infty}\int_{\phi=0}^{2\pi}\frac{-sz\, ds\, d\phi}{(s^2+z^2)^{3/2}}\hat{\boldsymbol{j}}\right]. \tag{i}$$

The first integral is

$$\int_{s=0}^{\infty}\frac{-s^2 ds}{(s^2+z^2)^{3/2}}\int_0^{2\pi}\sin\phi\, d\phi = 0.$$

The second integral is

$$\int_{s=0}^{\infty}\frac{-sz\, ds}{(s^2+z^2)^{3/2}}\int_{\phi=0}^{2\pi}d\phi = z\left|\frac{1}{\sqrt{s^2+z^2}}\right|_0^{\infty}\cdot 2\pi = -2\pi z\frac{1}{|z|}\cdot$$

$\sqrt{z^2}$ has been substituted by $|z|$ as only the positive square root is to be taken. For $z > 0$, the integral is -2π and for $z < 0$ it is $+2\pi$. From (i), the field at (9, 0, z) is

$$\begin{aligned}\boldsymbol{B} &= -\frac{\mu_0 K_0}{2}\hat{\boldsymbol{j}} \text{ for } z > 0\\ &= \frac{\mu_0 K_0}{2}\hat{\boldsymbol{j}} \text{ for } z < 0.\end{aligned} \tag{12.8}$$

The field is independent of z. So whether you are close to the surface current or far from it, the field is the same. This is because you have taken an infinite plane carrying a uniform current density. Recall that the electric field due to an infinite, plane charge layer is $\sigma/(2\varepsilon_0)$, which is also independent of the distance from the layer. In reality, you can only have a finite plane carrying the surface current, and Equation 12.8 is valid for points close to the plane and away from the edges.

The infinite plane surface current and the magnetic field lines have been shown in Figure 12.10 from a different perspective. The plane of the drawing is the y–z plane. The infinite x–y plane carrying the surface current is perpendicular to the plane of the drawing. The current is along the $\hat{\boldsymbol{i}}$-direction and is shown by the symbols ⊙ (coming out of the plane of the drawing). For $z > 0$, the magnetic field is in the $-\hat{\boldsymbol{j}}$-direction and for $z < 0$, it is in $+\hat{\boldsymbol{j}}$-direction. So the magnetic field changes discontinuously as you cross the plane $z = 0$. This is a general feature. Whenever you cross a surface current, the magnetic field changes discontinuously. More about this has been discussed later in the chapter.

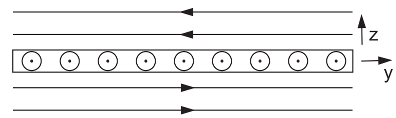


Fig. 12.10

The two expressions in Equation 12.8 may be combined to form the following single equation.

$$\boldsymbol{B} = \frac{1}{2}\mu_0 \boldsymbol{K}\times\hat{\boldsymbol{n}}, \tag{12.9}$$

where $\hat{\boldsymbol{n}}$ is the unit vector perpendicular to the plane of the surface current. The perpendicular should be drawn in the region where you are looking for the field. If you are looking for the field at a point in the region $z > 0$, the perpendicular should be drawn here and $\hat{\boldsymbol{n}}$ will be $\hat{\boldsymbol{k}}$. If you are looking for the field at a point in the region $z < 0$, $\hat{\boldsymbol{n}}$ will be $-\hat{\boldsymbol{k}}$. Check that, with this definition of $\hat{\boldsymbol{n}}$, Equation 12.9 gives the correct field for $z > 0$ as well as for $z < 0$.

12.3 Force between Two Current-carrying Wires

Suppose two wires carry currents I_1 and I_2. To take a concrete example, let the wires be thin, straight and placed parallel to each other along the z-direction at a separation d (Figure 12.11). Consider a small element dl in the second wire (carrying current I_2). The magnetic field due to the current I_1 in the first wire at the site of this element is

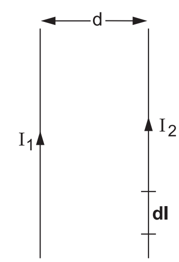


Fig. 12.11

$$\boldsymbol{B} = \frac{\mu_0 I_1}{2\pi d}\hat{\phi}.$$

The force due to this field on the element dl will be

$$\boldsymbol{F} = I_2 \boldsymbol{dl} \times \boldsymbol{B} = I_2 dl\,\hat{\boldsymbol{k}} \times \frac{\mu_0 I_1}{2\pi d}\hat{\phi} = -\frac{\mu_0 I_1 I_2}{2\pi d} dl\,\hat{\boldsymbol{s}}.$$

So each element will be attracted towards the first wire with a force $\frac{\mu_0 I_1 I_2}{2\pi d}$ per unit length.

This equation was the basis of the SI unit ampere of electric current till the 2019 revision of definition of SI units. If equal currents are maintained in two infinitely long, parallel wires at a separation of 1 m and each wire experiences a force of 2×10^{-7} newton per metre (which is numerically the same as $\frac{\mu_0}{2\pi}$), the current is called 1 ampere.

12.4 Force between Currents and Newton's Third Law

Consider two parallel wires carrying currents I_1 and I_2 placed at a separation d from each other. Consider an element dl_1 in one wire and another element dl_2 in the other wire (Figure 12.12). What is the force on the element dl_2 due to the current in dl_1? Using the Biot–Savart law, the magnetic field due to dl_1 at the site of dl_2 is

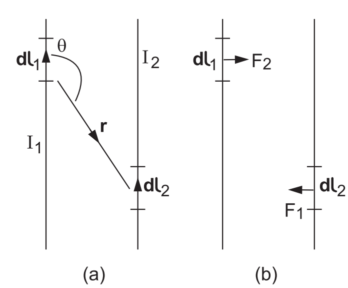


Fig. 12.12

$$\boldsymbol{B} = \frac{\mu_0 I_1 \boldsymbol{dl}_1 \times \boldsymbol{r}}{4\pi r^3},$$

where $\boldsymbol{r}$ is the vector joining dl_1 to dl_2. What is the direction of this field? Taking the first wire to be along the z-direction, this field is in the $\hat{\phi}$-direction and the magnitude is $\frac{\mu_0 I_1 dl_1 \sin\theta}{4\pi r^2}$, where θ is the angle between $\boldsymbol{dl_1}$ and $\boldsymbol{r}$. Call the field $B_1\hat{\phi}$.

The force on $\boldsymbol{dl_2}$ because of this field is

$$d\boldsymbol{F_2} = I_2 \boldsymbol{dl_2} \times \boldsymbol{B} = I_2 dl_2 \hat{\boldsymbol{k}} \times B_1 \hat{\phi} = I_2 B_1 dl_2 (-\hat{\boldsymbol{s}}).$$

The direction of the force on $\boldsymbol{dl_2}$ is towards the first wire. Similarly, the direction of the force $d\boldsymbol{F_1}$ on $\boldsymbol{dl_1}$ is towards the second wire.

You can work out that the magnitudes of $\boldsymbol{dF_1}$ and $\boldsymbol{dF_2}$ are the same.

The forces are equal in magnitude and opposite in direction, but they are not along the same line. Newton's third law is not valid here, at least not in its strong form.

You must have studied the principle of conservation of angular momentum. If there is no external torque, the angular momentum of a system remains constant. This principle depends crucially on the validity of Newton's third law in its strong form. If the forces exerted by two parts of a system on each other are not along the same line (apart from being equal and opposite), this law of conservation will not hold.

Is Newton's third law valid at least in its weak form? Work out the answer taking different configurations of two current elements. As an example, take one current element along the x-axis at the origin and the other along the y-axis at (0, y, 0). Do you find that the two forces are not equal even in magnitude?

If you remember what you have studied in mechanics, the law of conservation of linear momentum results from Newton's third law in the weak form and the law of conservation of angular momentum, from Newton's third law in the strong form. So, should we do away with conservation of linear momentum and that of angular momentum in case of current elements interacting through magnetic forces? Not really. You can talk about these momenta only when the elements move under these forces. Once they do that, the magnetostatic situation will not hold and all these equations will not work. There is no danger to the laws of conservation.

However, if you consider two closed-loop currents, the forces on each other will be equal and opposite, as shown in the next example.

EXAMPLE 12.2 **Consider two loops, one carrying current I_1 and the other, I_2. Show that the force acting on Loop 2 due to the current in Loop 1 is $\frac{\mu_1 I_1 I_2}{4\pi}\int \frac{(dl_1 \cdot dl_2)(r_2 - r_1)}{|r_2 - r_1|^3}$. This also shows that Newton's third law holds for forces between two current loops.**

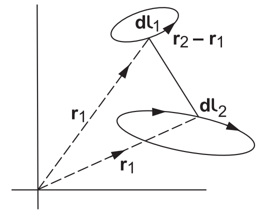


Fig. 12.13

Solution The magnetic field due to $\boldsymbol{dl}_1$ at the site of $\boldsymbol{dl}_2$ is

$$\boldsymbol{dB} = \frac{\mu_0 I_1}{4\pi}\frac{\boldsymbol{dl_1} \times (\boldsymbol{r_2} - \boldsymbol{r_1})}{|\boldsymbol{r_2} - \boldsymbol{r_1}|^3}.$$

The force on $\boldsymbol{dl_2}$ due to this field is

$$\boldsymbol{dF} = I_2 \boldsymbol{dl}_2 \times \frac{\mu_0 I_1}{4\pi}\frac{\boldsymbol{dl}_1 \times (\boldsymbol{r}_2 - \boldsymbol{r}_1)}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|^3}.$$

The force on Loop 2 due to Loop 1 is

$$\boldsymbol{F} = \frac{\mu_0 I_1 I_2}{4\pi}\oint\oint \frac{\boldsymbol{dl}_2 \times [\boldsymbol{dl}_1 \times (\boldsymbol{r}_2 - \boldsymbol{r}_1)]}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|^3}$$

$$= \frac{\mu_0 I_1 I_2}{4\pi}\left[\oint\oint \frac{(\boldsymbol{dl}_2 \cdot \boldsymbol{dl}_1)(\boldsymbol{r}_2 - \boldsymbol{r}_1)}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|^3} - \frac{[\boldsymbol{dl}_2 \cdot (\boldsymbol{r}_2 - \boldsymbol{r}_1)\boldsymbol{dl}_1}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|^3}\right].$$

The second integral is $I'' = \int\left[\int \frac{\boldsymbol{dl}_2 \cdot (\boldsymbol{r}_2 - \boldsymbol{r}_1)}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|^3}\right]\boldsymbol{dl}_1.$

Suppose you put a charge dq at the site of $\boldsymbol{dl}_1$. The electric field due to this charge at the location of $\boldsymbol{dl}_2$ will be

$$\boldsymbol{E} = \frac{dq}{4\pi\varepsilon_0}\frac{(\boldsymbol{r}_2 - \boldsymbol{r}_1)}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|^3}.$$

Its line integral over Loop 2 will be

$$\oint \boldsymbol{E} \cdot \boldsymbol{dl}_2 = \frac{dq}{4\pi\varepsilon_0}\oint \frac{\boldsymbol{dl}_2 \cdot (\boldsymbol{r}_2 - \boldsymbol{r}_1)}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|^3}.$$

But, for electrostatic fields,

$$\oint \boldsymbol{E} \cdot \boldsymbol{dl}_2 = 0.$$

Thus $\oint \frac{\boldsymbol{dl}_2 \cdot (\boldsymbol{r}_2 - \boldsymbol{r}_1)}{4\pi\varepsilon_0 |\boldsymbol{r}_2 - \boldsymbol{r}_1|^3} = 0$

or $I'' = 0$ and $\boldsymbol{F} = \frac{\mu_0 I_1 I_2}{4\pi}\oint\oint \frac{(\boldsymbol{dl}_2 \cdot \boldsymbol{dl}_1)(\boldsymbol{r}_2 - \boldsymbol{r}_1)}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|^3}.$

12.5 The Divergence of *B*

The magnetic field at a point P with position vector $\boldsymbol{r}$, due to a general current distribution described by a current density $\boldsymbol{J}$, is given by

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int_{\text{volume}} \frac{\boldsymbol{J}(\boldsymbol{r}') \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} d\tau', \tag{i}$$

where the integration is to be performed over the volume containing the current. The volume element $d\tau'$ is constructed at a position $\boldsymbol{r}'$ in the current distribution, and $\boldsymbol{J}(\boldsymbol{r}')$ is the current density at this location. Remember that any current distribution is a volume current distribution. Even if you think of an ideal linear or surface current distribution, it can be written as a volume current distribution with proper use of the Dirac delta function. So the results derived for a general volume current distribution are applicable to all kinds of current distributions.

$\boldsymbol{B}(\boldsymbol{r})$ in equation (i) is the magnetic field at a point $\boldsymbol{r}$. How does this field vary if you change to a nearby point $\boldsymbol{r}+d\boldsymbol{r}$? The current distribution remains the same. Now we are talking about a variation in the magnetic field from point to point. So we are now referring to the space derivatives of the function $\boldsymbol{B}(\boldsymbol{r})$. For a vector field, the two standard space derivatives are divergence and curl. First, let us discuss divergence. It involves some amount of vector calculus, which we will not go into here (see Appendix 4). The result is

$$\nabla \cdot \boldsymbol{B} = \frac{\mu_0}{4\pi} \int \left[\nabla \cdot \frac{\boldsymbol{J}(\boldsymbol{r}') \times (\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \right] d\tau' = 0.$$

Recall that the integration is on the volume containing the current distribution. The element $d\tau'$ is an element in this volume. The differentiations involved in calculating the divergence are with respect to the field point, that is, as $\boldsymbol{r}$ changes to $\boldsymbol{r}+d\boldsymbol{r}$. So, treat $\boldsymbol{r}'$ as a constant when you work out the divergence in the above expression and treat $\boldsymbol{r}$ as a constant when you do the volume integration.

The equation

$$\nabla \cdot \boldsymbol{B} = 0 \tag{12.10}$$

turns out to be universally true. Whether or not the currents are steady, whether the field is produced due to a current-carrying wire or a permanent magnet or by any other source, the divergence of $\boldsymbol{B}$ is always zero. In the case of electric fields, we have Gauss's law $\nabla \cdot \boldsymbol{E} = \rho/\varepsilon_0$, which is also universally true in all conditions. ρ describes the density of electric charge distribution, which produces the electric field. The divergence of a vector field is related to the point-like source (or sink) of that field. $\nabla \cdot \boldsymbol{B} = 0$ shows that you do not have point-like sources that can produce a magnetic field. You do not have special particles to produce magnetic fields. You do not have points from where the magnetic field may emerge in all directions.

Equation 12.10 is sometimes called Gauss's law for magnetism as it is similar to Gauss's law for electric fields.

You can write this now in integral form. Using a closed surface,

$$\oint \boldsymbol{B} \cdot d\boldsymbol{a} = \int \nabla \cdot \boldsymbol{B} d\tau = 0.$$

The flux of the magnetic field through a closed surface is always zero.

What about the curl of the magnetic field $\boldsymbol{B}$? Wait till the next chapter.

EXAMPLE 12.3 **A circular loop with its axis along the *z*-axis carries a current. The magnetic field at a point on the axis is given by**

$$\boldsymbol{B} = \frac{1\mu_0 IR^2}{2(R^2 + z^2)^{3/2}}\hat{k}.$$

Show that for points away from the axis but close to it, there must be an *s*-component of the field.

Solution There is ϕ-symmetry as the current is circular, the axis of the circle being along the z-axis. So, nothing will change if ϕ is changed keeping s and z fixed.

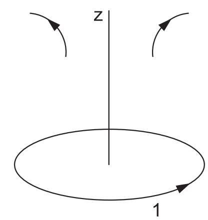


Fig. 12.14

$$\nabla \cdot \boldsymbol{B} = \frac{\partial B_s}{\partial s} + \frac{1}{s}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z} = 0.$$

As $$\frac{\partial B_\phi}{\partial \phi} = 0,$$

$$\frac{\partial B_s}{\partial s} = -\frac{\partial B_z}{\partial z}.$$

On the axis, $B_s = 0$ and B_z decreases as z increases. So $-\frac{\partial B_z}{\partial z}$ is positive. Therefore, $\frac{\partial B_s}{\partial s}$ has to be positive. This means that, as one goes away from the axis, there must be an s-component of $\boldsymbol{B}$.

Concepts at a Glance

1. Biot–Savart law: $\boldsymbol{dB} = \dfrac{\mu_0 I}{4\pi}\dfrac{\boldsymbol{dl} \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3}$

   or $\dfrac{\mu_0}{4\pi}\dfrac{\boldsymbol{K} \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3}da$

   or $\dfrac{\mu_0}{4\pi}\dfrac{\boldsymbol{J} \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3}d\tau.$

2. The magnetic field at a point P due to a current element $I\boldsymbol{dl}$ is perpendicular to the plane containing the element $\boldsymbol{dl}$ and the line joining the element to P.
3. The magnetic field due to a straight wire segment is zero at all points in the line of the segment.
4. The magnetic field due to a straight wire segment is

$$\boldsymbol{B} = \frac{\mu_0}{4\pi s}(\cos\theta_1 - \cos\theta_2)\hat{\phi}.$$

5. The magnetic field due to a plane surface current $\boldsymbol{K}$ is $\boldsymbol{B} = \frac{1}{2}\mu_0 \boldsymbol{K} \times \hat{\boldsymbol{n}}$. It is parallel to the plane but perpendicular to the current.
6. Newton's third law may not hold between two current elements when the force is mediated by the magnetic field in magnetostatic conditions.
7. $\nabla \cdot \boldsymbol{B} = 0, \oint \boldsymbol{B} \cdot d\boldsymbol{a} = 0.$
8. Magnetic field lines do not originate from any point. Nor do they terminate at any point.

EXERCISES

Based on Concept

1. A plastic ring carries a charge q distributed uniformly along its length. In which of the following cases will it make a steady current?
   (a) When it is rotated at a constant angular velocity about its axis
   (b) When it is rotated at a constant angular velocity about one of its diameters
   (c) When it slides in its own plane with a constant velocity
   (d) When it moves along its axis with a constant velocity
2. A constant current I flows through a wire with nonuniform cross section, as shown in Figure 12E.I. Is this an example of steady current?

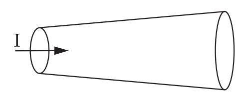


Fig. 12E.1

3. Consider a circular turn of a wire carrying a current. Take a small element of the wire. Will there be a magnetic force on this element due to the current in the turn? If so, what will be its direction?
4. In Section 12.3, we talked about the force on the element dl of the second wire due to the magnetic field produced by the current in the first wire. Why did we not consider the magnetic field due to the rest of the second wire (apart from dl)?
5. Two concentric, spherical shells of radii a and b $(a < b)$ are maintained at potentials zero and V_0 respectively. The space between these shells is filled by a weakly conducting material so that a steady, radial current flows between the shells. Why is the magnetic field due to this current zero everywhere?

Problems

1. A very small circular loop carries a current I. The magnetic field at a point on its axis at a distance of 20 cm from the loop is 48 μT. What is the field at a distance of 40 cm from the loop on the axis?
   [**Ans.** 6 μT]
2. You have to make a circular coil of radius 10 cm from a spool of insulated wire in such a way that passing 1 A of current in it produces a magnetic field of 4 mT at its centre. Assume that the insulated wire is of negligible thickness and find the length of the wire needed. [**Ans.** 400 m]

3. You have a wire of length 400 cm and diameter 0.1 cm including the insulation. Suppose you put all the turns in the same plane to make the coil, the adjacent turns touching each other and the innermost turn having a radius of 2.0 cm (Figure 12E.2). (a) What will be the radius of the outermost turn? (b) Calculate the magnetic field of this coil at the centre if a current of 1 A is passed through it. Make appropriate approximations and state them.

[**Ans.** (a) 4.1 cm, (b) 4.4×10^{-4} T]

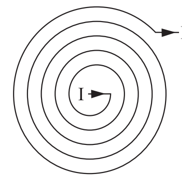


Fig. 12E.2

4. Consider the problem above. Suppose you make the coil by putting one turn above another (touching it) so that all the turns have the same radius 10 cm and the coil attains a certain height (Figure 12E.3). (a) What will be the height of the coil? (b) Calculate the magnetic field of this coil at the centre if a current of 1 A is passed through it. Make appropriate approximations and state them. [**Ans.** (a) 3.2 cm, (b) 7.9×10^{-5} T]

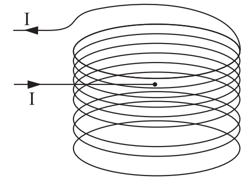


Fig. 12E.3

5. Figure 12E.4 shows three shapes made of a metallic wire of length L and connected to circuits so that the same current I passes through them. Apart from the small gap, the first shape is an equilateral triangle, the second one is a square and the third is a circle. Calculate the magnetic field at the centre of the shape in each case. Which of the three gives the largest magnetic field? [**Ans.** (a) $\frac{\sqrt{27}\mu_0 I}{2\pi L}$, (b) $\frac{8\sqrt{2}\mu_0 I}{\pi L}$, (c) $\frac{\pi\mu_0 I}{L}$]

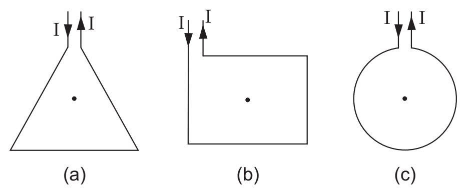


Fig. 12E.4

6. A current I passes through a regular polygon of n sides. The distance of each side from the centre is R. Find the magnetic field at a point on the symmetry axis (line through the centre perpendicular to the plane of the polygon) at a distance z from the centre. What is the limit of this expression as $n \to \infty$?

[**Ans.** $\dfrac{\mu_0 I R^2 \hat{k}}{2(R^2+z^2)\sqrt{R^2\left(1+\dfrac{\pi^2}{n^2}\right)+z^2}}$, $\dfrac{\mu_0 I R^2 \hat{k}}{2(R^2+z^2)^{3/2}}$]

7. Two semi-infinte wires of radius a each are separated by a distance d. They are joined by another wire AB at an end as shown in Figure 12E.5. A current I passes through the circuit. Find the force on the wire AB due to the currents in the two semi-infinite wires.

[**Ans.** $\frac{\mu_0 I^2}{4\pi} \ln \frac{d-a}{a}$]

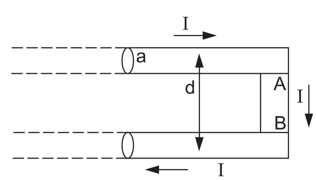


Fig. 12E.5

8. Assume that the earth's metallic core extends out to 3200 km. Suppose the earth's magnetic field 0.5 gauss at the earth's north pole is caused by a current flow around the "equator" of the core. Find the current needed. Radius of earth = 6400 km. [**Ans.** 5.7×10^9 A]

9. Two coaxial coils of radius a are separated by a distance a (such coils are called Helmholtz coils). Show that the axial field in the central region is given by $\frac{1}{2}\mu_0 nI\left(\frac{4}{5}\right)^{3/2}$ correct to the order $\left(\frac{x}{a}\right)^2$, where x is the distance along the axis from the central point. Here n is the number of turns in each coil and I is the common current.

10. A spherical shell of radius R carries a uniform charge distributed over its surface with surface charge density σ. It rotates about one of its diameters, taken as the z-axis, with a constant angular velocity ω. Using the Biot–Savert law for surface current distribution, find the magnetic field at the centre.
[**Ans.** $\frac{2}{3}\mu_0\sigma\omega R$]

11. A cylinder of length L and radius R carries a surface current $\boldsymbol{K} = K_0\hat{\phi}$. Find the magnetic field at the centre of the cylinder due to this current. [**Ans.** $\frac{\mu_0 K_0 L}{\sqrt{4R^2 + L^2}}$]

12. A wire is bent to make a two-turn coil of radius R, the turns being separated by a distance h (Figure 12E.6). A current I is passed through the wire. Assuming $h << R$, find the force on one turn due to the other. Evaluate the force if $h = 2$ mm, $R = 2$ cm and $I = 2$ A. Will there be a visible decrease in the separation? [**Ans.** $\frac{\mu_0 I^2 R}{h}$, 5×10^{-5} N, difficult]

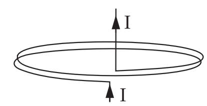


Fig. 12E.6

13. Electrons accelerated by a small potential difference V form a uniform cylindrical beam constituting a current I. The radius of the beam is r. Find the linear force on a peripheral electron due to the other particles in the beam. [**Ans.** $\mu_0 I\sqrt{2V\left(\frac{e}{2\pi r}\right)^3}$]

14. Suppose you have two straight, infinite line charges, each with charge density λ. They are parallel to each other, a distance d apart, moving along their lengths in the same direction at a constant speed v. What should be the value of v for the magnetic attraction to balance the electric repulsion? [**Ans.** $v = c$]

15. A cylindrical magnet is kept with its axis along the z-axis. Close to the axis, the z-component of the field outside of the magnet can be approximated as $B_z = B_0 e^{-\alpha z^2}$. From considerations of symmetry, the ϕ-component of the magnetic field is zero. Find the s-component of the field as a function of z for points close to the axis.
[**Ans.** $B_0 \alpha s z e^{-\alpha z^2}$]

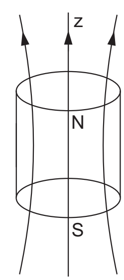


Fig. 12E.7

16. A wire carrying a current i is kept on the x–y plane along the curve $y = A\sin[2\pi x/\lambda]$. A magnetic field B exists in the z-direction. Find the magnitude of the magnetic force on the portion of the wire between $x = 0$ and $x = \lambda$. [**Ans.** $-i\lambda\hat{\boldsymbol{B}}$]

❑

13 Ampere's Law in Magnetostatics

In the previous chapter, we had discussed the Biot–Savart law, which gives a method for evaluating the magnetic field for a given current distribution in a magnetostatic situation. Suppose the magnetic field in all space is given and you are asked what kind of current distribution can produce this field. To answer this question, you can use Ampere's law, given by an equation that may be derived from the Biot–Savart law in a magnetostatic situation.

13.1 Ampere's Law

Suppose there is a steady current distribution described by a volume current density $\boldsymbol{J}(\boldsymbol{r})$. As mentioned earlier, any current distribution can be expressed as a volume current distribution and the results derived for a general volume current distribution are applicable to any current distribution. The magnetic field at $\boldsymbol{r}$ due to the given current distribution is

$$\boldsymbol{B}(\boldsymbol{r}) = \int \frac{\mu_0}{4\pi} \frac{\boldsymbol{J}(\boldsymbol{r}') \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} d\tau',$$

where $d\tau'$ is a small volume element in the current distribution at $\boldsymbol{r}'$. The integration is to be performed on a volume in which the entire current is distributed.

If you take the curl of this function, the result is strikingly simple. The calculation is quite tedious but the result is very interesting (see Appendix 5) and it is

$$\nabla \times \boldsymbol{B}(\boldsymbol{r}) = \mu_0 \boldsymbol{J}(\boldsymbol{r}). \tag{13.1}$$

This equation is known as Ampere's law for magnetostatics. It is a point-by-point relation. There is a steady current distribution extended in space in some region, and this produces a magnetic field $\boldsymbol{B}$. The field may vary from one space point to another and we can calculate $\nabla \times \boldsymbol{B}$ at a given point P. It involves the rate of change of $\boldsymbol{B}$ as we move a little in different directions from P. The value of $\nabla \times \boldsymbol{B}$ at P must be equal to the current density $\boldsymbol{J}$ at that point, multiplied by μ_0.

EXAMPLE 13.1 **The magnetic field in a region is given by $\boldsymbol{B} = ks\hat{\phi}$ in cylindrical coordinates. Find the current density in the region.**

Solution In cylindrical coordinates,

$$\nabla \times \boldsymbol{B} = \left[\frac{1}{s}\frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z}\right]\hat{s} + \left[\frac{\partial B_s}{\partial z} - \frac{\partial B_z}{\partial s}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(sB_\phi) - \frac{\partial B_s}{\partial \phi}\right]z.$$

In the given case, $B_s = B_z = 0$ and $B_\phi = ks$. So,

$$\nabla \times \boldsymbol{B} = \frac{1}{s}\frac{\partial}{\partial s}(ks^2)\hat{z} = 2k\hat{z}.$$

From Ampere's law, $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$. So,

$$\boldsymbol{J} = \frac{2k}{\mu_0}\hat{z}.$$

The current density $\boldsymbol{J}$ is constant in the region.

In the previous chapter, you learnt that $\nabla \cdot \boldsymbol{B} = 0$ everywhere in all situations. Here you have got to know that $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$ in magnetostatics. A vector field can be completely determined in a region if you know its divergence and curl everywhere in the region and also the value of the field at all points on the boundary of the region. Suppose you work with a current distribution confined to a finite region of space (in reality, this is always the case but sometimes we consider unrealistic, infinitely spread current distributions). Considering the whole space as the region, the value of $\boldsymbol{B}$ at the boundary, that is at infinity, will be zero and hence known. If you are given $\nabla \times \boldsymbol{B}$ everywhere, $\boldsymbol{B}$ is uniquely described as $\nabla \cdot \boldsymbol{B}$ is anyway known (zero).

13.2 Ampere's Law in Integral Form

Ampere's law can be written in integral form too. Consider a closed loop. It is only a geometrical construction and not a material wire loop. The loop encloses a surface area, which has two sides—call one side positive and the other negative. Put an arrow on the loop so that when the fingers of your right hand are along the loop in this direction, the stretched thumb points along the positive side of the surface.

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}.$$

So

$$\int (\nabla \times \boldsymbol{B}) \cdot \boldsymbol{da} = \mu_0 \int \boldsymbol{J} \cdot \boldsymbol{da}, \tag{i}$$

where integration is performed on the surface area bounded by the loop and $\boldsymbol{da}$ represents the area vector corresponding to an elementary area on the surface. The direction of $\boldsymbol{da}$ is along the positive side of the surface. Using Stokes' law,

$$\int (\nabla \times \boldsymbol{B}) \cdot \boldsymbol{da} = \oint \boldsymbol{B} \cdot \boldsymbol{dl}, \tag{ii}$$

where the line integral is to be evaluated on the loop bounding the surface area. The direction of $\boldsymbol{dl}$ is decided by the arrow on the loop. Using (i) and (ii),

$$\oint \boldsymbol{B}\cdot \boldsymbol{dl} = \mu_0 \int \boldsymbol{J}\cdot \boldsymbol{da}$$

or
$$\oint \boldsymbol{B}\cdot \boldsymbol{dl} = \mu_0 I_{\text{encl}}, \tag{13.2}$$

where I_{encl} is the total current going through the surface area from the negative side to the positive side. The subscript 'encl' stands for enclosed. This equation is called Ampere's law in integral form. The line integral on the left-hand side is called the circulation of the magnetic field over the loop.

As Ampere's law $\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{J}$ is applicable only to magnetic fields due to steady currents, so is the integral form given in Equation 13.2.

Please remember that on the left-hand side in Equation 13.2, we have the resultant magnetic field $\boldsymbol{B}$ due to all the currents whether or not they cross the surface area. On the right-hand side, however, you only have to count the currents crossing the surface. For example, suppose you calculate $\boldsymbol{B}$ due to all the four currents shown in Figure 13.1, and evaluate $\oint \boldsymbol{B}\cdot\boldsymbol{dl}$ on the curve shown. $\oint \boldsymbol{B}\cdot\boldsymbol{dl}$ will be equal to $\mu_0(I_2 + I_3)$.

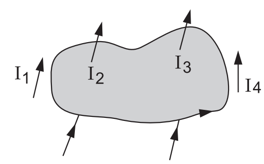


Fig. 13.1

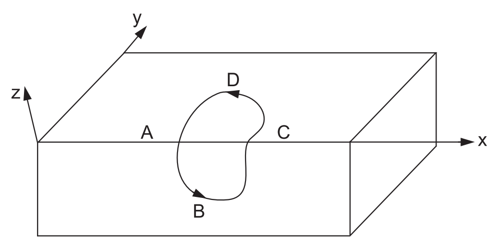


Fig. 13.2

Should the loop used in Ampere's law necessarily be in a plane? No, it need not be. Take a three-dimensional object such as a rectangular box, put your pencil somewhere on the box and move it anywhere you want on the six surfaces, but end at the starting point. You get a closed curve and can evaluate the circulation of the magnetic field over it. You should be careful while taking the positive and negative sides of the area enclosed. In Figure 13.2, ABCD is a closed loop. You can evaluate $\oint \boldsymbol{B}\cdot\boldsymbol{dl}$ on this. To get the current enclosed, you should know the positive side and negative side of the area. This is decided by the arrow on the loop. Check that for the part ABC, the positive side is in the $-\hat{\boldsymbol{j}}$-direction and for the part CDA, it is in the $+\hat{\boldsymbol{k}}$-direction. Any current that comes from inside the box and goes out through ABC or CDA will be counted in calculating the current enclosed.

Often you can get a nonplanar loop by deforming a planar loop. You can mentally decide the directions of $\boldsymbol{da}$ vectors on different parts of the planar loop and think that these $\boldsymbol{da}$ vectors remain tagged with the area elements as the loop is deformed to take the nonplanar shape.

Ampere's law in integral form is quite useful in calculating the magnetic field due to current distributions possessing certain symmetries. In certain cases, you may have to know the direction of the field before you can use Ampere's law. The loop used to apply Ampere's law is often called an "Amperian loop". We will now discuss some typical uses of this law.

13.3 Applications of Ampere's Law

Field due to a long, straight wire

A long, straight wire carries a current I. The magnetic field due to this current is needed at a point P, which is at a distance s from the wire. Consider any element $\boldsymbol{dl}$ on the wire. Using the Biot–Savart law, the direction of the field at P due to this element is along $\boldsymbol{dl'} \times (\boldsymbol{r} - \boldsymbol{r'})$. If you draw a circle passing through P with the wire as the axis, the field will have a direction tangential to this circle (Figure 13.3). Taking the z-axis to be along the wire, and the positive z-direction to be in the direction of the current, the magnetic field will be along the ϕ-direction. From considerations of symmetry, the field at all points on this circle will have the same magnitude B, and everywhere it will be tangential to the circle. Use Ampere's law on this circle to get the magnitude of the field. Put the arrow on the circle along the direction of the field.

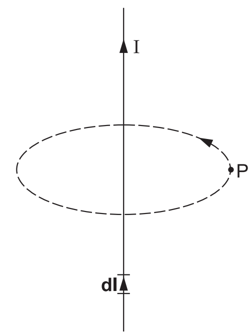


Fig. 13.3

Then,

$$\oint \boldsymbol{B} \cdot \boldsymbol{dl} = \mu_0 I$$

or $$\oint B dl = \mu_0 I \qquad \text{[angle between } \boldsymbol{B} \text{ and } \boldsymbol{dl} \text{ is 0]}$$

or $$B \oint dl = \mu_0 I \qquad \text{[the magnitude } B \text{ is constant on the circle]}$$

or $$2\pi s B = \mu_0 I$$

or $$B = \frac{\mu_0 I}{2\pi s}.$$

Thus, $$\boldsymbol{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi}.$$

You can see how much simpler it is to calculate the magnitude of $\boldsymbol{B}$ using Ampere's law than by using the Biot–Savart law. However, the direction of $\boldsymbol{B}$ has to be worked out from the Biot–Savart law.

A magnetic field due to an infinite-plane surface current

Suppose a uniform surface current $\boldsymbol{K} = k\hat{\boldsymbol{i}}$ exists on the entire x–y plane [Figure 13.4(a)] and we wish to find the magnetic field at a point in space due to this current distribution. We have already solved this problem using the Biot–Savart law. Let us take it up again to show you the simplicity of calculation provided by using Ampere's law if a certain symmetry exists and if we can work out the direction of the field beforehand.

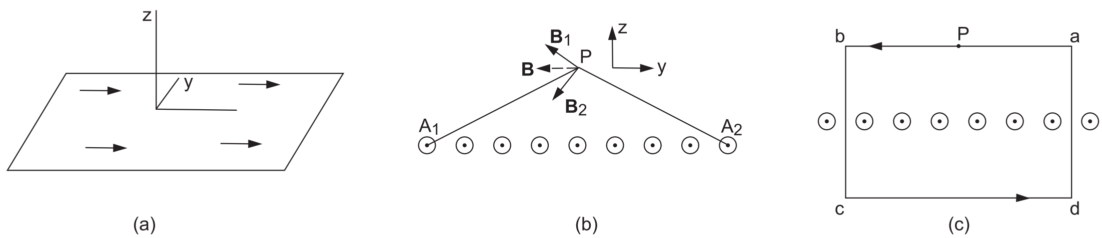


Fig. 13.4

If you look at the plane surface from a side so that currents come towards you, the situation will be as shown in Figure 13.4(b). Suppose the field is needed at the point P above the x–y plane ($z > 0$). Consider two strips in the x–y plane, one at A_1 and the other at A_2, both running parallel to the x-axis and equidistant from P. These strips may be treated as long, straight currents. The magnetic fields due to these strips at P are shown as $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ in Figure 13.4(b). The magnitudes of the two will be the same as the strips are equidistant from P. The resultant of these two will be in the $-\hat{\boldsymbol{j}}$-direction. The whole of the x–y plane may be divided into such symmetrical pairs of strips, and hence the net magnetic field is along the $-\hat{\boldsymbol{j}}$-direction. Note that if you take the point P to be below the x–y plane ($z < 0$), the field will be in the $+\hat{\boldsymbol{j}}$-direction.

Now that you know the direction of the magnetic field, you can find its magnitude using Ampere's law. Draw a rectangle *abcd* passing through P as shown in Figure 13.4(c) and use it as an Amperian loop. The sides *ab* and *cd* are drawn equidistant from the x–y plane. Half the loop is above the x–y plane and the other half is below it (assuming the z-direction to be upwards). The magnitude of the field at all points on the sides *ab* and *cd* will be the same as all these points are at the same distance from the x–y plane, which carries the surface current K.

$$\oint \boldsymbol{B}\cdot d\boldsymbol{l} = \int_a^b \boldsymbol{B}\cdot d\boldsymbol{l} + \int_b^c \boldsymbol{B}\cdot d\boldsymbol{l} + \int_c^d \boldsymbol{B}\cdot d\boldsymbol{l} + \int_d^a \boldsymbol{B}\cdot d\boldsymbol{l}.$$

For the segments *ab* and *cd*, the magnetic field $\boldsymbol{B}$ is in the same direction as $d\boldsymbol{l}$. For the segments *bc* and *da*, the field $\boldsymbol{B}$ is perpendicular to $d\boldsymbol{l}$. So,

$$\oint \boldsymbol{B}\cdot d\boldsymbol{l} = B\cdot l + 0 + B\cdot l + 0 = 2Bl,$$

where $l = ab = cd$. How much current goes through the area bounded by this rectangle? The surface current density is K in the x-direction. The rectangle cuts a length l in the x–y plane and a current Kl crosses it. So, using Ampere's law,

$$\oint \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 I_{\text{encl}}$$

or $$2Bl = \mu_0 Kl$$

or $$B = \frac{\mu_0 K}{2},$$

which is the same as that obtained earlier using the Biot–Savart law. Taking care of the directions, you can write,

$$\boldsymbol{B} = \frac{1}{2}\mu_0 \boldsymbol{K} \times \hat{n}, \tag{13.3}$$

where $\hat{n}$ is the unit vector perpendicular to the plane surface carrying current, drawn in the side where the field point P exists. In Figure 13.4, $\hat{n}$ for point P will be the same as $\hat{k}$. For points below the x–y plane, such as those on cd, $\hat{n}$ will be the same as $-\hat{k}$.

The magnetic field due to a long, tightly wound, current-carrying solenoid

What is a solenoid? Take a cylindrical tube and wind an enamelled wire around it, leaving no gap between the adjacent turns. Enamel is a nonconducting layer on the wire, which ensures that even when the adjacent turns touch each other, current cannot pass from one turn to another due to this contact. Current follows the length of the wire. Such a structure is called a solenoid. You will have two free ends of the wire wound, current will enter from one end, go through each turn, that is, through the entire length of the wire, and will come out at the other end. If you have wound N turns on a length L of the tube, the number of turns per unit length is $n = N/L$.

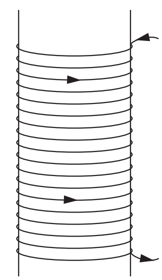


Fig. 13.5

Suppose the number of turns per unit length on a solenoid is n and the wire carries a current I. You need the magnetic field due to this current. Taking the axis of the solenoid as the z-axis, the current is in the $\hat{\phi}$ direction (to a very good approximation, if the solenoid is tightly wound).

Before you can use Ampere's law to find the magnitude of the magnetic field due to the current, you have to know the direction of the field. It turns out that the field inside the solenoid is along the axis of the solenoid, along the positive z-axis, whereas it is zero outside. Let us justify this statement.

First consider the region inside the solenoid. Consider a circle of some radius, coaxial with the solenoid [Figure 13.6(a)].

$$\int \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 I_{\text{encl}} = 0$$

or $$B_\phi 2\pi s = 0$$

or $\qquad B_\phi = 0.$

The field has no ϕ-component.

Now make a Gaussian pillbox as shown in Figure 13.6(b). It is a cylindrical box, coaxial with the solenoid.

$$\oint \boldsymbol{B}\cdot d\boldsymbol{a} = \int_{s_1} B_s da + \int_{s_2} B_z da,$$

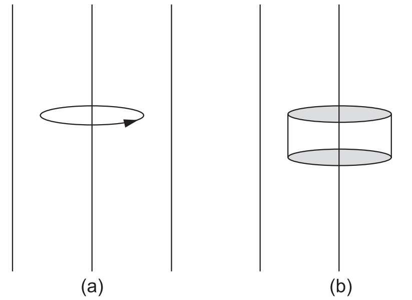


Fig. 13.6

where s_1 is the curved surface of the box and s_2 comprises the pair of flat surfaces of the box. Considerations of symmetry show that the field will not depend on z. So, the integrations $B_z da$ on the upper flat surface and lower flat surface will have the same magnitude. As the area vectors are in opposite directions, $\int_{s_2} B_z da = 0.$

Now, B_s on the cylindrical surface will have the same magnitude. So, $\int_{s_1} B_s da = B_s \cdot (2\pi sh).$

Therefore, $\oint \boldsymbol{B}\cdot d\boldsymbol{a} = B_s(2\pi sh).$

But $\qquad \oint \boldsymbol{B}\cdot d\boldsymbol{a} = 0. \qquad$ (As $\nabla\cdot\boldsymbol{B}$ everywhere)

So, $\qquad B_s = 0.$

It is left to you to show that B_s and B_ϕ are zero even outside the solenoid. While B_ϕ can be once again evaluated using $\oint \boldsymbol{B}\cdot d\boldsymbol{l} = \mu_0 I_{\text{encl}}$, you have to properly choose a Gaussian surface to show that $B_s = 0$. Thus, the magnetic field can be only in the z-direction.

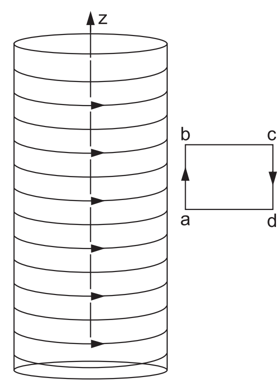


Fig. 13.7

First look at the magnetic field outside.

Take an Amperian loop as shown in Figure 13.7. As there is no current outside the solenoid,

$$\oint \boldsymbol{B}\cdot d\boldsymbol{l} = 0$$

or $\qquad B_1(ab) = B_2(cd)$

or $\qquad B_1 = B_2.$

The magnetic field has the same magnitude everywhere outside the solenoid. The direction has already been shown to be the z-direction. Accept without proof at this stage that this constant magnetic field outside is indeed zero—the proof is rather complex. But is it not obvious that B should be zero at an infinite distance from the solenoid? No, because the current distribution is not confined to a finite volume. For example, the magnetic field due to an infinite-plane surface current is not zero at an infinite distance from the plane.

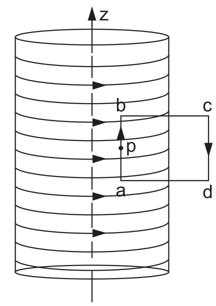


Fig. 13.8

To get the magnetic field at a point P inside the solenoid, at a distance *s* from the axis, draw a rectangular loop *abcd* passing through the point P [Figure 13.8]. The arm *ab* is parallel to the axis, lies inside the solenoid and has length *l*. The arm *cd* is outside the solenoid. Apply Ampere's law on this loop.

$$\oint \boldsymbol{B}\cdot d\boldsymbol{l} = \int_{ab} \boldsymbol{B}\cdot d\boldsymbol{l} + \int_{bc} \boldsymbol{B}\cdot d\boldsymbol{l} + \int_{cd} \boldsymbol{B}\cdot d\boldsymbol{l} + \int_{da} \boldsymbol{B}\cdot d\boldsymbol{l}$$

$$= Bl + 0 + 0 + 0 = Bl.$$

Remember, $\boldsymbol{B}$ inside is along the z-axis and outside is zero.

Look at the rectangular area *abcd*. Part of it is inside the solenoid and part of it is outside. The wire turns cross this area at the surface of the solenoid where the area cuts the solenoid. As the length here is l, a total of nl turns cross this rectangular area and hence the current enclosed is nli. Thus, using Ampere's law,

$$\oint \boldsymbol{B}\cdot d\boldsymbol{l} = \mu_0 I_{\text{encl}}$$

or $$Bl = \mu_0 nli \tag{13.4}$$

or $$B = \mu_0 ni.$$

This is the magnitude of the magnetic field inside the solenoid. You can see that it does not depend on the distance s from the axis of the solenoid. That means, in the whole of the solenoid, the magnetic field is uniform and along the z-axis. Thus,

$$\boldsymbol{B} = \mu_0 ni\hat{k}. \tag{13.5}$$

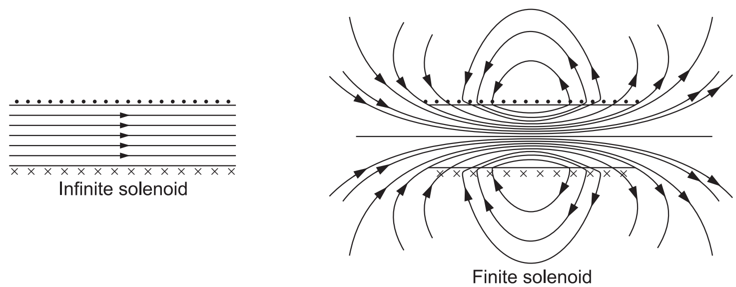


Fig. 13.9

The above result is for an infinitely long solenoid. In practical situations, if the length of the solenoid is much larger than its radius, and you are looking for the fields not close to the ends, you use Equation 13.5. The magnetic field lines for an assumed infinite solenoid and a finite solenoid are shown in Figure 13.9. The dots represent currents coming out of the plane of the drawing and the crosses represent those going into the plane of the drawing.

You can obtain the direction of the magnetic field inside the solenoid using the right-hand thumb rule. If you hold the solenoid in your right hand in such a way that the fingers curl along the direction of the current in the turns, the stretched thumb gives the direction of the field.

The magnetic field at a point on the axis of a long solenoid can be calculated to be $\mu_0 ni$ by using the Biot–Savart law. Ampere's law ensures that it is also $\mu_0 ni$ at all points inside the solenoid. All you have to do is to take the loop *abcd* of Figure 13.8 completely inside the solenoid. Now using a loop like that shown in Figure 13.8, you can prove that the field outside the long solenoid is zero.

Magnetic field due to a circumferential surface current on an infinitely long cylindrical surface

Consider an infinitely long cylindrical surface of radius R. Suppose the axis of this surface is the z-axis. The circumferential direction is along $\hat{\phi}$. Let a current with surface current density

$$\boldsymbol{K} = K\hat{\phi}$$

exist on the surface, where K is a constant. So if you draw a line of length l on the cylindrical surface, parallel to the axis, a current Kl will cross the line from one side to the other.

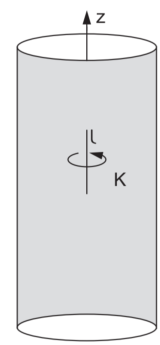


Fig. 13.10

This situation is similar to that of the solenoid with $K = ni$. The magnetic field due to such a surface current can be written using Equation 13.4. It is

$$\begin{aligned} \boldsymbol{B} &= \mu_0 K\hat{k} \qquad \text{for } s < R \\ &= 0 \qquad \text{for } s > R. \end{aligned} \tag{13.6}$$

Magnetic field due to a toroid

A solenoid is made by winding turns of wire on a long, straight tube. A toroid is made by winding turns of wire on a closed tube, like the tube of your bicycle (Figure 13.11). The cross section of the tube need not be circular, but in this example, we are taking it to be circular.

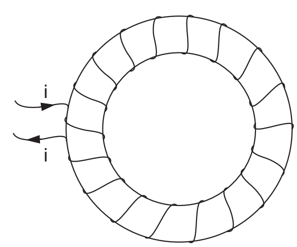


Fig. 13.11

You can think of a toroid to be made from a finite solenoid, by joining its two ends. If a current is passed through a long, straight solenoid, the magnetic field is along the axis inside the solenoid and is zero outside. For a finite solenoid, the field decreases near the ends. When you bend the solenoid and join the two ends, there are no ends, and you should expect the same field all along its length. However, the number of turns per unit length will not be the same on the inner and the outer sides. The periphery on the inner side will be shorter than that on the outer side. But the total number of turns on these peripheries will be the same. Thus, the number of turns per unit length n will be more on the inner side than on the outer side. So, you should expect that the magnetic field will depend on the distance s from the centre of the toroid but not on $\hat{\phi}$.

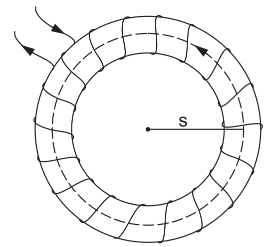


Fig. 13.12

It turns out that the magnetic field inside a closely wound toroid indeed depends on s only. To get this s-dependence, consider a closed circular curve of radius s, concentric with the toroid, as shown in Figure 13.12, and use Ampere's law $\oint \boldsymbol{B}\cdot d\boldsymbol{l} = \mu_0 I_{\text{encl}}$ on the curve. Each turn crosses the circular area πs^2 bounded by this circle (on the inner side of the toroid). At each such crossover, the current is in the same direction, coming out of the area in Figure 13.12. Thus

$$I_{\text{encl}} = NI,$$

where N is the total number of turns in the toroid. The integral

$$\oint \boldsymbol{B}\cdot d\boldsymbol{l} = \oint B dl = B\oint dl = 2\pi s B$$

in view of the ϕ-symmetry. Thus,

$$2\pi s B = \mu_0 NI$$

or

$$B = \frac{\mu_0 NI}{2\pi s}\cdot$$

Thus the magnetic field inside is

$$\boldsymbol{B} = \frac{\mu_0 NI}{2\pi s}\hat{\phi}. \tag{13.7}$$

Outside the toroid, the field is zero.

The right-hand thumb rule works well here too. If you hold the toroid at any place with your right hand so that the fingers curl along the current, the stretched thumb gives you the direction of the field.

13.4 Boundary Conditions on *B* across a Surface

The magnetic field $\boldsymbol{B}(\boldsymbol{r})$ in a magnetostatic situation satisfies

$$\nabla\cdot\boldsymbol{B} = 0$$

and

$$\nabla\times\boldsymbol{B} = \mu_0 \boldsymbol{J}.$$

In integral form, these equations are

$$\oint \boldsymbol{B} \cdot \boldsymbol{da} = 0$$

for any closed surface, and

$$\oint \boldsymbol{B} \cdot \boldsymbol{dl} = \mu_0 I_{\text{encl}}$$

for any closed loop. These allow us to write the boundary conditions on the normal and tangential components of $\boldsymbol{B}$ across a surface.

Normal component

Let Σ be a surface (Figure 13.13). Let A and B be two points on the two sides of Σ, close to each other. Call the side containing A Side 1 and the other Side 2. The magnetic field at A is $\boldsymbol{B}_1$ and that at B is $\boldsymbol{B}_2$. Let $\hat{\boldsymbol{n}}$ be the unit vector along the normal to the surface in Side 1.

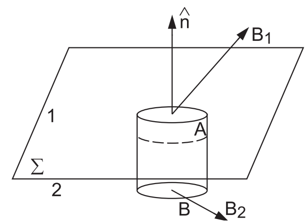


Fig. 13.13

Construct a pillbox-type Gaussian surface, as shown in Figure 13.13. The flat surfaces are parallel to Σ and have an area ΔA each. The flux of the magnetic field, $\oint \boldsymbol{B} \cdot \boldsymbol{da}$, through the pillbox must be zero. As the points A and B are close to each other, the thickness Δt of the pillbox is vanishingly small. Hence the flux through the curved part of the pillbox tends to zero as $\Delta t \to 0$. The flux through the flat area ΔA on Side 1 is

$$\boldsymbol{B}_1 \cdot (\hat{n}\Delta A) = B_{1n}\Delta A$$

and that through the flat area ΔA on Side 2 is

$$\boldsymbol{B}_2 \cdot (-\hat{n}\Delta A) = -B_{2n}\Delta A.$$

The net flux through the pillbox is, therefore,

$$(B_{1n} - B_{2n})\Delta A.$$

But this should be zero. So,

$$B_{1n} = B_{2n}.$$

The normal component of the magnetic field is continuous across the surface irrespective of whether or not there is any current on the surface.

Tangential component

Let there be a surface current $\boldsymbol{K}$ on the surface Σ. Take two mutually perpendicular directions on the surface Σ, one parallel to the surface current density $\boldsymbol{K}$ and the other along $\boldsymbol{K} \times \hat{n}$. Call the unit

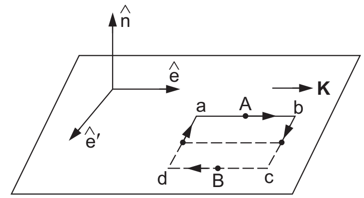


Fig. 13.14

vectors in these directions $\hat{e}$ and $\hat{e}'$ respectively. Here $\hat{n}$ is the unit vector perpendicular to Σ in Side 1. Both $\hat{e}$ and $\hat{e}'$ are tangential to the surface.

As usual, choose two points A and B, one on Side 1 and the other on Side 2, both very close to the surface. Draw a line *ab* through A and *cd* through B, both parallel to $\hat{e}$. Join *bc* and *da* (Figure 13.14). So, the line *ab* is on Side 1 and *cd* is on Side 2. These lines are parallel to the current density. The thickness $bc = da$ is vanishingly small. Let $ab = cd = \Delta l$.

As the current density $\boldsymbol{K}$ is parallel to the plane *abcd*, no current crosses the area bounded by this loop. So,

$$\oint \boldsymbol{B}\cdot d\boldsymbol{l} = \mu_0 I_{\text{encl}} = 0$$

or $$\boldsymbol{B}_1\cdot\hat{e}\,\Delta l + \boldsymbol{B}_2\cdot(-\hat{e})\Delta l = 0.$$

So, $$\boldsymbol{B}_1\cdot\hat{e} = \boldsymbol{B}_2\cdot\hat{e}$$

or $$B_{1t} = B_{2t}, \tag{13.8}$$

where B_{1t} and B_{2t} are components of the magnetic fields $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ along the surface current $\boldsymbol{K}$.

This means that if you take the tangential direction to be parallel to the surface current density, the tangential component of $\boldsymbol{B}$ is continuous across the surface.

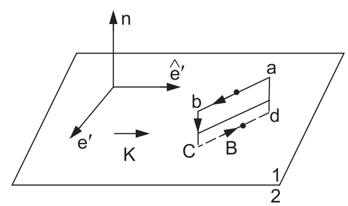


Fig. 13.15

Now consider the tangential direction $\hat{e}'$, i.e., along $\boldsymbol{K}\times\hat{n}$. Take a new rectangular loop *abcda*, *ab* on Side 1 and *cd* on Side 2, *ab* passing through *A* and *cd* through *B*. The lines *ab* and *cd* are parallel to $\hat{e}'$. The total current going through the area *abcda* is $K\Delta l$, where $ab = cd = \Delta l$.

So, $$\oint \boldsymbol{B}\cdot d\boldsymbol{l} = \mu_0 I_{\text{encl}} = \mu_0 K\Delta l$$

or $$\boldsymbol{B}_1\cdot\hat{e}'\Delta l + \boldsymbol{B}_2\cdot(-\hat{e}')\Delta l = \mu_0 K\Delta l$$

or $$\boldsymbol{B}_1\cdot\hat{e}' - \boldsymbol{B}_2\cdot\hat{e}' = \mu_0 K$$

or $$B_{1t} - B_{2t} = \mu_0 K. \tag{13.9}$$

If you take the tangential direction to be along $\boldsymbol{K}\times\hat{n}$, the tangential component of $\boldsymbol{B}$ suddenly changes by an amount $\mu_0 K$, where K is the surface current density.

Remember to draw $\hat{n}$ on Side 1. From here, you can determine the direction of $\hat{e}'$, which in turn determines the directions of the arrows on the loop *abcda*.

The actual magnetic field $\boldsymbol{B}$ may have a tangential component that is neither along $\hat{e}$ nor along $\hat{e}'$. But it can be written in components in these directions. As the component in the $\hat{e}$-direction is continuous, the change in the actual tangential component is the same as that in the $\hat{e}'$ component.

So, $$\boldsymbol{B}_{1t} - \boldsymbol{B}_{2t} = \mu_0 K \hat{e}'$$

or $$\boldsymbol{B}_{1t} - \boldsymbol{B}_{2t} = \mu_0 \boldsymbol{K} \times \hat{n}. \tag{13.10}$$

Here, $\boldsymbol{B}_{1t}$ and $\boldsymbol{B}_{2t}$ are the net tangential components of $\boldsymbol{B}$ on the two sides of the surface. As the normal component of $\boldsymbol{B}$ is continuous, you can also write

$$\boldsymbol{B}_1 - \boldsymbol{B}_2 = \mu_0 \boldsymbol{K} \times \hat{n}. \tag{13.11}$$

Concepts at a Glance

1. In magnetostatics, the magnetic field $\boldsymbol{B}(\boldsymbol{r})$ and the current density $\boldsymbol{J}$ are related by Ampere's law $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$ or $\oint \boldsymbol{B} \cdot \boldsymbol{dl} = \mu_0 I_{\text{encl}}$.
2. If the direction of $\boldsymbol{B}$ is known (from the Biot–Savart law or otherwise), Ampere's law in integral form can be used to find the magnitude of $\boldsymbol{B}$ for certain symmetric current distributions.
3. The magnetic field inside a long ideal solenoid is uniform. Its magnitude is $B = \mu_0 n i$, where n is the number of turns per unit length of the solenoid and i is the current. Its direction is along the axis of the solenoid.
4. For a planar surface current given by surface current density $\boldsymbol{K}$, the magnetic field is $\frac{1}{2}\mu_0 \boldsymbol{K} \times \hat{n}$.
5. The normal component of $\boldsymbol{B}$ is continuous across any surface.
6. The tangential component of $\boldsymbol{B}$ is discontinuous across a surface carrying a surface current density $\boldsymbol{K}$.
   $\boldsymbol{B}_{1t} - \boldsymbol{B}_{2k} = \mu_0 \boldsymbol{K} \times \hat{n}$.

EXERCISES

Based on Concepts

1. Why can't you use Ampere's law in integral form to calculate $\boldsymbol{B}$ due to a current in a straight wire of finite length?
2. It is given that the magnetic field outside a long, ideal solenoid is zero. Using boundary conditions on the normal and tangential components of the magnetic field, show that the magnetic field inside the solenoid is uniform and parallel to the axis of the solenoid.

3. Consider a very long solenoid carrying a current I. What is the magnetic field $\boldsymbol{B}_1$ at the centre of the solenoid due to half the solenoid on one side of the centre? What is the magnetic field at the centre of an end face of the solenoid?
4. A long, straight wire carries a current I. Consider a spherical surface of radius R with the wire along its diameter. What is the value of the surface integration $\oint \boldsymbol{B}\cdot d\boldsymbol{a}$ over this surface?
5. Check whether or not $\boldsymbol{B} = kz\hat{i}$ is a valid magnetic field. If it is, find the current density that can produce it.

Problems

1. Find the magnetic field at point P for the current loop shown in Figure 13E.1. The curved wires are in the form of circular arcs with P as the centre. The straight segments almost pass through P. The separation between the successive arcs is $a/4$. [**Ans.** $\dfrac{31\mu_0 I}{840a}$]

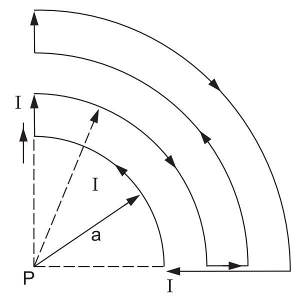


Fig. 13.E1

2. Figure 13E.2 shows a wire with two semi-infinite segments and a circular arc of radius R joining them. A current I passes through the wire. Find the magnetic field at the centre P of the arc.
[**Ans.** $\dfrac{(12+5\pi)\mu_0 I}{24\pi R}$]

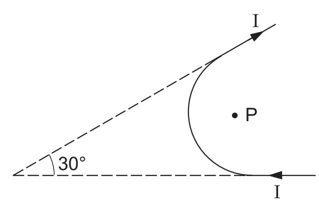


Fig. 13.E2

3. Two long solenoids with n_1 and n_2 turns, respectively, per unit length are kept coaxially, the first one inside the second. The inner solenoid has radius R_1, and carries a current I. The outer one has radius R_2 and also carries a current I but in the opposite sense. Find the magnetic field everywhere in space.
[**Ans.** $\mu_0(n_1 - n_2)I\hat{k}$ for $s < R_1$, $-\mu_0 n_2 I\hat{k}$ for $R_1 < s_1 R_2$, zero for $s > R$]
4. A cylindrical surface of radius R carries a surface charge density σ and is rotated about its axis with a constant angular speed ω. Find the magnetic field everywhere. [**Ans.** $\mu_0\sigma\omega R\hat{k}$ for $s < R_1$, zero outside]
5. Charge is distributed uniformly with density ρ in a long, cylindrical region of radius R. The whole distribution rotates with a constant angular speed ω about its axis. Find the magnetic field everywhere due to this rotating distribution. [**Ans.** $\dfrac{1}{2}\mu_0\rho\omega(R^2 - s^2)$ for $s < R$, zero for $s > R$]

6. The magnetic field inside a sphere (with the origin as its centre) of radius R is given by $\boldsymbol{B} = \frac{\alpha}{s}\hat{\varphi}$ in cylindrical coordinates. The field is zero outside this sphere. (i) Verify that this satisfies the criteria of being a valid magnetic field. (ii) Find the surface current density at the surface of the sphere.

   [**Ans.** (ii) $\frac{\alpha}{\mu_0 R}$]

7. Find the magnetic field everywhere due to the current given by $\boldsymbol{J} == J_0\hat{i}$ for $|z| < a/2$ and = 0 otherwise.

   [**Ans.** $\boldsymbol{B} = -\frac{\mu_0 J_0 a}{2}\hat{j}$ for $z > a/2$, $= \frac{\mu_0 J_0 a}{2}\hat{j}$ for $z < -a/2$, $= -\mu_0 J_0 z\hat{j}$ for $|z| < a/2$]

8. An infinitely long cylinder of radius a carries a current with current density $\boldsymbol{J} = \boldsymbol{J}_0\left(\frac{s}{a}\right)\hat{k}$. A thin wire placed along the axis of the cylinder carries a current in the opposite direction equal to that carried by the cylinder.
   (a) Compute the magnetic field $\boldsymbol{B}$ everywhere.
   (b) Show explicitly that the divergence of this magnetic field is zero. [**Ans.** (a) $\frac{\mu_0 J_0}{3sa}(s^3 - a^3)$]

9. A cylindrical cavity is drilled into a long, solid, cylindrical conductor (along the entire length of the cylinder). The axis of the cavity is parallel to that of the conductor and is displaced from it by a vector $\boldsymbol{a}$. A uniform current density $\boldsymbol{J}$ exists in the conductor and has a direction along the axis of the conductor. Find the magnetic field in the cavity. [**Ans.** $\frac{\mu_0}{2}(\boldsymbol{J} \times \boldsymbol{a})$]

10. An ideal battery of emf ε sends a current in a uniform wire of conductivity σ and length L. Find the curl of the magnetic field inside the wire. [**Ans.** $\mu_0\sigma\varepsilon/L$ along the current]

11. A magnetic field $\boldsymbol{B} = B_0\frac{R}{s}\hat{\phi}$ exists in the region $s \geq R$. Find the current density in the region. [**Ans.** zero]

12. The magnetic field in a region is given by $\boldsymbol{B} = ks\hat{\phi}$ for $s < R$ and $\boldsymbol{B} = \frac{kR^2}{s}\hat{\varphi}$ for $s > R$ in cylindrical coordinates. Write the current (in terms of k, R and μ_0) that will give this magnetic field. [**Ans.** $2k\hat{k}$, zero]

13. A surface current on the surface $s = R$ produces a magnetic field given by $\boldsymbol{B} = -B_0\hat{i}$ for $s < R$ and $\boldsymbol{B} = \frac{B_0 R^2}{s^2}(\cos\varphi\hat{s} + \sin\varphi\hat{\varphi})$ for $s > R$. (a) Show that it satisfies proper boundary conditions at $s = R$ to be a valid magnetic field. (b) Find the current that can produce this magnetic field.

    [**Ans.** (b) $K = \frac{2B_0}{\mu_0}\sin\varphi$ on the surface]

14. A solenoid is made by closely winding a long, thin wire around a cylindrical surface of radius R. The turns, however, are not perpendicular to the axis but make an angle θ with it. The number of turns per unit length of the cylinder is n. Find the magnetic field everywhere. Check that the boundary condition is satisfied on the surface of the cylinder. [**Ans.** $\mu_0 nI\hat{k}$ for $s < R\cos\theta$, $\mu_0 nI\sin\theta\hat{\phi}$ for $s > R$]

15. A tightly wound solenoid of radius r and length $l = 4r$ carries a current. Let B be the magnetic field at an axial point, a distance z from the solenoid and B_0 the field that would have been there if the solenoid were long.
    (a) Find B/B_0 at the centre of the solenoid.
    (b) Plot B/B_0 as a function of z/R. [**Ans.** (a) $\frac{4}{\sqrt{17}}$]

16. A tightly wound solenoid of length L and radius a is kept with its axis along the z-axis and the centre at the origin. The length of the solenoid is much larger than its radius. The magnetic field at the origin is $\boldsymbol{B}_0 \hat{k}$ when a current I is passed through the solenoid.

    (a) Find the magnetic dipole moment.

    (b) Find the magnetic field at $(0, 0, z_0)$ for $z_0 >> L$. [**Ans.** (a) πnIR^2L, (b) $\dfrac{\mu_0 nIR^2}{2z_0^3}$]

17. The current density in a long, cylindrical wire of radius R is $\boldsymbol{J} = J_0 e^{10\left(\frac{s-R}{R}\right)} \hat{k}$ with the z-axis along the axis of the wire. This means the current passes preferentially through the outer portions of the wire (skin effect). Find the magnetic fields at $s = R$ and $s = R/2$, and their ratio. [**Ans.** Ratio ≈ 330]

❑

14 The Relativistic Transformation of *E–B* Fields

As mentioned earlier, magnetism may be regarded as a display of relativistic phenomenon. A static charge produces only an electric field and no magnetic field. A moving charge produces an electric as well as a magnetic field. But what is static in one frame can be moving in another frame. So, what is a purely electric field in one frame can be a combination of electric and magnetic fields in some other frame. Also, what is a purely magnetic field in one frame can be a combination of electric and magnetic fields in some other frame. In general, the fields $\boldsymbol{E}, \boldsymbol{B}$ in one frame transform to $\boldsymbol{E}', \boldsymbol{B}'$ in some other frame according to certain transformation equations. In this chapter, we will briefly talk about these equations.

14.1 A Review of Some Relativistic Equations

The special theory of relativity was put forward by Albert Einstein in 1905 in his paper "On the Electrodynamics of Moving Bodies" published in German in the prestigious journal *Annalen der Physik*. Special relativity is a very interesting subject and you must go through a textbook on the topic to enjoy it. Here we shall discuss the main results, which will be needed to connect ***E–B*** fields in different frames.

The frames S and S′

Consider two frames, both inertial, moving with respect to each other with uniform velocities. Call one of the frames the S frame and the other, the S′ frame. Set up one Cartesian coordinate system with x–y–z axes fixed in S and another with x'–y'–z' axes fixed in S′ in such a way that the

(a) x-axis and x'-axis coincide with each other all the time, and

(b) y-axis and y'-axis are parallel to each other. You don't have to talk about the z-, z'-axis— they will anyway be parallel.

(c) The velocity of S′ with respect to S is $v\hat{\boldsymbol{i}}$.

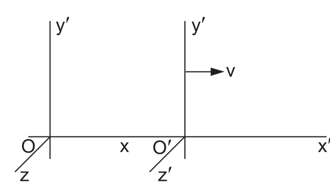


Fig. 14.1

There is another quantity that must be specified for the two frames and that is the zero of time. The rate of flow of time is decided by nature but $t = 0$ for a particular frame is decided by us.

We pick up an instant and call it $t = 0$. In our case, S′ moves with respect to S with a velocity $v\hat{\boldsymbol{i}}$. The instant at which the origin O' of S′ crosses the origin O of S is taken to be $t = 0$ for frame S and $t' = 0$ for frame S′.

The symbols S and S′ will always be used in this chapter to mean inertial frames with the axes and time origin defined as above.

Events

Any event, such as the breaking of a glass, the lighting of a match, dropping of a ball, etc., occurs at a particular location and a particular time. When one says that Netaji Subhash Chandra Bose was born on 23 July 1897 in Cuttack, one talks about an event and specifies its time of occurrence and its location (though not very precisely). While referring to an event during our discussion of relativity, we will assume that the location of the event can be specified precisely as a point and the time of occurrence of that event can be specified precisely as an instant. If an event is being described from the frame S, its location will be given as (x, y, z) and its time will be given as t. The same event can also be described from frame S′. The location and the time of occurrence will then be written as (x', y', z') and t'. These two sets of quantities are related to each other by what we call the Lorentz transformation:

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}},\ y' = y,\ z' = z,\ t' = \frac{t - \dfrac{xv}{c^2}}{\sqrt{1 - v^2/c^2}}. \tag{14.1}$$

Length contraction

One of the most fascinating results from the theory of special relativity is *length contraction*. The separation observed between two given points may be different from different frames. Thus the separation between two electric poles on a road is more in the road frame and less in the car frame if the car is moving along the road with some velocity. Similarly, the length of a metre stick is 1 m only if you are looking at it from a frame in which the stick is at rest. If you are in a frame in which the stick is moving along its length, the length of the metre stick will be less than 1 m. However, if the stick is moving perpendicular to its length, the length will remain 1 m.

Taking another example, suppose you have a square plate kept on your table. The edges are equal in length, say L_0. If this same plate starts moving along one of its edges, it will no more remain square. The two edges parallel to the motion will now have length less than L_0, whereas the two edges perpendicular to the motion will have length L_0.

The phenomenon of having different lengths of an object in different frames is called length contraction. The rules of length contraction may be summarized as follows.

(a) The length L_0 of an object is maximum when measured in a frame in which the object is at rest. This length is called the rest length of the object.

(b) The length of an object in the direction of its motion is contracted and is given by

$$L = L_0\sqrt{1 - v^2/c^2}, \tag{14.2}$$

where L_0 is the rest length of the object, v is the speed of the object in the frame from which it is viewed, and c is the speed of light in vacuum.

(c) The length of an object perpendicular to the direction of its motion remains unchanged.

Please do not think that length contraction is a material property. It is not that, because of velocity, some forces appear between the molecules of the object, which cause contraction. Consider two spaceships coasting in outer space, one behind the other, with the same uniform velocity v. The separation L between the two is the contracted distance. For an observer in one of the spaceships, both the spaceships are at rest and the separation L_0 between them is larger than the separation seen by an observer on the earth. There is no material between the spaceships, yet their separation is less when they move (as seen from the earth) than when they are at rest (as seen from one of the spaceships).

EXAMPLE 14.1 **A rod is kept at rest in frame S. It makes an angle θ with the x-axis. This same rod is viewed from the frame S′, which moves with respect to S along the common x–x' axis with velocity v. Find the angle made by the rod with the x'-axis as seen from S′.**

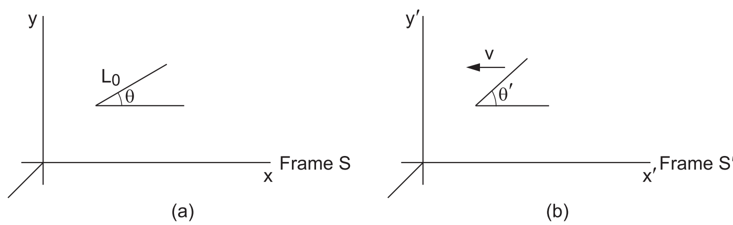


Fig. 14.2

Solution Let the rest length of the rod be L_0. The projection of the length of the rod along the x-axis is $L_0 \cos\theta$ and that perpendicular to the x-axis is $L_0 \sin\theta$. When seen from S′, the rod moves along the x'-axis (in the negative x'-direction) with speed v. The projection $L_0 \cos\theta$ is along the motion and will be contracted by the factor $\sqrt{1 - v^2/c^2}$. The projection $L_0 \sin\theta$ is perpendicular to the direction of motion and will hence remain unchanged.

Suppose the length of the rod as seen from S′ is L' and the angle it makes with the x'-axis is θ'. The projection along the x'-axis is $L'\cos\theta'$ and that perpendicular to the x'-axis is $L'\sin\theta'$. Thus,

$$L'\cos\theta' = (L_0\cos\theta)\sqrt{1 - v^2/c^2}$$

and $L' \sin\theta' = L_0 \sin\theta.$

This gives $\tan\theta' = \dfrac{\tan\theta}{\sqrt{1 - v^2/c^2}} \cdot$

Time dilation

Another fascinating result of the special theory of relativity is *time dilation*. The time interval between two events is not the same in all frames. Consider a high-speed spaceship moving in space along a straight path. Suppose it passes an object A and then another object B. In the earth's control centre, we time the two events and find that it took time Δt_1 for the spaceship to go from A to B. However, clocks are also placed in the spaceship itself. In the spaceship frame, the two events are described as

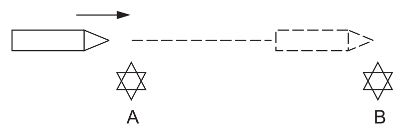


Fig. 14.3

(a) *A* coming from the right and crossing the spaceship, and

(b) *B* coming after some time and crossing it.

The time interval between the two events is Δt_2 as measured in the spaceship frame. It turns out that Δt_1 and Δt_2 are different and are related as $\Delta t_2 = \Delta t_1 \sqrt{1 - v^2/c^2}$. Time seems to run at different rates in different frames.

For some pairs of events, we can define a proper frame and a proper time interval. If in a frame the two events occur at the same location, that is, at the same point (x, y, z), that frame is the *proper frame* for this pair of events. The time interval between the two events measured in this frame is called the *proper time interval*. It turns out that the proper time interval is the smallest time interval for the given pair of events. Denote it by Δt_0. If you look at the same events from a frame moving with velocity v with respect to the proper frame, the time interval Δt between the events will be larger and will be given by

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}} \cdot \tag{14.3}$$

Think of the example of the spaceship crossing A and B once again. Which of Δt_1 and Δt_2 is the proper time interval and which is improper? Stop reading further, get your answer and then read the next sentence. The answer is that Δt_2 is the proper time interval. In the spaceship frame, A and B cross the spaceship at the same place. In the earth's frame, the spaceship crosses A and then moves to another location to cross B. The spaceship frame is, therefore, the proper frame for this pair of events and the earth's frame is improper. Thus, Δt_2 is smaller than Δt_1.

Velocity addition

Suppose two objects A and B move in a frame with velocities $\boldsymbol{u}_{\mathrm{A}}$ and $\boldsymbol{u}_{\mathrm{B}.}$ What is the velocity of B with respect to A? The nonrelativistic equation is $\boldsymbol{v}_{\mathrm{BA}} = \boldsymbol{v}_{\mathrm{B}} - \boldsymbol{v}_{\mathrm{A}}$. The relativistic equations are very different.

If the bodies are moving along the same line in the same direction,

$$v_{\mathrm{BA}} = \frac{v_{\mathrm{B}} - v_{\mathrm{A}}}{1 - \dfrac{v_{\mathrm{B}} v_{\mathrm{A}}}{c^2}}.$$

If they move in opposite directions, you have

$$v_{\mathrm{BA}} = \frac{v_{\mathrm{B}} + v_{\mathrm{A}}}{1 + \dfrac{v_{\mathrm{B}} v_{\mathrm{A}}}{c^2}}.$$

Next, suppose the velocity of an object in frame S is $\boldsymbol{u} = u_x \hat{\boldsymbol{i}} + u_y \hat{\boldsymbol{j}} + u_z \hat{\boldsymbol{k}}$. The velocity $\boldsymbol{u'}$ of this object as seen from S′ will be

$$u'_x = \frac{u_x - v}{1 - \dfrac{u_x v}{c^2}}, \quad u'_y = \frac{u_y \sqrt{1 - v^2/c^2}}{1 - \dfrac{u_x v}{c^2}}, \quad u'_z = \frac{u_z \sqrt{1 - v^2/c^2}}{1 - \dfrac{u_x v}{c^2}}.$$

Linear momentum

Newton's second law, as you have learnt in school, is $\boldsymbol{F} = m\boldsymbol{a}$, which is correct for nonrelativistic velocities. This can also be written as $\boldsymbol{F} = \dfrac{d\boldsymbol{p}}{dt}$, where $\boldsymbol{p}$ is the linear momentum defined as $\boldsymbol{p} = m\boldsymbol{v}$. It turns out that this equation $\boldsymbol{F} = \dfrac{d\boldsymbol{p}}{dt}$ is valid even in the case of relativistic velocities provided the linear momentum is redefined as $\boldsymbol{p} = \dfrac{m\boldsymbol{v}}{\sqrt{1 - v^2/c^2}}\cdot$ This is indeed taken as the definition of linear momentum. For $v << c$, it reduces to the more familiar equation $\boldsymbol{p} = m\boldsymbol{v}$. In many cases, substituting $\dfrac{m}{\sqrt{1 - v^2/c^2}}$ for the mass of a particle in equations derived for nonrelativistic situations gives correct equations for relativistic cases. But not always.

Force transformation

In a nonrelativistic description, when you go from one inertial frame to another, acceleration and forces do not change. For example, the gravitational force is $\boldsymbol{F} = G\dfrac{m_1 m_2}{r^2}\hat{\boldsymbol{r}}$. As the separation between the particles is the same in all inertial frames in nonrelativistic dynamics, the force $\boldsymbol{F}$ also remains the same. But that is not the case if you consider relativistic effects. The force also gets transformed as you change the frame. However, the law of physics, $\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt}$, remains valid in all inertial frames.

Suppose a particle has a velocity $\boldsymbol{u}$ in frame S at a certain instant and, at this instant, the resultant force on it is $\boldsymbol{F}$. What is the force on this particle at the corresponding time t' in frame S′ that moves at velocity $v\hat{\boldsymbol{i}}$ with respect to S? Again, we will only look at the results.

The transformation equations are as follows.

$$F'_x = \frac{F_x - \dfrac{v}{c^2}\boldsymbol{uF}}{1 - \dfrac{u_x v}{c^2}},$$

$$F'_y = \frac{F_y\sqrt{1 - v^2/c^2}}{1 - \dfrac{u_x v}{c^2}},$$

and
$$F'_z = \frac{F_z\sqrt{1 - v^2/c^2}}{1 - \dfrac{u_x v}{c^2}}.$$

These complicated equations get somewhat simplified if the particle is at rest in the frame S at the desired instant. Thus $\boldsymbol{u} = 0$ and one gets

$$F'_x = F,$$

$$F'_y = F_y\sqrt{1 - \frac{v^2}{c^2}},$$

and
$$F'_z = F_y\sqrt{1 - \frac{v^2}{c^2}}.$$

14.2 Simple Examples of *E–B* Transformations

We will first consider some simple cases of charge and current distributions and look at them from S and S′. Then we will use the standard methods to calculate $\boldsymbol{E}$ and $\boldsymbol{B}$ fields in S and S′. This will give you an idea about how $\boldsymbol{E}$ and $\boldsymbol{B}$ in one frame transform to $\boldsymbol{E}'$ and $\boldsymbol{B}'$ in another frame. Lastly, we will look at the general equations of transformation and show that the equations for special situations follow naturally from these general equations.

A line charge along the *x*-axis

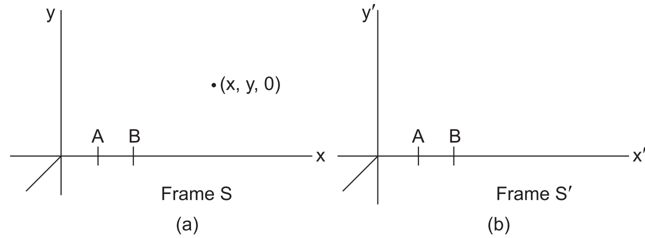


Fig. 14.4

Suppose there is an infinitely long line charge placed along the x-axis of frame S. Let the linear charge density λ be uniform. The charges are at rest in S and hence produce no magnetic field. Consider a point $(x, y, 0)$ [Figure 14.4(a)]. The electric field at this point is

$$\boldsymbol{E} = \frac{\lambda}{2\pi\varepsilon_0 y}\hat{\boldsymbol{j}}. \tag{i}$$

So $$E_x = 0, E_y = \frac{\lambda}{2\pi\varepsilon_0 y}, E_z = 0$$

$$B_x = 0, B_y = 0, B_z = 0.$$

Now look at the same situation from frame S′. Each charge on the x'-axis moves with a velocity $-v\hat{\boldsymbol{i}}$. Apart from this, the linear charge density is also changed due to length contraction. Consider two points A and B on this line charge. This length AB is moving in S′ but at rest in S. The length AB is therefore larger in S and smaller in S′. If the separation between A and B as seen in S and S′ be L_0 and L then

$$L = L_0\sqrt{1 - v^2/c^2}.$$

The charge q between A and B remains the same. Thus the linear charge density λ', as seen in the S′ frame, will be larger than λ by the factor $1/\sqrt{1 - v^2/c^2}$. Therefore,

$$\lambda' = \frac{\lambda}{\sqrt{1 - v^2/c^2}}.$$

So, in S′, there is a line charge of density λ' moving along the $-\hat{\boldsymbol{i}}$-direction with velocity v. The line charge will create an electric field according to Coulomb's law and the current resulting from the motion will create a magnetic field according to the Biot–Savart law.

The electric field at $(x', y', 0)$ will be

$$\boldsymbol{E}' = \frac{\lambda'}{2\pi\varepsilon_0 y'}\hat{\boldsymbol{j}} = \frac{\lambda}{2\pi\varepsilon_0 y\sqrt{1 - v^2/c^2}}\hat{\boldsymbol{j}}.$$

$y' = y$ according to the Lorentz transformation (Equation 14.1). Using (i),

$$\boldsymbol{E}' = \frac{\boldsymbol{E}}{\sqrt{1 - v^2/c^2}}. \tag{ii}$$

The current corresponding to the motion of the line charge will be $I' = \lambda' v$ in the negative x'-direction. The magnetic field at $(x', y', 0)$, according to the Biot–Savart law, is,

$$\boldsymbol{B}' = \frac{\mu_0 I'}{2\pi y'}(-\hat{\boldsymbol{k}}) = \frac{\mu_0 \lambda' v}{2\pi y}(-\hat{\boldsymbol{k}})$$

$$= \frac{\mu_0 \lambda v}{2\pi y\sqrt{1 - v^2/c^2}}(-\hat{\boldsymbol{k}}).$$

As we are looking for a relation between the fields, you should write λ in terms of E from (i). Then,

$$\boldsymbol{B}' = \frac{\mu_0 v \varepsilon_0}{\sqrt{1 - v^2/c^2}} E_y(-\hat{\boldsymbol{k}}) = \frac{-v/c^2}{\sqrt{1 - v^2/c^2}} E_y(\hat{\boldsymbol{k}}). \tag{iii}$$

In component form, the fields in S′ are

$$E'_x = 0, \ E'_y = \frac{E_y}{\sqrt{1 - v^2/c^2}}, \ E'_z = 0$$

$$B'_x = 0, \ B'_y = 0, \ B'_z = \frac{-\frac{v}{c^2} E_y}{\sqrt{1 - v^2/c^2}}.$$

The $\boldsymbol{E}$, $\boldsymbol{B}$ fields are written at $(x, y, 0)$ and $\boldsymbol{E}', \boldsymbol{B}'$ are written at the corresponding point $(x', y', 0)$. What is meant by 'the corresponding point'? These points are related through Lorentz transformation. As the fields do not depend on time in any of the two frames, and also do not depend on x, there is no confusion on which points and at which time we are relating the fields.

We expressed the electric field $\boldsymbol{E}'$ in the S′ frame by using Coulomb's law. Are we allowed to do that in view of the fact that the charges are moving in the S′ frame? As you know, Coulomb's law is valid for electrostatic situations only. Stop reading further for a while and try to answer the question yourself. Although the charges are moving, the charge density does not change with time. At any given point on the x'-axis, the charge density λ' remains constant as time passes. At any point other than those on the x'-axis, the charge density is zero and remains zero all the time. Thus it is indeed an electrostatic situation and we can use Coulomb's law. Another justification will be suggested in the Exercises.

An infinite-plane surface charge

Suppose the x–y plane of the S frame contains a uniform surface charge distribution with surface charge density σ. The whole charge distribution is at rest in the frame S. The charge will produce an electric field at any point (x, y, z), given by

$$\boldsymbol{E} = \frac{\sigma}{2\varepsilon_0}\frac{|z|}{z}\hat{\boldsymbol{k}}. \tag{i}$$

There will be no magnetic field as the charges are at rest. In component form,

$$E_x = 0,\ E_y = 0,\ E_z = \frac{\sigma}{2\varepsilon_0}\frac{|z|}{z}\hat{\boldsymbol{k}}$$

$$B_x = 0,\ B_y = 0,\ B_z = 0.$$

Now look at the same charge distribution from the frame S′. The whole charge on the x'–y' plane moves with velocity $-v\hat{\boldsymbol{i}}$ in this frame. Also, considering any area of the charged plane, its length parallel to the x'-axis will contract while its length parallel to the y'-axis will remain unchanged. Thus, the area will contract by a factor of $\sqrt{1-v^2/c^2}$ when seen from S′. Correspondingly, the charge density will increase by the same factor. If σ' be the surface charge density on the x'–y' plane, as seen from S′,

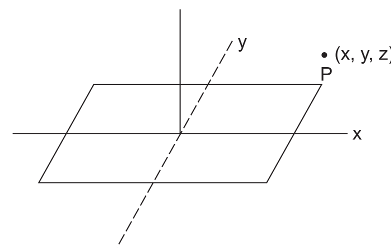


Fig. 14.5

$$\sigma' = \frac{\sigma}{\sqrt{1-v^2/c^2}}.$$

The electric field at (x', y', z') due to this surface charge distribution will be

$$\boldsymbol{E}' = \frac{\sigma'}{2\varepsilon_0}\frac{|z'|}{z'}\hat{\boldsymbol{k}} = \frac{\sigma}{2\varepsilon_0\sqrt{1-v^2/c^2}}\frac{|z|}{z}\hat{\boldsymbol{k}}$$

$$= \frac{\boldsymbol{E}}{\sqrt{1-v^2/c^2}}. \tag{ii}$$

The surface charge spread over the entire x'–y' plane moves with velocity $-v\hat{\boldsymbol{i}}$. This constitutes a surface current with surface current density

$$\boldsymbol{K}' = \sigma'(-v\hat{\boldsymbol{i}}) = \frac{\sigma v}{\sqrt{1-v^2/c^2}}(-\hat{\boldsymbol{i}}).$$

The magnetic field at (x', y', z') due to this current will be

$$\boldsymbol{B}' = \frac{1}{2}\mu_0\boldsymbol{K}'\times\hat{\boldsymbol{n}} = \frac{1}{2}\mu_0\boldsymbol{K}'\times\frac{|z'|}{z'}\hat{\boldsymbol{k}}$$

$$= \frac{1}{2}\mu_0\frac{\sigma v}{\sqrt{1-v^2/c^2}}\frac{|z'|}{z'}\hat{\boldsymbol{j}}.$$

Using (i) to eliminate σ,

$$\boldsymbol{B}' = \frac{\mu_0 \varepsilon_0 v E_z \hat{\boldsymbol{j}}}{\sqrt{1 - v^2/c^2}} = \frac{\frac{v}{c^2} E_z \hat{\boldsymbol{j}}}{\sqrt{1 - v^2/c^2}} \cdot \tag{iii}$$

In component form,

$$E'_x = 0, \; E'_y = 0, \; E'_z = \frac{E_z}{\sqrt{1 - v^2/c^2}}$$

$$B'_x = 0, \; B'_y = \frac{\frac{v}{c^2} E_z}{\sqrt{1 - v^2/c^2}}, \; B'_z = 0.$$

An infinitely long solenoid along the *x*-axis

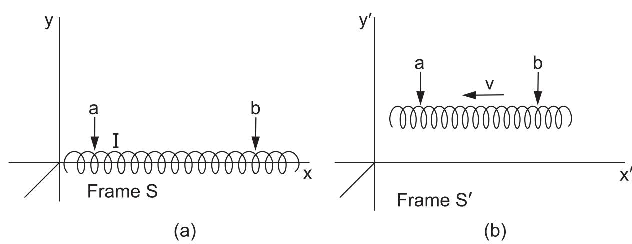


Fig. 14.6

Consider a tightly wound, infinitely long solenoid kept along the x-axis. In the frame S, the solenoid is at rest and carries a current I. The electric field inside the solenoid (not very close to the surface) is zero and the magnetic field is $\boldsymbol{B} = \mu_0 n I \hat{\boldsymbol{i}}$, where n is the number of turns per unit length of the solenoid. Why is the electric field zero? First of all, the current flows in a wire and the wire contains positive as well as negative charges. The charge density $\rho(\boldsymbol{r})$ is zero and hence there is no electric field. There may be a surface charge density, producing an electric field in the wire driving the current. But for points inside the solenoid, not very close to the wires, this will be zero. So the ***E, B*** fields inside the solenoid are

$$E_x = 0, \; E_y = 0, \; E_z = 0$$

$$B_x = \mu_0 n I, \; B_y = 0, \; B_z = 0.$$

Now look at the same situation from the S′ frame. In this frame, the solenoid moves towards the negative x'-axis with speed v. Now consider any two turns a and b of the solenoid. The separation between the two will be larger in the S frame and will contract by a factor of $\sqrt{1 - v^2/c^2}$ in the S′ frame. Thus the number of turns per unit length will be larger in S′ than in S by the factor $1/\sqrt{1 - v^2/c^2}$.

$$n' = \frac{n}{\sqrt{1 - v^2/c^2}} \cdot$$

If you have concluded that the magnetic field inside the solenoid is larger in S′ than in S as $n' > n$, you are in haste. You have to carefully see what happens to the current.

Consider a particular point A in the wire of the solenoid. First, let us discuss the situation from the perspective of frame S. A charge Δq crosses this point in a time interval Δt such that the current is $I = \Delta q/\Delta t$. Consider two events—(a) the beginning of this interval Δt when the leading edge of ΔQ has crossed the point A and (b) the end of Δt when all the charge ΔQ has crossed this point. As the point A is at rest in the frame S, the two events have occurred at the same location in S and hence Δt is the proper time interval between the events.

Now look at the same two events from S′ (Figure 14.7). Still look at the same point A of the solenoid and look at the times t_1' and t_2', when the charge Δq started crossing A and finished crossing A respectively. The two events occurred at different places, as the whole solenoid moved a distance $v(t_2' - t_1')$ in this interval. Thus the time interval $\Delta t' = (t_2' - t_1')$ is the improper time interval and is larger than Δt.

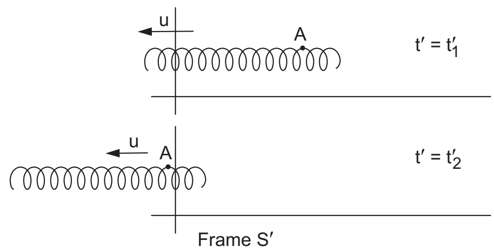


Fig. 14.7

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - v^2/c^2}}.$$

The current in the solenoid in frame S′ is

$$I' = \frac{\Delta q}{\Delta t'} = \frac{\Delta q}{\Delta t}\sqrt{1 - v^2/c^2} = \sqrt{1 - \frac{v^2}{c^2}}I.$$

The magnetic field inside the solenoid, as seen from S′, would be

$$\boldsymbol{B}' = \mu_0 n' I' \hat{\boldsymbol{i}} = \mu_0 n I \hat{\boldsymbol{i}} = \boldsymbol{B}.$$

The magnetic field inside the solenoid is the same in S and S′.

There is yet another aspect. The wire is taken to be neutral in frame S as positive and negative charge densities cancel each other. As you look at the solenoid from S′, the positive and negative charge densities change in different fashions. This is because they had different velocities in S (positive charge density at rest and the negative charge density moving to cause the current) and hence will have different velocities in S′. There could be a net surface charge density on the solenoid surface. But this does not cause any electric field inside the solenoid. Also, the entire surface charge density will also move in the negative x'-direction as seen in S′. This will constitute a surface current parallel to the x'-direction. But this current does not cause a magnetic field inside. So, the fields inside the solenoid, as seen from S′, are

$$\boldsymbol{E}' = 0, \ \boldsymbol{B}' = \boldsymbol{B}.$$

In component form,

$$E'_x = 0, \ E'_y = 0, \ E'_z = 0$$

$$B'_x = B_x, \ B'_y = B_y, \ B'_z = B_z.$$

14.3 Relativistic Transformation Equations

Let us now look at the general equations for transforming $\boldsymbol{E}$ and $\boldsymbol{B}$ fields in frame S to $\boldsymbol{E}'$ and $\boldsymbol{B}'$ in frame S′. Here, $\boldsymbol{E}$ and $\boldsymbol{B}$ are fields at a point (x, y, z) at time t, and $\boldsymbol{E}'$ and $\boldsymbol{B}'$ are fields at the corresponding point (x', y', z') at the corresponding time t'. The equations are

$$E'_x = E_x, \ E'_y = \frac{E_y - vB_z}{\sqrt{1 - v^2/c^2}}, \ E'_z = \frac{E_z + vB_y}{\sqrt{1 - v^2/c^2}}$$

$$B'_x = B_x, \ B'_y = \frac{B_y + \frac{v}{c^2}E_z}{\sqrt{1 - v^2/c^2}}, \ B'_z = \frac{B_z - \frac{v}{c^2}E_y}{\sqrt{1 - v^2/c^2}}. \tag{14.4}$$

The field components parallel to the motion of the frames do not change. The magnetic and electric field components in one frame, perpendicular to the motion of the frames, mix to give the perpendicular components in the other frame.

Remember that S′ moves with respect to S in the positive x-direction. The x- and x'-axes are coincident, and the y- and y'-axes are parallel to each other. The reverse transformation is easy to write. Interchange the primed and unprimed components, and replace v by $-v$. So,

$$E_x = E'_x, \ E_y = \frac{E'_y + vB'_z}{\sqrt{1 - v^2/c^2}}, \ E_z = \frac{E'_z - vB'_y}{\sqrt{1 - v^2/c^2}}$$

$$B_x = B'_x, \ B_y = \frac{B'_y - \frac{v}{c^2}E'_z}{\sqrt{1 - v^2/c^2}}, \ B_z = \frac{B'_z + \frac{v}{c^2}E'_y}{\sqrt{1 - v^2/c^2}}. \tag{14.5}$$

Check that the three examples given in Section 14.2 are consistent with the general transformation formulae.

14.4 Electric and Magnetic Fields due to a Point Charge Moving with Constant Velocity

Suppose a point charge q moves in frame S along the x-axis with a constant velocity $v\hat{\boldsymbol{i}}$. What are the electric and magnetic fields at a point (x, y, z) at time t? Remember this is certainly not an electrostatic or a magnetostatic situation. Neither the charge density nor the current density is constant in time. You cannot use Coulomb's law or the Biot–Savart law to get the fields.

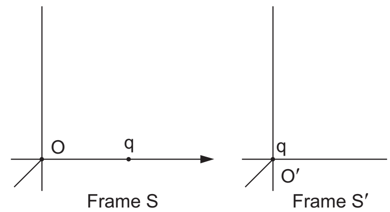


Fig. 14.8

However, if you go to frame S′, moving with respect to S, with a constant velocity v in the common x–x' direction, the charge is at rest in this frame. There is no current in this frame and the charge is at rest all the time. So the magnetic field is zero and the electric field is obtained by using Coulomb's law. If you take the position of the charge in the S′ frame to be the origin O′, the fields at (x', y', z') at any time t' are

$$\boldsymbol{E}' = \frac{q\boldsymbol{r}'}{4\pi\varepsilon_0 r'^3} \text{ and } \boldsymbol{B}' = 0.$$

In component form,

$$E'_x = \frac{qx'}{4\pi\varepsilon_0(x'^2 + y'^2 + z'^2)^{3/2}},\quad E'_y = \frac{qy'}{4\pi\varepsilon_0(x'^2 + y'^2 + z'^2)^{3/2}},\quad E'_z = \frac{qz'}{4\pi\varepsilon_0(x'^2 + y'^2 + z'^2)^{3/2}}$$

$$B'_x = 0,\ B'_y = 0,\ B'_z = 0.$$

You can use the transformation equations to get the fields in the S frame. These fields have to be time-dependent as the point charge keeps changing its position. To be specific, let us try to get the field at the point $(x, y, 0)$ at the instant the charge crosses the origin O in S. Take this instant as $t = 0$. At this same instant, the origin O′ of S′ also crosses O. We are also restricting ourselves to the x–y plane. So x and y are arbitrary but $z = 0$. This is not really a restriction, however, because the whole field will be symmetric about the x-axis (direction of motion of the charge).

What is the corresponding point of $(x, y, 0)$ in S′? The coordinates are given by the Lorentz transformation equation

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} = \frac{x}{\sqrt{1 - v^2/c^2}},\ y' = y \text{ and } z' = z = 0.$$

As the fields are independent of time in S′, you need not calculate the corresponding time in S′.

Electric field

The x-component of the electric field at $(x, y, 0)$ at time $t = 0$, as seen in frame S, is

$$E_x = E'_x = \frac{qx'}{4\pi\varepsilon_0(x'^2 + y'^2 + z'^2)^{3/2}}$$

$$= \frac{qx}{4\pi\varepsilon_0\sqrt{1-v^2/c^2}\left[\dfrac{x^2}{1-v^2/c^2}\right]^{3/2}}.$$

Similarly,

$$E_y = \frac{E'_y + vB'_z}{\sqrt{1-v^2/c^2}} = \frac{qy'}{4\pi\varepsilon_0\sqrt{1-v^2/c^2}\,(x'^2+y'^2+z'^2)^{3/2}}$$

$$= \frac{qy}{4\pi\varepsilon_0\sqrt{1-v^2/c^2}\left[\dfrac{x^2}{1-v^2/c^2}+y^2\right]^{3/2}}$$

and $$E_z = \frac{E'_z - vB'_y}{\sqrt{1-v^2/c^2}} = 0.$$

You can write the electric field at $t = 0$ as

$$\boldsymbol{E} = E_x\hat{\boldsymbol{i}} + E_y\hat{\boldsymbol{j}} + E_z\hat{\boldsymbol{k}}$$

$$= \frac{q(x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}})}{4\pi\varepsilon_0\sqrt{1-v^2/c^2}\left[\dfrac{x^2}{1-v^2/c^2}+y^2\right]^{3/2}}.$$

First look at the direction of the electric field. It is along $x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$, that is, radial from the origin. The field at any point P is radial as seen from the position of the point charge at that instant. This result is the same as that for a stationary charge. Wherever the charge goes, the electric field turns to become radial from the instantaneous position of the charge.

If you are wondering how the point P gets to know the instantaneous position of the charge so that the electric field is in the radial direction from that position, you are on the right track. The moment the charge moves ahead, the electric field at P changes its direction (Figure 14.9), as if no time is needed to pass the information to the point P about the movement of the charge. The special theory of relativity tells us that no information can be transmitted with speed more than c. But the observer at P can tell the position of the charge immediately by looking at the direction of $\boldsymbol{E}$ and extending it to cut the x-axis. Think of the statement of the problem and the conditions under which the expressions for the electric field components were derived. Do it carefully and

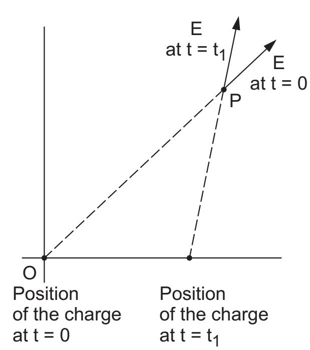


Fig. 14.9

you will understand how the field at P turns and becomes radial from the 'instantaneous' position of the charge. This expression is valid for a charge moving with a constant velocity $v\hat{\boldsymbol{i}}$ and crossing the origin at $t = 0$. Not only the instantaneous position but all future positions of the charge are also known. No wonder the field is radial from the instantaneous position.

Now come to the magnitude of the electric field. If OP makes an angle θ with the x-axis, $x = r\cos\theta$ and $y = r\sin\theta$, where $r = OP$ is the distance of P from O. Thus,

$$|\boldsymbol{E}| = \frac{qr}{4\pi\varepsilon_0\sqrt{1-v^2/c^2}\left[\dfrac{r^2\cos^2\theta}{1-v^2/c^2} + r^2\sin^2\theta\right]^{3/2}}$$

$$= \frac{q}{4\pi\varepsilon_0 r^2}\frac{1-v^2/c^2}{\left[\cos^2\theta + \left(1-\dfrac{v^2}{c^2}\right)\sin^2\theta\right]^{3/2}}$$

$$= \frac{q}{4\pi\varepsilon_0 r^2}\frac{1-v^2/c^2}{[1-(v^2/c^2)\sin^2\theta]^{3/2}}. \tag{14.6}$$

The electric field itself may be written as

$$\boldsymbol{E} = \frac{q(1-v^2/c^2)\boldsymbol{r}}{4\pi\varepsilon_0 r^3\left[1-(v^2/c^2)\sin^2\theta\right]^{3/2}}. \tag{14.7}$$

The magnitude of the electric field depends on θ. It continuously increases as θ goes from 0 to $\pi/2$ (and also from π to $3\pi/2$). For a given distance r from the charge, the field is maximum for $\theta = \pi/2$ and that is

$$E(\theta = \pi/2) = \frac{q}{4\pi\varepsilon_0 r^2}\frac{1}{\sqrt{1-v^2/c^2}}.$$

The field is minimum for $\theta = 0$, and equals

$$E(\theta = 0) = \frac{q}{4\pi\varepsilon_0 r^2}(1-v^2/c^2).$$

So there is more electric field in the transverse direction than in the longitudinal direction. Figure 14.10 shows the field lines qualitatively.

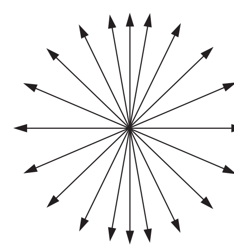


Fig. 14.10

Magnetic field

Now turn to the magnetic field at $(x, y, 0)$ at time $t = 0$ when the charge is at the origin, moving with velocity $v\hat{\boldsymbol{i}}$ in S. The field components are

$$B_x = B'_x = 0$$

$$B_y = \frac{B'_y - \frac{v}{c^2} E'_z}{\sqrt{1 - v^2/c^2}} = 0$$

$$B_z = \frac{B'_z + \frac{v}{c^2} E'_y}{\sqrt{1 - v^2/c^2}} = \frac{vqy'}{c^2 4\pi\varepsilon_0 [x'^2 + y'^2]^{3/2} \sqrt{1 - v^2/c^2}}$$

$$= \frac{\mu_0 qvy}{4\pi\sqrt{1 - v^2/c^2}\left[\frac{x^2}{1 - v^2/c^2} + y^2\right]^{3/2}}.$$

Write $x = r\cos\theta$ and $y = r\sin\theta$. Then,

$$B_z = \frac{\mu_0 qv(1 - v^2/c^2)\sin\theta}{4\pi r^2\left[1 - \frac{v^2}{c^2}\sin^2\theta\right]^{3/2}}.$$

So,

$$\boldsymbol{B} = \frac{\mu_0 qv(1 - v^2/c^2)\sin\theta}{4\pi r^2\left[1 - \frac{v^2}{c^2}\sin^2\theta\right]^{3/2}}\hat{\boldsymbol{k}}. \tag{14.8}$$

A point charge moving in a straight line does not constitute a steady current. But forget this fact for a moment and treat this moving charge as a current element $I = qv$ along the x-axis. What would the Biot–Savart law say about the magnetic field at $(x, y, 0)$? The current element is at the origin and takes the role of Idl in the basic Biot–Savart law. Thus, you would expect

$$\boldsymbol{B} = \frac{\mu_0 I\boldsymbol{dl} \times (\boldsymbol{r} - \boldsymbol{r}')}{4\pi\,|\boldsymbol{r} - \boldsymbol{r}'|^3} = \frac{\mu_0 (qv\hat{\boldsymbol{i}}) \times \boldsymbol{r}}{4\pi r^3}$$

$$= \frac{\mu_0 (qv\hat{\boldsymbol{i}}) \times (r\cos\theta\,\hat{\boldsymbol{i}} + r\sin\theta\,\hat{\boldsymbol{j}})}{4\pi r^3}$$

$$= \frac{\mu_0 qv\sin\theta}{4\pi r^2}\hat{\boldsymbol{k}}.$$

If you take the nonrelativistic limit $v^2/c^2 << 1$ in Equation 14.8, you get precisely this expression. The message is that for a slowly moving point charge, you can still apply the Biot–Savart law with qv in place of Idl to a good approximation.

An electric field due to a charge that starts moving suddenly

Consider a charge kept at rest for long time at point A (Figure 14.11). At $t = 0$, an impulsive force acts on it and the charge suddenly starts moving with velocity v along the x-direction. What will be the electric field at a later time t?

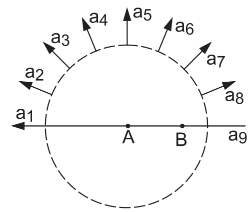


Fig. 14.11

First, think of region $r > ct$. The information that "something has happened to the charge at $t = 0$" has not reached this region. So this region will still have electric field radial from A and whose magnitude is the same for all θ, ϕ. If you draw electric field lines, they will be at equispaced angles. In Figure 14.11, nine such lines a_1 to a_9 have been shown, drawn from points on the circle $r = ct$ on the plane of the figure.

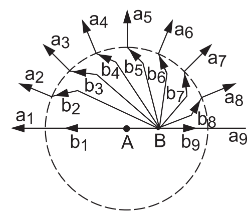


Fig. 14.12

Now consider the region $r < ct$. For this region, the charge had been moving with velocity v. What had happened earlier than $t = 0$ will not affect the field at time t at these points. So, this field will be radial from the instantaneous position B of the charge at time t. The distance AB will be equal to vt. Also, the field will be more intense in the transverse direction than in the radial direction. Again draw nine lines in this region. As electric field lines cannot originate or terminate in a charge-free region, the nine lines in the region $r < ct$ must join with the nine lines in the region $r > ct$. Figure 14.12 shows how the field lines bend at the surface $r = ct$ and a transverse field is created.

It has been assumed that the acceleration is infinite and the particle gains velocity v at time $t = 0$ itself. Thus there are two distinct regions $r > ct$ and $r < ct$ and the bending of the field line is along the path $r = ct$. If it takes time Δt to accelerate the particle, there will be a spherical shell $c(t - \Delta t) < r < ct$ in which the lines will bend. In Figure 14.12, the bent lines, slightly inside the circle $r = ct$, are shown. Though this is done for clarity, it is also realistic to some extent.

The radius of this disturbed zone is ct and hence it keeps expanding. It is like a wave front expanding from A with velocity c. The direction of expansion is radial and the electric field is transverse to this "direction of propagation". There is also a corresponding magnetic field in this disturbed region due to moving charge (during the acceleration), which will be perpendicular to the plane of the figure and hence perpendicular to the direction of propagation as well as to the direction of the electric field. The accelerated charge has created an electromagnetic wave pulse. If the charge stops at some time, another wave pulse will be generated but the first one will still continue to propagate radially with speed c. Once an electromagnetic wave is generated, it delinks from the source charge and propagates on its own, independent of what the source charge does afterwards.

Concepts at a Glance

1. The electric and magnetic fields in two frames moving with respect to each other are different.
2. The field components parallel to the direction of motion of the frame remain unchanged.
3. The transformation equations for standard descriptions of S and S′ are

$$E'_x = E_x\,,\; E'_y = \frac{E_y - vB_z}{\sqrt{1-v^2/c^2}}\,,\; E'_z = \frac{E_z + vB_y}{\sqrt{1-v^2/c^2}}\,.$$

$$B'_x = B_x\,,\; B'_y = \frac{B_y + \dfrac{v}{c^2}E_z}{\sqrt{1-v^2/c^2}}\,,\; B'_z = \frac{B_z - \dfrac{v}{c^2}E_y}{\sqrt{1-v^2/c^2}}\,.$$

4. The electric field due to a moving point charge is more along the transverse direction than along the direction of motion.
5. The magnetic field due to a slowly moving point charge can be obtained using the Biot–Savart Law by replacing $I\mathbf{dl}$ by $q\mathbf{v}$.
6. Accelerated charge produces electromagnetic waves.

EXERCISES

Based on Concepts

1. Charge is uniformly distributed in a spherical volume. If the distribution is seen from a different frame, moving at a uniform velocity with respect to the sphere, will the charge density remain uniform?
2. A capacitor is made of two circular plates of radii R placed a distance d apart. It is given a charge q. Neglect the fringing of the fields at the edges. The capacitor is observed from a frame moving with respect to it at a constant velocity v parallel to the plates. What is the charge density on each of the plates?
3. There exists an electric field in the lab frame. Can we have a frame in which the electric field is zero?
4. Suppose the magnetic field at some space–time point is zero in one system. Is it possible to find another system in which the electric field at that space–time point is zero?
5. In a frame, the electric field is zero but there is a magnetic field. If you use another frame, which of these is always true? (a) The magnetic field will have larger magnitude. (b) The magnetic field will have smaller magnitude. (c) The magnetic field will not have a larger magnitude.

Problems

1. A particle accelerator sends a beam of α-particles with velocity $c/2$. The number of particles per unit length is n. (a) What is the beam current? (b) The beam is seen from a frame moving with respect to the accelerator laboratory at a constant velocity $c/4$ in the direction of the beam. What is the number of particles per unit length and the beam current as seen from this frame? [**Ans.** (a) enc; (b) $\dfrac{7n}{2\sqrt{15}}, \dfrac{nec}{\sqrt{15}}$]

2. A long, cylindrical wire has n free electrons per unit volume. The wire is neutral because of fixed positive charges when no current passes through it. A current I is established in the wire along its length by setting an electric field which only moves the free electrons. What is the charge density in the wire? [**Ans.** $ne\left(\dfrac{nec}{\sqrt{n^2e^2c^2 - I^2}} - 1\right)$]

3. A capacitor consists of two rectangular parallel plates with a vertical separation of 2 cm. The east–west dimension of the plates is 20 cm and the north–south dimension is 10 cm. The capacitor is charged by connecting it temporarily to a battery of 300 V. A frame of reference S′ moves eastwards, relative to the laboratory frame, with speed 0.6c. Answer the following questions for S′. (a) What are the lengths of the edges of the plates and the separation between them? (b) What is the number of excess electrons on the negative plate? (c) What is the electric field strength between the plates? (d) Answer the same questions for a frame of reference S″ moving upwards with respect to the laboratory with speed 0.6c.

   [**Ans.** (a) East–West 16 cm, North–South 10 cm, separation 2 cm, (b) 1.6×10^9, (c) 190 V/cm]

4. A parallel-plate capacitor is at rest in S. Its plates are tilted at 45° to the lines joining their centres, taken as the x-axis (Figure 14.E1). The charge densities on the plates are $\pm\sigma_0$. Frame S′ is moving to the right at speed 0.6c relative to S. Find (a) the electric field $\boldsymbol{E}$ in S, (b) the electric field $\boldsymbol{E}'$ in S′, (c) the angle made by the plates with the x-axis in S′, and (d) the angle made by the electric field with the plates in S′.

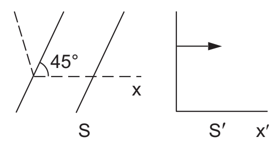


Fig. 14.E1

   [**Ans.** (a) $-\dfrac{\sigma_0}{2\varepsilon_0}(\hat{i} - \hat{j})$, (b) $\dfrac{\sigma}{\sqrt{2}\varepsilon_0}\left(\hat{i} - \dfrac{5}{4}\hat{j}\right)$, (c) $\tan^{-1}\left(\dfrac{5}{4}\right)$, (d) $\cos^{-1}\left(\dfrac{40}{41}\right)$]

5. A plane, parallel-plate capacitor with large plate area A is given a charge Q. The electric field in the capacitor is E. Look at this capacitor from a frame S′ moving at a constant velocity v, parallel to the plates. (a) What is the electric field in the capacitor? (b) What is the capacitance of the capacitor? (c) Give the current density $\boldsymbol{K}$ on each plate. (d) Work out the magnetic field due to these surface currents. (e) Check that the fields transform as expected from the transformation equations.

   [**Ans.** (d) $= \dfrac{\mu_0 Qu}{A\sqrt{1 - u^2/c^2}}\hat{j}$]

6. Consider a parallel-plate capacitor at rest in frame S′, charged to a surface charge density σ, with the plates parallel to the x'–y' plane. Frame S′ moves with respect to S with a velocity $v\hat{i}$. (a) Find the force $\boldsymbol{F}$ on a charged particle q placed between the plates in S′, assuming it to be in the z-direction. (b) From the force transformation equation, obtain the force on the particle in frame S. (c) Find the force on q, in S, due to the electric field. (d) Derive the remaining force and show that it is the same as one would expect from the magnetic field from moving surface charges. (e) Work out the magnetic field between the moving plates using field transformation equations.

7. Two very long parallel wires are at rest in frame S along the lines $y = 0$ and $y = 2a$ in the x–y plane. They carry uniform linear charge densities $+\lambda$ and $-\lambda$ respectively as seen in S. (a) What will be the electric and magnetic fields in frame S at a point P midway between the wires? (b) Using the field transformation equations, calculate the electric and magnetic fields measured in S′ at a point midway between the wires. (c) Determine the linear charge density λ' and current I' in the wires as seen by an observer in S′. Show that these values of λ' and I' lead to values of E' and B' obtained in (b).

8. A neutral conducting wire carries positive ions and conduction electrons, each with linear charge densities λ_0 when no current passes through it. (a) Find the drift speed v of the electrons when a current I passes through the wire. (b) Taking relativistic length contraction into account, derive the charge density appearing in the wire. (c) Suppose a trolley moves with speed u in the direction of the drifting electrons. Find the charge density in the wire in the frame of the body.

[**Ans.** (b) $-\lambda_0\left\{\dfrac{1}{\sqrt{1-v^2/c^2}}-1\right\}$, (c) $\lambda_0\left\{\dfrac{1}{\sqrt{1-v^2/c^2}}-1\right\}$]

9. A given region in a laboratory contains electric and magnetic fields $\boldsymbol{E} = E_0\hat{\boldsymbol{j}}$ and $\boldsymbol{B} = B_0\hat{\boldsymbol{k}}$ with $E_0/B_0 < c$. Consider a frame S′ moving with respect to the laboratory with a velocity $\boldsymbol{v}_0 = \dfrac{E_0}{B_0}\hat{\boldsymbol{i}}$. (a) Work out the electric field and the magnetic field in this frame. (b) A particle of charge q and mass m is released from rest in the laboratory frame. Will the path of this particle in S′ be circular?

[**Ans.** (a) $\boldsymbol{E}' = 0, \boldsymbol{B} = \boldsymbol{B}\sqrt{1-v^2/c^2}$; (b) Only if $\dfrac{\boldsymbol{E}_0}{\boldsymbol{B}_0} << c$.]

10. A region has electric and magnetic fields given by $\boldsymbol{E} = E_0\hat{\boldsymbol{j}}$ and $\boldsymbol{B} = \dfrac{1}{2c}E_0\hat{\boldsymbol{k}}$. A positively charged particle is released from rest at the origin at $t = 0$. Show that the y-coordinate keeps increasing with time. You may think of a frame moving with velocity $\dfrac{c}{2}\hat{\boldsymbol{i}}$.

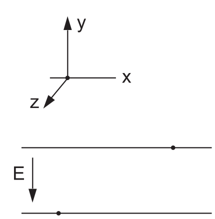


Fig. 14.E2

11. A capacitor with a separation d between the plates is placed in a laboratory and produces an electric field $\boldsymbol{E} = -E_0\hat{\boldsymbol{j}}$ between the plates. Electrons are emitted with negligible velocity from the negative plate. A uniform magnetic field $\boldsymbol{B} = B_0\hat{\boldsymbol{k}}$ is applied to prevent electrons from reaching the positive plate. Analyze the motion from the frame S′ moving with velocity $-\dfrac{E_0}{B_0}\hat{\boldsymbol{i}}$ and find the minimum value of B_0 needed, assuming $E_0 << B_0c$.

[**Ans.** $\sqrt{\dfrac{2mE_0}{qd}}$]

12. A plane electromagnetic wave moves in vacuum with speed c. As seen from an inertial frame S, the electric field in the wave is given by $\boldsymbol{E} = \boldsymbol{E}_0 \sin(kx - \omega t)$. The wave is looked at from the frame S′ moving with respect to S in the positive x-direction with a velocity v. The electric field as seen from this frame is $\boldsymbol{E}' = \boldsymbol{E}_0' \sin(k'x' - \omega' t')$. (a) Find k' and ω' in terms of k, ω, v and c. (b) Show that the speed of the wave as seen from S′ is also c.

[**Ans.** (a) $k' = \dfrac{k - \dfrac{\omega v}{c^2}}{\sqrt{1-v^2/c^2}}, \omega' = \dfrac{\omega - kv}{\sqrt{1-v^2/c^2}}$]

13. A charge q_1 moves along the x-axis with a velocity $0.5c$ and another charge q_2 is kept at rest on the x-axis at $x = a$. At $t = 0$, the first charge crosses the origin. Find (a) the force on q_1 at $t = 0$ due to the field of q_2, and (b) the force on q_2 at $t = 0$ due to the field of q_1. [**Ans.** (a) $\dfrac{q_1 q_2}{4\pi\varepsilon_0 a^2}(-\boldsymbol{i})$, (b) $\dfrac{3q_1 q_2}{16\pi\varepsilon_0 a^2}\hat{\boldsymbol{i}}$]

Fig. 14.E3

14. Consider an infinitely long line charge with linear charge density λ along the x-axis. The charge moves with a constant velocity $v\hat{\boldsymbol{i}}$. Take a small element of the charge and write the electric field due to it at a certain instant at $(x, y, 0)$. Integrate it over the line charge to get the net electric field at $(x, y, 0)$. Check that it is independent of v. [**Ans.** $\dfrac{\lambda}{2\pi\varepsilon_0 y}$]

15. Check that Gauss's law $\int \boldsymbol{E}\cdot d\boldsymbol{a} = \left(\dfrac{1}{\varepsilon_0}\right)Q_{\text{enc}}$ is obeyed by the field of a point charge in uniform motion, by integrating the field over a sphere of radius R centred on the charge.

16. Two large, plane charge sheets containing surface charge densities $+\sigma_0$ and $-\sigma_0$ are superposed on each other in the plane $z = 0$. The positive-charge sheet moves with a velocity $v\hat{\boldsymbol{i}}$ and the negative-charge sheet moves with a velocity $-v\hat{\boldsymbol{i}}$ in a frame S. A charge q is placed at rest at $(0, 0, h)$. (a) What is the current density in S? (b) What is the net force on q in S? (c) As seen from S′, what is the net charge density on the plane? (d) How much is the force on q due to the electric field in S′? (e) As the net force is zero, what is the magnetic force on the charge q in S′? (f) What is the current density in S′? (g) From the Biot–Savart law, find the magnetic field at q as seen in S′. (h) From $\boldsymbol{F} = q\boldsymbol{v}\times\boldsymbol{B}$, obtain the magnetic force on q in S′. Does it match with that obtained in part (e)?

[**Ans.** (a) $2\sigma v$ in $\hat{\boldsymbol{i}}$-direction, (b) zero, (c) $= \dfrac{2\sigma_0 \dfrac{v^2}{c^2}}{\sqrt{1 - v^2/c^2}}$, (d) $\dfrac{-q\mu_0\sigma_0 v^2}{\sqrt{1 - v^2/c^2}}\hat{\boldsymbol{k}}$, (e) $\dfrac{q\mu_0\sigma_0 v^2}{\sqrt{1 - v^2/c^2}}\hat{\boldsymbol{k}}$, (f) $\dfrac{2\sigma_0 v}{\sqrt{1 - v^2/c^2}}\hat{\boldsymbol{i}}$, (g) $\dfrac{-\mu_0\sigma_0 v}{\sqrt{1 - v^2/c^2}}\hat{\boldsymbol{i}}$]

17. A quantity is said to be relativistically invariant if its value remains the same as one goes from one inertial frame to other. Show that $(\boldsymbol{E}\cdot\boldsymbol{B})$ and $(E^2 - c^2B^2)$ are relativistically invariant.

❑

15 Magnetic Vector Potential

15.1 Introduction

The electric field $\boldsymbol{E}$ produced by a steady charge distribution satisfies the equations

$$\nabla \cdot \boldsymbol{E} = \rho/\varepsilon_0 \text{ and } \nabla \times \boldsymbol{E} = 0.$$

The second of these allows one to define an electric potential $V(\boldsymbol{r})$ so that $\boldsymbol{E} = -\nabla V$. The potential field is a scalar field and is easier to calculate for a given charge distribution than the electric field, which is a vector field. Also, the potential is directly related to electric potential energy, another important physical quantity.

The magnetic field $\boldsymbol{B}$ produced by a steady current distribution satisfies the equations

$$\nabla \cdot \boldsymbol{B} = 0 \text{ and } \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}.$$

As the curl of $\boldsymbol{B}$ is not zero (at places where a current exists), you cannot define a scalar potential from which $\boldsymbol{B}$ may be derived by taking the gradient. However, the divergence of $\boldsymbol{B}$ is always zero. This ensures that you can write $\boldsymbol{B}$ as the curl of another vector field. Remember that $\nabla \cdot \nabla \times \boldsymbol{A}(\boldsymbol{r})$ is always zero for any $\boldsymbol{A}(\boldsymbol{r})$. So, for any given $\boldsymbol{B}(\boldsymbol{r})$, you can find a vector field $\boldsymbol{A}(\boldsymbol{r})$ such that

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}. \tag{15.1}$$

Such a field $\boldsymbol{A}(\boldsymbol{r})$ is called the *magnetic vector potential* for the given magnetic field $\boldsymbol{B}$.

In fact, the word *magnetic* has been used here with some hesitation. In case of magnetostatics (steady currents), $\boldsymbol{A}$ is exclusively related to the magnetic field. However, for time-varying fields, $\boldsymbol{A}$ also gives part of the electric field, though the defining equation (Equation 15.1) is still valid. In this chapter, we will discuss only the cases related to steady currents and so we will continue to call $\boldsymbol{A}(\boldsymbol{r})$ the magnetic vector potential.

Several questions may be asked at this stage. First, how does it help? Since the magnetic vector potential is still a vector field, will defining such a vector field simplify the calculation of the field? The concern is genuine and in magnetostatics at least, there is not much simplification. The other question is: What is the physical significance of magnetic vector potential? Here too, the answer is not very encouraging. You don't relate magnetic vector potential to energy or some such thing. It simply represents magnetic field in another way. But once you start dealing with time-varying electric and magnetic fields, magnetic vector potential becomes important and useful. In this chapter, also in quantum electrodynamics, vector potential seems to be more fundamental than magnetic field. The following section will familiarize you with magnetic vector potential, using magnetostatic situations.

15.2 Magnetic Vector Potential and the Coulomb Gauge

For any given magnetic field $\boldsymbol{B}(\boldsymbol{r})$, if you can find a vector field $\boldsymbol{A}(\boldsymbol{r})$ so that

$$\nabla \times \boldsymbol{A}(\boldsymbol{r}) = \boldsymbol{B}(\boldsymbol{r}),$$

$\boldsymbol{A}(\boldsymbol{r})$ is a genuine magnetic vector potential in the region concerned. However, the magnetic vector potential is not unique. You know that the curl of the gradient of any function is zero. Suppose $\boldsymbol{A}$ is a vector potential for the given magnetic field $\boldsymbol{B}$, and λ is a scalar function of space points. Then,

$$\nabla \times (\boldsymbol{A} + \nabla\lambda) = \nabla \times \boldsymbol{A} + \nabla \times (\nabla\lambda) = \nabla \times \boldsymbol{A} = \boldsymbol{B}.$$

So $\boldsymbol{A} + \nabla\lambda$ is also a valid vector potential for the same magnetic field. It is up to you which of these infinite possible vector potentials you choose for yourself.

The electric potential function $V(\boldsymbol{r})$ also has a similar uncertainty. For a given electric field $\boldsymbol{E}$, if you have a potential V such that $\boldsymbol{E} = -\nabla V$, you can add any constant c to V and the new function $V(\boldsymbol{r}) + c$ will also satisfy $\boldsymbol{E} = -\nabla V$ and is a valid electric potential for the same $\boldsymbol{E}$. We then make a choice of 'zero' of potential to select one. We impose a condition on V that it should be zero at infinity or at some other chosen point. How does one choose magnetic vector potential?

A nice way to make this choice, for the case of a steady current distribution, is to impose the condition that the divergence of $\boldsymbol{A}$ is zero:

$$\nabla \cdot \boldsymbol{A} = 0. \tag{15.2}$$

Remember, this is a choice and not a law of physics. Similar is the situation with electric potential. It is said to be zero at infinity. Again, this is a choice agreed upon by physicists and not a law. Is it always possible to get a vector potential $\boldsymbol{A}$ for which the curl gives the given magnetic field and whose divergence is zero at all points? Calculations show that it is indeed always possible. Another question is that by putting up this condition: does the choice of $\boldsymbol{A}$ become unique? The answer is not a definite yes. If you also specify the value of $\boldsymbol{A}$ at infinity, or to be more precise, at the boundary surface of your region of interest, the choice becomes unique. For finite current distributions, you can put the condition that $\boldsymbol{A} = 0$ at infinity. Then there is only one function $\boldsymbol{A}$, with the conditions

$$\nabla \times \boldsymbol{A} = \boldsymbol{B},$$

$$\nabla \cdot \boldsymbol{A} = 0,$$

and $$\boldsymbol{A} = 0 \text{ at infinity.}$$

For currents extending to infinity (idealized mathematical problems), you still have many choices.

The condition $\nabla \cdot \boldsymbol{A} = 0$ is called the Coulomb gauge and is invariably used for magentostatic situations. Later, while considering time-varying fields, another choice of $\boldsymbol{A}$ will be introduced.

15.3 Procedures to Calculate *A*

A magnetic field is produced by electric currents. You calculate the magnetic field from the given current distribution using the Biot–Savart law or Ampere's law. Alternatively, you can calculate the vector potential $\boldsymbol{A}$ from the given current distribution and then get the magnetic field by using $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. So how should you get the vector potential from a given current distribution?

Suppose the current density for the given current distribution (assumed to be steady) is $\boldsymbol{J}(\boldsymbol{r})$. Then,

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$$

or $$\nabla \times (\nabla \times \boldsymbol{A}) = \mu_0 \boldsymbol{J}$$

or $$\nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = \mu_0 \boldsymbol{J}.$$

But $$\nabla \cdot \boldsymbol{A} = 0.$$

So, $$\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}. \tag{15.3}$$

In component form,

$$\nabla^2 A_x = -\mu_0 J_x, \nabla^2 A_y = -\mu_0 J_y, \nabla^2 A_z = -\mu_0 J_z. \tag{15.4}$$

Each of these has the same mathematical structure as that for the electric potential V for a given charge density $\rho(\boldsymbol{r})$, that is, $\nabla^2 V = -\rho/\varepsilon_0$. And you know how to get $V(\boldsymbol{r})$ for a given charge density $\rho(\boldsymbol{r})$.

$$V(\boldsymbol{r}) = \int \frac{\rho(\boldsymbol{r}')d\tau'}{4\pi\varepsilon_0 |\boldsymbol{r} - \boldsymbol{r}'|}$$

integrated over a volume containing all the charges. Write $\mu_0 J_x$ in place of ρ/ε_0 in this expression and you will get the expression for A_x. Similarly for A_y and A_z. Thus the expressions for A_x, A_y, A_z can be written as

$$A_x(\boldsymbol{r}) = \int \frac{\mu_0 J_x(\boldsymbol{r}')d\tau'}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}, \tag{15.5}$$

$$A_y(\boldsymbol{r}) = \int \frac{\mu_0 J_y(\boldsymbol{r}')d\tau'}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}, \tag{15.6}$$

and
$$A_z(\boldsymbol{r}) = \int \frac{\mu_0 J_z(\boldsymbol{r}')d\tau'}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}, \tag{15.7}$$

where the integrations are performed over a volume containing the whole current distribution and $d\tau'$ is the volume element in the distribution at $\boldsymbol{r}'$.

Combining these three equations (15.5, 15.6 and 15.7),

$$\boldsymbol{A}(\boldsymbol{r}) = \int \frac{\mu_0 \boldsymbol{J}(\boldsymbol{r}')d\tau'}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}.$$

If the current density $\boldsymbol{J}(\boldsymbol{r}')$ is in the same direction everywhere, $\boldsymbol{A}(\boldsymbol{r})$ will also be in the same direction.

In many cases, you don't really have to do all the integrations. You have done several such exercises in the chapters on electric potential. Directly use the expressions of $V(\boldsymbol{r})$ if you find that J_x or J_y or J_z has a form that you have solved with ρ. The trick is to treat the expression for $\mu_0\varepsilon_0 J_x$ as that of a charge density ρ and write the electric potential V for this ρ. This function will give A_x. Similarly for A_y and A_z.

An equivalent procedure will be to treat J_x itself as the charge density, write the expression for potential and replace ε_0 by $1/\mu_0$. This gives A_x. You can get A_y and A_z similarly.

Please note that $\mu_0\varepsilon_0 J$ does not have the same dimensions as those of charge density. Also, the electric potential V does not have the same dimensions as those of magnetic vector potential. Only the mathematical expressions are similar and not the physical quantities.

If you have a surface current density $\boldsymbol{K}$ then think in terms of surface charge density. Treat the expression for $\mu_0\varepsilon_0 K_x$ as that of a surface charge density σ, write the corresponding electric potential and you will get A_x. Obtain A_y and A_z along the same lines. Similarly, if you have a linear current I, think in terms of linear charge density. Treat $\mu_0\varepsilon_0 I$ as the linear charge density λ. Write the corresponding electric potential function and you will get the corresponding component of the magnetic vector potential. Let us take up some examples to make the procedure clearer.

EXAMPLE 15.1 **Consider a segment of a wire carrying a current I from $z = -a$ to $z = +a$. Find the vector potential corresponding to the magnetic field of this segment at the point (0, a, a).**

Solution The current is only along the z-direction. So $A_x = A_y = 0$. To get the expression for A_z, consider a linear charge density I spread over the segment $z = -a$ to $z = +a$ (Figure 15.1). The electric potential at the point $(0, a, a)$ is

$$V = \frac{1}{4\pi\varepsilon_0}\int_{-a}^{a}\frac{Idz'}{[a^2+(a-z')^2]^{1/2}}.$$

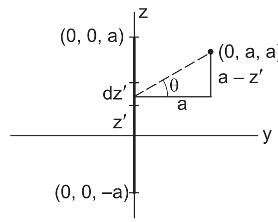


Fig. 15.1

From the figure,

$$a - z' = a\tan\theta$$

or $$dz' = -a\sec^2\theta d\theta.$$

And the distance $[a^2+(a-z')^2]^{1/2} = a\sec\theta$.

Thus, the integral is $-\frac{I}{4\pi\varepsilon_0}\int \sec\theta d\theta = -\frac{I}{4\pi\varepsilon_0}\ln(\sec\theta+\tan\theta)$.

At $z' = -a$, $\tan\theta = 2$, $\sec\theta = \sqrt{5}$.

At $z' = a$, $\tan\theta = 0$, $\sec\theta = 1$.

So, $$V = -\frac{I}{4\pi\varepsilon_0}[\ln 1 - \ln(2+\sqrt{5})]$$

$$= \frac{I}{4\pi\varepsilon_0}\ln(2+\sqrt{5}).$$

The vector potential due to the given current segment will be

$$\boldsymbol{A} = \frac{\mu_0 I}{4\pi}\ln(2+\sqrt{5})\hat{k}.$$

As usual, you can add any constant or expression to it for which the divergence is zero.

EXAMPLE 15.2 **A constant current I flows in a square loop of edge length a. The centre of the loop is at the origin and the edges are parallel to the x- and y-axis as shown in Figure 15.2. The current is anticlockwise as seen from the positive side of the z-axis. Find the magnetic vector potential at a point (x, y, z) far from the loop.**

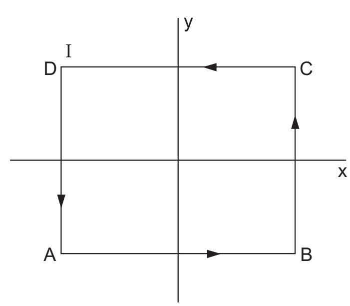


Fig. 15.2

Solution Let us first look for A_x. To do this, you should write J_x. Here $J_x = 0$ everywhere except at the lengths AB and CD, each of which carries a linear current. In AB the current is I and in CD it is $-I$ (you have to look at J_x, that is, the component along the positive x-axis). You should treat it as a linear charge density. So, the equivalent charge distribution is a linear charge density $\lambda_1 = \mu_0\varepsilon_0 I$ along AB and $\lambda_2 = -\mu_0\varepsilon_0 I$ along CD [Figure 15.3(a)]. As you have to calculate $\boldsymbol{A}$ at points far away from ABCD, you can treat the whole charge on AB as a point charge at (0, $-a/2$, 0) and the whole charge at CD as a point charge at (0, $a/2$, 0). The equivalent charge itself is $q = \lambda_1 a = \mu_0\varepsilon_0 Ia$ and $-q = \lambda_2 a = -\mu_0\varepsilon_0 Ia$ [Figure 15.3(b)].

So, what is the electric potential at (x, y, z) due to these point charges? The two point charges form an electric dipole of moment $\boldsymbol{p} = qa(-\hat{\boldsymbol{j}})$. The potential at a faraway point will be

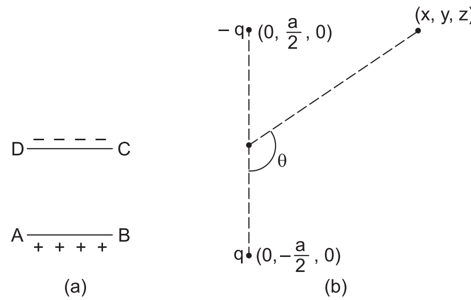


Fig. 15.3

$$V(\boldsymbol{r}) = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{qa\left(\frac{-y}{r}\right)}{4\pi\varepsilon_0 r^2}$$

or $$V(x, y, z) = -\frac{qay}{4\pi\varepsilon_0 r^3}.$$

Writing $\mu_0\varepsilon_0 Ia$ in place of q in this expression will give you A_x. So,

$$A_x = -\frac{\mu_0\varepsilon_0 Ia^2 y}{4\pi\varepsilon_0 r^3} = -\frac{\mu_0 Ia^2 y}{4\pi r^3}. \tag{i}$$

Now look at J_y. Doing a similar exercise as for J_x, you should take $\mu_0\varepsilon_0 I$ as the positive charge density along BC and $-\mu_0\varepsilon_0 I$ as the negative charge density along DA. For faraway points, this is equivalent to a charge $q = \mu\varepsilon_0 Ia$ at ($a/2$, 0, 0) and $-q = -\mu\varepsilon_0 Ia$ at (-$a/2$, 0, 0) as shown in Figure 15.4. The potential due to this distribution is

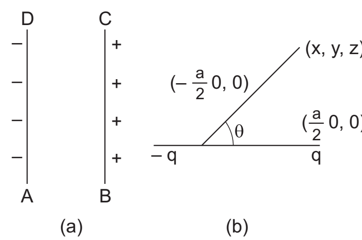


Fig. 15.4

$$V(\boldsymbol{r}) = \frac{p\cos\theta}{4\pi\varepsilon_0 r^3} = \frac{qa\left(\frac{x}{r}\right)}{4\pi\varepsilon_0 r^2}$$

or $$V(x, y, z) = \frac{qax}{4\pi\varepsilon_0 r^3}.$$

Writing $\mu\varepsilon_0 Ia$ in place of q in this expression gives A_y. So,

$$A_y = \frac{\mu_0 Ia^2 x}{4\pi r^3}. \tag{ii}$$

As there is no current anywhere in the z-direction, $J_z = 0$ and hence $A_z = 0$.

Therefore, the magnetic vector potential for this current loop is

$$\boldsymbol{A} = \frac{\mu_0 Ia^2}{4\pi r^3}(-y\hat{\boldsymbol{i}} + x\hat{\boldsymbol{j}}).$$

EXAMPLE 15.3 **There is a uniform surface current density $\boldsymbol{K} = K_0\hat{\boldsymbol{i}}$ along the entire x–y plane. Find the magnetic vector potential everywhere in space.**

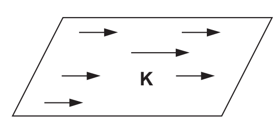


Fig. 15.5

Solution The surface current density is in the x–y plane and its x-component is K_0. To calculate A_x, treat $\mu_0\varepsilon_0 K_0$ as the surface charge density σ_0 spread over the entire x–y plane. You should write the expression for the electric potential due to the corresponding surface charge.

As the charge itself extends to infinity, the potential should not be taken to be zero at infinity. Take the potential at the x–y plane to be zero. The electric field is $\boldsymbol{E} = \frac{\sigma_0}{2\varepsilon_0}\hat{\boldsymbol{k}}$ for $z > 0$ and $-\frac{\sigma_0}{2\varepsilon_0}\hat{\boldsymbol{k}}$ for $z < 0$. The potential is then

$$V(x, y, z) = -Ez$$

$$= -\frac{\sigma_0}{2\varepsilon_0} z \text{ for } z > 0$$

$$\text{and } \frac{\sigma_0}{2\varepsilon_0} z \text{ for } z < 0.$$

To get A_x, write $\mu_0\varepsilon_0 K_0$ for σ_0 in the expression for $V(x, y, z)$. So,

$$A_x = -\frac{\mu_0 K_0}{2} z \text{ for } z > 0$$

$$= \frac{\mu_0 K_0}{2} z \text{ for } z < 0.$$

As $J_y = J_z = 0$ everywhere, the vector-potential components A_y and A_z are also zero. Thus,

$$\boldsymbol{A} = -\frac{\mu_0 K_0}{2} z\hat{\boldsymbol{i}} \text{ for } z > 0$$

$$= \frac{\mu_0 K_0}{2} z\hat{\boldsymbol{i}} \text{ for } z < 0$$

or $$\boldsymbol{A} = -\frac{\mu_0 |z| \boldsymbol{K}}{2}.$$

15.4 The Magnetic Field and Vector Potential due to a Magnetic Dipole

Look at the solution to Example 15.1. There is a square loop of edge length a carrying a current I. The magnetic dipole moment of this loop is $\boldsymbol{m} = Ia^2\hat{\boldsymbol{k}}$. For points far away from the loop, the magnetic vector potential is $\boldsymbol{A} = \frac{\mu_0 I a^2}{4\pi r^3}(-y\hat{\boldsymbol{i}} + x\hat{\boldsymbol{j}})$.

This can be written as $\boldsymbol{A} = \frac{\mu_0 m}{4\pi r^3}\hat{\boldsymbol{k}} \times (x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}})$

or $$\boldsymbol{A} = \frac{\mu_0 \boldsymbol{m} \times \boldsymbol{r}}{4\pi r^3}, \tag{15.8}$$

which is independent of the coordinate axes.

Using spherical polar coordinates,

$$\boldsymbol{A} = \frac{\mu_0 I a^2}{4\pi r^3}(-r\sin\theta\cos\phi\,\hat{\boldsymbol{i}} + r\sin\theta\sin\phi\,\hat{\boldsymbol{j}})$$

$$= \frac{\mu_0 I a^2}{4\pi r^3} r\sin\theta\,\hat{\phi}$$

or
$$\boldsymbol{A} = \frac{\mu_0 m}{4\pi}\frac{\sin\theta}{r^2}\hat{\phi}. \tag{15.9}$$

The magnetic field at a general point (r, θ, ϕ) far away from the loop is

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} = \frac{\mu_0 m}{4\pi}\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\sin\theta}{r^2}\right)\right]\hat{\boldsymbol{r}} + \frac{1}{r}\left[-\frac{\partial}{\partial r}\left(r\frac{\sin\theta}{r^2}\right)\hat{\theta}\right]$$

$$= \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{\boldsymbol{r}} + \sin\theta\,\hat{\theta}). \tag{15.10}$$

This can be written in a form independent of the coordinate system, as

$$\boldsymbol{B} = \frac{\mu_0}{4\pi r^3}[3(\boldsymbol{m}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{m}]. \tag{15.11}$$

The loop need not be square. For points far away from the loop, the shape is not important. For any current loop with magnetic moment $m\hat{\boldsymbol{k}}$, the field and the vector potential at faraway points will be given by Equations (15.10) and (15.11).

Compare Equation (15.10) with the expression for the electric field due to an electric dipole. The dependence on r, θ, ϕ is identical in the two cases.

<u>EXAMPLE 15.4</u> **A sphere of radius R has a uniform surface charge density σ. It rotates about one of its diameters, taken as the z-axis, with a uniform angular velocity $\omega\hat{\boldsymbol{k}}$. Find the magnetic vector potential and hence the magnetic field (a) inside the sphere and (b) outside the sphere.**

Solution (a) The rotating charge on the sphere is equivalent to a surface current on the surface of the sphere with a surface current density

$$\boldsymbol{K} = \sigma\boldsymbol{v} = \sigma\omega R\sin\theta\,\hat{\phi} = C\sin\theta\,\hat{\phi},$$

where $C = \sigma\omega R$ is a constant. As $\hat{\phi} = \cos\phi\,\hat{\boldsymbol{j}} - \sin\phi\,\hat{\boldsymbol{i}}$,

$$K_x = -C\sin\theta\sin\phi, \tag{i}$$

$$K_y = C\sin\theta\cos\phi, \tag{ii}$$

and $K_z = 0$.

First, look at the expression for K_x. Using the analogy between $\nabla^2 A_x = -\mu_0 I_x$ and $\nabla^2 V = -\rho/\varepsilon_0$, the component A_x can be obtained by writing the electric potential due to a surface charge density $\sigma = -C \sin\theta \sin\phi$ and replacing ε_0 by $1/\mu_0$.

Consider a sphere of radius R with uniform polarization $\boldsymbol{P} = -C\hat{\boldsymbol{j}}$. The surface charge density will be

$$\sigma = \boldsymbol{P} \cdot \hat{\boldsymbol{n}} = -C\hat{\boldsymbol{j}} \cdot \hat{\boldsymbol{r}} = -C \sin\theta \sin\phi. \tag{iii}$$

The electric field inside the sphere will be

$$\boldsymbol{E} = -\frac{\boldsymbol{P}}{3\varepsilon_0} = \frac{C\hat{\boldsymbol{j}}}{3\varepsilon_0}\cdot$$

Taking $V = 0$ at the centre, the electric potential giving this electric field will be

$$V = -\frac{Cy}{3\varepsilon_0}\cdot \tag{iv}$$

The expression for K_x [equation (i) above] is the same as that of the charge density of the polarized sphere [equation (iii) above]. Therefore, you can write the x-component of the vector potential due to the charged, rotating sphere using equation (iv) and change ε_0 with $1/\mu_0$, as

$$A_x = -\frac{1}{3}\mu_0 Cy = -\frac{1}{3}\mu_0 Cr \sin\theta \sin\phi.$$

To get A_y, consider a sphere of radius R polarized uniformly with polarization

$$\boldsymbol{P} = C\hat{\boldsymbol{i}}.$$

The charge distribution on the surface will correspond to a surface charge density

$$\sigma = \boldsymbol{P} \cdot \hat{\boldsymbol{n}} = C\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{r}} = C \sin\theta \cos\phi. \tag{v}$$

The electric field inside the sphere is

$$\boldsymbol{E} = -\frac{\boldsymbol{P}}{3\varepsilon_0} = -\frac{C}{3\varepsilon_0}\hat{\boldsymbol{i}}.$$

The corresponding electric potential is

$$V = \frac{Cx}{3\varepsilon_0}\cdot \tag{vi}$$

The expression for the surface charge density [equation (v) above] is the same as that for K_y. Thus the y-component of $\boldsymbol{A}$ is obtained by replacing ε_0 with $1/\mu_0$.

$$A_y = \frac{1}{3}\mu_0 Cx = \frac{1}{3}\mu_0 Cr \sin\theta \cos\phi.$$

As $K_z = 0$, A_z is also zero. Thus,

$$\boldsymbol{A} = \frac{1}{3}\mu_0 Cr \sin\theta(\cos\phi\hat{\boldsymbol{j}} - \sin\phi\hat{\boldsymbol{i}})$$

$$= \frac{1}{3}\mu_0 Cr \sin\theta\hat{\phi} = \frac{1}{3}\mu_0 \sigma\omega Rr \sin\theta\hat{\phi} = \frac{1}{3}\mu_0 \sigma R(x\hat{\boldsymbol{j}} - y\hat{\boldsymbol{i}}).$$

The magnetic field is

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \frac{2}{3}\mu_0 \sigma \omega R \hat{\boldsymbol{k}}.$$

It is interesting to note that the magnetic field inside the sphere is uniform.

(b) You can use the same polarized spheres to find the vector potential outside the sphere. For this region, a polarized sphere with polarization $\boldsymbol{P}$ gives a dipole field with moment $\boldsymbol{p} = \frac{4}{3}\pi R^3 \boldsymbol{P}$. The potential is given by $V(\boldsymbol{r}) = \frac{\boldsymbol{p} \cdot \hat{\boldsymbol{r}}}{4\pi\varepsilon_0 r^2}\cdot$ For A_x, A_y, write the appropriate expressions for this dipole potential and replace ε_0 by $\frac{1}{\mu_0}$, and you get the vector potential. The expressions are

$$A_x = -\frac{\mu_0 \omega \sigma R^4}{3r^2} \sin\theta \sin\phi,$$

$$A_y = -\frac{\mu_0 \omega \sigma R^4}{3r^2} \sin\theta \cos\phi,$$

and $A_z = 0.$

Thus
$$\boldsymbol{A} = \frac{\mu_0 \omega \sigma R^4}{3r^2} \sin\theta (\cos\phi \hat{\boldsymbol{j}} - \sin\phi \hat{\boldsymbol{i}})$$

$$= \frac{\mu_0 \omega \sigma R^4}{3r^2} \sin\theta \hat{\phi}.$$

15.5 Calculating *A* if *B* is Given

The vector potential $\boldsymbol{A}$ satisfies the equations

$$\nabla \times \boldsymbol{A} = \boldsymbol{B} \qquad \text{(by definition)}$$

and
$$\nabla \cdot \boldsymbol{A} = 0 \qquad \text{(by choice in magnetostatic situations).}$$

Compare these equations with

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$$

and
$$\nabla \cdot \boldsymbol{B} = 0.$$

The two sets of equations have an identical mathematical structure. This shows that $\boldsymbol{A}$ can be calculated from $\boldsymbol{B}$ in the same way as $\boldsymbol{B}$ is calculated from $\boldsymbol{J}$. If you are given a magnetic field $\boldsymbol{B}$, think that this function represents a current density. To be more precise, treat $\boldsymbol{B}/\mu_0$ as $\boldsymbol{J}$. Write the expression for the magnetic field due to this assumed current density. This expression will in fact be the expression for the magnetic vector potential for the given magnetic field. This procedure will be explained further using an example.

EXAMPLE 15.5 **A long, tightly wound solenoid of radius R and having n turns per unit length carries a current I. The axis of the solenoid is along the z-axis. Knowing that the magnetic field is $\mu_0 ni\hat{k}$ inside the solenoid and zero outside, find the vector potential for the field.**

Solution The magnetic field due to the solenoid is given by

$$\frac{\boldsymbol{B}}{\mu_0} = ni\hat{\boldsymbol{k}} \text{ for } s < R$$
$$= 0 \text{ for } s > R.$$

Let this expression represent a current density $\boldsymbol{J}$. Writing $ni = J_0$, this current density would be

$$\boldsymbol{J} = J_0 \hat{\boldsymbol{k}} \text{ for } s < R$$
$$= 0 \text{ for } s > R.$$

What will be the magnetic field for this current density? This is the case of a current $I = J_0 \pi R^2$ in a cylindrical wire of radius R uniformly distributed over its cross section. The magnetic field due to this current is, for $s > R$,

$$\boldsymbol{B}_1 = \frac{\mu_0 I}{2\pi s}\hat{\phi} = \frac{\mu_0 J_0 R^2}{2s}\hat{\phi}.$$

For $s < R$, the field is $\boldsymbol{B}_2 = \dfrac{\mu_0 I s}{2\pi R^2}\hat{\phi} = \dfrac{\mu_0 J_0 s}{2}\hat{\phi}.$

Writing ni for J_0 in these expressions, you get the magnetic vector potential for the given field.

So,
$$\boldsymbol{A} = \frac{\mu_0 n i R^2}{2s}\hat{\phi} \text{ for } s > R$$
$$= \frac{\mu_0 n i s}{2}\hat{\phi} \text{ for } s < R.$$

15.6 Vector Potential and Flux of the Magnetic Field

The flux of the magnetic field $\boldsymbol{B}$ through an area is

$$\Phi = \int \boldsymbol{B} \cdot d\boldsymbol{a},$$

where $d\boldsymbol{a}$ is an elementary area and the integration is performed on the whole area. The positive side of $d\boldsymbol{a}$ also defines the positive sense of the closed curve bounding the area. If $\boldsymbol{A}$ is the magnetic vector potential corresponding to the given field $\boldsymbol{B}$,

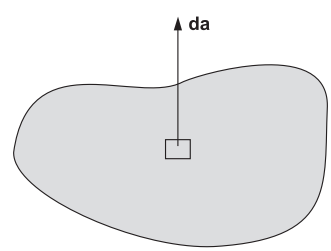


Fig. 15.6

$$\Phi = \int \boldsymbol{B} \cdot d\boldsymbol{a} = \int (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{a} = \oint \boldsymbol{A} \cdot d\boldsymbol{l},$$

where $d\boldsymbol{l}$ is the line element on the boundary curve and the integration is to be performed over the closed curve. Thus the magnetic flux through an area is equal to the circulation of the vector potential around the closed curve bounding the area.

This equation can also be used to calculate $\boldsymbol{A}$ if $\boldsymbol{B}$ is given. You will have to guess the direction of $\boldsymbol{A}$ from the given field expression for $\boldsymbol{B}$. If symmetry permits, you can evaluate $\oint \boldsymbol{A}\cdot d\boldsymbol{l}$ over a properly chosen closed curve. Equate it to $\int \boldsymbol{B}\cdot d\boldsymbol{a}$ over the area enclosed by that curve and get the magnitude of $\boldsymbol{A}$. Since you know the direction, you can get the expression for $\boldsymbol{A}$.

Concepts at a Glance

1. The vector potential $\boldsymbol{A}$ and the magnetic field $\boldsymbol{B}$ are related to each other as $\nabla\times\boldsymbol{A}=\boldsymbol{B}$.
2. In magnetostatics, we generally choose $\boldsymbol{A}$ in such a way that $\nabla\cdot\boldsymbol{A}=0$.
3. For a current distribution $\boldsymbol{J}(\boldsymbol{r}')$, the vector potential is given by

$$\boldsymbol{A}(\boldsymbol{r})=\frac{\mu_0}{4\pi}\int\frac{\boldsymbol{J}(\boldsymbol{r}')d\tau'}{|\boldsymbol{r}-\boldsymbol{r}'|}.$$

4. If you treat $\varepsilon_0\mu_0 J_x$ as a charge density and write the expression for the electric potential due to it, you get the expression for A_x. Similarly for A_y and A_z.
5. The vector potential $\boldsymbol{A}$ can be obtained from the magnetic field $\boldsymbol{B}$ by treating $\dfrac{\boldsymbol{B}}{\mu_0}$ as a current density $\boldsymbol{J}$ and writing the expression for the magnetic field from this $\boldsymbol{J}$.
6. The magnetic field due to a magnetic dipole $m\hat{\boldsymbol{k}}$ is $\dfrac{\mu_0 m}{4\pi r^2}(2\cos\theta\hat{\boldsymbol{r}}+\sin\theta\hat{\boldsymbol{\theta}})$.
7. The flux of a magnetic field through an area is given by $\Phi=\oint\boldsymbol{A}\cdot d\boldsymbol{l}$, evaluated on the curve bounding the area.

EXERCISES

Based on Concepts

1. What is the dimension of A/μ_0, where A is the magnitude of the vector potential?
2. With a given $\boldsymbol{B}(\boldsymbol{r})$, does $\boldsymbol{A}(\boldsymbol{r})$ get uniquely specified? With a given $\boldsymbol{A}(\boldsymbol{r})$, does $\boldsymbol{B}(\boldsymbol{r})$ get uniquely specified?
3. The magnetic vector potential due to a magnetic dipole kept at the origin is seen along the line $\theta=\phi=\pi/4$. If its magnitude is p at a distance r from the dipole, what will be its magnitude at $2r$?
4. From the equation $\boldsymbol{A}(\boldsymbol{r})=\dfrac{\mu_0}{4\pi}\displaystyle\int\frac{\boldsymbol{J}(\boldsymbol{r}')d\tau'}{|\boldsymbol{r}-\boldsymbol{r}'|}$, can you conclude that the direction of $\boldsymbol{A}$ at $\boldsymbol{r}$ is the same as that of $\boldsymbol{J}(\boldsymbol{r}')$, at least if $\boldsymbol{J}(\boldsymbol{r}')$ has same direction everywhere?

Problems

1. (a) The vector potential in a region is given by $\boldsymbol{A} = \frac{\mu_0 I z}{s}\hat{\boldsymbol{s}}$. Show that $\nabla \cdot \boldsymbol{A} = 0$. Find the magnetic field in the region. (b) Now suppose the vector potential is given by $\boldsymbol{A} = \mu_0 I \ln\left(\frac{a}{s}\right)\hat{\boldsymbol{k}}$. Find the magnetic field. (c) Look at the answers to the two parts. Do they show that for the same magnetic field, you can have two vector potentials, both satisfying $\nabla \cdot \boldsymbol{A} = 0$? [**Ans.** (a) $\frac{\mu_0 I}{s}\hat{\phi}$]

2. A circular loop of radius R is kept in the x–y plane with its centre at the origin. A current I passes through it in the $\hat{\phi}$-direction. (a) Find the vector potential at a point on the axis, taking it to be zero at large distances. (b) Find the vector potential at a point $(x, 0, 0)$ with $x >> R$. [**Ans.** (a) zero, (b) $\frac{\mu_0 I R^2}{4x^2}\hat{\boldsymbol{j}}$]

3. In a certain region, the vector potential is $\boldsymbol{A} = x\hat{\boldsymbol{j}}$. Find the current density here. [**Ans.** zero]

4. Current flows on a spherical surface. The surface current density is given by $\boldsymbol{K} = K_0 \sin\theta\hat{\phi}$, where the symbols have standard meanings. What is the magnetic field inside the sphere? [**Ans.** $\frac{2\mu_0 K_0}{3}\hat{\boldsymbol{k}}$]

5. A spherical shell of radius R carries a surface current $\boldsymbol{K} = K_0 \sin\theta\hat{\phi}$. Drawing an analogy from electrostatics and considering the equivalent surface charge distribution, calculate the magnetic vector potential outside the shell. [**Ans.** $\frac{\mu_0 K_0 R^3}{3r^2}\sin\theta\hat{\phi}$]

6. A sphere has a surface charge density σ_0 and rotates with a uniform angular velocity ω about one of its diameters, taken as the z-axis. Find the vector potential at the point with spherical polar coordinates $(2R, \pi/2, \pi/2)$. [**Ans.** $\frac{4}{3}\mu_0 R^2\omega\sigma(-\hat{\boldsymbol{i}})$]

7. The vector potential in a region is given as $\boldsymbol{A} = \boldsymbol{r} \times \boldsymbol{c}$ where $\boldsymbol{c}$ is a constant vector. Find the magnetic field in the region. [**Ans.** $-2\boldsymbol{c}$]

8. A magnetic field B_0 exists in the cylindrical region $r < R$. There is no field outside this region. Calculate the vector potential everywhere. [**Ans.** $\frac{B_0 R^2}{2s}$ for $s > R$ and $\frac{Bs}{2}\hat{\phi}$ for $s < R$]

9. In a region $a < s < b$, the magnetic vector potential is $\boldsymbol{A} = \mu_0 C\hat{\phi}$. Find the current density. [**Ans.** $\frac{C}{s^2}\hat{\phi}$]

10. The vector potential in a region is given by $\boldsymbol{A} = \frac{\mu_0 k s^2}{R}\hat{\boldsymbol{k}}$ for $s < R$ and $\boldsymbol{A} = \mu_0 k R \ln\left(\frac{s}{\phi R}\right)\hat{\boldsymbol{k}}$ for $s > R$ in cylindrical coordinates. Find the current density.
[**Ans.** $\left(\frac{-4k}{R}\hat{\boldsymbol{k}}\right)$ for $s < R$, 0 for $s > R$, surface current density $k\hat{\boldsymbol{k}}$ at $s = R$]

11. Consider a cylindrical wire of radius R carrying a current I uniformly distributed over its cross section. Find the vector potential (a) inside the wire and (b) outside the wire.
[**Ans.** (a) $\frac{\mu_0 I}{4\pi R^2}(R^2 - s^2)\hat{\boldsymbol{k}}$, (b) $-\frac{\mu_0 I}{2\pi}\ln\frac{s}{R}$]

12. A circular loop of radius 1 cm carrying a current of 1 A is placed in the x–y plane with its centre at the origin. Find the magnetic field at (20 cm, 0, 20 cm). [**Ans.** $\frac{\pi \times 10^{-8}}{16\sqrt{2}}(-\hat{\boldsymbol{k}})$ T]

13. A current distribution is confined between the planes $z = \pm a$, where it is given by $\boldsymbol{J} = J_0 \hat{\boldsymbol{i}}$. Find the vector potential at a general point (x, y, z). [**Ans.** $\frac{\mu_0 J_0 z^2}{2} \hat{\boldsymbol{i}}$ for $|z| < a$ and $-\mu_0 I_0 az \hat{\boldsymbol{i}}$ for $|z| > a$]

14. Consider a segment of wire $-a < z < a$ carrying a current I in the positive z-direction. Find the vector potential due to this segment at a point $(0, 0, a)$. [**Ans.** $\hat{\boldsymbol{k}} \frac{\mu_0 I}{4\pi} \ln (3 + 2\sqrt{2})$]

15. Two magnetic dipoles with magnetic moments m_1 and m_2 are widely separated from each other. They are brought to a separation R and oriented as shown in Figure 15E.1. Find the work done in this process.

[**Ans.** $\frac{\mu_0 m_1 m_2}{4\pi\varepsilon_0 R^3} (\sin \theta_1 \sin \theta_2 - 2 \cos \theta_1 \cos \theta_2)$]

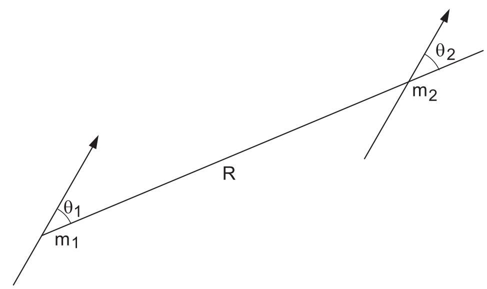


Fig. 15E.1

16. Consider a current element Idl. Write the vector potential due to it at a general point P. You can take the origin to be at dl and the z-axis along dl. Take the plane containing $\boldsymbol{dl}$ and $\boldsymbol{r}$ as the y–z plane. The coordinates of the point P will then be $(0, y, z)$. From the vector potential, find the magnetic field $\boldsymbol{dB} = \nabla \times \boldsymbol{A}$. Show that it is the same as $\boldsymbol{dB} = \frac{\mu_0}{4\pi} \frac{\boldsymbol{dl} \times \boldsymbol{r}}{r^3}$, as given by the Biot–Savart law.

❑

16 Motional emf

To maintain a current in a circuit, you need a source of emf. This source propels the electrons in a certain direction by a driving force that can be termed 'nonelectrostatic' at length scales of a few microns and above. A battery containing chemicals is a common example. The chemical reactions cause the electrons to reach the negative electrode and maintain the electric field in any circuit connected to it. A torch bulb does not glow if the battery in it becomes weak—the chemicals are exhausted and reactions do not occur.

What maintains the current in the table lamp, CFL tube, ceiling fan, refrigerator, television set and other appliances connected to the various power sockets and outlets fixed in the walls of your house? Where is the emf generated? In the electric power station, of course. How is this done? What drives the electrons to provide currents to the millions of circuits finally connected to the powerhouse through transmission-line wires? Almost always, it is the motion of a conductor in a magnetic field that does the job. A conductor moving in a magnetic field produces emf, which is used to drive the current in a circuit connected across this conductor. Such an emf is called *motional emf*.

16.1 The Electric Field in a Conducting Rod Moving in a Uniform Magnetic Field

As you know, a conductor has a large number of electrons that are more or less free to move anywhere in its body. Called free electrons, they scatter every now and then in the conductor, changing their direction of motion and magnitude of velocity randomly at each such scattering. Barring this random thermal motion, the free electrons may be treated to be at rest with respect to the conductor.

What happens when such a conductor moves in a magnetic field? Suppose a conducting rod of length L moves perpendicular to its length and enters a region where a uniform magnetic field $\boldsymbol{B}$ exists in a direction perpendicular to both the length of the rod and its velocity. Such a situation is shown in Figure 16.1. The velocity of the rod is $\boldsymbol{v} = v\hat{\boldsymbol{i}}$ and the magnetic field is $\boldsymbol{B} = -B\hat{\boldsymbol{k}}$.

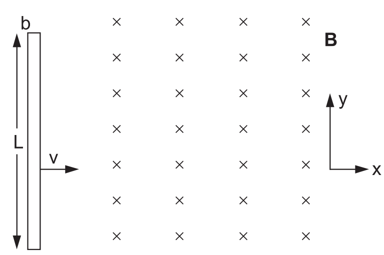


Fig. 16.1

The free electrons move together with the rod with velocity $v\hat{\boldsymbol{i}}$. The magnetic field exerts a force $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B} = (-e) \times (v\hat{\boldsymbol{i}}) \times (-B\hat{\boldsymbol{k}}) = -evB\hat{\boldsymbol{j}}$ on each such electron. The electrons are forced to bend in the downward $(-\hat{\boldsymbol{j}})$ direction. A free charge should move in a circular path when projected perpendicularly in a uniform magnetic field. However, here the situation is much more complicated. If you work out the radius and the time period of such a circular motion, you will find that for practical values of v, B, etc., the motion is in the scale of atomic dimensions. Neither our model of neglecting random thermal motion nor the model of "free electrons" with almost no interaction within the conductor will work.

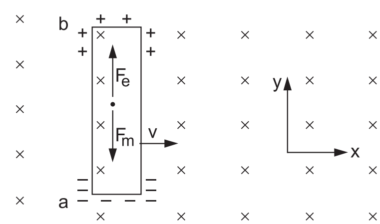


Fig. 16.2

Not attempting to analyze the transient motion of the free electrons when the rod enters the field, let us focus on the steady state. The magnetic field pushes the electrons down, and circular motion is not allowed by the edges of the rod and there are atomic interactions too. Remember, the electrons are not free to come out of the conductor. Ultimately, some of the free electrons end up on the surface of the conductor to create an electric field $\boldsymbol{E}_0$ inside the conductor (Figure 16.2). Denote the magnetic field inside the rod by $\boldsymbol{B}_0$. The electric force $\boldsymbol{F}_\mathrm{e} = -e\boldsymbol{E}_0$ opposes the magnetic force $\boldsymbol{F}_\mathrm{m} = -e\boldsymbol{v} \times \boldsymbol{B}_0$. In the steady state, when no further accumulation of charge takes place, the net force on the free electrons in the interior of the conductor should be zero. Then,

$$-e\boldsymbol{E}_0 = e\boldsymbol{v} \times \boldsymbol{B}_0$$

or

$$\boldsymbol{E}_0 = -\boldsymbol{v} \times \boldsymbol{B}_0. \tag{16.1}$$

In the situation shown in Figure 16.2, $\boldsymbol{E}_0 = -vB_0\hat{\boldsymbol{j}}$.

We started this section with the given magnetic field $\boldsymbol{B}$ but used a new symbol $\boldsymbol{B}_0$ for the magnetic field inside the rod. Is $\boldsymbol{B}_0$ different from the original field $\boldsymbol{B}$? The answer is yes. This is because the surface charges that appear on the rod move with a velocity v together with the rod, causing a current to flow that produces its own magnetic field. This field will be most pronounced in and around the rod and will diminish at points far from the rod. Thus the magnetic field far from the rod will still be $\boldsymbol{B}$, the same as the original field, but inside the rod and close to the rod it will be different.

It is easier to get the magnetic field $\boldsymbol{B}_0$ inside the rod by going to the rod frame, calculating the electric and magnetic fields inside the rod in this frame and transforming these fields back to the lab frame. Let us now do this.

Analysis from the frame of the rod

You can take the x'-, y'-, z'-axes on the rod itself, parallel to the corresponding axes in the lab frame. Suppose the x'-axis is made coincident with the x-axis. The rod frame is then the S′ frame and the ground frame is the S frame. First consider the field far from the rod, that is, the external field in which the rod is moving. The field components in the ground frame are

$$E_x = E_y = E_z = 0,\ B_x = B_y = 0,\ B_z = -B.$$

The corresponding field components in the rod frame are

$$E'_x = E_x = 0,\ E'_y = \frac{E_y - vB_z}{\sqrt{1-v^2/c^2}} = \frac{vB}{\sqrt{1-v^2/c^2}},\ E'_z = \frac{E_z + vB_y}{\sqrt{1-v^2/c^2}} = 0.$$

$$B'_x = B_x = 0,\ B'_y = \frac{B_y + \frac{v}{c^2}E_z}{\sqrt{1-v^2/c^2}} = 0,\ B'_z = \frac{B_z - \frac{v}{c^2}E_y}{\sqrt{1-v^2/c^2}} = \frac{-B}{\sqrt{1-v^2/c^2}}.$$

So there is an electric field in the y'-direction together with a magnetic field in the negative z'-direction. This is the external field that exists in space before the rod is put there. The conducting rod is at rest in this frame. In the steady state, there can be no electric field in the rod as it is a conductor at rest. But there is an external electric field $\boldsymbol{E}' = E'_y\hat{\boldsymbol{j}}$ in the region. Thus charges will get distributed on the surface of the rod in such a way that the electric field is zero everywhere inside the rod. In the steady state, when the redistribution on the rod is complete and the charges become stationary (on the average), the magnetic field plays no role. So the distribution of surface charge will be just the same as that produced when a conducting rod is kept in a uniform electric field $\boldsymbol{E}'$. The charge distribution will modify the electric field outside the rod too. Figure 16.3 shows the electric field lines qualitatively in the steady state.

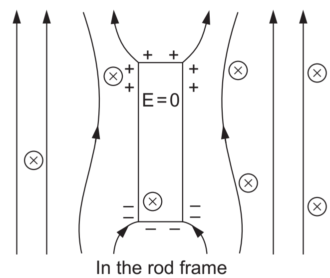


Fig. 16.3

Inside the rod, the electric field is zero and the magnetic field is the same as that outside.

Thus, in the rod frame, the field components inside the rod are

$$E'_{0x} = E'_{0y} = E'_{0z} = 0,\ B'_{0x} = B'_{0y} = 0,\ B'_{0z} = \frac{-B}{\sqrt{1-v^2/c^2}}.$$

The transformation equations can be used to get the fields inside the rod as seen in the lab frame after the steady state is reached. These fields will be

$$E_{0x} = E'_{0x} = 0,\ E_{0y} = \frac{E'_{0y} + vB'_{0z}}{\sqrt{1-v^2/c^2}} = -\frac{vB}{1-v^2/c^2},\ E_{0z} = \frac{E'_{0z} - vB'_{0y}}{\sqrt{1-v^2/c^2}} = 0.$$

$$B_{0x} = B'_{0x} = 0, B_{0y} = \frac{B'_{0y} - \frac{v}{c^2}E'_{0z}}{\sqrt{1-v^2/c^2}} = 0,$$

$$B_{0z} = \frac{B'_{0z} + \frac{v}{c^2}E'_{0y}}{\sqrt{1-v^2/c^2}} = -\frac{B}{1-v^2/c^2}.$$

Check that $\boldsymbol{E}_0 = -\boldsymbol{v}\times\boldsymbol{B}_0$

and $\boldsymbol{B}_0 = -\dfrac{\boldsymbol{B}}{1-v^2/c^2}.$

The magnetic field inside the rod is slightly larger than that far from it. Though $\boldsymbol{B}_0$ is different from $\boldsymbol{B}$, the difference is not much unless the rod moves at a very high speed. You can therefore approximate $\boldsymbol{B}_0$ to $\boldsymbol{B}$ and write

$$\boldsymbol{E} = -\boldsymbol{v}\times\boldsymbol{B} \tag{16.2}$$

for the electric field appearing in the rod when it is moved in a uniform magnetic field $\boldsymbol{B}$. The electric field is constant inside the rod as seen in the S frame. The charge accumulated at the surface creates this field. There is some electric field outside the rod too. The electric field lines are qualitatively shown in Figure 16.4.

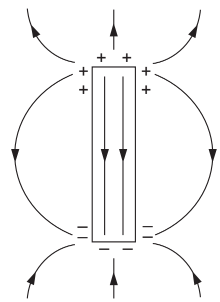


Fig. 16.4

The analysis remains unchanged even if you consider a conductor of any other shape.

16.2 The EMF due to the Motion of a Conducting Rod in a Magnetic Field

A magnetic field causes a charge separation on a moving conducting rod moving in it. There is a nonelectrostatic (magnetic) force that moves electrons preferentially to one side of the rod. In other words, an emf is produced. In Figure 16.2, the emf between a and b is

$$\varepsilon_{ab} = \int_a^b \frac{\boldsymbol{F}}{q}\cdot d\boldsymbol{l} = \int_a^b \frac{q(\boldsymbol{v}\times\boldsymbol{B})}{q}\cdot d\boldsymbol{l} = \int_a^b (\boldsymbol{v}\times\boldsymbol{B})\cdot d\boldsymbol{l}. \tag{16.3}$$

Taking $\boldsymbol{v} = v\hat{\boldsymbol{i}},\ \boldsymbol{B} = -B\hat{\boldsymbol{k}},\ d\boldsymbol{l} = dl\hat{\boldsymbol{j}},$

$$\varepsilon_{ab} = vBl. \tag{16.4}$$

The moving rod acts as a source of emf vBl. Equation 16.4 is applicable when the length of the rod, its velocity and the magnetic field are all mutually perpendicular to each other. If they are not, use Equation 16.3. Equation 16.3 or 16.4 will be generally used to get the magnitude of the emf. The sign can be easily determined by looking at the direction in which the magnetic force $q\boldsymbol{v} \times \boldsymbol{B}$ acts in the rod.

The emf generated in a conductor moving in a magnetic field is called *motional emf*.

The charges appearing on the rod create an electric field $\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B} = -vB\hat{j}$ in the rod. Hence there is a potential difference between a and b.

$$V_b - V_a = -\int_a^b \boldsymbol{E} \cdot d\boldsymbol{l} = vBl.$$

As usual with any source of emf, when no circuit is connected to the moving rod and no current is drawn, the potential difference across the rod is equal to the emf.

EXAMPLE 16.1 **A metallic rod PQ of length L and placed parallel to the y-axis moves parallel to itself with a velocity $v = v_0\left(\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right)$. A magnetic field $-B_0\hat{k}$ exists in the region. Find the potential difference between P and Q. Which of the two ends will be at higher potential?**

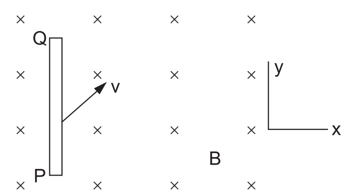


Fig. 16.5

Solution The force on a free electron due to the magnetic field will be $-e\boldsymbol{v} \times \boldsymbol{B}$. It will have a component along the rod towards P. The electrons will be pushed towards P and hence the potential will be higher at Q and lower at P. The motional emf generated will be

$$\varepsilon = \int_P^Q (\boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{l}$$

$$= \int_0^L v_0\left(\frac{1}{2}\hat{i} \times \frac{\sqrt{3}}{2}\hat{j}\right) \times (-B_0\hat{k}) \cdot dl\hat{j} = \frac{v_0 B_0 L}{2}.$$

When no current is drawn, this is the same as the potential difference across the ends. So the potential difference between P and Q is $v_0 B_0 L/2$.

16.3 The Flux Rule

The emf in a rectangular loop moving in a magnetic field

Suppose a conducting rod ab of length l moves with velocity v in a perpendicular magnetic field B as shown in Figure 16.6. The axes x, y, z are also shown in the figure. The ends a and b of the rod touch the arms of a fixed conducting frame *cdef*. Thus, these ends are connected by the conducting path *bdea*. The lengths *ae* and *bd* keep decreasing with time as the rod moves towards the right. The moving rod acts like a battery of emf $\varepsilon = Bvl$. The end b is at a higher potential than a. Since these ends are connected by a conducting path, the circuit is complete and there will be a current in it. The parts *bd, de* and *ea* are at rest and hence there is no emf in these sections. The emf in the circuit is therefore vBl. If the resistance of the circuit be R, the current will be $I = vBl/R$.

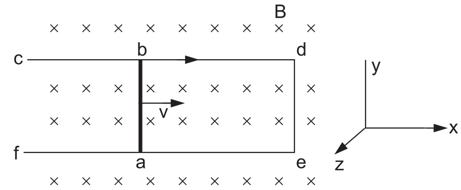


Fig. 16.6

The intention in this section is to tell you about a very simple expression for the emf in a closed loop produced by a moving conductor. In Figure 16.6, the circuit *bdeab* forms a closed loop. The area of the surface *bdea* bounded by the loop at time t is

$$A = (ab)(bd).$$

The arrow on the loop indicates the positive sense. If you move a pencil along the loop *bdeab* in the direction of the arrow, the pencil moves clockwise. By convention, the area vector corresponding to the surface *bdeab* will point away from us, that is, in the $-\hat{k}$-direction in Figure 16.6. The magnetic field is also in the same direction. So the flux of the magnetic field through the area *bdeab* is

$$\Phi = \boldsymbol{B} \cdot \boldsymbol{A} = B(ab)(bd).$$

Differentiating with respect to time,

$$\frac{d\Phi}{dt} = B(ab)\frac{d(bd)}{dt} = B \cdot l(-v) = -\varepsilon.$$

Thus, $$\varepsilon = -\frac{d\Phi}{dt} \cdot \tag{16.5}$$

This equation is called the *flux rule*. The emf developed in a closed loop is equal to the negative rate of change of the magnetic flux through the surface bounded by the loop. Here, the equation has been derived for the simple case of a rectangular loop where one arm is moving. But it is very general. Whatever be the shape of the loop and in whatever way its parts are moving, Equation 16.5 gives the emf in the circuit. We will discuss the proof for a more general situation but before that, let us take up another simple example related to the flux rule, and especially look at the interpretation of the negative sign in the equation for this rule.

EXAMPLE 16.2 **Figure 16.7 shows a situation in which a rectangular frame ABCD, containing a resistor with resistance R and thick wires, is moved with a constant velocity $v\hat{i}$. The sides AB and CD of the frame are parallel to the x-axis and have a length l each. The sides BC and DA are parallel to the y-axis and have a length b each. A magnetic field $B = -B_0\hat{k}$ exists in the region $x > 0$. Find the current in the resistor when (a) the arm BC of the frame is in the magnetic field but AD is not, and (b) the whole frame is in the magnetic field.**

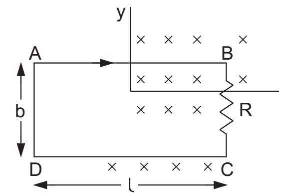


Fig. 16.7

Solution (a) Suppose the distance of BC from the y-axis is x at time t. As the frame is moving with velocity $v\hat{i}$, x will keep increasing with time and dx/dt will be equal to v. The flux of the magnetic field through the area of the circuit, taking ABCDA as the positive sense and hence $-\hat{k}$ as the positive direction of the area vector, is

$$\Phi = (-B_0\hat{k})\cdot(bx)(-\hat{k}) = B_0 bx.$$

Thus, $$\varepsilon = -\frac{d\Phi}{dt} = -B_0 b\frac{dx}{dt} = -B_0 bv.$$

The emf turns out to be negative. The negative emf means that the current will be driven in the opposite sense, that is, along ADCBA. The current itself will be

$$I = \frac{B_0 bv}{R},$$

using Ohm's law.

(b) When the whole frame is in the magnetic field, the flux of the magnetic field through the area of the circuit is $\Phi = B_0 lb$, and does not change with time as the frame moves ahead. Thus the emf generated is zero and there is no current in the circuit.

The emf in a conductor of arbitrary shape moving in a magnetic field

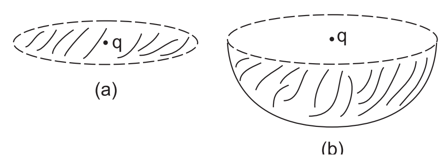


Fig. 16.8

There is an interesting question regarding the calculation of the flux through the surface bounded by a loop. In the example above, the loop was rectangular and we took the rectangular surface bounded by the loop to calculate the flux of the magnetic field. But there can be other surfaces with the same boundary. Does the flux depend on the surface chosen? Let us take the example of the flux of an electric field. Suppose there is a charge q placed at the centre of a circle. Consider the circular disk bounded by the circle [Figure 16.8(a)]. What is the flux of the electric field of this charge through this surface? It is zero because the direction of the field everywhere will be parallel to the surface on this area. Now consider the hemispherical surface made with the same circle as the boundary [Figure 16.8(b)]. This is also a surface bounded by that circle. What is the flux of the electric field through this surface? It is $\frac{q}{2\varepsilon_0}$. (If you complete the sphere, the flux will be q/ε_0, from Gauss's law). So the flux does depend on the surface taken even though it is bounded by the same loop.

However, the case of magnetic flux is different. Again consider two surfaces, a circular disk A and a hemispherical surface B bounded by the same circle. If you put disk A on hemisphere B, you get a closed surface S, as shown in Figure 16.9(a). As $\nabla \cdot \boldsymbol{B} = 0$, $\oint \boldsymbol{B} \cdot \boldsymbol{da} = 0$ on any closed surface and hence on S. Hence,

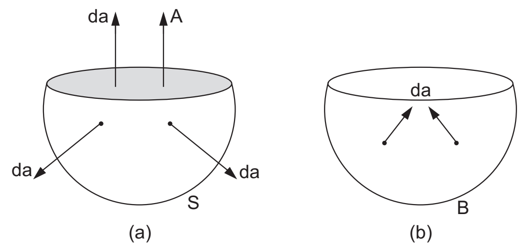


Fig. 16.9

$$\int_A \boldsymbol{B} \cdot \boldsymbol{da} + \int_B \boldsymbol{B} \cdot \boldsymbol{da} = 0.$$

So $$\int_A \boldsymbol{B} \cdot \boldsymbol{da} = -\int_B \boldsymbol{B} \cdot \boldsymbol{da}.$$

In this equation, $\boldsymbol{da}$ is taken along the outward normal, as shown in Figure 16.9(a). If you invert the arrows on surface B for the area vectors, as suggested in Figure 16.9(b), $\int_A \boldsymbol{B} \cdot \boldsymbol{da} = \int_B \boldsymbol{B} \cdot \boldsymbol{da}$.

So, the magnetic flux through different areas bounded by the same closed curve is the same. However, you have to take the positive sides of the areas carefully. You can think of getting these different areas by gradually deforming one into other, keeping the boundary the same. The normals to the areas also rotate gradually.

Now let us prove the flux rule in the more general case of a closed loop. Consider a closed loop moving in a magnetic field. The loop may be of arbitrary shape, moving in any way. In Figure 16.10, the solid line represents the loop at time t and the dotted line represents the loop at time $t + dt$. We wish to calculate the change in the magnetic flux through the loop in time dt. Call these loops C_1 and C_2 respectively. Put an arrow on the loop to define the positive sense of the loop and also the positive side of the area. Take the surface S_1 as that shown shaded and bounded by C_1, to calculate the flux at time t. The arrows for $\boldsymbol{da}$ at all points of S_1 are towards the left. Now carefully make the surface S_2, bounded by C_2, on which we will calculate the flux at time $t + dt$. Join the corresponding points of C_1 and C_2 by small, straight lines. If the point a goes to a', join a to a', and so on. Then you will get a structure like a strip. The original surface S_1 together with this strip makes a surface in the shape of an open box with C_2 as the boundary. Call this surface S_2 (Figure 16.11). You have to calculate the flux through the loop at time $t + dt$, that is, through the

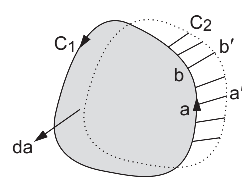


Fig. 16.10

surface S_2. How do you take the direction of the ***da*** vectors? Think of a plane surface parallel to S_1 and bounded by C_2. Think of pushing this area to deform it into the shape of S_2. If the ***da*** vector on S_1 to calculate $\Phi(t)$ is taken towards the left, as in Figure 16.11, the area vectors for $\Phi(t+dt)$ should also be towards the left and on S_1 and outward on the strip as shown in Figure 16.11. Thus,

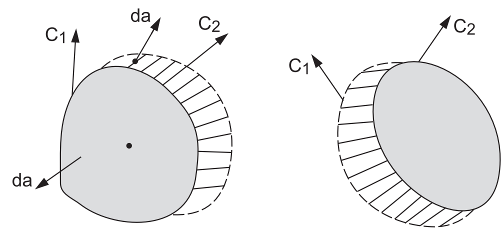


Fig. 16.11

$$\Phi(t+dt) + \Phi(\text{on } S_1) + \Phi(\text{on the strip})$$

or $$d\Phi = \Phi(\text{on the strip}).$$

To calculate the flux through the strip, consider a small length element ab of the loop (Figure 16.10). It moves from ab to $a'b'$ in time dt. The area vector for $abb'a'a$ can be written as

$$\boldsymbol{da} = \boldsymbol{bb'} \times \boldsymbol{aa'} = (\boldsymbol{v}dt) \times \boldsymbol{dl}.$$

Check that it gives the right direction of the area vector on the strip, as shown in Figure 16.11.

If the magnetic field at the site of the element ab is $\boldsymbol{B}$, the flux through this area is

$$\delta\Phi = \boldsymbol{B}\cdot\boldsymbol{da} = \boldsymbol{B}\cdot(\boldsymbol{v}\times\boldsymbol{dl})dt = (\boldsymbol{B}\times\boldsymbol{v})\cdot\boldsymbol{dl}\,dt.$$

The net flux on the strip is

$$d\Phi = \oint_{C_1} (\boldsymbol{B}\times\boldsymbol{v})\cdot\boldsymbol{dl}\,dt$$

or $$-\frac{d\Phi}{dt} = \oint_{C_1} (\boldsymbol{v}\times\boldsymbol{B})\cdot\boldsymbol{dl}.$$

But the right-hand side is the net motional emf ε in the loop C_1 at time t. Then,

$$\varepsilon = -\frac{d\Phi}{dt}\cdot$$

EXAMPLE 16.3 **A circular coil of radius a, made of N turns, rotates with a uniform angular velocity ω about one of its diameters, which is kept fixed along the y-axis. A uniform magnetic field $B_0\hat{i}$ exists in space. Find the magnitude of the emf when the coil is in the x–y plane.**

Solution If the plane of the coil makes an angle θ with the x–y plane,

$$\theta = \omega t + \phi$$

and the flux of the magnetic field through the coil at time t is

$$\Phi = N\boldsymbol{B}\cdot\boldsymbol{A} = NB\pi a^2 \sin(\omega t + \phi). \quad \text{(i)}$$

Note that each turn contributes to the flux and hence the multiplication by N. The emf generated is

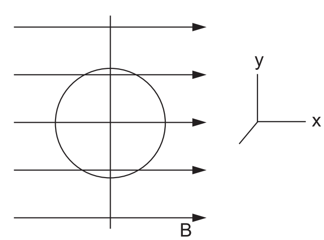


Fig. 16.12

$$\varepsilon = -\frac{d\Phi}{dt} = \omega NB\pi a^2 \sin(\omega t + \phi). \tag{ii}$$

When the coil is in the x–y plane, the flux itself is zero. This is because the area vector and the magnetic field $\boldsymbol{B}$ will be perpendicular to each other in this case. Since the flux is zero, $\cos(\omega t + \phi) = 0$ and hence $\sin(\omega t + \phi) = 1$ or -1. Thus,

$$|\varepsilon| = \omega NB\pi a^2.$$

16.4 Lenz's Law

The equation for the flux rule has a minus sign in it, which is crucial to determining the direction of current in the loop. Start by putting an arrow on the loop to fix the positive sense. This determines the positive side of the area. The simple way to do this is to use the right-hand rule. If you curl the fingers of the right hand around the loop in the positive sense, the stretched thumb gives the positive side of the area. Another way is to look at the loop from the front. If the positive sense of the loop looks anticlockwise, the positive side of the area is towards you. If the positive sense of the loop looks clockwise to you, the positive side of the area is away from you. The area vector is to be drawn in the positive side of the area. This rule has been mentioned earlier but no harm repeating it. Once the direction of the area vector is determined, calculate Φ and then $-d\Phi/dt$. If this quantity comes out to be positive, the current in the loop due to the emf will be in the same direction as suggested by the arrow that you had put on it. If $-d\Phi/dt$ turns out to be negative, the current will be opposite to that suggested by your arrow on the loop.

There is a much simpler way to determine the direction of the current in a loop moving in a magnetic field. The procedure is as follows.

Look for the magnitude of the flux of the magnetic field and check whether it is increasing or decreasing with time. If the flux (magnitude) is increasing with time, the current in the loop will produce its own magnetic field opposite to the original magnetic field. If the flux is decreasing, the current in the loop will produce its own magnetic field in the same direction as the original magnetic field. Thus if the flux is increasing, the current tries to reduce it (by creating its own field opposite to the original field). And if the flux is decreasing, the current tries to increase it (by creating its own field in the direction of the original field). The principle is summarized in the following statement.

The current in a loop corresponding to a changing magnetic flux opposes the change in the flux itself. This is known as Lenz's law.

This is one way of stating Lenz's law. There are other ways too to state it. A popular version is "the current induced in a loop opposes the cause that has produced it".

EXAMPLE 16.4 **A magnet is fixed in a vertical position with the north pole pointing downwards (Figure 16.13). A metallic ring is kept in a horizontal plane so that the magnet is along its axis. The ring now moves downwards. Looking at the set-up from above, is the current produced in the ring clockwise or anticlockwise? What happens if the ring moves towards the magnet?**

Solution As the ring moves downwards, it goes away from the magnet and hence encounters a weaker field. The flux (magnitude) of the magnetic field therefore decreases with time. According to Lenz's law, the current in the loop will be such that it produces its own magnetic field in the same direction as the original field, that is, vertically downwards. Looking at the situation from above the ring, the current should be clockwise so that the field is downwards.

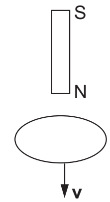


Fig. 16.13

If the ring moves towards the magnet, the flux increases with time. Thus the field produced by the current in the ring should be opposite to that by the magnet, that is, upwards. Hence, the current will be anticlockwise as seen from above.

16.5 Ohm's Law

For metallic objects, Ohm's law is generally stated as $\boldsymbol{J} = \sigma \boldsymbol{E}$. However, if you consider the part BC of the loop in Example 16.2, the current is from C to B whereas the electric field in it is from B to C. Thus $\boldsymbol{J}$ and $\boldsymbol{E}$ are in opposite directions. If a conductor moves in a magnetic field, you have an additional force $[-e\boldsymbol{v} \times \boldsymbol{B}]$ on the free electrons apart from $(-e\boldsymbol{E})$. Ohm's law gets modified to $\boldsymbol{J} = \sigma[\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}]$.

EXAMPLE 16.5 **A metallic wire AB of length L moves in a magnetic field B_0 with speed v_0 (Figure 16.14). The length, velocity and the field are perpendicular to each other. The wire is a part of a circuit so that a current I flows in it from the end A to the end B. Find the electric field developed in the wire.**

Solution Suppose the wire has a cross-sectional area S and conductivity σ. By Ohm's law,

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

or

$$\boldsymbol{E} = \frac{\boldsymbol{J}}{\sigma} - \boldsymbol{v} \times \boldsymbol{B}$$

$$= \frac{J\hat{\boldsymbol{i}}}{\left(\frac{L}{SR}\right)} - v_0\hat{\boldsymbol{i}} \times B_0(-\hat{\boldsymbol{k}})$$

$$= \frac{JSR - v_0 B_0 L}{L}\hat{\boldsymbol{j}} = \frac{IR - v_0 B_0 L}{L}\hat{\boldsymbol{j}}.$$

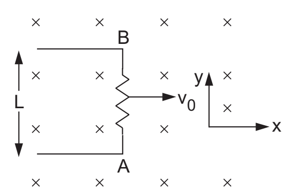


Fig. 16.14

Concepts at a Glance

1. When a metallic object moves through a magnetic field, the free electrons experience a magnetic force.

   Due to this force, charges are redistributed on the surface of the metal, and a uniform electric field $\boldsymbol{E} \approx -\boldsymbol{v} \times \boldsymbol{B}$ is produced inside the metal.

   Due to this electric field, often an emf is developed between certain parts of the metal, known as motional emf.
2. The motional emf across the length of a metallic strip is given by $\varepsilon = \int (\boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{dl}$.
3. For a rod of length L moving in a magnetic field $\boldsymbol{B}$ perpendicular to both its length and the magnetic field, the emf between the ends is vLB.
4. The motional emf in a closed loop is given by $\varepsilon = -\frac{d\Phi}{dt}$, where $\Phi = \int \boldsymbol{B} \cdot \boldsymbol{da}$ is the flux of the magnetic field through the area bounded by the loop.
5. The direction of the current induced in a loop due to the motion of the loop in a magnetic field can be obtained from Lenz's law, which states that the direction of current is such that it opposes the change in the magnetic flux through the loop.

EXERCISES

Based on Concepts

1. For a conducting rod moving in a magnetic field, the electric field was derived to be $\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B}$. If a conductor of some other shape moves in the magnetic field, can we still use the same expression for the electric field?
2. A long copper plate is moved at a speed v along its length as suggested in Figure 16E.1. A magnetic field exists perpendicular to the plate in a cylindrical region cutting the plate in a circular region. A and B are two fixed conducting brushes which maintain contact with the plate as the plate slides past them. These brushes are connected by a conducting wire.

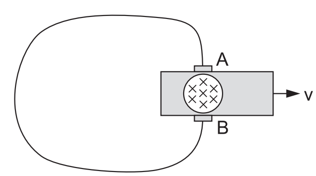
   

   Fig. 16E.1

   Is there a current in the wire? If yes, in which direction?

   [**Ans.** Yes, from A to B]
3. A circular metallic loop is placed with its axis along the length of a bar magnet as shown in Figure 16E.2. The loop moves away from the magnet with a velocity $v\boldsymbol{i}$ with the axes as shown.

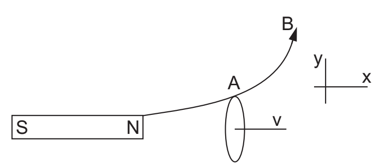
   

   Fig. 16E.2

   (a) From $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$, find the direction of the force a free electron will experience at A. (b) As seen from the side of the magnet, will the current induced be clockwise or anticlockwise? (c) Check that you get the same result using Lenz's law.

4. A circular conducting disk of radius R is moved at a constant velocity u in its plane. A uniform magnetic field B exists in the perpendicular direction as suggested in Figure 16E.3. (a) Is the field the same everywhere in the disk? (b) What is the minimum potential difference between diametrically opposite points? (c) What is the maximum potential difference between diametrically opposite points?

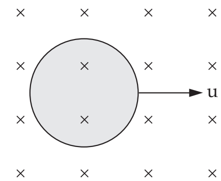


Fig. 16E.3

Problems

1. A copper rod of length 10 cm is moved at a constant velocity of 2.5 m/s in a uniform magnetic field of 0.5 T. The rod is along the x-axis, the velocity is in the $\sqrt{\frac{1}{2}}(\hat{i}+\hat{j})$ direction and the field is in the $\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})$ direction. Find the electric field inside the rod. [**Ans.** $\frac{2.25}{\sqrt{6}}(\hat{i}-\hat{k})$ V/m]

2. A conducting rod moves in a uniform magnetic field with a constant velocity of 3 cm/s. The length of the rod, the velocity and the field are mutually perpendicular. By what fraction will the magnetic field in the rod differ from that outside at a large distance? [**Ans.** 10^{-20}]

3. An annular conducting ring of inner and outer radii a and b rotates at a constant angular velocity ω about its axis (Figure 16E.4). A uniform magnetic field B exists perpendicular to the plane of the ring. Find the electric field in the ring.

   [**Ans.** Taking the z-axis along the axis, the field is $\boldsymbol{E}=-\omega sB\hat{\boldsymbol{s}}$ in cylindrical coordinates]

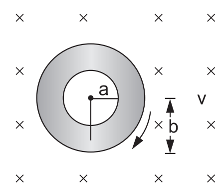


Fig. 16E.4

4. A region contains a magnetic field that has no $\hat{\phi}$-component as described in cylindrical coordinates. The s-component of the magnetic field has a value B_0 at $s=a$. The loop formed by a circular wire is at $z=0, s=a$ at time $t=0$, and moves along the z-axis with a uniform velocity v. (a) Find the emf produced in the loop due to its motion. (b) Find the change in magnetic flux through the loop as it moves through a distance z. [**Ans.** (a) $2\pi avB_0$, (b) $2\pi aB_0 z$]

5. A region contains a magnetic field with the following characteristics. $B_\phi=0, \frac{\partial B_z}{\partial s}=\frac{\partial B_z}{\partial \phi}=0, \frac{\partial B_s}{\partial z}=\frac{\partial B_s}{\partial \phi}=0$. The field at $s=a, z=z_0$ is $B_1\hat{\boldsymbol{k}}+B_2\hat{\boldsymbol{s}}$. Find $\frac{\partial B_z}{\partial z}$ and $\frac{\partial B_s}{\partial s}$ at this point. [**Ans.** $\frac{-2B_2}{a}, \frac{B_2}{a}$]

6. A spherical, conducting shell of radius R rotates about a diameter taken as the z-axis, with angular velocity ω in a uniform magnetic field $B=B_0\hat{\boldsymbol{k}}$. The centre of the shell is at the origin. Calculate the emf developed between the points with $z=0$ and $z=R$. [**Ans.** $=\frac{\omega R^2 B_0}{2}$]

7. A circular coil of 240 turns with a radius of 10 cm is rotated at 1200 rpm about its diameter, which is fixed along the x-axis. A uniform magnetic field of 0.30 T exists in the direction of $\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}+\hat{\boldsymbol{k}}$. Find the maximum emf produced in the coil. [**Ans.** 38.6 mV]

8. Figure 16E.5 shows a wire frame *abcde*, of which the portion *bcd* is semicircular and *ab* and *de* are straight segments along the diameter *bd*. The length $ab = bd = de = D$. A magnetic field B exists in the plane of the paper perpendicular to *ab*. The frame *abcde* is rotated at angular speed ω about the line *abde*. At time $t = 0$, the frame is in the plane of the paper. Find the potential difference between a and e as a function of time.

[**Ans.** $\dfrac{\pi D^2 \omega B}{8} \cos \omega t$]

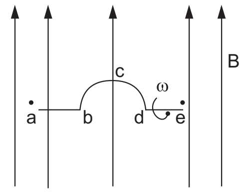


Fig. 16E.5

9. A uniform magnetic field $\boldsymbol{B} = -B_0 \hat{\boldsymbol{k}}$ exists in the region $0 < x < R$ (Figure 16E.6). There is no field outside this region. A wire loop in the shape of an equilateral triangle of height R is placed in the x–y plane as shown in the figure, with one of its edges parallel to the y-axis. The loop is moved parallel to the x-axis with a constant velocity $v_0 \hat{\boldsymbol{i}}$. At time $t = 0$, the vertex just enters the field. Find the emf in the loop as a function of time and sketch it for $0 < t < 2R/v$.

[**Ans.** $\dfrac{4Bv^2}{\sqrt{3}} t$ for $0 < t < \dfrac{R}{v}$, $-\dfrac{4Bv^2}{\sqrt{3}}\left(t - \dfrac{R}{v}\right)$ for $\dfrac{R}{v} < t < \dfrac{2R}{v}$]

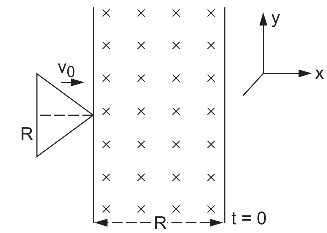


Fig. 16E.6

10. A conducting rod of length l and mass m slides on two parallel horizontal rails, connected at one end by a resistance R (Figure 16E.7). The rails and the rod are assumed to have no resistance and no friction. At $t = 0$, the rod has a velocity v_0. (a) Find the velocity of the rod at time t. (b) Find the power P dissipated in the resistor at time t. (c) Calculate $\int_0^\infty P dt$.

[**Ans.** (a) $v_0 e^{-\frac{B^2 l^2}{mR} t}$, (b) $\dfrac{B^2 l^2 v_0^2}{R} e^{-\frac{2B^2 l^2}{mR}}$, (c) $\dfrac{1}{2} m v_0^2$]

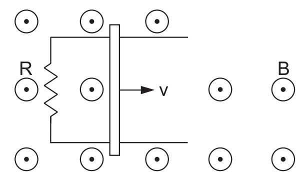


Fig. 16E.7

11. A square loop of wire (side L) lies on a table near a very long, straight wire that carries a current I, as shown in Figure 16E.8. The nearer side of the loop is at a distance a from the wire. (a) Determine the flux of $\boldsymbol{B}$ through the loop. (b) The loop is now pulled away from the wire with a speed v along the y-axis. Determine the emf and the direction of the current generated in the loop. (c) How will your answer change if the loop is pulled to the right (parallel to the x-axis) with a speed v? [**Ans.** (b) $\dfrac{\mu_0 I L^2 v}{2\pi a(a + L)}$]

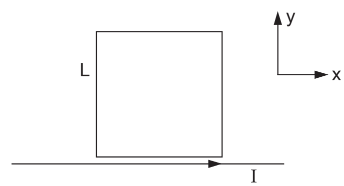


Fig. 16E.8

12. Consider the wire loop *abcdefa* shown in Figure 16E.9. The parts *abc* and *def* are concentric circles with radii R_1 and R_2. The part *dc* is along a radial line. The part *af* is quite small and goes over *dc*. The loop is always parallel to the x–y plane, but moves in the z-direction with a velocity v. A magnetic field B exists in the region with its z-component given by $B_z = B_0 e^{-\alpha z^2}$. Find the emf induced in the loop as it passes through the plane $z = z_0$.

[**Ans.** $2\pi B_0 \alpha z_0 v_0 (R_1^2 + R_2^2) e^{-a z_0^2}$]

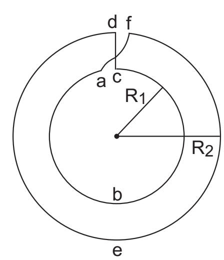


Fig. 16E.9

13. A metallic sphere of radius R is moved with a constant velocity $v_0\hat{\boldsymbol{i}}$ in a uniform magnetic field $B_0\hat{\boldsymbol{k}}$. (a) Find the surface charge density appearing on the sphere at the point $y = R$. (b) What is the potential difference between the points $y = R$ and $y = -R$? Assume $v << c$. [**Ans.** (a) $3\varepsilon_0 v_0 B_0$, (b) $2v_0B_0R$]

14. A magnet NS is placed along the axis of a circular conducting ring of radiu/s a and total resistance R. The ring itself moves along its axis, taken as the z-axis. At an instant (Figure 16E.10), the velocity of the ring is v_0 and the magnetic field at the periphery of the ring is $\boldsymbol{B} = B_1\hat{\boldsymbol{s}} + B_2\hat{\boldsymbol{k}}$. Find the induced current. [**Ans.** $\frac{2\pi v_0 B_1 a}{R}$]

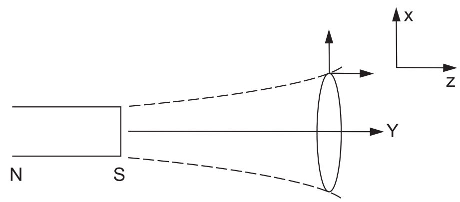


Fig. 16E.10

15. A rectangular loop of conducting wire (Figure 16E.11) is being pulled out from a region containing a magnetic field $\boldsymbol{B} = -B_0\hat{\boldsymbol{k}}$. The velocity is maintained at $\boldsymbol{v} = v_0\hat{\boldsymbol{i}}$. Consider a free electron in the segment AB. Suppose it moves in the segment with drift speed v_d. (a) Find the x-component of the magnetic force on this electron. What is the force F balancing this component? (b) Find the work done by this balancing force F as the electron traverses the length AB and also as it goes around the circuit.

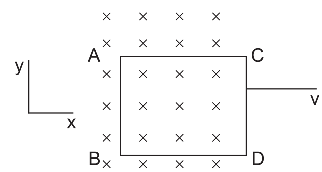


Fig. 16E.11

[**Ans.** (a) eB_0v_d; (b) $-eB_0v_0(AB)$, 0]

16. Two thick, parallel metallic rods make up an inclined track on which a metallic bar slides down (Figure 16E.12). The separation between the rods is L and the inclination of the track from the horizontal is α. There is a uniform magnetic field B perpendicular to the plane of the track, going into the plane in the figure. To start with, the track is open on both sides. The resistance and the friction are to be neglected everywhere. The mass of the bar is m. (a) The ends A and D of the track are joined by a resistance R. (i) Find the velocity as a function of time. (ii) Show that the bar acquires a terminal velocity and find its value. (b) Instead of the resistance, A and D are now joined by a capacitor of capacitance C. Show that the rod slides down with a constant acceleration and find its value.

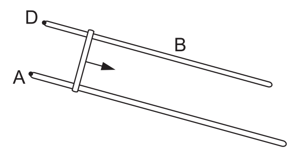


Fig. 16E.12

[**Ans.** (a) (ii) $\frac{mgR\sin\alpha}{B^2L^2}$, (b) $a = \frac{mg\sin\alpha}{m + C^2B^2L^2}$]

17. A conductor moves in the x-direction with a velocity $\boldsymbol{v} = 0.6c\hat{\boldsymbol{i}}$, as seen from a frame S. The surface of the conductor at time t is given by $\left(\frac{5x - 3ct}{4}\right)^2 + y^2 + z^2 = R^2$. There is an otherwise uniform magnetic field $\boldsymbol{B} = B_0\hat{\boldsymbol{j}}$ in the region. Find the total charge appearing on the part of the conductor defined by $z > 0$. [**Ans.** $\frac{9}{4}\varepsilon_0 cB_0\pi R^2$]

18. Consider the situation shown in Figure 16E.13. The fixed resistor has resistance R_1 and the sliding rod has resistance R_2. The magnetic field is B. Find the electric field existing in the rod. [**Ans.** $\frac{R_1}{R_1 + R_2}uB$]

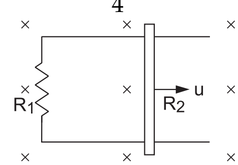


Fig. 16E.13

❑

17 Faraday's Law of Induction

17.1 The Current in a Circuit due to a Moving Magnet

In the previous chapter, we talked about motional emf and the flux rule. If a conductor moves in a magnetic field, a force $q\boldsymbol{v}\times\boldsymbol{B}$ acts on the free electrons. This causes a charge distribution on the surface of the conductor and an emf. If the circuit is complete, as in the case of a conducting loop, current flows in it.

In the last chapter, we took the example of a horizontal metallic ring and a bar magnet placed along its axis. When the ring is moved away from or towards the magnet, a current flows in the ring. You can look at this experiment from the frame of the ring. Here, the ring is at rest and the magnet moves away from or towards the ring. In this frame also, you will find a current in the ring. Consider any cross section of the ring. If electrons are crossing it in the lab frame, they are indeed crossing it in the ring frame too. You can also cut the ring and join the two ends to the terminals of a galvanometer. When current passes through the ring, the needle of the galvanometer is deflected. If this deflection can be seen from the lab frame, it can surely be seen from the ring frame, confirming that there is current in the ring in the ring frame also.

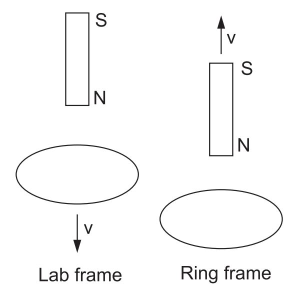


Fig. 17.1

Of course, you can do the experiment in the lab frame itself. Keep a ring or a coil with a galvanometer fixed and move a magnet towards or away from the ring, and see that there is a current in the circuit. Michael Faraday performed such experiments in 1831 and concluded that it is the relative motion between the coil and the magnet that causes a current in the coil. It does not matter whether the magnet is at rest and the coil is moving or if the coil is at rest and the magnet is moving. As mentioned earlier, this formed the basis for the special theory of relativity presented by Albert Einstein in 1905.

17.2 Induced Electric Field

Now that it has been established that a magnet moving near a fixed conducting loop causes a current in the loop, the question that arises is "what starts the motion of the free electrons in the ring, causing the current". A magnet moving towards a coil is shown in Figure 17.2. To start with, suppose the magnet is also kept at rest. There is no current in the ring—the free electrons do not exhibit any systematic motion. There is a magnetic field due to the magnet. Now start moving the magnet. The magnetic field at the site of the electrons, and also everywhere else, changes with time. Simultaneously, a current starts to flow in the ring. This means that the free electrons get pushed in a particular direction. What has pushed the electrons? The magnetic field, constant or variable, cannot exert a force on the electrons at rest. There must be an electric field to start the motion of these electrons. And it is the moving magnet that produces this electric field.

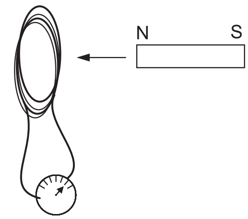


Fig. 17.2

The electric field due to a moving magnet

Suppose a magnet SN moves along the x-axis with a velocity v [Figure 17.3(a)]. Let the frame be called S. What is the electric field at $(x, y, 0)$ at time t? Go to the frame S′ that moves with respect to S with velocity v [Figure 17.3(b)]. Get the point x', y', z' corresponding to this set x, y, z, t using the Lorentz transformation

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \; y' = y, \; z' = z.$$

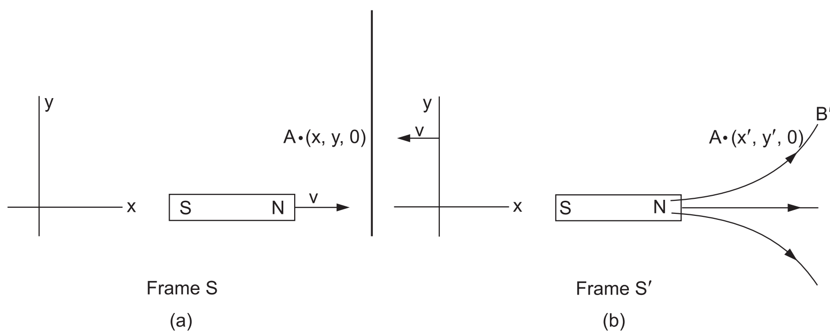


Fig. 17.3

Since the magnet is at rest in this frame, it will not produce an electric field. You know the magnetic field due to a stationary magnet. At a point A with coordinates $(x', y', 0)$, the magnetic field will have an x-component B_1, a y-component B_2 and no z-component. Thus the field at A in S′ can be written as

$$E'_x = 0, E'_y = 0, E'_z = 0, B'_x = B_1, B'_y = B_2, B'_z = 0.$$

This is independent of time t' as the magnet is at rest in this frame.

Now make the field transformation from S′ to S. The electric field at $(x, y, 0)$ will be

$$E_x = E'_x = 0,\ E_y = \frac{E'_y + vB'_z}{\sqrt{1-v^2/c^2}} = 0,\ E_z = \frac{E'_z - vB'_y}{\sqrt{1-v^2/c^2}} = \frac{-vB_2}{\sqrt{1-v^2/c^2}}.$$

Thus, in the frame S, where the magnet is moving, you have an electric field in the z-direction (or the negative z-direction). Remember, this is not a constant electric field. As time passes, the values of x' will change and hence B_2 will also change.

Consider the magnet–coil experiment again. Suppose the magnet is moving along the x-axis with velocity v and a coil is placed with its axis along the x-axis (Figure 17.4). Look at the points a and b of the coil, which are in the x–y plane. In the given situation, if you go to the S′ frame, B_2 will be positive for point a and negative for point b.

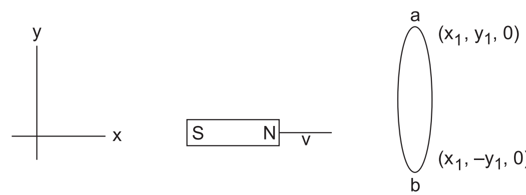


Fig. 17.4

Looking at the equation $E_z = \dfrac{-vB_2}{\sqrt{1-v^2/c^2}}$, the electric field at a will be in the negative z-direction, i.e, it will go into the plane. The electric field at b will come out of the plane. The currents induced in the coil at a and b are also in the same directions. Thus, looking at the situation from where the magnet is, the current is anticlockwise. The magnetic field due to the induced current is towards the magnet, that is, opposite to the original magnetic field of the magnet. This is consistent with Lenz's law, introduced in the previous chapter, as the induced current opposes the change in the magnetic flux.

The magnetic field in the frame S is given by

$$B_x = B'_x,\ B_y = \frac{B'_y - \dfrac{v}{c^2}E'_z}{\sqrt{1-v^2/c^2}} = \frac{B_2}{\sqrt{1-v^2/c^2}},\ B_z = \frac{B'_z + \dfrac{v}{c^2}E'_y}{\sqrt{1-v^2/c^2}} = 0.$$

Note that $\quad E_z = \dfrac{-vB_2}{\sqrt{1-v^2/c^2}} = -vB_y.$

Also $\boldsymbol{E} = E_z\hat{\boldsymbol{k}}$, $\boldsymbol{B} = B_y\hat{\boldsymbol{j}}$ and $\boldsymbol{v} = v\hat{\boldsymbol{i}}$. This gives $\boldsymbol{E} = -\boldsymbol{v}\times\boldsymbol{B}$.

A moving magnet produces a magnetic field $\boldsymbol{B}$ and an electric field $\boldsymbol{E}$ related by $\boldsymbol{E} = -\boldsymbol{v}\times\boldsymbol{B}$. If $v << c$, the magnetic fields in S and S′ do not differ much. You can still employ the usual expressions for $\boldsymbol{B}$ and calculate $\boldsymbol{E}$ using $\boldsymbol{E} = -\boldsymbol{v}\times\boldsymbol{B}$.

Is it the property of a magnet to create an electric field when in motion? Not really. You can set up an experiment as suggested in Figure 17.5. Put two coils C_1 and C_2 close to and also parallel to each other. Join the ends of C_1 to a battery and a switch, and those of C_2 to a galvanometer. Now keep watching the galvanometer and close the switch. You will find that there is a current in C_2 momentarily. Open the switch and again you find a momentary current in C_1. There are no moving magnets here. Yet an electric field is generated when the current in C_1 changes.

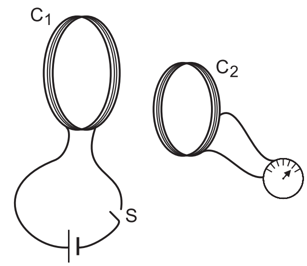


Fig. 17.5

Any charge distribution or source that creates a time-varying magnetic field also produces an electric field.

You can say that it is the changing magnetic field that creates the electric field. When you move a magnet, the magnetic field at any given point changes with time. When you close the switch S in the situation shown in Figure 17.5, suddenly there is a current in C_1 and this current sets up a magnetic field. The change in magnetic field from zero to the final value is responsible for the electric field. In whichever manner you change the magnetic field, you will create an electric field.

The electric field generated due to a changing magnetic field (you can also take it as being caused by sources producing the changing magnetic field) is called *induced electric field* and the current caused by such an electric field is called *induced current*.

The nature of an induced electric field is very different from that of the electric field produced by a static charge distribution. The most striking difference is that the curl of an induced electric field is not always zero. In other words, if you calculate $\oint \boldsymbol{E}_{\text{ind}} \cdot d\boldsymbol{l}$ for the induced electric field over a closed loop, it is not necessarily zero. The physics of induced electric fields, in general, is altogether different and is contained in what we call Faraday's law of induction.

17.3 Faraday's Law

A changing magnetic field causes an induced electric field. This is well established from experiments and is also a requirement for the special theory of relativity. If moving a coil near a fixed magnet produces a current in the coil, moving a magnet near a fixed coil must also produce a current in the coil, showing that a changing magnetic field has to produce an electric field. This we have seen from the field transformation equations. What is the rule or equation that connects the induced electric field with the changing magnetic field? Coulomb's law or Gauss's law gives the equation connecting the electrostatic field and its source—charge distribution. Similarly, there has to be some rule connecting the induced electric field $\boldsymbol{E}_{\text{ind}}$ with its source, the changing magnetic field. And that rule is given by the equation

$$\nabla \times \boldsymbol{E}_{\text{ind}} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{17.1}$$

which is known as Faraday's law of induction. If the magnetic field at a point changes with time, the curl of the electric field there is not zero and is given by the negative rate of change of the magnetic field at that point.

Where does Equation 17.1 come from? From an absolutely new kind of physics, not from anything stated so far for electric and magnetic fields. So take it as a fundamental equation and don't try to derive it from more fundamental rules (unless you use the theory of relativity). However, let us do something which will look like a sequence of logical steps to reach this law.

An induced electric field has no point sources. The field occurs due to a changing magnetic field and does not originate from any point. Thus the divergence of this field should be zero. Let us assume that the curl of this field is also zero everywhere and show that this is wrong. Take all the sources to be confined within a finite volume, so that the induced field at infinity is zero. As you know, a vector field in a region is completely determined if the divergence of the field, the curl of the field and the value at the boundary are given. With all three zero, the only possibility is $\boldsymbol{E}_{\text{ind}} = 0$ everywhere. But that is not the case. Hence,

$$\nabla \times \boldsymbol{E}_{\text{ind}} \neq 0.$$

If $\nabla \times \boldsymbol{E}_{\text{ind}} \neq 0$, what will it depend on? Quite obviously on $\frac{\partial \boldsymbol{B}}{\partial t}$ as it is the changing magnetic field that assures us that there is an induced electric field $\boldsymbol{E}_{\text{ind}}$.

$$\nabla \times \boldsymbol{E}_{\text{ind}} = f\left(\frac{\partial \boldsymbol{B}}{\partial t}\right). \tag{i}$$

Any regular function $f(x)$ can be written as $f(x) = \sum_n c_n x^n$ with c_n as constants and n taking positive, negative or zero values. Now suppose $f(x)$ has the same dimension as x. Then you can have no other term in $f(x) = \sum c_n x^n$ except for $n = 1$, and hence $f(x) = cx$.

Now look at the dimensions of $\nabla \times \boldsymbol{E}$ and $\frac{\partial \boldsymbol{B}}{\partial t}\cdot$ You will find them to be the same. So what is the first guess about the functional form in (i)? The guess would be that $\nabla \times \boldsymbol{E}_{\text{ind}} = k\frac{\partial \boldsymbol{B}}{\partial t}$, where k is a dimensionless constant. The value of k can be obtained by considering a particularly simple situation that will be given as a problem, and you will find that $k = -1$, giving Equation 17.1.

17.4 Calculation of Induced Electric Field

An induced electric field is related to a changing magnetic field. The relation given by Faraday's law is

$$\nabla \times \boldsymbol{E}_{\text{ind}} = -\frac{\partial \boldsymbol{B}}{\partial t}. \tag{i}$$

An induced electric field has no point sources. Hence the divergence of this field is zero.

$$\nabla \cdot \boldsymbol{E}_{\text{ind}} = 0. \tag{ii}$$

The divergence and curl of a vector field (together with the boundary conditions) completely specify the vector field. Compare (i) and (ii) with the equations of magnetostatics

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \tag{iii}$$

and

$$\nabla \cdot \boldsymbol{B} = 0. \tag{iv}$$

Equations (i) and (ii) have the same mathematical structures as (iii) and (iv) respectively. Thus, the induced electric field can be found from $\left[-\frac{1}{\mu_0}\frac{\partial \boldsymbol{B}}{\partial t}\right]$ in the same way as the magnetic field is found from the current density $\boldsymbol{J}$. You can develop a working procedure as follows. Given $\frac{\partial \boldsymbol{B}}{\partial t}$, write the expression for $\left[-\frac{1}{\mu_0}\frac{\partial \boldsymbol{B}}{\partial t}\right]$ and think that it is the expression for a current density. Write what would be the magnetic field for this current density. The expression you write will be that for the induced electric field.

EXAMPLE 17.1 **An infinitely long cylindrical region of radius R contains a magnetic field pointing along its axis. The field is uniform everywhere in the region but its magnitude increases with time as $B = \alpha t$ (Figure 17.6). There is no magnetic field outside the region. Find the induced electric field everywhere.**

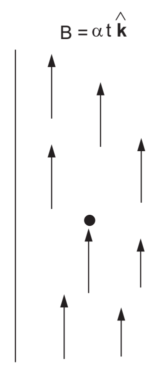


Fig. 17.6

Solution Taking the axis of the cylindrical region to be the z-axis,

$$\boldsymbol{B} = \alpha t \hat{\boldsymbol{k}} \qquad \text{for } s < R$$

and

$$= 0 \qquad \text{for } s > R.$$

Thus

$$-\frac{1}{\mu_0}\frac{\partial \boldsymbol{B}}{\partial t} = -\frac{\alpha}{\mu_0}\hat{\boldsymbol{k}} \qquad \text{for } s < R$$

and

$$= 0 \qquad \text{for } s > R.$$

Now think of a current distribution given by $\boldsymbol{J} = -\left(\frac{\alpha}{\mu_0}\right)\hat{\boldsymbol{k}}$ for $s < R$, $= 0$ for $s > R$. The situation is similar to that of a long cylindrical wire of radius R carrying a uniformly distributed current in the negative z-direction (Figure 17.7). You know the magnetic field due to such a current. What is the direction of the magnetic field in this situation? It is $-\hat{\phi}$. Thus, the induced electric field in the given problem will also be in the $-\hat{\phi}$ direction. The magnetic field lines due to a cylindrical current are circles coaxial with the current. In the given problem, the electric field lines for $\boldsymbol{E}_{\text{ind}}$ will also be circles coaxial with the cylindrical region carrying the magnetic field.

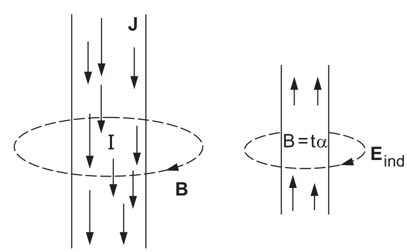


Fig. 17.7

Now come to the magnitude of the induced electric field. If you have an infinitely long, cylindrical wire of radius R carrying a uniformly distributed current I, the magnetic field is

$$B = \frac{\mu_0 I}{2\pi s} \qquad \text{for } s > R$$

and $$= \frac{\mu_0 I s}{2\pi R^2} \qquad \text{for } s < R.$$

Writing in terms of the current density $J = \frac{1}{\pi R^2}$, these expressions are

$$B = \frac{\mu_0 J R^2}{2s} \qquad \text{for } s > R$$

and $$= \frac{\mu_0 J s}{2} \qquad \text{for } s < R.$$

Writing α/μ_0 for J, these same expressions will give the magnitude of the induced electric field for the given problem. So

$$E_{\text{ind}} = \frac{\alpha R^2}{2s} \qquad \text{for } s > R$$

and $$= \frac{\alpha s}{2} \qquad \text{for } s < R.$$

Considering the directions,

$$\boldsymbol{E}_{\text{ind}} = -\frac{\alpha R^2}{2s}\hat{\phi} \quad \text{for } s > R$$

and $$= -\frac{\alpha s}{2}\hat{\phi} \qquad \text{for } s < R.$$

....

Make an important and interesting observation from Example 17.1. The magnetic field is confined to the cylindrical region $s < R$. There is no changing magnetic field outside this region. So $\frac{\partial \boldsymbol{B}}{\partial t} = 0$ for $s > R$. You still have an induced electric field for $s > R$. The changing magnetic field in the cylindrical region $s < R$ has also given rise to an electric field outside the cylindrical region. For the points outside, $\nabla \times \boldsymbol{E}_{\text{ind}}$ will be zero but $\boldsymbol{E}_{\text{ind}}$ will not be zero.

Faraday's law (Equation 17.1) was written in terms of an induced electric field. Indeed, this law is all about such an electric field. However, we need not always carry the subscript ind in this equation. If you write the electric field produced by the static charges as $\boldsymbol{E}_{\text{ch}}$, you have $\nabla \times \boldsymbol{E}_{\text{ch}} = 0$. The electric field at a point may be due to the charges placed somewhere, due to the magnetic field changing somewhere or due to both these sources. In general, you can write the net electric field as

$$\boldsymbol{E} = \boldsymbol{E}_{\text{ch}} + \boldsymbol{E}_{\text{ind}}.$$

Thus, $$\nabla \times \boldsymbol{E} = \nabla \times \boldsymbol{E}_{\text{ch}} + \nabla \times \boldsymbol{E}_{\text{ind}}$$

or $$\nabla \times \boldsymbol{E} = \nabla \times \boldsymbol{E}_{\text{ind}}$$

or $$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}\cdot \tag{17.2}$$

Feel free to use Equation 17.2 as Faraday's law, without worrying where the electric field is coming from.

17.5 The Integral Form of Faraday's Law

Consider any closed curve with an arrow on it to indicate its positive sense (Figure 17.8). This also decides the positive side of the area bounded and hence the direction of any elemental area vector ***da*** on the bounded area. There may be electric and magnetic fields of all varieties in the region. At any point, you will always have

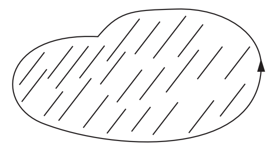


Fig. 17.8

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}\cdot$$

Integrating over the area enclosed by the curve,

$$\int (\nabla \times \boldsymbol{E})\cdot d\boldsymbol{a} = \int \left(-\frac{\partial \boldsymbol{B}}{\partial t}\right)\cdot d\boldsymbol{a}. \tag{i}$$

Using Stokes' theorem,

$$\int (\nabla \times \boldsymbol{E})\cdot d\boldsymbol{a} = \oint \boldsymbol{E}\cdot d\boldsymbol{l},$$

where the line integral is evaluated along the closed curve, that is along the periphery of the area. Thus, equation (i) becomes

$$\oint \boldsymbol{E}\cdot d\boldsymbol{l} = -\frac{\partial}{\partial t}\int \boldsymbol{B}\cdot d\boldsymbol{a}$$

or $$\oint \boldsymbol{E}\cdot d\boldsymbol{l} = -\frac{d}{dt}\Phi, \tag{17.3}$$

where $\boldsymbol{\Phi}$ is the flux of the magnetic field through the area. If this flux is changing with time, you will get a nonzero value of the line integral of the electric field. Equation 17.3 gives you the integral form of Faraday's law. Notice that Equation 17.3 contains $\frac{d}{dt}$ and not $\frac{\partial}{\partial t}$ because $\boldsymbol{\Phi}$ itself is an integral over the area bounded by the loop. So the space-dependence of B is already integrated and $\boldsymbol{\Phi}$ is a function of time only.

17.6 Induced emf

Consider, once again, the magnet–coil experiment described in the beginning of this chapter. As the magnet is moved near the coil, there is an electric current in the coil. It was argued that the magnetic field itself does not exert a force on the free electrons at rest and hence the moving magnet must have produced an electric field which drove the current in the coil. Faraday's law gives the quantitative relation between the changing magnetic field and the electric field produced by it.

The induced electric field $\boldsymbol{E}_{\text{ind}}$ is non-electrostatic in nature and if there is a conductor placed in it, it will drive the charges in a particular direction. Thus it is a source of emf. The driving force on a charge q will be $\boldsymbol{F}_{\text{d}} = q\boldsymbol{E}_{\text{ind}}$.

A conductor, such as a rod, placed in an induced electric field (that is, near a source of a changing magnetic field), therefore, acts like a battery. Charges are separated and maintained at the ends of the conductor due to the force exerted by this induced electric field. The accumulated charges will produce an electrostatic field $\boldsymbol{E}_{\text{ch}}$ inside the conductor and this field will oppose the induced electric field, and at equilibrium, the net electric field inside the conductor will be zero. But there will still be a potential difference between the ends corresponding to the electrostatic field due to the charges accumulated. The value of the emf is given by

$$\varepsilon = \int \frac{\boldsymbol{F}_{\text{d}}}{q} \cdot d\boldsymbol{l} = \int \boldsymbol{E}_{\text{ind}} \cdot d\boldsymbol{l},$$

evaluated along the conductor. Now consider a conducting loop, such as the coil of the magnet–coil experiment. Each portion of the loop is in the induced electric field and hence an emf is created there. The net emf in the circuit is therefore

$$\varepsilon = \oint \boldsymbol{E}_{\text{ind}} \cdot d\boldsymbol{l}.$$

But this quantity is equal to $-d\Phi/dt$, where Φ is the flux of the magnetic field through the area enclosed by the loop (Equation 17.3). Thus, the emf created in the conducting loop due to the changing magnetic field is

$$\varepsilon = -\frac{d\Phi}{dt},$$

which is nothing but the flux rule derived for motional emf in the previous chapter. The emf created due to the induced electric field is called *induced emf*.

If the flux is calculated through a coil having more than one turn, you should multiply $\int \boldsymbol{B} \cdot d\boldsymbol{a}$. by the number of turns.

It is amazing that the flux rule works even if no part of the conducting loop (or the circuit) moves. The physics of the emf is totally different in the two cases. In one case, the conductor moves in a magnetic field and the magnetic force $q\boldsymbol{v}\times\boldsymbol{B}$ results in the emf. In the other case, a nonelectrostatic electric field is produced by a changing magnetic field and it is the electric force $q\boldsymbol{E}_{\text{ind}}$ that results in the emf. But in both the cases, the emf is equal to the negative of rate of change of the magnetic flux through the loop. This makes life simple. You don't have to worry how the emf is produced. The moment you see that the flux through a circuit is changing, you can calculate the emf using the flux rule. Electrical engineers call the emf produced due to a changing magnetic field 'transformer emf'.

EXAMPLE 17.2 **The magnetic field inside a bar magnet is maximum at its centre and remains close to this value except near the ends, where the field diverges. The magnet passes perpendicularly through a coil with a uniform speed [Figure 17.9(a)]. Plot the emf induced in the coil as a function of time as the magnet crosses the coil.**

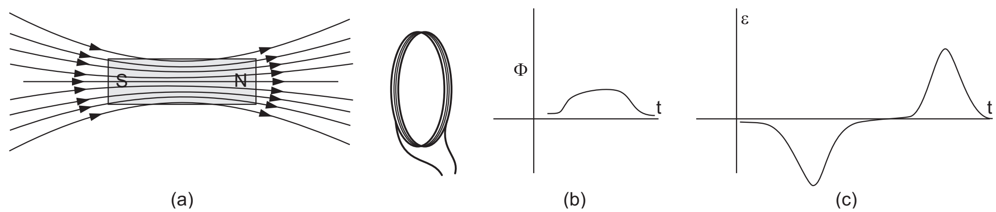


Fig. 17.9

Solution When the magnet is far from the coil, the field at points in the plane of the coil is very small and hence the magnetic flux Φ is also small. As the magnet approaches the coil, the magnetic flux increases slowly in the beginning. When the front edge is close to the coil and enters it, the flux will increase rapidly. After this, the flux will remain almost the same as the field inside the magnet is nearly constant. Again, when the rear end of the magnet comes close to the coil, the field and so the flux decreases rapidly to small values. The plot of the magnetic flux is shown in Figure 17.9(b). The induced emf $\varepsilon = \frac{d\Phi}{dt}$ is obtained from the slope of the $\Phi - t$ curve. Figure 17.9(c) shows the shape of the $\varepsilon - t$ curve.

EXAMPLE 17.3 **A long, tightly wound solenoid with n turns per unit length and radius a carries a current $i = i_0 \sin \omega t$. Surrounding the solenoid is a closed circuit made by joining a bulb to a conducting wire (Figure 17.10). The net resistance of the circuit is R. Find the current through the bulb.**

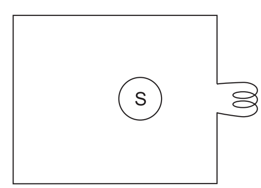


Fig. 17.10

Solution Look at the area enclosed by the circuit. Assume it to be plane and perpendicular to the axis of the solenoid. There is a magnetic field $B = \mu_0 n i$ in the area πa^2 where the solenoid cuts the plane of the circuit. There is no magnetic field outside the area πa^2. So the flux of the magnetic field through the area enclosed by the circuit, at time t, is

$$\Phi = (\mu_0 ni)(\pi a^2) = \mu_0 ni_0 \pi a^2 \sin \omega t.$$

The emf induced at time t is

$$\varepsilon = -\frac{d\Phi}{dt} = -\mu_0 ni_0 \pi a^2 \omega \cos \omega t$$

and the current in the circuit, and hence through the bulb, is

$$I = \frac{\varepsilon}{R} = -\frac{\mu_0 ni_0 \pi a^2 \omega \cos \omega t}{R}.$$

In this example, the current in the circuit will itself produce a magnetic field and that too will change with time. This has not been taken into consideration. In practical situations, this second-generation field is often small. We shall talk about inductance in the next chapter, where the effect of a changing current in a circuit will be discussed.

As the flux rule holds for induced currents caused by a changing magnetic field, so does Lenz's law. The direction of the induced current in the circuit should be such that it opposes the change in the flux of the magnetic field.

Consider the example of a bar magnet moving towards a fixed coil, as shown in Figure 17.11. The north pole is facing the coil. The magnetic field at the site of the coil due to the bar magnet is largely downwards. As the magnet moves down, the magnetic field and hence the magnetic flux through the area increases. According to Lenz's law, a current will be induced in the coil in such a direction that it creates its own magnetic field opposite to the original magnetic field. Thus, this field should be upward in the situation shown in Figure 17.11. Looking at the set-up from above, the current in the coil should be anticlockwise.

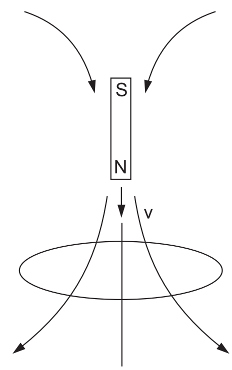


Fig. 17.11

What is the force on the magnet due to the induced current in this situation? The magnetic field lines due to the induced current in the coil have been shown in Figure 17.12. The field will get weaker as you go away from the coil. This inhomogeneous magnetic field exerts a force on the magnet. If you take the pole picture (Chapter 11), the force on the north pole will be larger than that on the south pole, as the north pole is closer to the coil. Also, the force on the north pole is towards the magnetic field (that is, upwards) and the force on the south pole is opposite to the magnetic field (that is, downwards). Thus the net force on the bar magnet due to the current in the coil will be upwards. The coil repels the magnet.

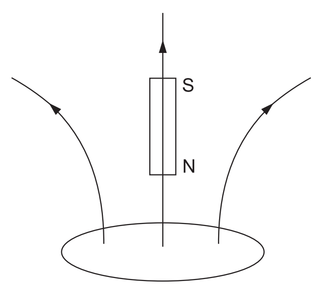


Fig. 17.12

This leads to another interpretation of Lenz's law. A current is induced in the coil because the magnet is moving towards it. The induced current will oppose this cause, meaning that it will repel the magnet and resist its movement towards the coil.

Now suppose the bar magnet has crossed the coil and is going away from it (Figure 17.13). The field at the coil's area, due to the magnet, is still largely downwards, and the flux decreases with time [Figure 17.13(a)]. Thus the current in the coil should be in such a direction that it produces a magnetic field in the direction of the original field, that is downwards. The field lines due to the induced current in the coil are sketched in Figure 17.12(b). The force on the south pole, which is nearer, will be larger than that on the north pole, which is farther from the coil. Also, the force on the south pole will be opposite to the magnetic field (that is, upwards) and that on the north pole will be in the direction of the field (that is, downwards). Hence, the force on the magnet will be upwards. The coil attracts the magnet. Once again, Lenz's law can be used in terms of the force. The current is induced due to the magnet going away from the coil. The induced current will oppose this cause, meaning that it will attract the magnet to resist its going away.

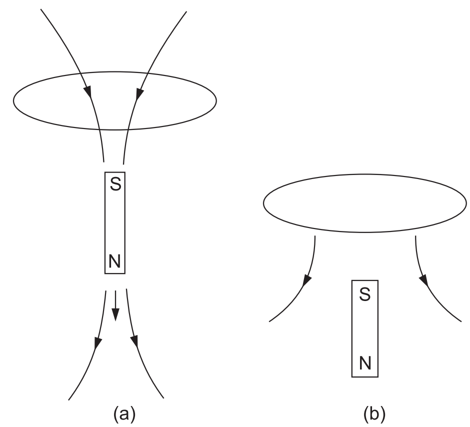


Fig. 17.13

17.7 A Magnet Falling through a Conducting Tube

When a magnet is allowed to fall through a vertical conducting tube, it slows down considerably. This effect has been used for a long time as a dramatic classroom demonstration of Lenz's law. To perform the experiment, you need a straight conducting tube and a short but strong magnet. A copper tube of thickness of a few millimetres and a length of about 25 cm is good enough. Aluminium tubes used for hanging window curtains will also do. The magnet should be able to produce a field of 0.2–0.5 T near its poles. Rare-earth or Fe–Ni–B magnets are easily available, and have the required strength.

Hold the tube vertically in one hand. Drop some unmagnetized objects from the upper end and estimate the time they take to cross the tube and come out from the lower end. For 25 cm of tube, these objects will take about one fifth of a second. Ignoring air resistance, the time is given by $x = \frac{1}{2}gt^2$. Now drop a magnet through the tube. You will be amazed to find that it takes much longer to cross the tube. If your magnet is strong, it may take 10–15 times longer than the unmagnetized objects.

Why does the magnet slow down while falling through the conducting tube? As the magnet moves in the tube, the magnetic field at any given place changes with time. This changing magnetic field creates an electric field and this electric field drives currents in the tube. The magnetic field changes largely in the vertical direction (say $\hat{k}$) and hence, the induced electric field will be largely in the circumferential direction (say $\hat{\phi}$) and will drive currents in the tube in this direction. You can consider the tube to be a stack of conducting rings. At any given instant, there will be some rings below the magnet and some above it. The rings above the magnet will have induced currents which will attract the magnet as the magnet moves from these rings. This means the portion of the tube above the magnet exerts an upward force on the magnet. For the rings below magnet, the magnet is coming closer to the rings. Hence current will be induced in these rings to repel the magnet. This means the portion of the tube below the magnet also exerts an upward force on the magnet. In both cases, the force is upwards and hence the motion is slowed down.

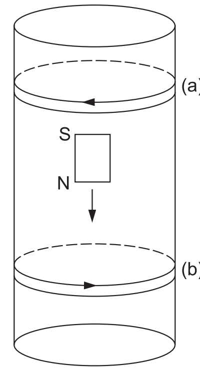


Fig. 17.14

While it is quite easy to qualitatively understand the slowing down of a magnet falling in a vertical conducting tube, finding an expression for the time of fall in terms of the parameters of the tube and the magnet is a challenging task.

17.8 Eddy Currents

In most of the examples taken in this and the previous chapters, we provided a well-defined circuit in which current was induced due to the changing magnetic field or due to the motion of a part of the circuit. But there are situations where emf is produced in some portion of a conductor that provides an infinite number of conducting paths for current to flow. The system then decides the current distribution in the conductor, which is not easy to visualize. The currents in such situations are commonly called *eddy currents*. An interesting demonstration experiment involving eddy current is given below.

Take an aluminium plate and paste an A4 paper on it. Take a ply board of the same dimensions and paste a similar A4 paper on it. Using some supports, place them side by side at the same inclination. Take a strong cylindrical magnet and place it with a flat end on the inclined ply board. If the magnet does not slide or slides slowly, increase the angle of inclination so that it slides reasonably fast. Keep the other board also at the same inclination.

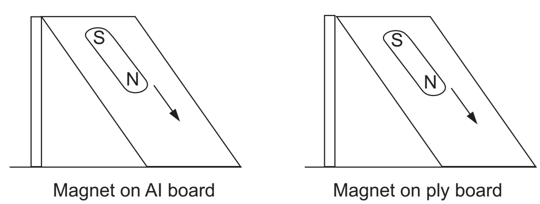


Fig. 17.15

Now put the magnet on the aluminium incline. Enjoy watching the magnet coming down quite slowly.

This slowing down is because of eddy currents induced in the aluminium plate. As the magnet slides, it creates an electric field that drives currents in the plate. There currents produce their own magnetic field that exerts a force on the magnet opposing its downward sliding. A4 papers were pasted on both the boards to ensure the same friction coefficient in the two cases.

17.9 Potentials for Time-Varying Fields

Faraday's law is

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} = -\frac{\partial (\nabla \times \boldsymbol{A})}{\partial t} = -\nabla \times \left(\frac{\partial \boldsymbol{A}}{\partial t}\right).$$

or
$$\nabla \times \left(\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t}\right) = 0.$$

This shows that $\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t}$ can be written as the gradient of a scalar function. Writing this function as $-V(\boldsymbol{r})$,

$$\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} = -\nabla V$$

or
$$\boldsymbol{E} = -\nabla V - \frac{\partial \boldsymbol{A}}{\partial t}.$$

So the electric and magnetic fields can be obtained from a scalar potential V and a vector potential $\boldsymbol{A}$. While $\boldsymbol{B}$ is obtained from $\boldsymbol{A}$ alone, to get $\boldsymbol{E}$ you need both V and $\boldsymbol{A}$. The part of the electric field coming from charges is given by the $(-\nabla V)$ term and the one coming from the changing magnetic field is given by $\left(-\frac{\partial \boldsymbol{A}}{\partial t}\right)$.

EXAMPLE 17.4 **Solve the problem stated in Example 17.1 by getting the vector potential and differentiating it.**

Solution The magnetic field is given by

$$\begin{aligned} \boldsymbol{B} &= \alpha t \hat{\boldsymbol{k}} \quad \text{for } s < R \\ &= 0 \quad \text{for } s > R. \end{aligned}$$

To get the vector potential $\boldsymbol{A}$ from $\boldsymbol{B}$, you can treat $\frac{\boldsymbol{B}}{\mu_0}$ as a current density and write the expression of the magnetic field. This expression will actually be the expression for the vector potential. Now consider a current density

$$\boldsymbol{J} = \left(\frac{\alpha t}{\mu_0}\right)\hat{\boldsymbol{k}} \quad \text{for } s < R$$
$$= 0 \qquad \text{for } s > R.$$

It is equivalent to a current $I = \dfrac{\alpha t \pi R^2}{\mu_0}$ in a cylindrical wire of radius R. The magnetic field for this current, for $s < R$, is

$$\boldsymbol{B} = \frac{\mu_0 I s}{2\pi R^2}\hat{\phi} = \frac{\alpha t}{2} s\hat{\phi}$$

and for $s < R$,

$$\boldsymbol{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi} = \frac{\alpha t R^2}{2s}\hat{\phi}.$$

Thus the vector potential for the given situation is

$$\boldsymbol{A} = \frac{\alpha s}{2} t\hat{\phi} \text{ for } s < R$$
$$= \frac{\alpha R^2}{2s} t\hat{\phi} \text{ for } s > R.$$

The induced electric field is

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t}$$
$$= -\frac{\alpha s}{2}\hat{\phi} \text{ for } s < R$$
$$= -\frac{\alpha R^2}{2s}\hat{\phi} \text{ for } s > R.$$

EXAMPLE 17.5 **The scalar potential and vector potential in a region are given by $V(\boldsymbol{r}, \boldsymbol{t}) = \dfrac{K_1 \cos\theta}{r^2}$ and $A(\boldsymbol{r}, \boldsymbol{t}) = \dfrac{K_2}{r^2} \cos \omega t \sin\theta\hat{\phi}$, respectively, as expressed in spherical coordinates. Find the electric and magnetic fields at r at time t.**

Solution

$$E = -\nabla V - \frac{\partial \boldsymbol{A}}{\partial t} = \frac{K_1}{r^3}(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\theta}) + \frac{K_2\omega}{r^2}\sin\omega t \sin\theta\hat{\phi}.$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = K_2 \cos\omega t\left[\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\sin^2\theta\right)\frac{1}{r^2}\hat{\boldsymbol{r}} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\sin\theta}{r^2}\right)\hat{\theta}\right]$$

$$= K_2\cos\omega t\left[\frac{2\cos\theta}{r^3}\hat{\boldsymbol{r}} + \frac{\sin\theta}{r^3}\hat{\theta}\right].$$

17.10 Can Voltage Depend on the Position of the Meter?

Figure 17.16 shows a PVC pipe and a solenoid. Put as many bicycle spokes in the pipe as you can. If AC voltage is applied between the two ends of the solenoid, a sinusoidal magnetic field will be created inside it. Because of the spokes, the field will be stronger and will be extended with less divergence beyond the solenoid.

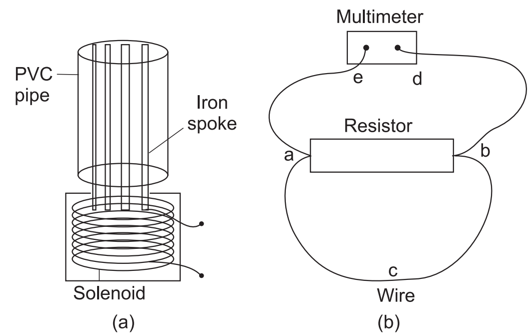


Fig. 17.16

Next take a resistor, say of 100 Ω, and connect its ends to those of a small wire *acb* [Figure 17.16(b)]. Now connect the junctions *a* and *b* to the multimeter with two long wires. It would be best to solder the wires at the junctions.

After that, put this resistance–wire loop around the PVC pipe—the multimeter should be on the resistance side [Figure 17.17(a)]. Fix the loop with the help of cellophane tapes.

Put the multimeter in AC-voltage mode. Switch on the power for a few seconds so that there is AC current in the solenoid and note the voltage displayed on the multimeter. Turn off the power.

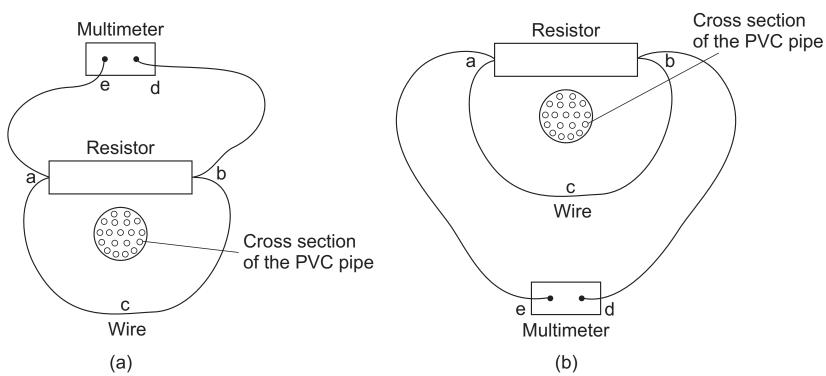


Fig. 17.17

Now, keeping the resistance–wire loop fixed in position, lift the multimeter and take it to the other side of the pipe [Figure 17.17(b)]. Switch on the power to give the same AC current to the solenoid for a few seconds and note the voltage displayed on the multimeter. You will find a much smaller voltage displayed this time.

The solenoid, current and everything else are the same. The points *a* and *b* are fixed. The multimeter is connected to the same points. Only by changing the position of the multimeter is the voltage displayed changed considerably.

Explanation When the multimeter is connected to the points a and b through the wires ae and bd, it measures $\int_a^b \boldsymbol{E}\cdot d\boldsymbol{l}$ on the path $aedb$. In the situation shown in Figure 17.17(a), the loop $aedba$ encloses an area where there is no magnetic field. Hence

$$\oint_{aedba} \boldsymbol{E}\cdot d\boldsymbol{l} = 0$$

or $$\oint_{aedb} \boldsymbol{E}\cdot d\boldsymbol{l} = \int_{ab} \boldsymbol{E}\cdot d\boldsymbol{l}.$$

But $$\oint_{ab} \boldsymbol{E}\cdot d\boldsymbol{l} = IR_1.$$

So, $$\int_{aedb} \boldsymbol{E}\cdot d\boldsymbol{l} = IR_1,$$

where R_1 is the resistance of the resistor together with the connecting wires ea and bd, and I is the current.

Thus, IR_1 is the voltage displayed in the situation shown in Figure 17.16(a).

Now consider the situation shown in Figure 17.16(b). Look at the closed loop $aedbca$. There is no magnetic field in the area bounded by this loop. So

$$\oint_{aedbca} \boldsymbol{E}\cdot d\boldsymbol{l} = 0$$

or $$\int_{aedb} \boldsymbol{E}\cdot d\boldsymbol{l} = \int_{acb} \boldsymbol{E}\cdot d\boldsymbol{l}.$$

If the wire acb has a resistance R_2,

$$\int_{acb} \boldsymbol{E}\cdot d\boldsymbol{l} = IR_2.$$

So, $$\int_{aedb} \boldsymbol{E}\cdot d\boldsymbol{l} = IR_2.$$

This is the voltage displayed in the situation represented by Figure 17.16(b). As the resistance R_2 is much smaller than R_1, the voltage displayed in the situation shown in Figure 17.17(b) is much smaller than that shown in Figure 17.17(a).

17.11 Extension of Kirchhoff's Law

Kirchhoff's loop law is essentially formulated for circuits having a conservative electric field for which potential can be defined. In that case the sum of potential drops across different elements in a closed circuit is zero. The potential drop across a resistor is Ri, across a capacitor is q/C, across a voltage source (such as a battery) is equal (in magnitude) to its emf ε, and so on. The law is stated as

$$\sum_i (\Delta V)_i = 0 \tag{i}$$

for a closed loop. You can separate the voltage drops on the voltage sources and other elements and write the law as

$$\sum_j (\Delta V_j) = \sum_k \varepsilon_j, \tag{ii}$$

where k runs over the voltage sources in the loop and j runs over the other elements such as resistors and capacitors.

If the loop has only resistors and voltage sources, you can write equation (ii) as

$$\sum_i R_j i_j = \sum_k \varepsilon_j$$

or

$$\sum_i R_j i_j = \varepsilon, \tag{17.4}$$

where ε is the total emf in the loop. If you have capacitors, appropriate terms like q/c are added on the left-hand side of this equation.

Kirchhoff's loop law in the form of Equation 17.4 is also applicable if you have a time varying magnetic field producing a nonconservative electric field which results in an induced emf $\oint \vec{E} \cdot \vec{dl}$ in the loop.

Take the example of the circuit shown in Figure 17.18, where a time-varying magnetic field exists inside the loop. You cannot define potential drop across the resistor AB as $\oint \vec{E} \cdot \vec{dl}$ is not path independent. If you go from A to B via the path ACB, $\int \vec{E} \cdot dl$ is zero. But if you go through the resistor, it is Ri. If you still treat Ri as voltage drop across resistor, the total voltage drop as you traverse the loop ABCA, will not be zero as demanded by the usual form of Kirchhoff's loop law [equation (i)]. But you can safely apply Equation 17.4 and write

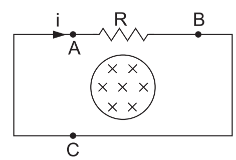


Fig. 17.18

$$Ri = -\frac{d\phi}{dt}.$$

You can use Equation 17.4 to represent Kirchhoff's loop law and apply it freely even if there are changing magnetic fields.

Concepts at a Glance

1. A changing magnetic field produces an electric field called induced electric field.
2. Induced electric field $\vec{E}$ satisfies the equation $\nabla \times \boldsymbol{E}_{\text{ind}} = -\frac{\partial \boldsymbol{B}}{\partial t}$.
3. Faraday's law: $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$.
4. $\oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d}{dt}\Phi$, where $\Phi = \int \boldsymbol{B} \cdot d\boldsymbol{a}$.
5. The emf in a loop due to an induced electric field is given by $\varepsilon = \oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d\Phi}{dt}$.
6. A magnet moving near a metal causes eddy currents in the metal, and these currents slow it down.

EXERCISES

Based on Concepts

1. Which of the following statements is always true for an induced electric field?
   (a) $\oint \boldsymbol{E} \cdot d\boldsymbol{l} = 0$ (b) $\oint \boldsymbol{E} \cdot d\boldsymbol{l} \neq 0$ (c) $\oint \boldsymbol{E} \cdot d\boldsymbol{l}$ may be zero on some closed curves and nonzero on some other closed curves.
2. If a strong magnet is dropped through a vertical solenoid (carrying no current) whose ends are joined, will the magnet slow down?
3. What is the value of $\oint \left(\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} \right) \cdot d\boldsymbol{l}$ for a closed loop?
4. A circular, metallic loop lies in the x–y plane with its centre at the origin. Take the positive normal to the loop in the z-direction. A coil is carrying a current in the plane $z = z'$ with its centre at $(0, 0, z')$. An observer stationed at $(0, 0, 2z')$ finds the current in the coil to be anticlockwise. The coil is slowly pushed towards the origin. (a) Is the flux $\Phi = \int \boldsymbol{B} \cdot d\boldsymbol{a}$ through the area enclosed by the loop positive or negative? (b) Is $|\Phi|$ increasing or decreasing? (c) Is $\frac{d\Phi}{dt}$ positive or negative? (d) Is the induced emf ε positive or negative? (e) Is the current induced in the loop clockwise or anticlockwise, as seen by the observer?
5. At time $t = 0$, a magnet SN is situated along the axis of a conducting loop of resistance R. The loop itself moves with a very small velocity v along the axis, as shown in Figure 17E.1. Taking the normal towards the right to be positive, the flux of the magnetic field through the ring at $t = 0$ is Φ_0. Neglect any relativistic change in time, length, etc. (a) Draw a diagram showing the magnet and the ring at time $t = \Delta t$. Let the flux of the magnetic field through the ring be Φ.

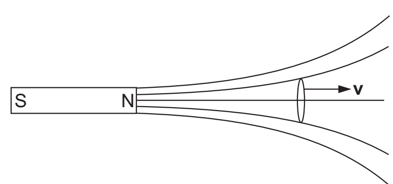


Fig. 17E.1

(a) Is the current in the ring clockwise or anticlockwise, as seen from the right?

(b) Consider the same situation from the ring frame and draw a diagram showing the ring and the magnet at $t = \Delta t$. The current in the ring and the resistance remain the same in this frame for small values of v. What is the emf driving this current?

(c) In the ring frame, the emf is given by $\oint \boldsymbol{E}_{\text{ind}} \cdot d\boldsymbol{l}$. Using the result obtained in part (b), show that $\oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d\Phi}{dt}$.

(d) From the result obtained in part (c), show that $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$.

Problems

1. A circular coil of radius a having n turns is located in the field of an electromagnet. The field is perpendicular to the coil and its strength B_0 is uniform over the area. The coil is connected to an external resistor. The total resistance of the circuit is R. Find the total charge Q passing through the resistor when the electromagnet is turned off. [**Ans.** $\frac{n\pi a^2 B_0}{R}$]

2. A long solenoid of radius a has n turns per unit length and is surrounded by a loop of wire connected to a resistance R (Figure 17E.2). (a) If the current in the solenoid is increased at a constant rate $dI/dt = k$, determine the current flowing in the loop. (b) If the current I in the solenoid is held constant but the solenoid is pulled out of the loop and reinserted from the opposite direction, what is the total charge that passed through R? [**Ans.** (a) $\mu_0 nk\pi a^2/R$, (b) $2\mu_0 nI\pi a^2/R$]

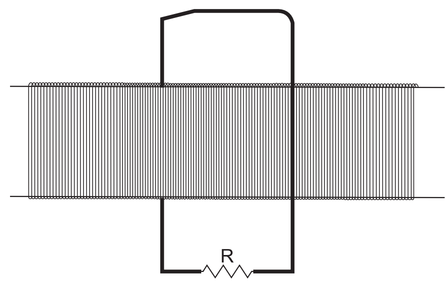


Fig. 17E.2

3. A short magnet of magnetic moment M is moving at a constant velocity v along its axis. A conducting ring of radius R is kept coaxially with the magnet (Figure 17E.3). Consider the situation in which the distance x between the ring and the magnet is much larger than R and find the emf induced in the ring. [**Ans.** $\frac{3\mu_0 R^2 M}{2x^4} v$]

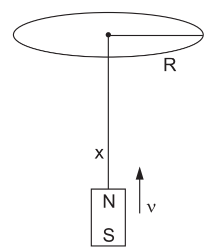


Fig. 17E.3

4. A battery of emf ε_0 is connected to a resistance R, as shown in Figure 17E.4. The circuit encloses a solenoid of number of turns n per unit length and area of cross section A. The current I in the solenoid produces a magnetic field going into the plane of the circuit. At what rate should the current in the solenoid change so that there is no current in the circuit? Should the current be increased or decreased? [**Ans.** $\frac{\varepsilon_0}{\mu_0 nA}$, decreased]

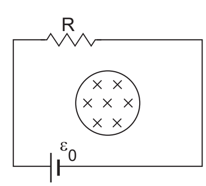


Fig. 17E.4

5. A rectangular, conducting loop with edge lengths l and b is rotated about the line midway between the longer edges (length l), as shown in the figure. This line is taken as the x-axis. A magnetic field $\boldsymbol{B} = B_0 \sin \omega t \hat{\boldsymbol{j}}$ exists in the region. At $t = 0$, the loop lies in the x–z plane. Find the induced emf in the loop. [**Ans.** $B_0 lb\omega \cos 2\omega t$]

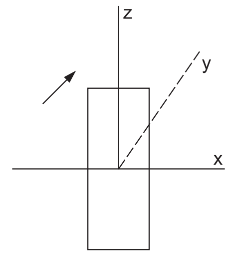


Fig. 17E.5

6. A long solenoid of radius R having n turns per unit length carries a current $I = I_0(1 + \alpha t)$. Find the induced electric field everywhere. [For $s < R$, $-\dfrac{nI_0\alpha s}{2}\hat{\phi}$]

7. Two coaxial solenoids have the same number of turns n per unit length and radii a and $2a$. The innner solenoid has a current varying with time as $I_0 kt$ and the outer solenoid has one equalling $2I_0 kt$ going in the same sense. Find the electric field at a distance (a) $\dfrac{a}{2}$, (b) $\dfrac{3a}{2}$, (c) $\dfrac{5a}{2}$ from the common axis.

[**Ans.** (a) $\dfrac{-9}{8}\alpha\hat{\phi}$, (b) $\dfrac{-33}{12}\alpha\hat{\phi}$, (c) $\dfrac{27}{10}\alpha\hat{\phi}$, where $\alpha = \mu_0 I_0 ka^2$]

8. The magnetic field in the cylindrical region $s < R$ is given by $\boldsymbol{B} = B_0(1 - e^{-t/\tau})\hat{\boldsymbol{k}}$, where τ is a positive constant. There is no magnetic field outside the cylinder. P_1 and P_2 are two points at distances s and $2s$ from the axis of the cylinder and outside it. (a) Find the direction of the induced electric field $\boldsymbol{E}$ at P_1 due to this magnetic field. Answer in terms of $\hat{\boldsymbol{s}}, \hat{\phi}, \hat{\boldsymbol{k}}$ with proper signs. (b) Find the ratio $\dfrac{|E| \text{ at } P_1}{|E| \text{ at } P_2}$ at time t. (c) Find the ratio $\dfrac{|E| \text{ at } t = 0}{|E| \text{ at } t = \tau}$ at point P_1. [**Ans.** (a) $-\hat{\phi}$, (b) 2, (c) e]

9. A magnetic field $\boldsymbol{B} = B_0 \cos \omega t \hat{\boldsymbol{k}}$ exists in a long, cylindrical region $s \le a$. A circular, conducting loop of radius b surrounds the field region coaxially. Though the thickness of the wire forming the loop is uniform, the conductivity is σ_1 for half the loop $(0 < \phi < \pi)$ and σ_2 for the other half $(\pi < \phi < 2\pi)$ (Figure 17E.6). Find the electric field in the ring (neglecting any self-inductance).

[**Ans.** $\boldsymbol{E} = \dfrac{\sigma_2 K}{\sigma_1 + \sigma_2}\hat{\phi}$ for $0 < \phi < \pi$, and $\dfrac{\sigma_1 K}{\sigma_1 + \sigma_2}\hat{\phi}$ for $\pi < \phi < 2\pi$, where $K = \dfrac{a^2 B_0 \omega \sin \omega t}{b}$]

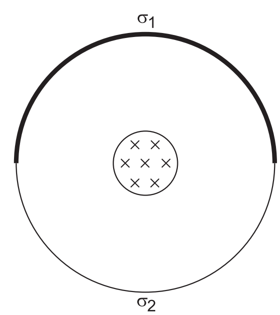


Fig. 17E.6

10. A magnetic field exists in a cylindrical region of radius 5 cm and increases at a rate of 0.1 T/s. AB is a line of length 50 cm, at a closest distance of 25 cm from the axis of the cylindrical region as shown in Figure 17E.7. Find $\int_A^B \boldsymbol{E} \cdot \boldsymbol{dl}$. [**Ans.** $\approx 2.0 \times 10^{-4}$ V]

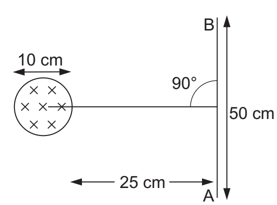


Fig. 17E.7

11. A magnetic field $\boldsymbol{B} = B_0 \cos \omega t \hat{\boldsymbol{k}}$ exists in a long, cylindrical region $s \le a$. Consider a rectangular path *ABCD* in the *x–y* plane, with the centre of the rectangle at the origin (Figure 17E.8). Let the length $AB = l$ and $BC = b$. Find (a) $\int_A^B \boldsymbol{E} \cdot \boldsymbol{dl}$, (b) $\int_B^C \boldsymbol{E} \cdot \boldsymbol{dl}$, (c) $|\boldsymbol{E}|$ at *A*.

[**Ans.** (a) $K \tan^{-1}\left(\frac{l}{b}\right)$, (b) $K \tan^{-1}\left(\frac{b}{l}\right)$, (c) $\frac{K}{\sqrt{l^2 + b^2}}$, where $K = B_0 a^2 \omega \sin \omega t$]

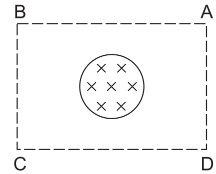


Fig. 17E.8

12. Consider the situation described in the previous problem. Suppose a uniform frame made of copper is placed to fit *ABCD*. What is the value of $\int_A^B \boldsymbol{E} \cdot \boldsymbol{dl}$? Why is it different from that found in the previous problem? Will $\int \boldsymbol{E} \cdot \boldsymbol{dl}$ over the rectangle *ABCD* be the same or different in the two cases?

[**Ans.** $\frac{\pi K l}{2(l + b)}$, the same]

13. A long, cylindrical solenoid carries a slowly varying current and produces a magnetic field in the negative *z*-direction. A uniform conducting square loop *ABCD* of side 2*L* is placed symmetrically around it. The rate of change of the magnetic flux through the square is λ. Consider the point *P* on the loop at a distance *y* from the midpoint of the side *BC*, as shown in Figure 17E.9. Take the axes as shown in the figure. (a) Find the induced electric field $\boldsymbol{E}_1$ at *P*. (b) Find the net electric field $\boldsymbol{E}_2$ at *P*. (c) Write the residual field $\boldsymbol{E}_r = \boldsymbol{E}_2 - \boldsymbol{E}_1$. What is the source of this field? (d) Show that $\oint \boldsymbol{E}_r \cdot \boldsymbol{dl} = 0$. [**Ans.** (a) $\frac{\lambda(-y\hat{\boldsymbol{i}} + L\hat{\boldsymbol{j}})}{2\pi(L^2 + y^2)}$, (b) $\frac{\lambda}{8L}\hat{\boldsymbol{j}}$]

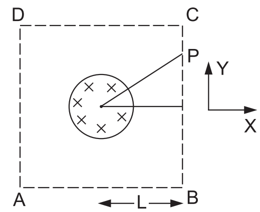


Fig. 17E.9

14. An infinite surface current confined to the *x–y* plane is given by $\boldsymbol{K} = k_0(1 + \alpha t)\hat{\boldsymbol{j}}$. Find the induced electric field everywhere. [**Ans.** $\frac{\mu_0 k_0 \alpha |z|}{2}\hat{\boldsymbol{j}}$]

15. Two large metal sheets are placed at $z = a/2$ and $z = -a/2$. The first sheet carries a surface current $\boldsymbol{K} = K_0(1 - e^{-t/\tau})\hat{\boldsymbol{i}}$ and the other, $\boldsymbol{K} = -K_0(1 - e^{-t/\tau})\hat{\boldsymbol{i}}$. Find the induced electric field everywhere.

[**Ans.** $-\frac{\lambda a}{2}\hat{\boldsymbol{i}}$ for $z > \frac{a}{2}$, $\frac{\lambda a}{2}\hat{\boldsymbol{i}}$ for $z < \frac{a}{2}$, and $\lambda z \hat{\boldsymbol{i}}$ for $-\frac{a}{2} < z < \frac{a}{2}$, where $\lambda = \frac{\mu_0 K_0}{\tau}$]

16. A toroidal coil with inner radius *a*, outer radius *b*, and height *h* has a rectangular cross section. It has a total of *N* tightly wound turns and carries a current $I = I_0 + \alpha t$ and is placed horizontally. If the width and the height are both much less than *a*, find the electric field at a point *z* above the centre of the toroid.

17. A circular wire loop of radius 4*a* has a resistance 4*R*. A long solenoid of radius *a* carrying a current is placed coaxially with the wire loop. The magnetic field in the solenoid varies as $\boldsymbol{B} = kt\hat{\boldsymbol{k}}$. Two points *A* and *B* of this loop are connected to a resistance 2*R* through thick wires (Figure 17E.10).

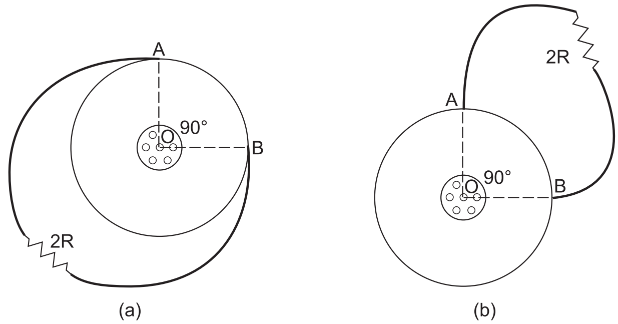


Fig. 17E.10

The arc AB makes an angle of 90° at the centre. Find the current I in the resistance $2R$ in the two situations shown. [**Ans.** (a) $\frac{3k\pi a^2}{11R}$ (b) $\frac{k\pi a^2}{11R}$]

18. A long, straight wire of radius R carries a current I, which varies slowly with time. Find the induced electric field. You can write the vector potential and from that get the induced electric field.

[**Ans.** $\frac{-\mu_0(R^2 - s^2)}{4\pi R^2}\frac{dI}{dt}\hat{k}$ for $s < R$ and $\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln\left(\frac{s}{R}\right)\hat{k}$ for $s > R$]

19. The magnetic field in a region is given by $\boldsymbol{B} = \frac{B_0}{s}\hat{\phi}$. Consider a square loop of edge a and resistance R placed coplanarly with the z-axis, with its near edge at a distance of a from the z-axis. If the magnetic field is suddenly made zero, how much charge flows through any cross section of the loop? In which direction does the current flow? [**Ans.** $\frac{B_0 a^2 \ln 2}{R}$]

20. A very long solenoid of a small radius a, with n turns per unit length, carries a current $I = I_0 + \alpha t$. A circular ring of a much larger radius b, having a resistance R, is placed coaxially with the solenoid. Find the value of $\int \boldsymbol{E}\cdot\boldsymbol{J}\,d\tau$ over the volume of the ring. Here $\boldsymbol{E}$ is the electric field and $\boldsymbol{J}$ is the current density in the ring. How does the value of this integral compare with the Joule heating of the ring?

[**Ans.** $\frac{(\alpha\mu_0 n\pi a^2)^2}{R}$]

21. A spherical shell of radius R carries a surface current $\boldsymbol{K}(t) = (K_0 + \alpha t)\sin\theta\hat{\phi}$ in spherical coordinates with the centre of the shell as the origin. Assume α to be small so that the current varies slowly. (a) Find the vector potential $\boldsymbol{A}(t)$. (b) Find the induced electric field.

[**Ans.** (b) $-\frac{1}{3}\mu_0\alpha r\sin\theta\hat{\phi}$ for $r < R$ and $-\frac{1}{3}\mu_0 R^3\alpha\frac{\sin\theta}{r^2}\hat{\phi}$ for $r > R$]

22. A strong but short magnet is dropped down a metallic pipe. Assume that the magnet acquires its terminal velocity at a very short distance. It is found that the time of fall is inversely proportional to the fourth power of the inner diameter of the pipe. Do a dimensional analysis and suggest how it should depend on the thickness of the pipe.

23. A ring of radius a carries a uniformly distributed charge Q. A magnetic field perpendicular to the plane of the ring is switched on. The field is confined to a coaxial cylindrical volume of radius $a/3$ and its final value is B_0. Assuming that only electromagnetic forces exist, find the angular momentum the ring acquires about its axis. [**Ans.** $\frac{QBa^2}{18}$, opposite of the direction B_0]

24. A long, vertical solenoid of radius R_1 and n turns per unit length carries a current I in the anticlockwise direction as seen from above. An annular disk of inner and outer radii R_2 and R_3 is joined to the solenoid surface through light spokes and can rotate freely about the central axis. The disk has a mass M and carries a uniformly distributed charge Q on its surface. The current is switched off. Find the angular velocity of the disk. Will it rotate clockwise or anticlockwise? [**Ans.** $\frac{\mu_0 nIqR_1^2}{2M(R_3^2+R_2^2)}$, anticlockwise]

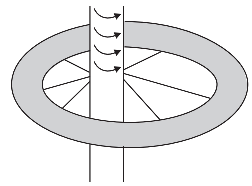


Fig. 17E.11

25. A thick conducting wire with a radius a is connected to a resistance R making a loop which is pulled at a constant velocity v by applying a force on one side as shown in the figure. One arm of the loop moves through a uniform magnetic field B_0 perpendicular to the loop as shown. (a) If there are n free electrons per unit volume in the wire, find the drift speed of these electrons in the loop. (b) Find the horizontal and vertical components of magnetic force on a free electron drifting in the segment ab with this drift speed. [**Ans.** (a) $\frac{B_0 vl}{Rne\pi a^2}$, (b) $\frac{B_0^2 vl}{Rn\pi a^2}$, evB_0]

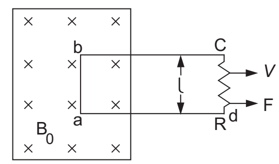


Fig. 17E.12

26. A sphere of radius R has a uniform surface charge density σ on its surface. It rotates about a diameter (taken as the z-axis) with a slowly varying angular velocity ω. Find the net electric field inside the sphere. [**Ans.** $-\frac{\mu_0 R\sigma r \sin\theta}{3}\frac{d\omega}{dt}\hat{\phi}$]

27. A coaxial cable has an inner wire of radius a and an outer shell of inner radius b. The current I passes along the inner wire and returns along the outer surface. Both currents are distributed uniformly on the surfaces. I varies slowly with $\alpha = \frac{dI}{dt}\cdot$ Find the induced electric field.

[**Ans.** $E = \frac{\mu_0\alpha}{2\pi}\ln\frac{s}{b}$ for $a < s < b$, $\frac{\mu_0\alpha}{2\pi}\ln\frac{a}{b}$ for $s < a$ and $s > b$]

❑

18 Inductance and Magnetic Field Energy

The flux rule tells us that whenever the magnetic flux through a closed loop changes, an emf is induced in the loop. If you change the current in a circuit, the corresponding magnetic field changes. Hence the flux through this circuit and also through any nearby circuit changes. This induces emfs in these circuits. The emfs depend on the geometrical shapes of these circuits and how they are placed, among other factors. This phenomenon is used in all kinds of transformers, tuning circuits, and so on. Let us discuss this in detail in the following sections.

18.1 Mutual Inductance

Consider two closed loops 1 and 2 placed near each other as shown in Figure 18.1. If you pass a current I_1 in Loop 1, say by connecting a battery in the loop, it will produce a magnetic field. This magnetic field will have a flux $\Phi_2 = \int \boldsymbol{B} \cdot \boldsymbol{da}_2$ through Loop 2. If you change I_1, the magnetic field will change and hence the flux Φ_2 will also change, and an emf will be produced in Loop 2, driving a current in it.

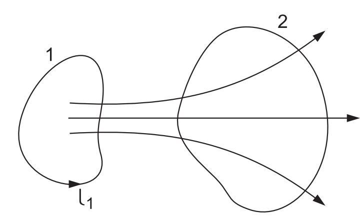


Fig. 18.1

According to the Biot–Savart law, the magnetic field at any point due to I_1 will be proportional to I_1. Hence the flux Φ_2 through Loop 2 will also be proportional to I_1. The proportionality constant between the two is written as M_{21}. Thus

$$\Phi_2 = M_{21} I_1. \tag{18.1}$$

You can get a nice expression for M_{21} using magnetic vector potential. Write the magnetic field due to I_1 as $\boldsymbol{B}_1(\boldsymbol{r})$ and the corresponding vector potential as $\boldsymbol{A}_1(\boldsymbol{r})$. Then,

$$\boldsymbol{B}_1(\boldsymbol{r}) = \nabla \times \boldsymbol{A}_1(\boldsymbol{r}).$$

The flux through Loop 2 is

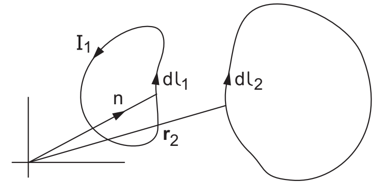


Fig. 18.2

$$\Phi_2 = \int \boldsymbol{B}_1(\boldsymbol{r}_2) \cdot \boldsymbol{da}_2 = \int [\nabla \times \boldsymbol{A}_1(\boldsymbol{r}_2)] \cdot \boldsymbol{da}_2$$

$$= \oint A_1(\boldsymbol{r}_2) \cdot \boldsymbol{dl}_2. \tag{i}$$

The last integral is a line integral evaluated over Loop 2. The line element $\boldsymbol{dl}_2$ is taken along the loop itself and $\boldsymbol{r}_2$ represents the point on Loop 2 where the line element $\boldsymbol{dl}_2$ is taken.

The vector potential $\boldsymbol{A}_1(\boldsymbol{r})$ corresponds to the current I_1 in Loop 1. You know how to relate them. You can write

$$\boldsymbol{A}_1(\boldsymbol{r}) = \oint_1 \frac{\mu_0}{4\pi}\frac{I_1 \boldsymbol{dl}_1}{|\boldsymbol{r}-\boldsymbol{r}_1'|},$$

where $\boldsymbol{dl}_1$ is the line element on Loop 1 and the integration is to be performed on the whole of Loop 1. Thus, the flux through Loop 2 due to the current I_1 in Loop 1 is, from (i),

$$\Phi_2 = \oint_2 \left(\oint_1 \frac{\mu_0}{4\pi}\frac{I_1 \boldsymbol{dl}_1}{|\boldsymbol{r}_2-\boldsymbol{r}_1|} \right)\cdot \boldsymbol{dl}_2$$

$$= \frac{\mu_0 I_1}{4\pi}\oint_2\oint_1 \frac{\boldsymbol{dl}_1\cdot\boldsymbol{dl}_2}{|\boldsymbol{r}_2-\boldsymbol{r}_1|}.$$

Using the definition of M_{21}, that is, $\Phi_2 = M_{21}I_1$,

$$M_{21} = \frac{\mu_0}{4\pi}\oint_2\oint_1 \frac{\boldsymbol{dl}_1\cdot\boldsymbol{dl}_2}{|\boldsymbol{r}_2-\boldsymbol{r}_1|}. \tag{18.2}$$

Considering the same set-up as shown in Figure 18.1, suppose you send a current I_2 in Loop 2 and look at the flux Φ_1 of the magnetic field, through Loop 1. This flux will be proportional to I_2 and you can write the proportionality constant as M_{12}. So,

$$\Phi_1 = M_{12}I_2.$$

The roles of 1 and 2 are interchanged and you can do the same analysis as above, to get an expression for M_{12} similar to that in Equation 18.2. Interchanging 1 and 2 in this equation,

$$M_{12} = \frac{\mu_0}{4\pi}\oint_1\oint_2 \frac{\boldsymbol{dl}_2\cdot\boldsymbol{dl}_1}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}. \tag{18.3}$$

The expressions for M_{12} and M_{21} are equal because $\boldsymbol{dl}_1\cdot\boldsymbol{dl}_2 = \boldsymbol{dl}_2\cdot\boldsymbol{dl}_1$ and $|\boldsymbol{r}_1-\boldsymbol{r}_2| = |\boldsymbol{r}_2-\boldsymbol{r}_1|$. The order of integration does not matter either as the two loops are independent of each other for the purpose of integration. Thus,

$$M_{21} = M_{12}. \tag{18.4}$$

Pass a current I in Loop 1 and look at the flux Φ of the magnetic field through Loop 2. Now switch off this current and pass the same current in Loop 2. Equation 18.4 says that you will get the same flux Φ through Loop 1.

The proportionality constant $M_{21} = M_{12} = M$ is called the *mutual inductance* between the two loops. If a current I is sent in any of the loops, the flux through the other loop will be

$$\Phi = MI. \tag{18.5}$$

It is not easy to make calculations using the expression for mutual inductance given in Equation 18.2 or 18.3. However, it helps us reach an important conclusion—that the mutual inductance between two loops depends only on geometrical parameters.

What is the SI unit of mutual inductance? Flux is measured in weber and so M has the SI unit weber/ampere. This unit is given the name "henry" and is denoted by the symbol H.

If the current I in one of the loops varies with time, so does the flux Φ in the other loop and hence an emf ε is induced in this other loop. If the current does not vary rapidly, this emf is given by

$$\varepsilon = -\frac{d\Phi}{dt} = -M\frac{dI}{dt} \cdot \tag{18.6}$$

Why the condition that the current should not vary fast? Because only for steady current distributions can one surely say that the magnetic field is proportional to the current. If the current variation is pretty fast, one has to consider the effect on the magnetic field due to time-varying fields.

EXAMPLE 18.1 **Two circular loops of radii a and b are placed coaxially with a separation d between their centres as shown in Figure 18.3. Take $d >> a >> b$. Find the mutual inductance between the loops.**

Solution Suppose a current I is sent in the bigger loop. It will produce a magnetic field

$$B = \frac{\mu_0 I a^2}{2\pi(a^2 + d^2)^{3/2}}$$

at the centre of the smaller loop. The field will be perpendicular to the planes of the loops. As $b << a$, you can assume that the magnetic field is uniform over the area of the smaller loop and hence the flux through it is

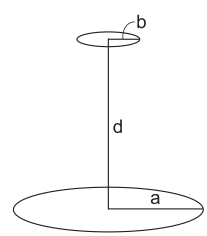


Fig. 18.3

$$\Phi \approx BA = \frac{\mu_0 I a^2}{2\pi(a^2 + d^2)^{3/2}} \cdot \pi b^2.$$

The mutual inductance between the loops is, therefore,

$$M = \frac{\Phi}{I} = \frac{\mu_0 a^2 b^2}{2(a^2 + d^2)^{3/2}} \cdot$$

Also, it is given that $d >> a$. Neglecting a^2 in comparison to d^2 in the denominator,

$$M \approx \frac{\mu_0 a^2 b^2}{2d^3} \cdot$$

What happens if instead of a single loop, you have a coil of several turns? As usual, when you calculate the flux, you take care of each turn and the flux gets multiplied by the number of turns. The mutual inductance is correspondingly enhanced.

18.2 Self-Inductance

If you pass a current in a loop, the current produces a magnetic field. This magnetic field has a flux through the area bounded by the loop itself. The magnetic field at any point is proportional to the current in the loop. Hence the flux is also proportional to the current. If I be the current in the loop and Φ be the flux through the loop,

$$\Phi = LI, \tag{18.7}$$

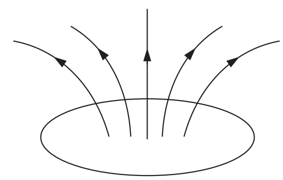


Fig. 18.4

where L is a constant for the given loop. This constant L is called the *self-inductance* of the loop or, in short, the inductance of the loop. It depends only on the geometrical parameters of the loop.

If the current in the loop is changed, the magnetic field also changes and so does the flux. This induces an emf in the loop given by

$$\varepsilon = -\frac{d\Phi}{dt}$$

or

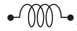
$$\varepsilon = -L\frac{dI}{dt}\cdot \tag{18.8}$$

Once again, this relation is valid only when the fields and currents do not vary too fast. Obviously, the SI unit of inductance is also henry.

All current-carrying circuits are closed loops. Whenever the current increases in a circuit, the inductance of the circuit causes an extra emf, given by Equation 18.8, which tries to reduce the current (ε is negative). Thus the inductance opposes the original current. On the other hand, if the current in the circuit decreases, the induced emf ε given by Equation 18.8 is positive and supports the original current. In either case, self-induction slows down the rate of change of current. This emf is also called *back emf*.

As usual, if there are several turns in the closed loop, the flux Φ gets a contribution from each turn and it is this total flux that should be considered in Equation 18.8.

The inductor

The self-inductance of a closed loop depends only on geometrical factors. If you take a wire and make a square loop out of it, the self-inductance will be smaller than what you get if you make a multi-turn coil out of it. Thus the self-inductance of the loop  in Figure 18.5(a) is much smaller than that of the coil in Figure 18.5(b) although it is the same wire and the same total length. An element that has high self-inductance and low resistance is called an inductor, and is shown by the symbol . The symbol suggests the construction of an inductor—it is a conducting wire wrapped as a multi-turn coil or small solenoid with two ends open. Some magnetic material is also put inside the turns to increase the inductance; don't worry about the actual design.

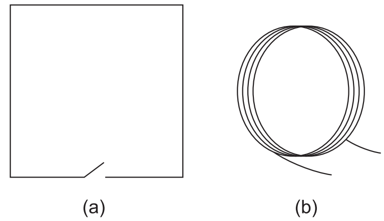


Fig. 18.5

You may wonder how a coil with open ends will have inductance, as inductance is defined only for closed loops. Suppose you take an inductor and connect the two open ends to a battery to complete the circuit. When current I goes through the circuit, the flux through the circuit (which is now a closed loop) consists of that through the areas bounded by the turns of the inductor and also through the other parts of the area, such as those shown shaded in Figure 18.6. It turns out that the flux through the turns of the inductor is much larger than that through the other parts. The self-inductance is defined for the entire circuit (which forms a closed loop), but since most of the flux is contributed by the inductor, the self-inductance of the circuit is attributed to the self-inductance L of the inductor. When you pass a current I in the circuit, the inductor will give rise to the flux $\Phi = LI$. There may be more flux through the circuit, but that will be quite small.

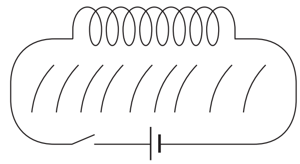


Fig. 18.6

EXAMPLE 18.2 **Find the inductance of a tightly wound solenoid of radius a, length l and number of turns per unit length n. Assume l to be large compared to a.**

Solution Suppose a current I is passed through the solenoid. The magnetic field inside, not close to the ends, will be

$$B = \mu_0 nI. \tag{i}$$

Close to the ends, the field will decrease and right at the centre of the face at the end it will become almost $\mu_0 nI/2$ for a long solenoid. But the field decreases very fast as one approaches the end and for most of the solenoid, B is given by (i). We will therefore neglect the decrease in the field B that occurs close to the ends while calculating the total flux through the solenoid. The number of turns in the solenoid is nl and hence the magnetic flux in it is

$$\Phi = (nl)(\mu_0 nI)\pi a^2.$$

The inductance is

$$L = \frac{\Phi}{I} = \pi\mu_0 n^2 a^2 l.$$

What is the dimension of n^2a^2l? It is same as length. You will always find the expression for inductance to be μ_0 times a length.

Try making an inductor by using typical values of the parameters in the expression derived in Example 18.2. For example, suppose the inductor is 10 cm long and 1 cm wide. You may wrap, say, 1000 turns on this length so that $n = 10^4$ per metre. The inductance for this inductor will be

$$L = 3.14 \times 4 \times 3.14 \times 10^{-7} \times 10^8 \times (0.005)^2 \times (0.1)$$
$$\approx 10^{-3}\ \text{H}.$$

This is 1 millihenry. We have taken a large length and large radius. The practical inductors are much smaller. But they put some magnetic material inside to enhance the inductance. Typically, commercial inductors have inductances of the order of millihenry.

An *L–R* circuit with a battery

Growth of current in an *L–R* circuit Consider a circuit with a resistor, an inductor, a battery and a switch (Figure 18.7). The resistor has resistance R, the inductor has inductance L and the battery has emf ε. The inductor has negligible resistance and the inductance of the circuit is assumed to be the same as L. Suppose we press the switch at $t = 0$ to complete the circuit.

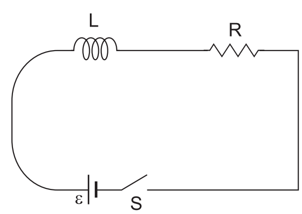


Fig. 18.7

As the circuit is completed, a current starts to flow at $t = 0$. But as the current increases from zero, a back emf is induced which slows down the rise of current. Let us derive the time-dependence of the current in this circuit.

The emf in the circuit is ε due to the battery and $-L\dfrac{dI}{dt}$ due to the back emf. The net emf is therefore $\varepsilon - L\dfrac{dI}{dt}\cdot$ Using Ohm's law,

$$\varepsilon - L\frac{dI}{dt} = RI \qquad \text{(i)}$$

or $$L\frac{dI}{dt} = \varepsilon - RI$$

or $$\frac{dI}{\varepsilon - RI} = \frac{dt}{L}\cdot$$

At $t = 0$, $I = 0$ and at time t, the current becomes I.

So, $$\int_0^I \frac{dI}{\varepsilon - RI} = \int_0^t \frac{dt}{L}$$

or $$-\frac{1}{R}\ln\frac{\varepsilon - RI}{\varepsilon} = \frac{t}{L}$$

or $$I = \frac{\varepsilon}{R}(1 - e^{-\frac{R}{L}t}). \qquad (18.9)$$

Writing $\tau = L/R$, and $I_0 = \dfrac{\varepsilon}{R}$,

$$I = I_0(1 - e^{-t/\tau}). \qquad (18.10)$$

You can see that τ has the dimension of time. It is called the time constant of the L–R circuit. The dependence of the current I on time is shown graphically in Figure 18.8. In one time constant the current reaches the value $I_0(1 - 1/e)$, which is about 63% of I_0. The maximum value of the current is I_0, which is achieved, in principle, at $t = \infty$. However, after a few time constants, the current is almost equal to I_0.

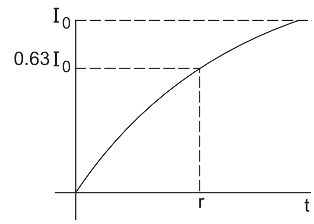


Fig. 18.8

It is interesting to note that $\oint \vec{E}\cdot\vec{dl}$ in the circuit is not zero as the magnetic flux through the circuit is time-varying. The electric field E in the inductor is zero if it is assumed to be resistance-less. $\vec{E}\cdot\vec{dl}$ along the resistor is equal to RI (from Ohm's law $\vec{J} = r\,\vec{E}$) and that along the cell is $-\varepsilon$ as we traverse it from the negative terminal to the positive terminal. Thus, $\oint \vec{E}\cdot\vec{dl} = RI - \varepsilon$ which should be equal to $-\dfrac{d\phi}{dt} = -L\dfrac{dI}{dt}$ according to Faraday's law. Thus, $RI - \varepsilon = -L\dfrac{dI}{dt}$ as stated in equation (i).

EXAMPLE 18.3 **An L–R circuit is connected to a battery at $t = 0$. The inductance is 20 mH and the resistance is 1 kΩ. How much time will it take the current to reach 99.99% of its maximum value?**

Solution The time constant is

$$\tau = L/R = \frac{20\times10^{-3}}{10^3}\text{ s} = 2\times10^{-5}\text{ s}.$$

If the maximum current is I_0,

$$I = I_0(1 - e^{-t/\tau}).$$

So, $$0.9999I_0 = I_0(1 - e^{-t/\tau})$$

or $$e^{-t/\tau} = 0.0001$$

or $$\frac{t}{\tau} = 9.2$$

or $$t = 9.2 \times 2 \times 10^{-5}\ \text{s} = 1.84 \times 10^{-4}\ \text{s}.$$

In less than a millisecond, the current will reach 99.99% of its maximum value.

Decay of current in an *L–R* circuit Consider the circuit shown in Figure 18.9(a). If the switch is connected for a long time, the current in the inductor will be $I_0 = \dfrac{\varepsilon}{R_1}$ and that in R_2 will be $I_1 = \dfrac{\varepsilon}{R_2}\cdot$ Suppose the switch is disconnected at $t = 0$ [Figure 18.9(b)].

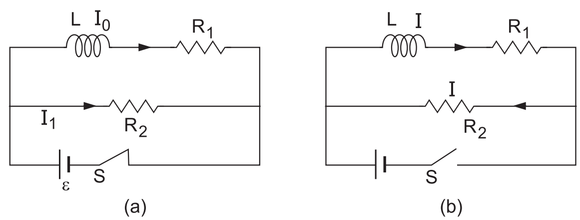


Fig. 18.9

Now the battery is disconnected from the circuit. If the inductor were not there, the current would have dropped to zero at $t = 0$ itself. But in the present situation, the decreasing current will produce an emf $-L\dfrac{dI}{dt}$, which will oppose the decrease in current. Thus the current will decrease slowly. Let us work out the mathematics. Figure 18.9(b) shows the situation at time t. If the current at this instant is I,

$$-L\frac{dI}{dt} = (R_1 + R_2)I = RI$$

or $$\frac{dI}{I} = -\frac{R}{L}dt$$

or $$\ln\frac{I}{I_0} = -\frac{R}{L}t$$

or $$I = I_0 e^{-\frac{R}{L}dt}$$

or $$I = I_0 e^{-t/\tau}, \tag{18.11}$$

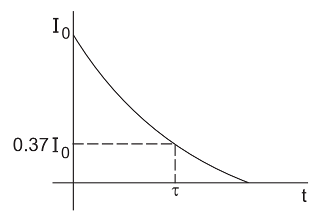


Fig. 18.10

where $\tau = \dfrac{L}{R}$ is the time constant. The current decreases to $\dfrac{1}{e} \approx 0.37$ of its initial value in one time constant. The dependence of the current on time is shown in Figure 18.10.

Consider the circuit shown in Figure 18.11. The two-way switch S can be connected to either a or b. Suppose it is connected to a for a long time and then at $t = 0$ it is connected to b. How will the current change after this?

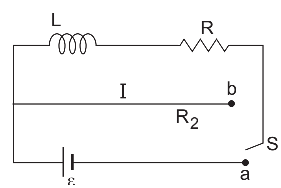


Fig. 18.11

As the switch leaves point a, the resistance of the circuit becomes infinite. The time constant $\frac{L}{R}$ thus becomes zero. The current drops to zero. When the switch is connected to b, the current is already zero and hence there is no further charge.

A large emf is generated as the switch leaves a. This may create a large electric field, which may ionize the air and you may see sparks.

An *L–R* circuit with an AC voltage source

The emf of a battery remains constant over a long period. There can also be an alternating-current (AC) voltage source, whose emf changes periodically. The power supply in your home is of this type. Electricity is produced in power stations and is transmitted to our houses through transmission lines. One can have independent compact AC voltage sources by using batteries, transistors and proper electric circuits. Such a source is often called an AC power supply. The emf between the terminals of the power supply is called AC voltage, and is given by

$$V = V_0 \sin(\omega t + \delta). \tag{i}$$

By choosing $t = 0$ properly, you can set $\delta = 0$ and then $V = V_0 \cos \omega t$. Here V_0 is the maximum emf. The angular frequency ω is related to the frequency ν as $\omega = 2\pi\nu$. The household supply comes at $\nu = 50$ Hz.

Suppose an inductor of inductance L, a resistor of resistance R and an AC voltage source having an emf given by (i) are connected in a circuit as shown in Figure 18.12. The inductor will provide a back emf $-L\frac{dI}{dt}$, where I is the current in the circuit at time t. Kirchhoff's law gives

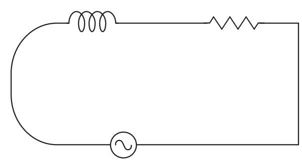


Fig. 18.12

$$V - L\frac{dI}{dt} = RI$$

or $$V_0 \cos \omega t - L\frac{dI}{dt} = RI$$

or $$L\frac{dI}{dt} + RI = V_0 \cos \omega t.$$

This equation can be solved using standard procedure and the steady-state solution turns out to be

$$I = I_0 \cos(\omega t + \phi), \tag{18.12}$$

where $$I_0 = \frac{V_0}{\sqrt{R^2 + L^2\omega^2}} \tag{18.13}$$

and $$\phi = \tan^{-1}\left(-\frac{L\omega}{R}\right). \tag{18.14}$$

What is meant by 'steady-state solution'? The circuit must have been connected at a certain instant. Immediately after the circuit was connected, there was a variation of I with time, which may not have been the same as that given by Equation 18.12. But after a sufficient time, much larger than the time constant L/R, the current will vary in the manner given by Equation 18.12.

There are several lessons to learn from these equations. The current has a maximum value, the current amplitude, given by Equation 18.13. It depends on the inductance and also on the frequency of the voltage source. Writing $Z = \sqrt{R^2 + L^2\omega^2}$,

$$I_0 = \frac{V_0}{Z},$$

which resembles Ohm's law with Z playing the role of resistance. In fact, Z is called the *impedance* of the circuit. The quantity $L\omega$ is called *reactance* and is often denoted by X. The higher the frequency, the greater are X and Z, and the less is the current amplitude.

Another important observation is that the current and the emf are not in phase. They differ by a phase difference of ϕ, as given by Equation 18.14. When the emf is maximum, the current is not at its maximum. The variation of ε and I are shown on the same plot in Figure 18.13. Remember that ϕ is negative. The current reaches its maximum a little after the emf reaches its maximum. The current is therefore said to lag behind the emf.

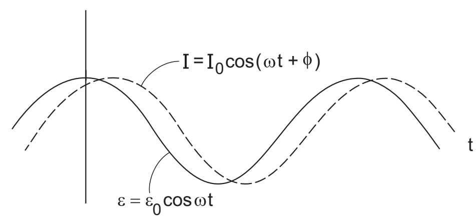


Fig. 18.13

What happens if $R << L\omega$? In this case $\phi \approx -\pi/2$. The current reaches its maximum value when the emf is zero and becomes zero when the emf is maximum.

An inductive loop partially in a magnetic field

Suppose a loop having an inductor of inductance L is placed perpendicular to a uniform magnetic field $\boldsymbol{B} = -B_0\hat{\boldsymbol{k}}$. The loop is partially inside and partially outside the field, as shown in Figure 18.13. At time $t = 0$, the left end of the loop, whose length is l, is at $x = 0$ and the loop is given a velocity $v = v_0\hat{\boldsymbol{i}}$. What will be the subsequent motion of the loop?

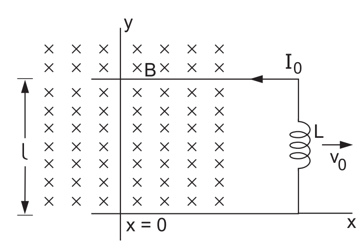


Fig. 18.14

Let us take the anticlockwise sense as positive for the current in the loop.

At time t, the flux of the magnetic field through the loop is

$$\Phi = \int \boldsymbol{B} \cdot d\boldsymbol{a} = -B_0 \hat{\boldsymbol{k}}(d-x)l\hat{\boldsymbol{k}},$$

where d is the initial length of the loop inside the field and x is the position of the left end at time t. The induced emf is

$$\varepsilon = -\frac{d\Phi}{dt} = -B_0 vl.$$

The equation for the current in the loop is

$$\varepsilon - L\frac{dI}{dt} = IR,$$

where R is the resistance of the loop. Ignoring the resistance,

$$\varepsilon = L\frac{dI}{dt}$$

or $$-B_0 vl = L\frac{dI}{dt}\cdot \tag{i}$$

The force on the loop is

$$\boldsymbol{F} = I\boldsymbol{l} \times \boldsymbol{B} = Il(-\hat{\boldsymbol{j}}) \times B_0(-\hat{\boldsymbol{k}}) = IlB_0\hat{\boldsymbol{i}}$$

or $$m\frac{dv}{dt} = IlB_0. \tag{ii}$$

From (i) and (ii),

$$-B_0 l\left(\frac{IlB_0}{m}\right) = L\frac{d^2I}{dt^2}$$

or $$\frac{d^2I}{dt^2} = -\left(\frac{B_0^2 l^2}{mL}\right)I = -\omega^2 I$$

or $$I = I_0 \cos(\omega t + \phi), \tag{iii}$$

where $$\omega = \frac{B_0 l}{\sqrt{m}}\cdot \tag{iv}$$

At $t = 0$, the current $I = 0$. Also, from equation (i),

$$\frac{dI}{dt} = -\frac{B_0 vl}{L}\cdot$$

At $t = 0$,

$$\frac{dI}{dt} = -\frac{B_0 v_0 l}{L}\cdot$$

Thus, $0 = \cos\phi$ and $-\dfrac{B_0 v_0 l}{L} = -I_0 \omega \sin\phi.$

This gives $\phi = \pi/2$, and

$$I_0 = \frac{B_0 v_0 I}{L\omega}\cdot \tag{v}$$

Hence, $I = -I_0 \sin \omega t,$

where ω and I_0 are given by equations (iv) and (v). To get the motion of the loop, you can now use equation (ii).

$$m\frac{dv}{dt} = -I_0 l B_0 \sin \omega t$$

or
$$[v]_{v_0}^{v} = \frac{I_0 l B_0}{m\omega}[\cos \omega t]_0^t$$

or
$$v = v_0 - \frac{I_0 l B_0}{m\omega}[1 - \cos\omega t] = v_0 - v_0[1 - \cos \omega t] \qquad \text{[using (v)]}$$
$$= v_0 \cos \omega t.$$

This gives $x = \dfrac{v_0}{\omega}\sin \omega t = \dfrac{v_0\sqrt{mL}}{B_0 l}\sin\left[\dfrac{B_0 l}{\sqrt{mL}}t\right]\cdot$

The loop undergoes simple harmonic oscillations.

18.3 Magnetic Energy

When a current flows in a circuit, energy is dissipated in the resistance in the form of heat. The current also produces a magnetic field and this magnetic field also carries energy. To get a relation between the magnetic field and the magnetic energy associated with it, consider an inductor and a resistor connected to a battery through a switch, as shown in Figure 18.7. Before you press the switch, there is no current and no magnetic field. Press the switch at $t = 0$ and current starts to flow in the circuit. At any time t,

$$\varepsilon = L\frac{dI}{dt} + RI. \tag{i}$$

In the next time interval dt, a charge Idt flows through the circuit. As this charge passes through the battery, the battery does work $\varepsilon(Idt)$. This much chemical energy is released. Where does it appear? Multiply Equation (i) by Idt to get

$$\varepsilon Idt = LIdI + I^2 Rdt.$$

The left-hand side represents the work done by the battery. The term I^2Rdt on the right gives the thermal energy developed in the circuit. But not all the work done by the battery (εIdt) goes into heating the wire. $EIdt$ is not equal to I^2Rdt. Where does the rest go? As the current increases from I to $I+dI$, the magnetic field increases everywhere. There is a magnetic energy associated with the current in the circuit. The increase in current in the circuit increases the magnetic energy too. The part left over, *LIdI*, is the increase in magnetic energy. Writing the magnetic energy as U_m,

$$dU_m = LIdI.$$

As the current increases from 0 to *I*, the magnetic energy increases from 0 to U_m. Thus,

$$U_m = \int_0^I LIdI = \frac{1}{2}LI^2. \tag{18.15}$$

This expression was derived by taking a simple circuit which can be associated with an inductance. What would be the magnetic energy for a more general current distribution given by a volume current density $\boldsymbol{J}(\boldsymbol{r})$ in a given volume τ? Another question is whether the energy is associated with the magnetic field or with the current itself. In the form given by Equation 18.15, the magnetic energy seems to be linked to the current in the circuit. However, it is possible to write the magnetic energy in terms of the magnetic field $\boldsymbol{B}(\boldsymbol{r})$ alone without referring to the current *I* or any parameter of the circuit. We will do this first for the simple case of a long solenoid carrying a current, and then derive a general expression for the magnetic energy in terms of the field quantities alone.

Magnetic energy in a long solenoid

Consider a long, tightly wound solenoid. Let the length of the solenoid be *l*, its radius be *a* and the number of turns per unit length be *n*. Suppose a current *I* is passed in it.

Neglecting the end effects, the inductance of the solenoid is

$$L = \pi\mu_0 n^2 a^2 l,$$

as calculated in Example 18.1. If a steady current *I* goes through the circuit, the magnetic energy associated with the solenoid is

$$U_m = \frac{1}{2}LI^2 = \frac{1}{2}\pi\mu_0 n^2 a^2 l I^2.$$

If the solenoid is very long (as compared to its radius), most of the magnetic field is confined to the inside of the solenoid. If you think of an infinitely long solenoid, the magnetic field outside is zero. The field inside is

$$B = \mu_0 nI$$

so that the magnetic energy in the length l of the solenoid is

$$U_m = \frac{1}{2\mu_0} B^2 \pi a^2 l.$$

But $\pi a^2 l$ is the volume of the solenoid in length l. Thus magnetic energy is written in terms of magnetic field and volume. One can interpret the energy to be distributed in the volume and associate it with the field. As the field is uniform inside the solenoid, it is natural to assume that the magnetic energy is also uniformly distributed over its volume. You can then define magnetic field energy per unit volume,

$$u_m = \frac{U_m}{\pi a^2 l} = \frac{B^2}{2\mu_0}\cdot \tag{18.16}$$

This quantity is called *magnetic energy density*.

You may see several loopholes in these arguments. The magnetic field lines are continuous and have to come out of the solenoid at the ends. So, there has to be a magnetic field and magnetic energy outside the solenoid. We have totally neglected these. Near the ends of the solenoid, the field inside is less than $\mu_0 ni$, but we took it to be $\mu_0 ni$ everywhere. However, the errors nullify each other and Equation 18.16 turns out to be the exact expression for magnetic energy density.

Magnetic energy density in a general case

The magnetic energy corresponding to a circuit is given by

$$\begin{aligned} U_m &= \frac{1}{2}LI^2 = \frac{1}{2}(LI)I = \frac{1}{2}\Phi I \\ &= \frac{1}{2}\oint \boldsymbol{A}\cdot \boldsymbol{dl}I \\ &= \frac{1}{2}\oint \boldsymbol{A}\cdot (I\boldsymbol{dl}). \end{aligned} \tag{i}$$

Here $\boldsymbol{A}$ is the vector potential corresponding to the magnetic field due to the current and the integration is performed over the circuit. In this form, you can interpret the magnetic energy as coming from the interaction between the current element $I\boldsymbol{dl}$ and the vector potential $\boldsymbol{A}$ present there. The contribution is $\frac{1}{2}\boldsymbol{A}\cdot(I\boldsymbol{dl})$ due to this current element. When you take contributions from all current elements in the circuit, you get the total magnetic energy.

The generalization to a general current distribution is easy. Suppose you have a current distribution given by volume current density $\boldsymbol{J}(\boldsymbol{r})$ in a volume τ. You know that this is the most general distribution. Any line current or surface current can also be written as a volume current distribution with the proper use of delta functions. Magnetic field energy can be interpreted as an interaction between the volume current element $\boldsymbol{J}(\boldsymbol{r})d\tau$ and the vector potential $\boldsymbol{A}(\boldsymbol{r})$ existing there—indeed, the contribution is to be taken from all the current elements. Thus the generalization of (i) is

$$U_m = \frac{1}{2}\int \boldsymbol{A}(\boldsymbol{r})\cdot \boldsymbol{J}(\boldsymbol{r})d\tau, \tag{18.17}$$

where the integration has to be performed over a volume that contains the entire current distribution. In place of the current element $Id\boldsymbol{l}$ in (i), we have put $\boldsymbol{J}(\boldsymbol{r})d\tau$, which is the current element at $d\tau$. The magnetic vector potential here is $\boldsymbol{A}(\boldsymbol{r})$ and the magnetic energy associated with this current element is $\frac{1}{2}\boldsymbol{A}(\boldsymbol{r})\cdot \boldsymbol{J}(\boldsymbol{r})d\tau$. Integrating over the volume containing the entire current distribution, you get the total magnetic energy. So contributions to U_m come only from regions where current exists. If the current in a particular region is zero, the contribution to the integral in Equation 18.17 from that region is also zero.

You can write the expression in Equation 18.17 entirely in terms of fields, without any reference to the current. Writing $\boldsymbol{J} = \frac{1}{\mu_0}\nabla\times\boldsymbol{B}$ (assuming steady currents),

$$U_m = \frac{1}{2\mu_0}\int \boldsymbol{A}\cdot(\nabla\times\boldsymbol{B})d\tau. \tag{ii}$$

For any vector fields $\boldsymbol{A}$ and $\boldsymbol{B}$,

$$\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\nabla\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\nabla\times\boldsymbol{B}).$$

So, $$\boldsymbol{A}\cdot(\nabla\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\nabla\times\boldsymbol{A}) - \nabla\cdot(\boldsymbol{A}\times\boldsymbol{B}).$$

Using this in (ii),

$$\begin{aligned}
U_m &= \frac{1}{2\mu_0}\left[\int \boldsymbol{B}\cdot(\nabla\times\boldsymbol{A})d\tau - \nabla\cdot(\boldsymbol{A}\times\boldsymbol{B})d\tau\right] \\
&= \frac{1}{2\mu_0}\int \boldsymbol{B}\cdot\boldsymbol{B}d\tau - \frac{1}{2\mu_0}\int \nabla\cdot(\boldsymbol{A}\times\boldsymbol{B})d\tau \\
&= \int \frac{B^2}{2\mu_0}d\tau - \frac{1}{2\mu_0}\oint_s (\boldsymbol{A}\times\boldsymbol{B})\cdot d\boldsymbol{a}.
\end{aligned} \tag{18.18}$$

In the second term, Gauss's divergence theorem has been used to convert the volume integral to the surface integral over the surface that bounds the volume τ. The surface S encloses the entire current distribution.

So you take a volume τ that contains all the currents. Look at the magnetic field everywhere inside this volume and perform the integration in the first term of Equation 18.18. To evaluate the second term, look at the vector potential $\boldsymbol{A}$ and the magnetic field $\boldsymbol{B}$ "only at the surface" of this volume and perform the surface integration. Thus the energy gets a contribution from the volume containing the current including its surface.

What happens if you start with a larger volume that contains the entire current distribution. In some portion of this volume, you will not have any current. Here $\boldsymbol{J}$ will be zero. But that does not change the RHS of Equation 18.17. The contribution to U_m comes only from volume elements where $\boldsymbol{J} \neq 0$. Hence enlarging the volume will not change U_m. Therefore, in Equation 18.18 also, you can take any volume τ that contains the current distribution and you will get the same value of U_m. If you enlarge the volume, both the integrals in this equation will change but the net value of U_m will remain the same.

Take the entire space as the volume τ. Look at the surface integral in Equation 18.18. You need to look at $\boldsymbol{B}$ and $\boldsymbol{A}$ only at the surface, which is now at infinity. For any given finite current distribution, there is no magnetic monopole moment. For magnetic dipoles the vector potential $\boldsymbol{A}$ falls as $1/r^2$ and the magnetic field $\boldsymbol{B}$ falls as $1/r^3$ at large distances. So $\boldsymbol{A} \times \boldsymbol{B}$ will fall as $1/r^5$ as r tends to infinity. The surface area itself will increase as r^2 as r increases. So the surface integral $\oint_s (\boldsymbol{A} \times \boldsymbol{B}) \cdot d\boldsymbol{a}$ will fall as $1/r^3$ and will tend to zero as r tends to infinity.

Therefore, Equation 18.18 can be rewritten as

$$U_m = \int_{\text{all space}} \frac{B^2}{2\mu_0} d\tau. \tag{18.19}$$

You can get the magnetic energy associated with a current distribution by just integrating $(B^2/2\mu_0)d\tau$ over all space. There is no reference to current any more. Wherever you have a magnetic field, you get a contribution to U_m according to Equation 18.19. This allows you to visualize the magnetic energy stored in the magnetic field. In the volume $d\tau$ the magnetic field energy is $(B^2/2\mu_0)d\tau$, whether or not there is a current here. When you integrate over all space containing the magnetic field, you get the net magnetic energy. The quantity

$$U_m = \frac{B^2}{2\mu_0} \tag{18.20}$$

gives the magnetic energy per unit volume, that is, magnetic energy density.

Where does magnetic energy reside? Equation 18.17 suggests that it exists wherever you have nonzero current. The contribution to this energy comes only from those volume elements where $\boldsymbol{J} \neq 0$. On the other hand, Equation 18.19 suggests that it resides wherever the magnetic field $\boldsymbol{B}$ is nonzero. In magnetostatics, we cannot decide on this question. Both interpretations are equally correct and you can choose any of them. However, while discussing electromagnetic waves, it will be far more appropriate to assume that magnetic energy resides in the magnetic field. If you remember, a similar situation exists in electrostatics. You can assume that electrostatic energy comes from the interaction of charges with potential fields and resides with the charges, or you may assume that the energy is stored in the electric field.

Magnetic energy for more than one circuit

Suppose you have two circuits carrying currents I_1 and I_2, placed close to each other. Let the self-inductances of the circuits be L_1 and L_2. If they were far apart, they would have had magnetic energies $\frac{1}{2}LI_1^2$ and $\frac{1}{2}LI_2^2$. But since they are close, they will have a mutual inductance. The magnetic field at a point is the sum of the fields due to the two circuits. But energy is not a linear function of field and hence does not add up that way.

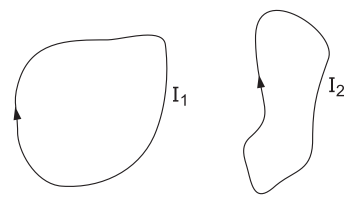


Fig. 18.15

Start with Equation 18.17.

$$U = \frac{1}{2}\int \boldsymbol{A}(\boldsymbol{r})\cdot \boldsymbol{J}(\boldsymbol{r})d\tau.$$

For the two-circuit combination shown in Figure 18.15, this becomes

$$\begin{aligned} U &= \frac{1}{2}\int \boldsymbol{A}(\boldsymbol{r})\cdot I_1 \boldsymbol{dl}_1 + \frac{1}{2}\int \boldsymbol{A}(\boldsymbol{r})\cdot I_2 \boldsymbol{dl}_2 \\ &= \frac{1}{2}\left[I_1 \int \boldsymbol{A}(\boldsymbol{r})\cdot \boldsymbol{dl}_1 + I_2 \int \boldsymbol{A}(\boldsymbol{r})\cdot \boldsymbol{dl}_2\right] \\ &= \frac{1}{2}[I_1\Phi_1 + I_2\Phi_2], \end{aligned} \tag{i}$$

where Φ_1 is the magnetic flux through Circuit 1 and Φ_2 is that through Circuit 2.

The flux of the magnetic field through Circuit 1 is

$$\Phi_1 = L_1 I_1 + M I_2$$

and that through Circuit 2 is

$$\Phi_2 = L_2 I_2 + M I_1.$$

Thus the magnetic energy associated with the two circuits together, from (i), is

$$U = \frac{1}{2}[L_1 I_1^2 + 2MI_1 I_2 + L_2 I_2^2].$$

What if the direction of current in one of the loops is reversed? The middle term within the square brackets will get an extra negative sign. Or you can say that M will change sign.

18.4 Transformers

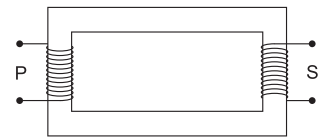


Fig. 18.16

Mutual inductance allows us to make transformers which are one of the most widely used electrical appliances. While AC voltages of the order of several tens of kilovolts are transmitted through high-tension wires over large distances, the household supply needs 220 V AC. Different kinds of chargers need only a few volts of direct current. To get such a DC voltage, one has to first convert 220 V AC to a low-voltage AC, which is then converted to the required low-voltage DC. Conversion of AC from one voltage to the other is done by transformers. We will only discuss the basic principle of the transformer.

Suppose you have two coils P and S arranged in such a manner that the magnetic flux through each turn of coil P is the same as the magnetic flux through each turn of S. How should we make such an arrangement? By using ferromagnetic materials. We will talk about these materials in a later chapter. For now, let us say that they are strongly attracted by magnets. Iron is the most common ferromagnetic material. The coils P and S are wound on a frame made of properly designed sheets of a ferromagnetic material. This frame is called the core.

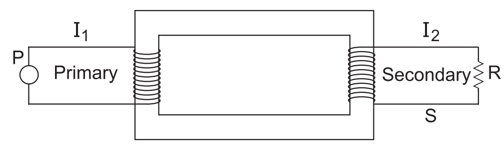


Fig. 18.17

Connect the initial AC voltage (which you wish to convert) source to one of the coils and the required AC voltage appears in the other coil. The first is called the primary coil and the other, the secondary coil. In Figure 18.17, P and S are respectively the primary and the secondary.

Suppose an AC source giving voltage $v = v_{\text{in}} \cos \omega t$ is connected to the primary and a load resistance R_1 (say a filament bulb) is connected to the secondary. To keep the mathematics simple, assume that the coils themselves have negligible resistance. Also the AC voltage will produce some AC current in the primary and the secondary circuits. This will create a changing magnetic field, which in turn will produce an eddy current in the core. Assume that the conductivity of the core is so low and its geometry is designed in such a way that the eddy current can be neglected. In fact, neglect any kind of energy loss in the transformer.

Suppose the primary coil has N_1 turns and self-inductance L_1. These quantities for the secondary coil are N_2 and L_2. Also suppose the mutual inductance of the two coils is M.

Assume the current in the primary circuit to be I_1 and that in the secondary circuit to be I_2. Suppose the flux through each turn of the primary and secondary coils is Φ. Assume that your ideal transformer is designed in such a way that the flux through each turn of the two coils is equal.

The total flux through the primary coil can also be written as $L_1I_1 + MI_2$. So,

$$L_1I_1 + MI_2 = N_1\Phi$$

or $$\left(\frac{L_1}{N_1}\right)I_1 + \frac{M}{N_1}I_2 = \Phi.$$

Similarly, the total flux through the secondary coil will be $L_2I_2 + MI_1$.

So, $$L_2I_2 + MI_1 = N_2\Phi$$

or $$\left(\frac{L_2}{N_2}\right)I_2 + \frac{M}{N_2}I_1 = \Phi$$

or $$\left(\frac{L_1}{N_1}\right)I_1 + \frac{M}{N_1}I_2 = \left(\frac{L_2}{N_2}\right)I_2 + \frac{M}{N_2}I_1.$$

By changing the value of the load resistance, I_1 and I_2 can be changed but the above equation will still remain valid. This can only be possible if

$$\frac{L_1}{N_1} = \frac{M}{N_2}$$

and $$\frac{M}{N_1} = \frac{L_2}{N_2}.$$

From these equations,

$$M^2 = L_1L_2.$$

Voltage

Suppose the flux through each turn of the primary and also of the secondary is Φ. The net flux through the primary is $N_1\Phi$ and that through the secondary is $N_2\Phi$. As the resistance of the primary coil is assumed to be small, Kirchhoff's equation gives

$$v_{\text{in}} - N_1\frac{d\Phi}{dt} = 0$$

or $$v_{\text{in}} = N_1\frac{d\Phi}{dt}. \quad \text{(i)}$$

The voltage induced across the secondary coil is

$$v_{\text{out}} = N_2\frac{d\Phi}{dt}. \quad \text{(ii)}$$

Thus $$\frac{v_{\text{out}}}{v_{\text{in}}} = \frac{N_2}{N_1}. \quad (18.21)$$

If the secondary has fewer turns than the primary ($N_2 < N_1$), the voltage obtained across the secondary coil is smaller than the input voltage. Such a transformer is called a step-down transformer. In the opposite case, the output voltage is more than the input voltage, and such a transformer is called a step-up transformer.

Current

Now look at the currents. The net emf in the primary $v_{\text{in}} - N_1 \dfrac{d\Phi}{dt}$ is equal to zero. But the primary can have a current as we have assumed the resistance to be zero. The value of this current depends on the value of current in the secondary, which in turn depends on the load resistance R joined to this part of the circuit.

Suppose the source voltage is given by

$$v_{\text{in}} = V_1 \cos \omega t.$$

For the primary circuit, Kirchhoff's equation gives

$$V_1 \cos \omega t - L_1 \frac{dI_1}{dt} - M \frac{dI_2}{dt} = 0. \tag{i}$$

And for the secondary circuit,

$$-L_2 \frac{dI_2}{dt} - M \frac{dI_1}{dt} = R_L I_2. \tag{ii}$$

Eliminate $\dfrac{dI_2}{dt}$ from these equations. You can multiply (i) by L_2, (ii) by M, and subtract.

$$L_2 V_1 \cos \omega t - L_1 L_2 \frac{dI_1}{dt} + M^2 \frac{dI_1}{dt} = -R_L M I_2.$$

But $M^2 = L_1 L_2$. So, $L_2 V_1 \cos \omega t = -R_L M I_2$

or $$I_2 = -\frac{L_2 V_1 \cos \omega t}{R_L M}. \tag{18.22}$$

From (i), $$L_1 \frac{dI_1}{dt} = V_1 \cos \omega t - M \frac{L_2 V_1 \omega \sin \omega t}{R_L M}$$

or $$I_1 = \frac{V_1}{L_1}\left[\frac{\sin \omega t}{\omega} + \frac{L_2}{R_L} \cos \omega t\right]. \tag{18.23}$$

Power

The power delivered by the source to the transformer is

$$P_{\text{in}} = V_{\text{in}} I_1 = (V_1 \cos \omega t)\left(\frac{V_1}{L_1}\right)\left(\frac{\sin \omega t}{\omega} + \frac{L_2}{R_L}\cos \omega t\right)$$

$$= \frac{V_1^2}{L_1}\left(\frac{\sin \omega t \cos \omega t}{\omega} + \frac{L_2}{R_L}\cos^2 \omega t\right). \tag{iii}$$

The power delivered by the transformer to the load is (using Equation 18.21)

$$P_{\text{out}} = I_2^2 R_L = \left(\frac{L_2^2 V_1^2 \cos^2 \omega t}{R_L M^2} \right). \tag{iv}$$

You can see that $P_{\text{in}} \neq P_{\text{out}}$. It has been assumed that there is no energy loss in the transformer. Then why is the power given to the transformer by the AC source not equal to the power delivered by the transformer to the load resistor?

You can get the clue to the answer from the expressions for P_{in} and P_{out}. The second term in (iii) is the same as P_{out}. The first term has a factor $\sin \omega t \cos \omega t$, which alternately becomes positive and negative. So the output power is more than the input power at some instants and less than the input power at some other instants. So power is not lost, it is converted to some other form (when $P_{\text{out}} < P_{\text{in}}$) and is given back to the load resistance (when $P_{\text{out}} > P_{\text{in}}$). It is easy to identify this "some other form" as magnetic field energy. As the magnetic field changes sinusoidally with time, the magnetic field energy in the transformer also changes with time and hence this inequality between P_{in} and P_{out}.

The average of $\sin \omega t \cos \omega t$ over a time period is zero as

$$\int_0^T \sin \omega t \cos \omega t \, dt = -\left.\frac{\cos 2\omega t}{4}\right|_0^T = \frac{1}{4}[\cos 4\pi \cos 0] = 0.$$

Thus $\langle P_{\text{in}} \rangle = \langle P_{\text{out}} \rangle$,

where the angular brackets denote averaging over a time period.

Actual transformers will essentially have energy losses and the analysis will be much more involved.

18.5 An Interesting Demo

Take an FM radio set which can be connected to an external speaker. Take a cylindrical plastic box and wrap a copper wire around it to form a coil. Wind a similar wire around another plastic box.

Connect the terminals for the external speaker to the free ends of the first coil as suggested in Figure 18.18. Connect the free ends of the second coil to a speaker. Keep the two coils in front of each other with a common axis. Switch on the radio and enjoy the music.

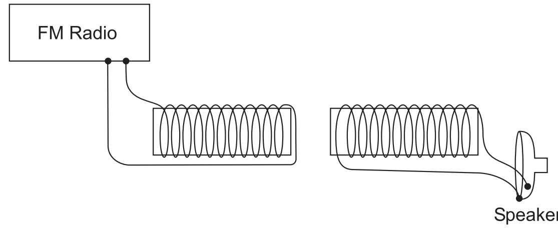


Fig. 18.18

The radio sends a voltage across the external speaker points, which drives the current in the first coil. Because of this current, there is a magnetic field that has a flux through the second coil. With the modulation of sounds, the voltage changes and hence, so does the current in the first coil. Accordingly, a current is induced in the second coil, which drives the speaker and you hear the sound.

Increase and decrease the separation between the coils and notice the change in the intensity of sound. Place one coil above the other or change the orientation and observe. What do you learn from your observations?

Concepts at a Glance

1. Two nearby circuits or loops have a mutual inductance defined as $M = \frac{\mu_0}{4\pi}\oint_1\oint_2 \frac{\boldsymbol{dl}_1 \cdot \boldsymbol{dl}_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|}$.
2. If the current in one of two nearby circuits is I, the magnetic flux in the other circuit is $\Phi = I$.
3. Each circuit has a self-inductance L.
4. The SI unit of M and L is henry.
5. When current in a circuit changes, a back emf $\varepsilon_b = -L\frac{dl}{dt}$ is induced.
6. In an L–R circuit connected to a battery, the current builds up as $I = \frac{\varepsilon}{R}\left(1 - e^{\frac{-R}{L}t}\right)$.
7. The time constant of an L–R circuit is given by $\tau = L/R$.
8. An inductor in an AC circuit offers a reactance $L\omega$. In an L–R circuit, the total impedance is $\sqrt{R^2 + (L\omega)^2}$.
9. Current and applied AC voltage differ in phase in an L–R circuit; the phase difference is $\phi = \tan^{-1}(-L\omega/R)$.
10. The magnetic energy stored in an inductor is given by $U_m = \frac{1}{2}LI^2$.
11. The energy density of a magnetic field is $U_m = B^2/2\mu$.
12. The magnetic energy of two nearby circuits is $v = \frac{1}{2}[L_1I_1^2 + 2MI_1I_2 + L_2I_2^2]$.
13. For an ideal transformer, $\frac{v_{\text{out}}}{v_{\text{in}}} = \frac{N_2}{N_1}$.

EXERCISES

Based on Concepts

1. You are given a copper wire of a certain length. In which of the following three cases will the self-inductance be maximum and in which will it be minimum? (a) The wire is bent to form a square. (b) The wire is bent to form a rectangle. (c) The wire is bent to form a circle.

2. Two circular wire loops lie in a single plane. If one of the loops is rotated about the line joining the centres, will the mutual inductance increase or decrease?

3. An long insulated wire AB is folded at its midpoint C and the two halves are kept almost in contact. Starting from C, this double wire is wrapped on a cylinder to make a coil. What will be the self-inductance of the coil?

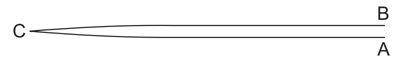


Fig. 18E.1

4. Can mutual inductance be negative? Can self-inductance be negative?

Problems

1. Two long metal ribbons, each of width b, are placed parallel to each other at a very small separation d, one above the other. These ribbons form a circuit so that current flows along the length of a ribbon and returns along the other. The current density on each ribbon is uniform. Find (a) the capacitance C per unit length, (b) self-inductance L per unit length, and (c) the product LC. Assume that almost all the magnetic field is confined between the ribbons. [**Ans.** (a) $\frac{\varepsilon_0 b}{d}$, (b) $\frac{\mu_0 d}{b}$, (c) $\varepsilon_0\mu_0$]

2. A coaxial cable consists of two very-thin-walled conducting cylinders of radii r_1 and r_2. Current I goes in one direction down the inner cylinder and comes back, up the outer cylinder. Find (a) the capacitance C per unit length, (b) the self-inductance L per unit length, and (c) the product LC.

[**Ans.** (a) $\frac{\mu_0}{2\pi}\ln\frac{b}{a}$, (b) $\frac{2\pi\varepsilon_0}{\ln\frac{b}{a}}$, (c) $\mu_0\varepsilon_0$]

3. Two circular loops carry currents I_1 and I_2 as shown in the figure. The radius a of the first loop is much larger than the radius b of the smaller loop. They are kept coaxially and at a separation z. Let the self-inductances of the two loops be L_1 and L_2 respectively. (a) Write the expression for the magnetic energy U of this pair of circular currents. (b) By using $F = -\frac{dU}{dz}$, find the magnetic force between the loops.

[**Ans.** (a) $\frac{1}{2}L_1I_1^2 + \frac{1}{2}L_2I_2^2 + MI_1I_2$, where $M = \frac{\mu_0\pi a^2b^2}{2(a^2+z^2)^{3/2}}$, (b) $I_1I_2\cdot\frac{3\mu_0\pi a^2b^2z}{(a^2+z^2)^{5/2}}$]

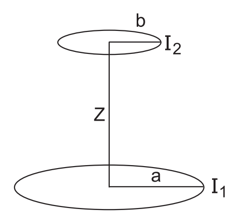


Fig. 18E.2

4. A circular, conducting loop of radius a is placed inside a much longer solenoid. The axis of the loop makes an angle θ with the axis of the solenoid. The solenoid has n turns per unit length. Find the mutual inductance between the two. [**Ans.** $\mu_0 n\pi a^2\cos\theta$]

5. Determine the mutual inductance between a triangular conducting loop and a very long straight wire, as shown in Figure 18E.3. [**Ans.** $\frac{\sqrt{3}\mu_0 d}{2\pi}(\ln 4 - 1)$]

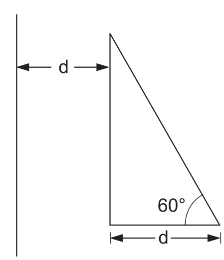


Fig. 18E.3

6. Determine the inductance per unit length of a coaxial transmission line that has a solid inner conductor of radius a and a very thin outer conductor of inner radius b. The space between the conductors is air. You can use the expression of magnetic energy stored in a magnetic field.

7. Two small circular loops, each of radius a, are placed coplanarly with their centres separated by a distance d which is much larger than a. Find their mutual inductance. [**Ans.** $\frac{\mu_0 \pi a^4}{2d^3}$]

8. A cylindrical coil with its axis along the z-axis carries a current I. There is a circular wire loop of radius R placed at a certain distance with its axis also along the z-axis (Figure 18E.4). The mutual inductance between the two circuits is M. Find the ϕ-component of the magnetic vector potential due to the current I in the coil at the circular wire loop. [**Ans.** $\frac{M}{2\pi R}$]

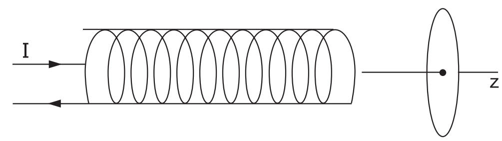


Fig. 18E.4

9. Circuits A and B are placed close to each other. Circuit A carries a current I that changes with time. When the current I changes at a rate α, an emf e_0 is produced in Circuit B. Find the value of $\oint\oint \frac{\boldsymbol{dl}_1 \cdot \boldsymbol{dl}_2}{\boldsymbol{r}_1 - \boldsymbol{r}_2}$, where $\boldsymbol{dl}_1$ and $\boldsymbol{dl}_2$ are elements at positions $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ on the two circuits. [**Ans.** $\frac{4\pi e_0}{\mu_0 \alpha}$]

10. Show that two inductances L_1 and L_2 (a) connected in series will have an equivalent inductance $L_1 + L_2$, (b) connected in parallel will have an equivalent inductance $\frac{L_1 L_2}{L_1 + L_2}$.

11. The circuit shown in Figure 18E.5 is switched on at $t = 0$. (a) What is the maximum current given by the battery? (b) When the battery gives maximum current (almost), what is the current through the 12-mH inductor? (c) At what time does the battery give a current equal to half the maximum?

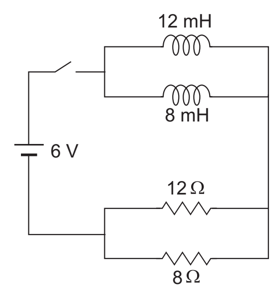


Fig. 18E.5

12. The self-inductance of Coil b is twice that of Coil a. When they are connected as in Figure 18E.6 (i), the net inductance of the combination is 80 mH. When they are connected as in Figure 18E.6 (ii), the net inductance is 40 mH. Find (a) the mutual inductance between a and b, and (b) the self-inductance of Coil a. [**Ans.** (a) 10 mH (b) 20 mH]

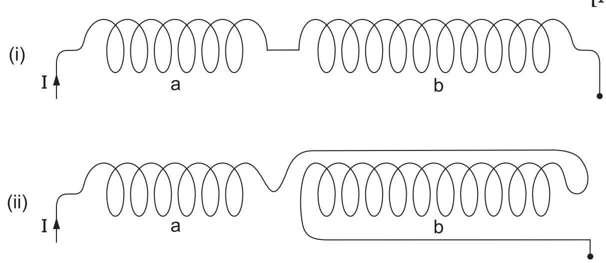


Fig. 18E.6

13. In an AC L–R circuit with angular frequency ω, the inductance is $100\ \Omega/\omega$ and the resistance R is $100\ \Omega$. The steady-state current is $\sqrt{5}$ A. What is the phase difference between the applied voltage and the current? If the inductor is replaced by another one, of inductance $200\ \Omega/\omega$, what will be the steady-state current? [**Ans.** $45°$, $\sqrt{2}$ A]

14. When the current through a coil is changed at the rate of 1 A/s, an emf of 1 mV is induced in it. What is the magnetic field energy when the current in it is 1 A? [**Ans.** 0.5 mV]

15. A long straight wire carries a current I. Consider a coaxial cylindrical region of height h between $r = 1$ cm and $r = 2$ cm. Find the magnetic field energy contained in this region. [**Ans.** $\dfrac{\mu_0 I^2 h \ln 2}{4\pi}$]

16. The vector potential in a region is given by $\boldsymbol{A} = \dfrac{\mu_0 k s^2}{R}\hat{\boldsymbol{k}}$ for $s < R$ and $\boldsymbol{A} = \mu_0 k R \ln\left(\dfrac{s}{R}\right)\hat{\boldsymbol{k}}$ for $s > R$ in cylindrical coordinates. Find the magnetic energy contained in the volume given by $0 < z < z_0$ and $s < 5R$. [**Ans.** $\mu_0 \pi z_0 k^2 R^2 (1 + \ln 2)$]

17. A circular loop of radius a is placed perpendicular to a uniform magnetic field B_0. The loop has a self-inductance L and a resistance R. At $t = 0$, the field starts decreasing linearly, $B(t) = B_0 - kt$ till it becomes zero. Find the current in the loop as a function of time.

[**Ans.** For $0 < t < B_0/k$, $I(t) = \dfrac{\pi a^2 k}{R}(1 - e^{-Rt/L})$; for $t > B_0/k$, $I(t) = \dfrac{\pi a^2 k}{R}\left(1 - e^{-\frac{RB_0}{kL}}\right)e^{-\frac{R}{L}\left(t - \frac{B_0}{k}\right)}$]

18. A rectangular metallic loop of negligible resistance and inductance L is placed in a magnetic field as shown in Figure 18.E7. The height of the loop is h and its mass is M. Part of the loop is in the field $\boldsymbol{B} = -B_1\hat{\boldsymbol{k}}$ and another part is in the field $\boldsymbol{B} = -B_2\hat{\boldsymbol{k}}$. At $t = t_0$, the loop is given a small velocity v_0. Show that the loop executes simple harmonic motion and find the frequency and amplitude of this motion. [**Ans.** $\dfrac{|B_1 - B_2|\, l}{2\pi\sqrt{ML}}, \dfrac{v_0\sqrt{ML}}{|B_1 - B_2|}$]

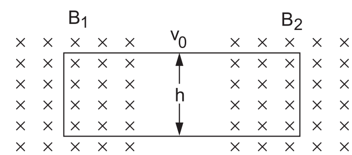


Fig. 18.E7

19. Suppose the loop in the above question has a small resistance R. Set up a differential equation for the current in it. (a) Show that the current oscillates with decreasing amplitude. (b) Find the time in which the current amplitude decreases by a factor of e. (c) Find the angular frequency of the oscillation of the current. [**Ans.** (b) $\dfrac{2L}{R}$, (c) $\sqrt{\dfrac{B_0^2 l^2}{m} - \dfrac{R^2}{4L^2}}$]

20. A rectangular loop abcd of width d and height h is placed close to a long straight wire. The wire is in the plane of the loop with the side ad parallel to it. The near arm ad is at a distance x from the wire. The resistance of the loop is R and its self-inductance is L. (a) Find the mutual inductance between the wire and the loop. (b) A constant current i_0 is switched on at $t = 0$. Find the current induced in the loop. [**Ans.** (a) $M = \dfrac{\mu_0 h}{2\pi}\ln\left(1 + \dfrac{d}{x}\right)$ (b) $-\dfrac{M}{L} i_0 e^{-Rt/L}$]

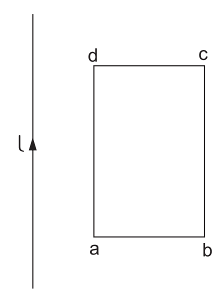


Fig. 18.E8

❑

19 Magnetic Fields Produced by Time-Varying Electric Fields

As you know, a magnetic field is produced by electric currents, and the relation between the field $\boldsymbol{B}$ and the current density $\boldsymbol{J}$ is $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$. But this is true only in magnetostatics. Only for steady currents do you have $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$. So, what happens if you have non-steady currents, that is, if charges move in such a way that the electric fields are time-dependent? You know that a time-dependent magnetic field is accompanied by an electric field and the equation $\nabla \times \boldsymbol{E} = 0$, which is valid for the electrostatic case, is generalized to $\nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}$ in the case of a time-varying magnetic field. In this sense, $\dfrac{\partial \boldsymbol{B}}{\partial t}$ can be taken as a source of an electric field. The other equation $\nabla \cdot \boldsymbol{E} = \dfrac{\rho}{\varepsilon_0}$ remains valid even for time-varying magnetic fields. The first question is, "If a time-varying magnetic field can be a source of an electric field, can a time-varying electric field be a source of a magnetic field?" The answer, based on experimental findings, is yes. You have already seen that a moving charge produces an electric field as well as a magnetic field, and a moving magnet produces a magnetic field as well as an electric field. A time-varying electric field is indeed accompanied by a magnetic field. The next question is, "How do the basic laws for a magnetic field, $\nabla \cdot \boldsymbol{B} = 0$ and $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$, get modified if time-varying electric fields are present?"

19.1 Modification in Ampere's Law

A changing electric field acts as a source of a magnetic field. However, it is not a point-like source and hence the divergence of the field generated by the changing electric field is still zero, that is, the equation $\nabla \cdot \boldsymbol{B} = 0$ remains valid even if the magnetic field corresponds to a changing electric field. But the curl equation gets modified. The magnetic field produced by a current is governed by the equation $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$. Now, it is $\partial \boldsymbol{E}/\partial t$ which is the source of the magnetic field. Can you guess what the relation between $\nabla \times \boldsymbol{B}$ and $\partial \boldsymbol{E}/\partial t$ is? Faraday's law can give you a hint. The electric field produced by a changing magnetic field obeys the equation $\nabla \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$.

With this information, a natural guess is that the curl of the magnetic field corresponding to $\partial \boldsymbol{E}/\partial t$ is proportional to $\partial \boldsymbol{E}/\partial t$. And that is indeed the case. So, write

$$\nabla \times \boldsymbol{B} = k \frac{\partial \boldsymbol{E}}{\partial t}. \tag{i}$$

Do some dimensional analysis to get more information about the proportionality constant k.

$$\text{Dim}\,[\nabla \times \boldsymbol{B}] = \text{Dim}\,[\mu_0 J] = \text{Dim}\,[\mu_0]\,\text{Dim}\left[\frac{q}{tA}\right],$$

where q is charge, t is time and A is area. Current density is current/area and current is charge/time. Also,

$$\text{Dim}\left[\frac{\partial \boldsymbol{E}}{\partial t}\right] = \text{Dim}\left[\frac{q}{\varepsilon_0 r^2 t}\right] = \text{Dim}\left[\frac{q}{\varepsilon_0 tA}\right].$$

Hence, from (i),

$$\text{Dim}\,[\mu_0]\text{Dim}\left[\frac{q}{tA}\right] = \text{Dim}\,[k]\text{Dim}\left[\frac{1}{\varepsilon_0}\right]\text{Dim}\left[\frac{q}{tA}\right]$$

or $$\text{Dim}\,[k] = \text{Dim}\,[\mu_0 \varepsilon_0].$$

Thus, you can write $k = k'\varepsilon_0\mu_0$, where k' is a dimensionless constant. So,

$$\nabla \times \boldsymbol{B} = k'\mu_0\varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}.$$

This gives the magnetic field produced by the changing electric field. If you also have steady currents, the general equation will be

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + k'\mu_0\varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}. \tag{ii}$$

If there is no time-varying electric field, this equation is reduced to Ampere's law for magnetostatics. The constant k' appearing in equation (ii) can also be worked out. To do this, take the divergence on both sides of this equation.

$$\nabla \cdot (\nabla \times \boldsymbol{B}) = \mu_0 \nabla \cdot \boldsymbol{J} + k'\mu_0\varepsilon_0 \frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{E}). \tag{iii}$$

The divergence of the curl of any field is zero, and so the LHS is zero. Remember the equation of continuity that was derived in the chapter on electric currents. It follows from the conservation of charge and gives

$$\nabla \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0.$$

Also, Gauss's law gives

$$\nabla \cdot \boldsymbol{E} = \rho/\varepsilon_0.$$

Using these in equation (iii),

$$0 = \mu_0\left(-\frac{\partial\rho}{\partial t}\right) + k'\mu_0\varepsilon_0\frac{\partial}{\partial t}\left(\frac{\rho}{\varepsilon_0}\right)$$

or $$k' = 1.$$

Thus, the general equation for curl $\boldsymbol{B}$ becomes

$$\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{J} + \mu_0\varepsilon_0\frac{\partial\boldsymbol{E}}{\partial t}\cdot \tag{19.1}$$

Remember that this equation has not really been derived here. We have only made an intelligent guess to arrive at it. The second term on the right-hand side represents a new kind of physics, and cannot be worked out from any previously known results. The credit for discovering this new kind of physics goes to James Clerk Maxwell, who worked out this term in the 1860s. Equation 19.1 is also called Maxwell–Ampere's law. However, we will continue to call it Ampere's law.

As we will discuss in later chapters, this new term in Ampere's law helped physicists conceive of the existence of electromagnetic waves, which play such an important role in our lives. More on this later.

But Maxwell did not arrive at the modification of Ampere's law in the same way as done here nor were SI units prevalent at that time. In fact, Maxwell arrived at this result taking a very different route, doing some theoretical work on the mechanical properties of a hypothetical substance, which he modelled to carry electric and magnetic fields in terms of various vortices and displacements. You can find more details in the article "Maxwell and displacement current" in *Physics Education*, January 1975 by A F Chalmers. Later, scientists realized that there was no need to assume the existence of any such material. However, the inclusion of $\mu_0\varepsilon_0\partial\boldsymbol{E}/\partial t$ in Ampere's law was perfect. The equations relating $\boldsymbol{E}$ and $\boldsymbol{B}$ fields, charges and currents are all given by the following four equations in the most general case.

$$\nabla\cdot\boldsymbol{E} = \rho/\varepsilon_0$$

$$\nabla\cdot\boldsymbol{B} = 0$$

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}$$

$$\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{J} + \mu_0\varepsilon_0\frac{\partial\boldsymbol{E}}{\partial t}\cdot$$

These equations are collectively known as Maxwell's equations.

An you can see, $\varepsilon_0\dfrac{\partial\boldsymbol{E}}{\partial t}$ plays a role similar to that of current density $\boldsymbol{J}$ in producing a magnetic field. For example, a current density in the z-direction produces a magnetic field in the $\hat{\phi}$-direction. The same is true for $\frac{\partial \boldsymbol{E}}{\partial t}$. If you have an electric field in the z-direction whose magnitude changes with time, the corresponding magnetic field will be in the $\hat{\phi}$-direction. Physicists call $\varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$ $\boldsymbol{J}_{\mathrm{d}}$, the displacement current density.

19.2 Integral Form of Ampere's Law

Consider a closed curve C bounding a surface area S. As usual, put an arrow on the curve to show its positive sense, which also determines the positive side of the area. If you take a small area da on the surface, the corresponding area vector $\boldsymbol{da}$ should be drawn in the positive side.

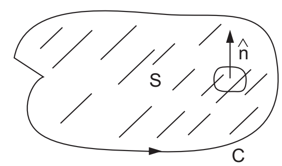


Fig. 19.1

There may be some currents in space, and also a time-varying electric field. Correspondingly, there could be a magnetic field, which too may be time-varying. Integrating both sides of Ampere's law on the surface chosen,

$$\int_S (\nabla \times \boldsymbol{B}) \cdot \boldsymbol{da} = \int_S \mu_0 \boldsymbol{J} \cdot \boldsymbol{da} + \int_S \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \cdot \boldsymbol{da}$$

or
$$\oint_C \boldsymbol{B} \cdot \boldsymbol{dl} = \mu_0 I_{\mathrm{encl}} + \mu_0 \varepsilon_0 \frac{d}{dt} \int_S \boldsymbol{E} \cdot \boldsymbol{da}$$

or
$$\oint \boldsymbol{B} \cdot \boldsymbol{dl} = \mu_0 I_{\mathrm{encl}} + \mu_0 \varepsilon_0 \frac{d}{dt} \Phi_E = \mu_0 \left[I_{\mathrm{encl}} + \varepsilon_0 \frac{d\Phi_E}{dt} \right], \tag{19.2}$$

where Φ_E is the flux of the electric field through the area bounded by the closed curve C and I_{encl} is the current piercing through this area, going from the negative side of the surface to the positive side. This is the integral form of Ampere's law.

You see that $\varepsilon_0 \frac{d\Phi_E}{dt}$ adds to the current I_{encl}. Call it displacement current I_{d} through the loop.

As you know, Ampere's law in the integral form can be used to find the magnetic field due to a current if symmetry permits evaluation of the line integral. Similar is the case with a changing electric field. If symmetry permits, you can use Ampere's law in the integral form to calculate the magnetic field due to a changing electric field. Here is an example.

EXAMPLE 19.1 **A parallel-plate capacitor with large circular plate areas, A for each plate, is connected to an AC voltage source with a small angular frequency ω. As a result, the charge on the capacitor varies with time as $Q = Q_0 \sin(\omega t + \phi)$. Consider a point P close to the centre of the capacitor, at a distance r from its axis. You can assume that the connecting wires attached to the capacitor plates are along the axis and only after large distances do they turn and connect to the voltage source. Find the magnetic field at P.**

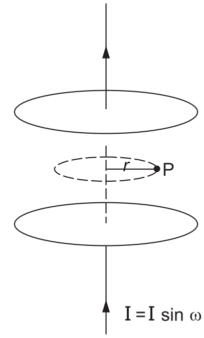


Fig. 19.2

Solution As the bends in the wires and the voltage source are quite far from the capacitor, their effect will be neglected while calculating the magnetic field at P. The situation is then adequately represented by Figure 19.2. Draw a circle of radius r through P, with its axis the same as that of the capacitor. The circular surface bounded by this curve is parallel to the plates and has an area πr^2. We shall use this curve and this area to apply Ampere's law in the integral form.

$$\oint \boldsymbol{B} \cdot \boldsymbol{dl} = \mu_0 I_{\text{encl}} + \mu_0 \varepsilon_0 \frac{\partial \Phi_E}{\partial t}. \tag{i}$$

In the present case, no current passes through the area considered. From the voltage source, a current flows through the wires and on the plates, and all of these are outside the area. So, $I_{\text{encl}} = 0$. Calculate $\dfrac{\partial \Phi_E}{\partial t}$. The charge on the positive plate of the capacitor at time t is

$$Q = Q_0 \sin(\omega t + \phi).$$

The charge density is

$$\sigma = \frac{Q}{A} = \frac{Q_0}{A} \sin(\omega t + \phi).$$

The plate area is large and P is close to the axis. So, we can safely neglect the edge effect (fringing of the electric field near the edges of the capacitor) and write

$$\boldsymbol{E} = \frac{\sigma}{\varepsilon_0} \hat{\boldsymbol{k}} = \frac{Q_0}{\varepsilon_0 A} \sin(\omega t + \phi) \hat{\boldsymbol{k}},$$

where $\hat{\boldsymbol{k}}$ is taken along the axis in the upward direction. So, the flux through the area πr^2 is

$$\Phi_E = \frac{Q_0 \pi r^2}{\varepsilon_0 A} \sin(\omega t + \phi)$$

and
$$\frac{\partial \Phi_E}{\partial t} = \frac{Q_0 \pi r^2 \omega}{\varepsilon_0 A} \cos(\omega t + \phi).$$

A current density in the z-direction produces a magnetic field in the $\hat{\phi}$-direction. Here also, $\dfrac{\partial \boldsymbol{E}}{\partial t}$ is in the z-direction and hence the corresponding magnetic field is in the $\hat{\phi}$-direction.

Now, look at Figure 19.2 carefully. The magnetic field at any point on the circular loop is in the $\hat{\phi}$-direction. Also, there is a cylindrical symmetry in the current and the electric field. So the magnitude B of the magnetic field will be the same at all the points. Thus $\boldsymbol{B} = B\hat{\phi}$ and hence,

$$\oint \boldsymbol{B} \cdot \boldsymbol{dl} = \oint B\hat{\phi} \cdot rd\phi\hat{\phi} = 2\pi rB.$$

Thus, equation (i) becomes

$$2\pi rB = \mu_0 \frac{Q_0 \pi r^2 \omega}{A} \cos(\omega t + \phi)$$

or
$$B = \frac{\mu_0 Q_0 \omega r}{2A} \cos(\omega t + \phi).$$

If you are happy with the given solution to Example 19.1, answer a question. The magnetic field $\boldsymbol{B}$ turns out to be time-dependent. This time-varying magnetic field will have its own electric field, which will also be time-varying. Why did we not take into account this electric field while calculating the net magnetic field? Calculate $\nabla \times \boldsymbol{B}$ and compare it with $\mu_0 \varepsilon_0 \dfrac{\partial \boldsymbol{E}}{\partial t}$. The $\boldsymbol{E}$ and $\boldsymbol{B}$ fields obtained are consistent with each other and hence are the final fields.

EXAMPLE 19.2 **A capacitor is being charged by connecting its plates to a constant-current source. Consider an open surface (Figure 19.3) in the shape of a balloon, which goes around one of the plates of the capacitor, and has curve a as the boundary. The curve is far from the capacitor plates. Show that the equation $\oint \boldsymbol{B} \cdot \boldsymbol{dl} = \mu_0 \boldsymbol{I}_{\mathbf{encl}} + \mu_0 \varepsilon_0 \int \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{t}} \cdot \boldsymbol{da}$ is valid on this surface.**

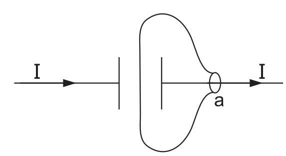


Fig. 19.3

Solution First consider the plane area bounded by the curve a. As the curve is far from the capacitor, there is no electric field here. Remember that, because of fringing, some electric field is present close to the edges of the capacitor, even outside it. But here this can be ignored as the curve a is far away. The current I in the connecting wire goes through the area bounded by the curve a, and so, using Ampere's law for magnetostatics on this loop,

$$\oint_a \boldsymbol{B} \cdot \boldsymbol{dl} = \mu_0 I. \tag{i}$$

Was the positive sense on the curve for calculating the line integral indicated? It was, implicitly. The current I in Figure 19.3 goes from left to right through the plane area bounded by the loop. This is the positive direction of the area used in (i). The sense on the curve a will be anticlockwise, as seen from the right.

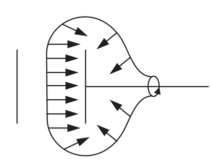


Fig. 19.4

Now, consider the balloon-shaped surface together with the plane surface bounded by the given curve a. This forms a closed surface. If you take all the normals to the different portions of this surface to point inwards, by Gauss's law,

$$\oint \boldsymbol{E}\cdot d\boldsymbol{a} = \frac{q}{\varepsilon_0},$$

where q is the charge on the capacitor. Note that the closed surface encloses the negative plate of the capacitor and the normals are taken inward. As the flat part enclosed by the curve a is far from the capacitor and there is no electric field there, $\int \boldsymbol{E}\cdot d\boldsymbol{a}$ on this part is zero. Hence,

$$\int\limits_{\text{balloon}} \boldsymbol{E}\cdot d\boldsymbol{a} = \frac{q}{\varepsilon_0}$$

or $$\mu_0\varepsilon_0 \frac{d}{dt}\int \boldsymbol{E}\cdot d\boldsymbol{a} = \mu_0\varepsilon_0 \frac{q}{\varepsilon_0}\frac{d}{dt},$$

where the integration is only on the balloon-shaped surface.

Thus, $$\mu_0\varepsilon_0 \int \frac{\partial \boldsymbol{E}}{\partial t}\cdot d\boldsymbol{a} = \mu_0 I. \tag{ii}$$

Comparing (i) and (ii),

$$\oint\limits_a \boldsymbol{B}\cdot d\boldsymbol{l} = \mu_0\varepsilon_0 \int \frac{\partial \boldsymbol{E}}{\partial t}\cdot d\boldsymbol{a}.$$

As $I_{\text{encl}} = 0$ for the balloon-shaped surface,

$$\int\limits_a \boldsymbol{B}\cdot d\boldsymbol{l} = \mu I_{\text{encl}} + \mu_0\varepsilon_0 \int \frac{\partial \boldsymbol{E}}{\partial t}\cdot d\boldsymbol{a}.$$

19.3 The Poynting Vector

The vector $\boldsymbol{S} = \dfrac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0}$ is a very important quantity and is related to the flow of electromagnetic energy (electric field energy and magnetic field energy) at the point where it is calculated. It is known as the Poynting vector. Let us see how it is related to energy flow.

First consider a region free of any charge or current, meaning that $\rho = 0$ and $J = 0$ in this region. Thus, Maxwell's equations are

$$\nabla\cdot\boldsymbol{E} = 0, \qquad \nabla\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla\cdot\boldsymbol{B} = 0, \qquad \nabla\times\boldsymbol{B} = \varepsilon_0\mu_0 \frac{\partial \boldsymbol{E}}{\partial t}\cdot$$

The Poynting vector is $\boldsymbol{S} = \dfrac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0}$ by definition. Taking the divergence on both sides,

$$\begin{aligned}
\nabla\cdot\boldsymbol{S} &= \nabla\cdot\frac{\boldsymbol{E}\cdot\boldsymbol{B}}{\mu_0} \\
&= \frac{1}{\mu_0}[(\nabla\times\boldsymbol{E})\cdot\boldsymbol{B} - (\nabla\times\boldsymbol{B})\cdot\boldsymbol{E}] \\
&= \frac{1}{\mu_0}\left[\left(-\frac{\partial\boldsymbol{B}}{\partial t}\right)\cdot\boldsymbol{B} - \left(\mu_0\varepsilon_0\frac{\partial\boldsymbol{E}}{\partial t}\right)\cdot\boldsymbol{E}\right] \qquad \text{(we are talking about free space, so } \boldsymbol{J} = 0\text{)} \\
&= \frac{1}{\mu_0}\left[\left(-\frac{1}{2}\frac{\partial}{\partial t}(\boldsymbol{B}\cdot\boldsymbol{B})\right) - \left(\mu_0\varepsilon_0\frac{1}{2}\frac{\partial}{\partial t}\boldsymbol{E}\cdot\boldsymbol{E}\right)\right] \\
&= -\left[\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu_0}\right) + \frac{\partial}{\partial t}\left(\frac{\varepsilon_0 E^2}{2}\right)\right] \\
&= -\frac{\partial}{\partial t}\left[\frac{B^2}{2\mu_0} + \frac{1}{2}\varepsilon_0 E^2\right]
\end{aligned}$$

or
$$\nabla\cdot\boldsymbol{S} = -\frac{\partial}{\partial t}[u_{\text{EM}}], \tag{19.3}$$

where u_{EM} is the combined energy density corresponding to electric and magnetic fields.

Remember a similar equation worked out while discussing electric currents. The equation of continuity for charges is

$$\nabla\cdot\boldsymbol{J} = -\frac{\partial}{\partial t}\rho.$$

ρ is the charge density and $\boldsymbol{J}$ is the current density, giving the rate at which charge flows. In Equation 19.3, u_{EM} is the energy density and hence $\boldsymbol{S}$ gives the rate at which energy flows, that is, energy flowing per unit area per unit time.

Now, consider any closed surface and integrate both sides of Equation 19.3 over the volume enclosed. Then,

$$\int(\nabla\cdot\boldsymbol{S})d\tau = -\frac{\partial}{\partial t}\int u_{\text{EM}}d\tau$$

or
$$\oint\boldsymbol{S}\cdot d\boldsymbol{a} = -\frac{\partial}{\partial t}[U_{\text{EM}}]. \tag{19.4}$$

What is there on the right-hand side? $U_{\text{EM}} = \int u_{\text{EM}}d\tau$ is the total electromagnetic energy in the volume enclosed by the closed surface. Thus, the right-hand side of Equation 19.4 gives the rate at which energy is going out of the volume. It must be going through the surface. There is nothing in the volume that can create or absorb energy in any other form. We are talking about a volume of free space having no charge, current or any other object. The only form of energy is electromagnetic field energy. Then, $\oint \boldsymbol{S}\cdot \boldsymbol{da}$ is the total energy per unit time crossing the entire closed surface. It is natural to assume that $\boldsymbol{S}\cdot\boldsymbol{da}$ is the energy crossing *da* per unit time, which is true in many cases but not always.

In equations 19.3 and 19.4, it is assumed that electric and magnetic fields exist in free space, having no charges or currents. Consider a more general case, where there may be charges and currents. Let us see how these equations get modified.

$$\nabla\cdot\boldsymbol{S} = \nabla\cdot\frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0} = \frac{1}{\mu_0}[(\nabla\times\boldsymbol{E})\cdot\boldsymbol{B} - (\nabla\times\boldsymbol{B})\cdot\boldsymbol{E}]$$

$$= \frac{1}{\mu_0}\left[-\boldsymbol{B}\cdot\frac{\partial\boldsymbol{B}}{\partial t} - \boldsymbol{E}\cdot\left(\mu_0\boldsymbol{J} + \mu_0\varepsilon_0\frac{\partial\boldsymbol{E}}{\partial t}\right)\right]$$

$$= -\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu_0} + \frac{1}{2}\varepsilon_0 E^2\right) - \boldsymbol{E}\cdot\boldsymbol{J}.$$

Thus, $$\int\nabla\cdot\boldsymbol{S}d\tau = -\frac{\partial}{\partial t}\int u_{\mathrm{EM}}d\tau - \int\boldsymbol{E}\cdot\boldsymbol{J}d\tau$$

or $$\oint\boldsymbol{S}\cdot\boldsymbol{da} = -\frac{\partial}{\partial t}[U_{\mathrm{EM}}] - \int\boldsymbol{E}\cdot\boldsymbol{J}d\tau. \tag{19.5}$$

What is $\int\boldsymbol{E}\cdot\boldsymbol{J}d\tau$? We have assumed charges and currents in the volume. The force due to the electric field $\boldsymbol{E}$ on a charge element $\rho d\tau$ is

$$\boldsymbol{F} = (\rho d\tau)\boldsymbol{E}$$

and in a small displacement $\boldsymbol{dr}$, the work done by this force is

$$dW = \boldsymbol{F}\cdot\boldsymbol{dr} = \rho d\tau\boldsymbol{E}\cdot\boldsymbol{dr} = \rho d\tau\boldsymbol{E}\cdot\boldsymbol{v}dt$$

$$= (\rho\boldsymbol{v})\cdot\boldsymbol{E}d\tau\, dt = \boldsymbol{E}\cdot\boldsymbol{J}d\tau\, dt.$$

No work is done on this charge due to the magnetic force $\boldsymbol{F} = (\rho d\tau)\boldsymbol{v}\times\boldsymbol{B}$. The total work done on all the charges in the volume, in time dt, is therefore equal to $(\int\boldsymbol{E}\cdot\boldsymbol{J}d\tau)dt$. Thus, $\int\boldsymbol{E}\cdot\boldsymbol{J}d\tau$ is the work done on all the charges in the given volume per unit time. The work done on a system increases its kinetic energy. In the case of charge carriers in materials, this kinetic energy is finally distributed in the whole of the material due to collisions and hence appears as thermal energy. Let us check this using a simple example.

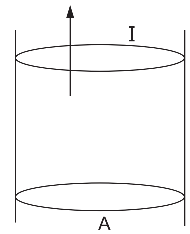


Fig. 19.5

EXAMPLE 19.3 **A cylindrical wire carries a current I that is uniformly distributed over its cross section A. Consider a length l of this wire. Show that $\int \boldsymbol{E}\cdot\boldsymbol{J}d\tau$ over this length of wire is equal to I^2R, where R is the resistance of this portion.**

Solution If the conductivity of the wire is σ,

$$\boldsymbol{J} = \sigma\boldsymbol{E}.$$

So,
$$\int \boldsymbol{E}\cdot\boldsymbol{J}d\tau = \int \frac{\boldsymbol{J}}{\sigma}\cdot\boldsymbol{J}d\tau = \int \frac{I^2}{A^2\sigma}d\tau = \frac{I^2}{A^2\sigma}\cdot Al$$

$$= I^2\left(\frac{1}{\sigma}\frac{l}{A}\right) = I^2R.$$

Thus, $\int \boldsymbol{E}\cdot\boldsymbol{J}d\tau$ is the rate at which thermal energy is produced.

As the thermal energy of the charge system comes from the work done by the electromagnetic field on the system, the energy of the electromagnetic field itself decreases by the same amount.

You can write Equation 19.5 as

$$\oint \boldsymbol{S}\cdot\boldsymbol{da} + \int \boldsymbol{E}\cdot\boldsymbol{J}d\tau = -\frac{\partial}{\partial t}[U_{\mathrm{EM}}]. \tag{19.6}$$

You can interpret it easily. The first term $\oint \boldsymbol{S}\cdot\boldsymbol{da}$ gives the electromagnetic field energy going out of the volume. The second term $\int \boldsymbol{E}\cdot\boldsymbol{J}d\tau$ gives the loss in electromagnetic energy per unit time due to the work done on the charges. Therefore, the left-hand side gives the net rate of loss of electromagnetic field energy from the volume. The right-hand side is just the same. Thus, the energy balance is complete.

You can interpret this in a different way too. Write Equation 19.5 as

$$-\oint \boldsymbol{S}\cdot\boldsymbol{da} = \oint \boldsymbol{E}\cdot\boldsymbol{J}d\tau + \frac{\partial}{\partial t}[U_{\mathrm{EM}}]. \tag{19.7}$$

The LHS tells us how much energy is going into the volume per unit time. The first term on the RHS, $\oint \boldsymbol{E}\cdot\boldsymbol{J}d\tau$, is the energy contributing to thermal energy per unit time. The rest is the increase in the EM energy of the volume per unit time.

Equations 19.5, 19.6 and 19.7 represent the Poynting theorem, which is always valid for a closed surface. But the interpretation that $\boldsymbol{S}\cdot\boldsymbol{da}$ is the energy crossing $\boldsymbol{da}$ per unit time for each element $\boldsymbol{da}$ may not work in certain cases.

The flow of energy in a battery–resistance circuit

Consider a resistance connected to a battery by thick wires of negligible resistance (Figure 19.6). A constant current $I = \varepsilon/R$ passes through the circuit. The thermal energy produced in the resistance is I^2R per unit time. The same is the work done by the battery. So, the battery supplies an amount of energy per unit time equalling I^2R, and this energy is absorbed by the resistance. The thermal energy of the resistance increases by the same amount. The question is, "How does energy travel from the battery to the resistor?"

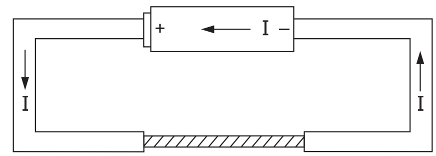


Fig. 19.6

What we will now describe is based on an article in *American Journal of Physics*, volume 73, p. 140, 2005.

Let us first look at the battery itself. Assume it to have a cylindrical volume of radius a and let the current pass uniformly through this volume. The electric field is from the positive terminal to the negative terminal, that is, from left to right as shown in Figure 19.7. Taking the z-axis to be towards the right, the electric field $\boldsymbol{E}$ *in the battery volume is in the* $\hat{\boldsymbol{k}}$*-direction*. The current inside the battery is from right to left, that is, in the $-\hat{\boldsymbol{k}}$-direction. The magnetic field is therefore in the $-\hat{\phi}$-direction.

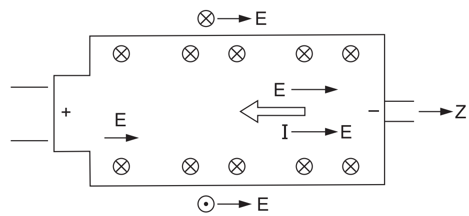


Fig. 19.7

The Poynting vector in the volume of the battery is

$$\boldsymbol{S} = \frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0} = \frac{E\,\hat{\boldsymbol{k}}\times B(-\hat{\phi})}{\mu_0} = \frac{EB}{\mu_0}\hat{\boldsymbol{s}}.$$

This extends up to the surface of the battery. Thus, energy flows out of the battery from the cylindrical surface axially. As the tangential component of the electric field is continuous, just outside the surface of the battery too, the Poynting vector is given by the same expression. Here

$$E = \frac{V}{L} \text{ and } B = \frac{\mu_0 I}{2\pi a}\cdot$$

So,
$$\boldsymbol{S} = \frac{VI}{2\pi aL}\hat{\boldsymbol{s}}.$$

The total energy going out of the battery per unit time is

$$\oint \boldsymbol{S}\cdot d\boldsymbol{a} = \frac{VI}{2\pi aL}\cdot 2\pi aL = VI.$$

So, in our model of current uniformly going into the cylindrical volume of the battery and the existence of a uniform electric field in this volume, the energy supplied by the battery leaves it through the cylindrical surface in the axial direction.

Now, consider the connecting wires. First, take the wire to the left of the battery. As we have neglected its resistance, the potential does not drop along it and hence there is no electric field in the wire. The current produces a magnetic field in the $-\hat{\phi}$-direction both inside and outside the wire (Figure 19.8). The Poynting vector inside the wire is zero as there is no electric field. So, energy does not travel in the wire along with the current.

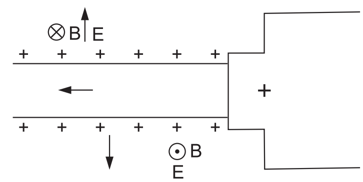


Fig. 19.8

However, there could be an electric field just outside the wire. This is because there could be a surface charge density on the wire. As the tangential electric field is continuous across the surface, the field just outside the wire is perpendicular to the surface, that is, in the axial direction. This will be the case if the charge density on the surface is uniform. If λ be the charge per unit length on the wire, the electric field outside the wire is

$$\boldsymbol{E} = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{\boldsymbol{s}}.$$

Why is a charge density needed on the wire, even away from any bend and with no electric field? Since the electric field in the wire is zero, the wire need not have charges on the surface, but it is connected to the resistor and battery. The boundary conditions of the fields require that some charge density exists on the surface of the thick connecting wires too.

The Poynting vector outside the wire is

$$\boldsymbol{S} = \frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0} = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{s}\times\frac{I}{2\pi s}(-\hat{\phi}) = \frac{\lambda I}{4\pi^2 s^2\varepsilon_0}(-\hat{\boldsymbol{k}}).$$

Energy flows parallel to the wire but outside it.

Now, look at the connecting wire to the right of the battery (Figure 19.9). If the charge appearing on the wire on the left is positive, that appearing on the wire on the right should be negative. After all, the entire system is charge-neutral. If the arrangement is symmetrical as suggested in Figure 19.6, the charge density here will be $-\lambda$ per unit length.

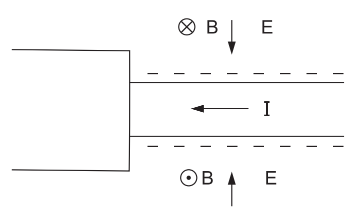


Fig. 19.9

The electric field inside the wire is zero and that outside is

$$\boldsymbol{E} = \frac{\lambda}{2\pi\varepsilon_0 s}(-\hat{s}).$$

The magnetic field is

$$\boldsymbol{B} = \frac{\mu_0 I}{2\pi s}(-\hat{\phi}).$$

The Poynting vector is

$$\boldsymbol{S} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0} = \frac{\lambda}{2\pi\varepsilon_0 s}\frac{I}{2\pi s}\hat{\boldsymbol{k}} = \frac{\lambda I}{4\pi^2 s^2 \varepsilon_0}\hat{\boldsymbol{k}}.$$

The Poynting vector is in the $\hat{\boldsymbol{k}}$-direction. Energy is flowing opposite to the direction of current. Similar will be the case with the wire in the perpendicular direction. With all this information, you can draw a qualitative path for the flow of energy from the battery to the resistance (Figure 19.10).

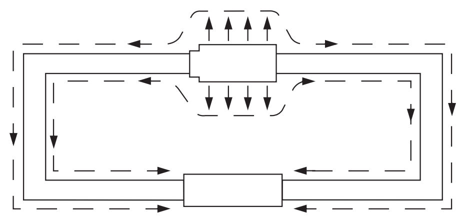


Fig. 19.10

Now come to the resistance. Assume this too to be a cylindrical wire of resistivity ρ and radius R.

The electric field inside the wire is

$$\boldsymbol{E} = \frac{\boldsymbol{J}}{\sigma} = \rho\boldsymbol{J} = \frac{\rho I}{\pi R^2}\hat{\boldsymbol{k}}.$$

Just outside the wire, the electric field has the same tangential component. It will also have a normal component outside as there will be a surface charge density. To produce an electric field along the $\hat{\boldsymbol{k}}$-direction inside the wire, there should be a varying charge density on the wire. Figure 19.11 shows such a charge distribution.

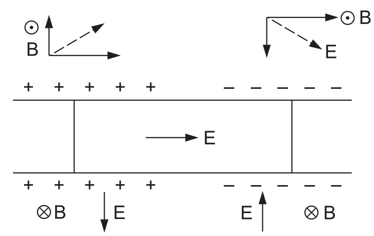


Fig. 19.11

The charge density λ is positive and the highest, λ_0, at the left end. As you go towards the right, λ gradually decreases and becomes zero in the middle. It becomes negative, increases and becomes $-\lambda_0$ at the right end.

The normal component of the electric field outside the resistive wire varies as suggested in Figure 19.11. The magnetic field is in the $\hat{\phi}$-direction as the current is in the $\hat{\boldsymbol{k}}$-direction. Thus, the Poynting vector has an axial component going towards the wire. The tangential component is directed towards the centre of the wire. Energy thus enters the wire from the surface on the side, as suggested in Figure 19.12.

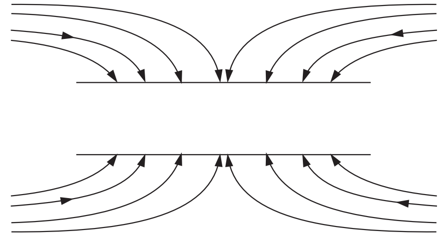


Fig. 19.12

Concepts at a Glance

1. A changing electric field acts as a source of a magnetic field.
2. The equation $\nabla \cdot \boldsymbol{B} = 0$ remains unchanged even if time-varying fields are present.
3. The curl equation is modified to $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \dfrac{\partial \boldsymbol{E}}{\partial t}$.
4. The energy density in a magnetic field is $u_B = \dfrac{B^2}{2\mu_0}$.
5. The Poynting vector is defined as $\boldsymbol{S} = \dfrac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0}$. It represents the rate at which electromagnetic energy flows across a surface.

EXERCISES

Based on Concepts

1. Show that the electric field in a charged parallel-plate capacitor cannot abruptly become zero when the capacitor plates have uniform charge. Can it do so if the charge on the capacitor is varying?
2. A displacement current produces a magnetic field like a conventional current. Can a displacement current produce heat in a wire in the same way as a conventional current?
3. A magnet is placed between the two plates of a charged capacitor. There is an electric field due to the capacitor plates and a magnetic field due to the magnet, giving a nonzero Poynting vector. Is energy flowing in the region?

Problems

1. Starting from the Maxwell–Ampere law $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \dfrac{\partial \boldsymbol{E}}{\partial t}$, derive the equation of continuity for electric charges.
2. Consider a parallel-plate capacitor immersed in sea water and driven by a voltage $V_0 \cos(\omega t)$ at a frequency $\nu = 4 \times 10^8$ Hz. Sea water has a permittivity $\varepsilon = 81\varepsilon_0$, permeability $\mu = \mu_0$, and resistivity $\rho = 0.23\ \Omega$ m. What is the ratio of the rms conduction current to the rms displacement current? [**Ans.** 2.4]
3. A voltage source $V = V_0 \sin \omega t$ is connected between two concentric conducting spheres of radii $r = a$ and $r = b$ $(b > a)$. The volume in the region $b > r > a$ is filled with a material of dielectric constant K. Find the total displacement current I_d and the free current I_f through the dielectric material.

   [**Ans.** $I_\mathrm{d} = -\dfrac{4\pi\varepsilon_0 K a b \omega}{(b-a)} \nu_0 \sin \omega t,\ I_\mathrm{f} = -\dfrac{4\pi\sigma a b \omega \nu_0 \sin \omega t}{b-a}$]
4. Consider a parallel-plate capacitor in which the space between the plates is filled with a poorly conducting nonmagnetic material (not a dielectric). The capacitor is charged and then the battery is disconnected. Ignoring edge effects (fringing) show that there is no magnetic field inside.

5. A thin, cylindrical wire ab of length l, area of cross section A and resistance R is connected to a battery of emf V and an inductor of inductance L. The switch is put on at $t = 0$. Find the magnitude of $\nabla \times \boldsymbol{B}$ at a point inside the wire as a function of time.

[**Ans.** $\dfrac{\mu_0 \varepsilon}{RA}\left\{1 - e^{-t\frac{R}{L}}\left(1 - \dfrac{\varepsilon_0 A R^2}{L l_0}\right)\right\}$]

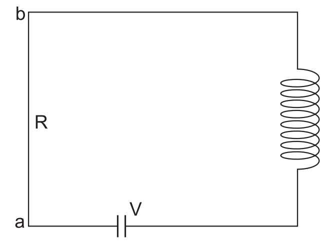


Fig. 19E.1

6. A positive charge q is moving along the positive x-axis with a velocity $v << c$. Determine the displacement current density J_d at a point (a) $x = x_0$ on the x-axis, (b) $y = y_0$ on the y-axis when the charge q is at the origin. [**Ans.** (a) $\dfrac{2qv}{4\pi\varepsilon_0 x_0^3}\hat{\boldsymbol{i}}$, (b) $\dfrac{qv}{4\pi\varepsilon y_0^3}$]

7. An alternating current $i = i_0 \cos \omega t$ passes along a long, straight wire of radius a and returns along the surface of a coaxial cylindrical shell of radius ea. (a) Find the displacement current density $\boldsymbol{J}_d$. (b) Integrate it to get the total displacement current $I_d = \int \boldsymbol{J}_d \cdot d\boldsymbol{a}$ through a cross section. (c) Suppose the wire is 2 mm in diameter. Estimate the frequency for which $I_d/I = 0.01$.

[**Ans.** With $A = \dfrac{i_0 \omega^2 \cos \omega t}{2c^2}$, (a) $A \ln \dfrac{ea}{s}\hat{\boldsymbol{k}}$ for $s > a$, $A\dfrac{(3a^2 - s^2)}{2a^2}\hat{\boldsymbol{k}}$ for $s < a$, (b) $\dfrac{\pi a^2 A}{4}(7 + 2e^2)$, (c) $\approx 10^{10}$ Hz]

8. A long, straight wire of conductivity 6×10^7 S/m and cross-sectional area 1 mm^2 carries a time-varying current for a short period. At $t = 0$, the current is 2 A and changes at a constant rate to 5 A in 1 ms. Calculate the displacement current through a cross section of the wire. Assume that the current is uniformly distributed over the cross section. [**Ans.** 1.48×10^{-16} A]

9. A parallel-plate capacitor is getting charged at such a rate that its voltage is increased at 300 V/s. The plates are circular with a radius of 10 cm and are separated by 1 cm. Calculate the displacement current density j_d and the magnetic field B at a distance of 5.0 cm from the axis of the capacitor, in the space between the plates. [**Ans.** 8.2×10^{-7} T]

10. An air-filled parallel-plate capacitor has plates of radius 10 cm that are separated by a distance of 2 mm. If this capacitor is connected to a 50-V peak, l-MHz sinusoidal voltage source, calculate the magnitude of the maximum displacement current density. [**Ans.** 1.4 A/m^2]

11. The electric field in a large region varies as follows. $\boldsymbol{E}(\boldsymbol{r}, t) = \dfrac{k}{r^2}\hat{\boldsymbol{r}}$ if $r < vt$, $= 0$ if $r > vt$. (a) Find the charge density everywhere. (b) Find the displacement current density at (r, θ, ϕ) as a function of time t.

[**Ans.** (a) $4\pi k \delta(\boldsymbol{r})$, (b) $\dfrac{\varepsilon_0 k}{r^2}\delta\left(t - \dfrac{r}{v}\right)\hat{\boldsymbol{r}}$]

12. A parallel-plate capacitor is made by placing two circular plates, each of radius R, at $z = -\dfrac{a}{2}$ and $z = \dfrac{a}{2}$. The capacitor is being charged by connecting wires along the z-axis and sending a constant current I in it. Find the magnetic field at $(s, \phi, 0)$, where $s >> R$. [**Ans.** $\dfrac{\mu_0 I}{2\pi s}\hat{\phi}$]

13. Two long, straight wires carry a constant current I in the z-direction to a parallel-plate capacitor having circular plates (Figure 19E.2). Take the centre of the capacitor as the origin. Each plate has a radius R which is much larger than the separation between the plates. The wires are connected to the centres of the plates. Assume that the surface charge is uniform over the surface of the plates at any given time, being zero at $t = 0$. (a) What is the displacement current $\int \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \cdot d\boldsymbol{a}$ in the space between the plates? (b) Consider a circle of radius s (< R) in the plane z = 0 (upper part of the figure). Get the magnetic field at the periphery of this circle from the Ampere–Maxwell equation $\oint \vec{B} \cdot d\vec{l} = \mu_0 \int \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{a}$ using this circle as the closed curve and the flat circular area enclosed by it as the surface area bounded by it. (c) Now consider an open cylindrical surface, as shown in the lower part of the figure, with the circle of radius a as the periphery. Explicitly evaluate the right side of the Ampere–Maxwell equation on this surface and see that it is the same as that calculated in part (b). Neglect any fringing effect. [**Ans.** (a) I, (b) $\frac{\mu_0 Is}{2\pi R^2}$]

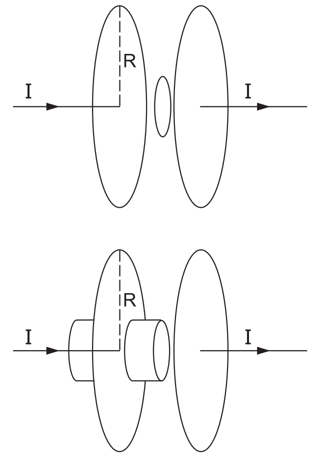


Fig. 19E.2

14. A parallel-plate capacitor with plate area A (assumed to be large) and capacitance C is being charged by connecting it to a battery of emf ε and resistance R. Find the value of $\nabla \times \boldsymbol{B}$ at the centre of the capacitor. [**Ans.** $\frac{\mu_0 \varepsilon}{AR} e^{-t/RC}$]

15. Suppose a cylindrical conductor of length d and area of cross section A is connected to two thin wires at the end faces as shown in Figure 19.E3. Suppose the wires carry a current I, which varies with time. Also suppose the electric field in the conductor is maintained by charges Q and $-Q$ appearing at the end faces.

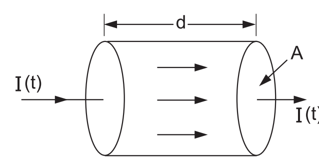


Fig. 19E.3

(a) Show that $\varepsilon_0 \rho \frac{dI_1}{dt} + I_1 = I$, where I_1 is the conduction current in the conductor.

(b) Show that the first term in the above equation represents the displacement current through a cross section of the conductor.

16. Consider a parallel-plate capacitor being charged with a current I. Let the plates be circular, the circles being of radius R, and let the separation between the plates be d, which is much smaller than R. Take the z-axis to be along the axis of the capacitor. Consider a cylindrical surface ($s = R$) extending from one plate to the other. Neglect any fringing. (a) Calculate the Poynting vector $\boldsymbol{S}$ at the surface at a time t when the charge on the capacitor is q. (b) Integrate this over the cross secion to find the total electromagnetic energy ΔU flowing into the space between the plates in the next time interval Δt. (c) Knowing that the energy of a capacitor is $U = \frac{Q^2}{2C}$, calculate the change in energy ΔU of the capacitor in this time interval Δt.

17. A very long solenoid of radius a, with n turns per unit length, carries a current I. A circular ring of much larger radius b and having a resistance R is placed coaxially with the solenoid. The current in the

solenoid is gradually increased and this induces a current in the ring. (a) Find the induced electric field just outside the solenoid. (b) Find the magnetic field just outside the solenoid, produced by the current in the ring. Remember that it is a function of the height above the plane of the ring. (c) Determine the power loss in the ring. (d) Find the Poynting vector just outside the solenoid and from this the total energy flowing away from the solenoid. Compare it with the answer to part (c). (e) Sketch the direction of energy flow at various places.

18. A long, cylindrical wire of radius a is surrounded by a coaxial cylindrical tube of radius b (Figure 19.E4). Assume the two to be resistanceless. They are connected to a battery of emf ε_0 at one end and to a resistor of resistance R at the other end. Consider the region between the wire and the tube, far away from the ends. (a) Find the surface charge density appearing on the inner wire. (b) What is the electric field in the space between the two, i.e., $a < s < b$? (c) Work out the magnetic field in the region $a < s < b$. (d) Find the Poynting vector $\boldsymbol{S}$ in the region $a < s < b$. (e) Consider a cross section, $a < s < b$, in the space between the two conductors. How much is the energy flowing through this area per unit time?

[**Ans.** (a) $\dfrac{\varepsilon_0 V}{a \ln \frac{b}{a}}$, (b) $\dfrac{V}{s \ln \frac{b}{a}}\hat{\boldsymbol{s}}$, (c) $\dfrac{\mu_0 V}{2\pi r s R}\hat{\phi}$, (d) $\dfrac{V^2}{2\pi R s^2 \ln \frac{b}{a}}(-\hat{\boldsymbol{k}})$, (e) $\dfrac{\varepsilon_0^2}{R}$]

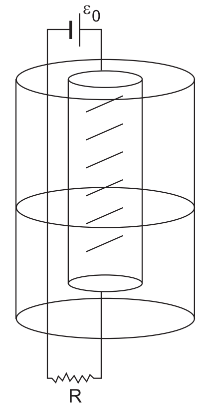


Fig. 19E.4

19. A charge q is going in the x-direction at speed v, not very small as compared to c. Calculate the Poynting vector $\boldsymbol{S}$ at the point $(x, y, 0)$ at the instant q passes the origin. What is the shape of the field lines of $\boldsymbol{S}$ in the x–y plane at this instant? [**Ans.** Circular]

20. Let V and $\boldsymbol{A}$ be the scalar and vector potentials corresponding to a given field combination. Assuming that $\nabla \cdot \boldsymbol{A} + \mu_0\omega_0 \dfrac{\partial V}{\partial t} = 0$, prove that $\nabla^2 \boldsymbol{A} - \mu_0\varepsilon_0 \dfrac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu_0 \boldsymbol{J}$.

❑

20 Electromagnetic Waves in Free Space

Electric and magnetic fields and their relations with charges and currents are given by the four equations presented by Maxwell. We have discussed electrostatics and magnetostatics, where the fields do not vary with time. These fields are produced by charges and currents respectively. Even though the sources are confined to a finite volume, the fields can extend to other regions. We have also discussed time-varying fields and found that a time-varying electric field can be treated as a source of magnetic field, and that a time-varying magnetic field can be treated as a source of electric field. The characteristics of electric and magnetic fields depend on the source distribution.

We shall now discuss a combination of time-varying electric and magnetic fields that is self-sustaining. Once created, electric and magnetic fields can exist without charges and currents. A time-varying electric field creates a magnetic field and a time-varying magnetic field produces an electric field. This way they keep each other going. Such a combination of fields is called *electromagnetic radiation* or an *electromagnetic wave*.

Electromagnetic waves are always around us. When you use television sets, radios, mobile phones and the Internet, you use electromagnetic waves. Even if you don't use modern electronic gadgets, you are surrounded by electromagnetic waves in the form of light. Also, all hot objects emit electromagnetic waves, which we feel as radiant heat.

20.1 General Concepts of Waves

Before talking about the theory of electromagnetic waves, let us recall a few general concepts of wave motion. The waves we are most familiar with are perhaps sound waves and waves on a string. In each of these, there is a medium, and the wave corresponds to some kind of disturbance in the medium. This disturbance is described in terms of a wave quantity that has an equilibrium value and any change from this value is a disturbance. For sound waves, this quantity is pressure in air. There is an equilibrium pressure p_0 and as the pressure increases or decreases from p_0, we say that a disturbance has occurred. The quantity $\Delta p = p - p_0$ is the wave quantity. Similarly, a displacement from the mean position is the wave quantity for a transverse wave on a string.

When a disturbance created in one part of the medium propagates and reaches other parts of it, we say that a wave has gone through the medium.

An Interesting Experiment

Take a cellotape of width about 1 cm, and spread a length of about 3 m of the tape on the floor, with the sticky part facing up. Put some weights at the ends so that the tape remains in place. Now take some drinking straws or long toothpicks and make them lie on the tape perpendicularly at regular intervals of 2–3 cm. Draw another length of the cellotape and put it over the straws, the sticky part facing down. The two sticky parts thus join and the straws get sandwiched between the two layers of tape.

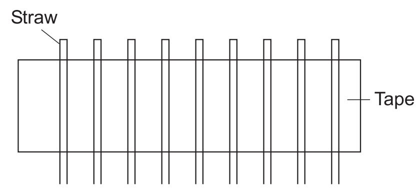


Fig. 20.1

Now, hold one end of the tapes and ask your friend to hold the other end. Stretch the tapes from both sides so that they remain tight and the straws are vertical. Jerk one of the straws with your fingers so that it tilts in one direction and comes back. You will see that the disturbance travels down the line. Each successive straw tilts in the same direction and comes back. Each straw remains in place, except when it gets tilted for a short time. The tilting (disturbance) is passed on to the next straw. The motion of the disturbance down the line is wave motion.

If the wave travels at a particular speed v, it moves a distance vt in time t. Suppose it moves in the x-direction. Then, whatever is the value of the wave quantity Φ at the point $x-vt$ at time $t=0$, the same will be the value at the point x at time t. In other words,

$$\Phi(x, t) = \Phi(x - vt, 0).$$

This means that, for a wave moving in the x-direction, x and t occur in the function Φ in the combination $x - vt$. For example, it may be

$$\Phi(x, t) = A \sin k(x - vt),\ A \cos k(x - vt),\ Ae^{-k(x-vt)},\ \text{etc.}$$

So, an equation $\Phi = f(x - vt)$ represents a wave going in the positive x-direction with a wave speed v. Similarly, $\Phi = f(x + vt)$ represents a wave travelling in the negative x-direction with speed v. Both these functions satisfy the partial differential equation

$$\frac{\partial^2 \Phi(x, t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \Phi(x, t)}{\partial t^2}. \tag{20.1}$$

Let us see this for the function $\Phi(x, t) = f(x - vt)$.

$$\frac{\partial \Phi}{\partial x} = \frac{\partial f}{\partial x} = \frac{\partial f(x - vt)}{\partial (x - vt)} \cdot \frac{\partial (x - vt)}{\partial x} = \frac{\partial f(x - vt)}{\partial (x - vt)}$$

or $$\frac{\partial^2\Phi}{\partial x^2} = -\frac{\partial}{\partial x}\left(\frac{\partial f(x-vt)}{\partial(x-vt)}\right) = \left[\frac{\partial^2 f(x-vt)}{\partial(x-vt)^2}\cdot\frac{\partial(x-vt)}{\partial x}\right] = \frac{\partial^2 f(x-vt)}{\partial(x-vt)^2}\cdot \tag{i}$$

Also, $$\frac{\partial\Phi}{\partial t} = \frac{\partial f(x-vt)}{\partial(x-vt)}\cdot\frac{\partial(x-vt)}{\partial t} = -v\frac{\partial f(x-vt)}{\partial(x-vt)}$$

or $$\frac{\partial^2\Phi}{\partial t^2} = -v\frac{\partial^2 f(x-vt)}{\partial(x-vt)^2}\cdot\frac{\partial(x-vt)}{\partial t} = v^2\,\frac{\partial^2 f(x-vt)}{\partial(x-vt)^2}\cdot \tag{ii}$$

From (i) and (ii),

$$\frac{\partial^2\Phi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\Phi}{\partial t^2}\cdot$$

This equation is often called the wave equation. Of the many possible functional forms, the sinusoidal function is the most important from the theoretical point of view. You can write it as

$$\Phi = A\cos\,(kx - \omega t + \delta). \tag{20.2}$$

This represents a wave in which the wave quantity Φ is periodic in both x and t. At a given x, Φ is periodic in time with a periodicity

$$T = \frac{2\pi}{\omega}\cdot \tag{20.3}$$

The frequency of the wave is $\nu = \frac{1}{T} = \frac{\omega}{2\pi}$

or $$\omega = 2\pi\nu. \tag{20.4}$$

At a given t, Φ is periodic in x with a periodicity

$$\lambda = \frac{2\pi}{k}\cdot \tag{20.5}$$

The constant k is called *propagation constant* and ω is called *angular frequency*. You can write

$$kx - \omega t + \delta = k\left(x - \frac{\omega}{k}t\right) + \delta,$$

which shows that $\frac{\omega}{k}$ is the wave velocity v.

$$v = \frac{\omega}{k}\cdot \tag{20.6}$$

Also, $$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\omega}\cdot\frac{\omega}{k} \quad \text{or} \quad \lambda = vT. \tag{20.7}$$

The constant δ appearing in the expression for Φ (Equation 20.2) is sometimes called phase constant. It depends on how you choose $x = 0$ and $t = 0$. Normally, if you are just talking about a single wave, you can choose $x = 0$ and $t = 0$ so as to make $\delta = 0$.

The equation $\Phi = A\cos(kx - \omega t + \delta)$ tells us that the wave quantity depends only on x and not on y or z. If the medium carrying the wave is itself one-dimensional and along the x-axis, this has a natural interpretation. A wave moving on a long string stretched along the x-axis is an example because there is no y or z. However, if you think of a sound wave travelling in air, the medium is three-dimensional and the pressure varies at all the points in the medium. You can still have a wave given by $\Delta p = \Delta p_0 \cos(kx - \omega t + \delta)$. This represents a sound wave going everywhere in the x-direction. Consider a plane $x = x_1$, perpendicular to the x-axis. At any given time, the pressure is the same at all points in the plane. At the same time t, the pressure could be different in another plane $x = x_2$ but again be the same at all points in this plane (Figure 20.2). A surface on which the value of the wave quantity is the same everywhere at a given instant is called a *wavefront*. In this case the wavefronts are planes (perpendicular to the direction of propagation) and thus the wave is called a plane wave.

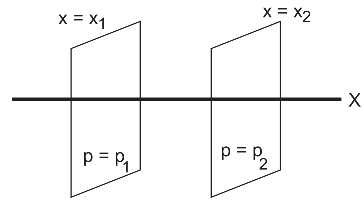


Fig. 20.2

How do you generate a plane wave? A small source of sound generally sends waves in all directions. However if you are far from the source, the wave will travel in almost the same direction in your vicinity and will be more or less like a plane wave.

The quantity A gives the maximum value of Φ, and is called the *wave amplitude*.

To generalize, think of a three-dimensional space in which the points are described by the position vector $\boldsymbol{r}$. A sinusoidal wave of wavelength λ and angular frequency ω moves in a fixed direction given by the unit vector $\hat{\boldsymbol{n}}$. Then, the wave is described by

$$\Phi = A\cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \delta), \tag{20.8}$$

where $\boldsymbol{k} = k\hat{\boldsymbol{n}} = \frac{2\pi}{\lambda}\hat{\boldsymbol{n}}$ is called the propagation vector. The wave speed is, as usual, $v = \frac{\omega}{k}\cdot$ Thus, if the wave moves in the x-direction, $\boldsymbol{k} = k\hat{\boldsymbol{i}}$ and Equation 20.8 becomes

$$\Phi = A\cos[k\hat{\boldsymbol{i}}\cdot(x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}) - \omega t + \delta] = A\cos(kx - \omega t + \delta),$$

which is the same as Equation 20.2. For a wave moving in the negative x-direction, $\boldsymbol{k} = k(-\hat{\boldsymbol{i}})$ and the wave equation is

$$\Phi = A\cos[(-k\hat{\boldsymbol{i}})\cdot(x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}) - \omega t + \delta] = A\cos(-kx - \omega t + \delta)$$

or $$\Phi = A\cos(kx + \omega t + \delta'), \text{ where } \delta' = \delta + \pi.$$

EXAMPLE 20.1 **A sinusoidal wave of amplitude A moves in the direction of the vector $3\hat{i} - 4\hat{j} + 5\hat{k}$. The wavelength is λ and the frequency is ν. Write the expression for the wave quantity Φ.**

Solution The unit vector in the direction of propagation is $\hat{n} = \dfrac{3\hat{i} - 4\hat{j} + 5\hat{k}}{5\sqrt{2}}$, so that the propagation vector is $\boldsymbol{k} = \dfrac{2\pi}{\lambda}\dfrac{(3\hat{i} - 4\hat{j} + 5\hat{k})}{5\sqrt{2}}$.

The angular frequency is $\omega = 2\pi\nu$. Taking $\delta = 0$,

$$\begin{aligned}\Phi &= A\cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\\ &= A\cos\left[\frac{2\pi}{5\sqrt{2}\lambda}(3\hat{i} - 4\hat{j} + 5\hat{k})\cdot(x\hat{i} + y\hat{j} + z\hat{k}) - \omega t\right]\\ &= A\cos\left[\frac{\sqrt{2}\pi}{5\lambda}(3x - 4y + 5z) - 2\pi\nu t\right].\end{aligned}$$

Sine representation We have expressed the sinusoidal wave as

$$\Phi = A\cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \delta).$$

You can choose the value of the constant δ to suit your calculation. It depends on which instant you call $t = 0$ and which point in space you call $\boldsymbol{r} = 0$. Suppose you have chosen the origin. The value of Φ at this point will vary with time in a sinusoidal fashion. Suppose you choose $t = 0$ to be an instant when $\Phi = 0$ at the origin and it is increasing. Verify that $\delta = -\pi/2$. In this case,

$$\Phi = A\sin(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t).$$

In general, you can write the equation for Φ as

$$\Phi = A\sin(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \delta).$$

As you have chosen δ, you can describe any sinusoidal wave using either a sine function or a cosine function.

Wave equation in three dimensions The differential wave equation as written in Equation 20.1 is for one dimension. The wave quantity Φ depends only on x but not on y or z. In general, Φ can depend on all the three coordinates. The differential wave equation in such cases is

$$\nabla^2\Phi(\boldsymbol{r}, t) = \frac{1}{v^2}\frac{\partial^2\Phi(\boldsymbol{r}, t)}{\partial t^2}. \tag{20.9}$$

20.2 Electromagnetic Waves

Let us come back to electromagnetic waves. First consider free space—no materials, no charges and no currents. There are no boundaries to reflect or absorb the waves. If ***E***, ***B*** fields exist in such a space, they must satisfy

$$\nabla \cdot \boldsymbol{E} = 0$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}.$$

Why were the equations not written in terms of ***D*** and ***H***? Because we are writing the equations for free space and there are no bound currents and bound charges. Thus, we can work comfortably with ***E*** and ***B*** fields alone.

Let us take the curl on both sides of the $\nabla \times \boldsymbol{E}$ equation.

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\frac{\partial}{\partial t}(\nabla \times \boldsymbol{B}) = -\frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\right) = -\mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2}.$$

But $\nabla \times (\nabla \times \boldsymbol{E}) = \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} = -\nabla^2 \boldsymbol{E}$.

In writing this, we have made use of the fact that $\nabla \cdot \boldsymbol{E} = 0$. So,

$$\nabla^2 \boldsymbol{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

or

$$\nabla^2 \boldsymbol{E} = \frac{1}{c^2} \frac{\partial^2 \boldsymbol{E}}{\partial t^2}, \tag{20.10}$$

where $c = \sqrt{\dfrac{1}{\mu_0 \varepsilon_0}}$.

Equation 20.10 is a combination of the three equations

$$\nabla^2 E_x = \frac{1}{c^2} \frac{\partial^2 E_x}{\partial t^2},$$

$$\nabla^2 E_y = \frac{1}{c^2} \frac{\partial^2 E_y}{\partial t^2}$$

and

$$\nabla^2 E_z = \frac{1}{c^2} \frac{\partial^2 E_z}{\partial t^2}.$$

Each of these is a differential wave equation, the wave speed being represented by c (Equation 20.9). So, you can have an electric field in the form of waves and given by the equations

$$E_x = E_{x0} \cos (\boldsymbol{k} \cdot \boldsymbol{r} - \omega t + \delta),$$

$$E_y = E_{y0} \cos (\boldsymbol{k} \cdot \boldsymbol{r} - \omega t + \delta)$$

and $$E_z = E_{z0} \cos (\boldsymbol{k} \cdot \boldsymbol{r} - \omega t + \delta).$$

Or $$\boldsymbol{E} = \boldsymbol{E}_0 \cos (\boldsymbol{k} \cdot \boldsymbol{r} - \omega t + \delta). \tag{i}$$

You can analyze the magnetic field in a similar fashion.

$$\nabla \times (\nabla \times \boldsymbol{B}) = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} (\nabla \times \boldsymbol{E})$$

or $$\nabla(\nabla \cdot \boldsymbol{B}) - \nabla^2 \boldsymbol{B} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}\left(-\frac{\partial \boldsymbol{B}}{\partial t} \right)$$

or $$-\nabla^2 \boldsymbol{B} = -\mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2}$$

or $$\nabla^2 \boldsymbol{B} = \frac{1}{c^2} \frac{\partial^2 \boldsymbol{B}}{\partial t^2}. \tag{20.11}$$

You can therefore have a magnetic field wave given by

$$\boldsymbol{B} = \boldsymbol{B}_0 \cos (\boldsymbol{k} \cdot \boldsymbol{r} - \omega t + \delta). \tag{ii}$$

Can we have only an $\boldsymbol{E}$ wave and no $\boldsymbol{B}$ wave? No, we cannot. This is because a changing $\boldsymbol{E}$ field must be accompanied by a $\boldsymbol{B}$ field and if the $\boldsymbol{E}$ field is changing sinusoidally, the $\boldsymbol{B}$ field will also vary sinusoidally. Thus, you cannot have only an $\boldsymbol{E}$ wave. Similarly, you cannot have only a $\boldsymbol{B}$ wave. Thus $\boldsymbol{E}$, $\boldsymbol{B}$ waves always coexist and this combination is called an electromagnetic (EM) wave.

Equations (i) and (ii) represent plane waves. Since a plane wave involves a single λ, it is also a monochromatic wave. Thus, we call such waves plane monochromatic waves. You also have waves which are not plane waves and/or not monochromatic waves. These also satisfy Equations 20.10 and 20.11. In most of the discussions in this chapter, we will talk only about plane monochromatic waves.

We will now talk about some general properties of electromagnetic waves in free space. To keep the mathematics simple, take the x-axis to be in the direction of propagation of the wave.

20.3 General Properties of EM Waves

EM waves do not need a medium

To get Equations 20.10 and 20.11, we started with Maxwell's equations for ***E, B*** fields, which correspond to free space or vacuum. These equations involve no material property. The waves can therefore exist in vacuum. Sound waves, waves on water, waves on a string, etc., proceed because of the properties of the medium. The speed of a wave depends on the elastic and inertia properties of the medium. A compressed layer of air exerts a force on the adjacent layer to compress it and a rarefied layer exerts a force on the adjacent layer to rarefy it. This is how a disturbance produced in one part of the medium propagates to other parts of the medium. But in an electromagnetic wave, you don't need a medium. ***E, B*** fields produced in one region of space can pass on to another without any intermolecular force coming into the picture. Understanding how ***E, B*** fields propagate in vacuum in the form of waves is not simple. But electromagnetic waves including light do travel in vacuum. That is how light from the sun reaches the earth. A large part of the space between the earth and the sun is vacuum, and light crosses it easily.

Scientists of that era, including Maxwell himself, could not reconcile with this situation and assumed that what we call vacuum is, in fact, filled with a material of very low density but extremely high elastic constant. They named this imagined medium 'aether' and lots of theories were developed to understand the behaviour of aether when light and other EM waves supposedly passed through it. Experimentalists too were keen to detect the presence of aether. However, all their efforts led to the conclusion that no such medium is needed for EM waves to travel. It is just the particular combination of time-varying ***E, B*** fields that by itself propagates in space. As no mechanical force is needed for EM waves to travel, these waves are called *nonmechanical* waves.

EM waves are transverse

Suppose an EM wave is moving in the x-direction. The $\boldsymbol{E}$ field of an EM wave will be of the form $\boldsymbol{E} = \boldsymbol{E}_0 \cos(kx - \omega t + \delta)$. In free space, $\nabla \cdot \boldsymbol{E} = 0$. Thus,

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0. \tag{i}$$

But, for the assumed electric field, nothing changes with y or z. Hence, the $\partial/\partial y$, $\partial/\partial z$ terms are zero. So,

$$\frac{\partial E_x}{\partial x} = 0. \tag{ii}$$

Write $\quad \boldsymbol{E}_0 = E_{0x}\hat{\boldsymbol{i}} + E_{0y}\hat{\boldsymbol{j}} + E_{0z}\hat{\boldsymbol{k}}$ so that

$$\boldsymbol{E} = (E_{0x}\hat{\boldsymbol{i}} + E_{0y}\hat{\boldsymbol{j}} + E_{0z}\hat{\boldsymbol{k}})\cos(kx - \omega t),$$

$E_x = E_{0x}\cos(kx - \omega t + \delta)$, from which

$$\frac{\partial E_x}{\partial x} = -E_{0x}k\sin(kx - \omega t + \delta).$$

Thus, from (ii),

$$E_{0x}k\sin(kx - \omega t + \delta) = 0.$$

As this is true for all x and t, E_{0x} must be zero.

Thus, $\quad \boldsymbol{E}_0 = E_{0y}\hat{\boldsymbol{j}} + E_{0z}\hat{\boldsymbol{k}}.$

This means that $\boldsymbol{E}$ is in the y–z plane, which is perpendicular to the direction of propagation. Thus, the electric field in an electromagnetic field propagating in free space is transverse to the direction of propagation. Similarly, from $\nabla \cdot \boldsymbol{B} = 0$, you can conclude that $\boldsymbol{B}$ is also transverse to the direction of propagation.

Wave speed

The speed of EM waves in free space is

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \tag{20.12}$$

or

$$c = \frac{1}{\sqrt{8.85 \times 10^{-12} \times 4\pi \times 10^{-7}}}\ \text{m/s} \approx 3 \times 10^8\ \text{m/s}.$$

This is the same as the speed of light in free space.

It is now common knowledge that light is an electromagnetic wave. Indeed ε_0 happens to be equal to $\frac{1}{\mu_0 c^2}$, where μ_0 is equal to $4\pi \times 10^7$ Tm/A and c is by definition 299,792,458 m/s.

But think of Maxwell's time. Scientists knew about electricity, the behaviour of charged balloons, capacitors with and without dielectric materials, forces between charges, and so on. A constant ε_0 appeared in many of the equations derived governing electrical behaviour. (In those times, the system of units was very different, and equations were not written in terms of ε_0, μ_0 as done now. But never mind, it is not the units but the phenomena that are important. Whatever units the scientists used, the fact that the constants involved in electric and magnetic phenomena combined to give the speed of light came as a big surprise.) Scientists also knew about magnetism—the force on a moving charged particle because of currents, and **B**-fields due to solenoids, coils, the force between two magnets, and so on. The experiments they did told them that a constant μ_0 appears in equations governing magnetic behaviour. The relation between electric and magnetic phenomena was also known by the time Maxwell appeared on the scene. Many scientists from Benjamin Franklin to Coulomb, Gauss, Ampere, Fleming, Faraday and others worked hard with charges, currents and magnets. Also, several others were more interested in understanding light, and were busy experimenting with slabs, prisms, mirrors, candles, sunlight, starlight, and so on. Scientists like Newton, Huygens, Romer, Bradley, Young, Fraunhofer, Fresnel, Airy, Poisson and others studied optical behaviour. Bradley had measured the speed of light based on astronomical observations in 1726 and had obtained a value close to 3×10^8 m/s. Electromagnetism had nothing to do with optics prior to Maxwell.

And all of sudden, Maxwell came up with a theory concerning electric and magnetic fields, and showed that the speed of light could be obtained from the waves of these fields in terms of purely electrical and magnetic phenomena. It was a big surprise that the constants involved in these phenomena were related to a characteristic property of light. The natural inference is that light is also a wave of electric and magnetic fields. When light travels in space, electric and magnetic fields travel. Wherever there is light, **E** and **B** fields are there. Maxwell thus integrated optics with electromagnetism.

E and *B* are related

Let $$\boldsymbol{E} = E_0 \hat{\boldsymbol{j}} \cos (kx - \omega t + \delta). \tag{i}$$

We can write the plane wave in this form without loss of generality. The x-axis is chosen along the direction of propagation. As the EM wave is transverse, the electric field must be in the y–z plane. We can choose the y-axis to be along the direction of the **E**-field. Using this equation,

$$\nabla \times \boldsymbol{E} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & E_0 \cos (kx - \omega t + \delta) & 0 \end{vmatrix}$$

$$= -\hat{\boldsymbol{k}} E_0 \sin (kx - \omega t + \delta).$$

Remember that $\hat{\boldsymbol{k}}$ is the unit vector along the z-axis and $k = 2\pi/\lambda$. Don't think that $\hat{\boldsymbol{k}}$ is the unit vector in the direction of the propagation vector $\boldsymbol{k}$, which is $k\hat{\boldsymbol{i}}$.

Now, $$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}\cdot$$

Thus, $$\frac{\partial \boldsymbol{B}}{\partial t} = \hat{\boldsymbol{k}} E_0 k \sin (kx - \omega t + \delta)$$

or $$\boldsymbol{B} = \hat{\boldsymbol{k}} E_0 \frac{k}{\omega} \cos(kx - \omega t + \delta)$$

or $$\boldsymbol{B} = \hat{\boldsymbol{k}} B_0 \cos(kx - \omega t + \delta),$$

where $$B_0 = \frac{E_0 k}{\omega}$$

or $$B_0 = E_0/c. \tag{20.13}$$

You can think of an integration constant, but we are not interested in any static field. You can see that the magnetic field $\boldsymbol{B}$ is in the z-direction. The electric field $\boldsymbol{E}$ was taken to be in the y-direction, and the direction of propagation in the x-direction. Thus, $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{k}$ are mutually perpendicular. You can check that

$$\boldsymbol{B} = \frac{\boldsymbol{k} \times \boldsymbol{E}}{\omega}. \tag{20.14}$$

This form of the relation among $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{k}$ is valid even if the coordinate axes are not chosen as done in the above derivation.

Also, see that $\boldsymbol{E}$ and $\boldsymbol{B}$ are in the same phase.

EXAMPLE 20.2 **The electric field in an electromagnetic wave moving in free space is given by $\boldsymbol{E} = \boldsymbol{E_0} \hat{\boldsymbol{k}} \sin(ky - \omega t)$. Find the expression for the magnetic field.**

Solution The wave is travelling along the positive y-direction with propagation constant k. Thus, the propagation vector is $\boldsymbol{k} = k\hat{\boldsymbol{j}}$.

$$\boldsymbol{B} = \frac{\boldsymbol{k} \times \boldsymbol{E}}{\omega} = \frac{k\hat{\boldsymbol{j}} \times E_0 \hat{\boldsymbol{k}} \sin(ky - \omega t)}{\omega}$$

$$= \frac{E_0}{c} \hat{\boldsymbol{i}} \sin(ky - \omega t).$$

Direction of polarization

The direction of the electric field in an electromagnetic wave is called the direction of polarization of the wave. Thus, if $\boldsymbol{E} = E_0 \hat{\boldsymbol{j}} \sin(kx - \omega t)$, the wave is polarized in the y-direction. We do not distinguish between the $+y$-direction and $-y$-direction. Indeed, the electric field reverses its direction periodically and remains along the $+y$-direction half the time and the $-y$-direction the other half of the time. An electromagnetic wave with an electric field along a specific direction is said to be linearly polarized. The plane containing the direction of the electric field and that of propagation is called the plane of polarization.

Light is an electromagnetic wave, and has electric and magnetic fields. Light emanating from ordinary sources like hot bodies, filament bulbs, LEDs, etc., does not have an electric field in a specific direction. The direction of the electric field keeps changing rapidly and randomly with time in the plane perpendicular to the direction of propagation. So does the magnetic field, in keeping with the relation with the electric field, as required by Maxwell's law. Such an electromagnetic wave is called an unpolarized wave.

There are certain plastic sheets in which the molecules are connected as long chains, and the chains can be set into longitudinal vibrations by oscillating electric fields, especially with optical frequencies. If light is passed through such sheets perpendicularly, the component of the electric field parallel to the chains gets absorbed in setting these chains into vibration. The component of the ***E***-field perpendicular to the chains does not get affected much. If the sheet is fabricated in such a way that all the chains are parallel to each other, and an unpolarized light wave is made to pass through it, you will get a linearly polarized light on the other side with the ***E***-field perpendicular to the direction of the chain. An instrument that converts unpolarized light to linearly polarized light is called a polarizer. The direction in which the ***E***-field is allowed by a polarizer is called its transmission axis. Light from most of the available laser sources is linearly polarized to a large extent. But to carry out sophisticated experiments where linearly polarized light is needed, it is further passed through polarizers.

EXAMPLE 20.3 **The electric field in a plane monochromatic electromagnetic wave moving in free space is given by $\boldsymbol{E} = (2\hat{\boldsymbol{k}} - 3\hat{\boldsymbol{j}}) \times 10^{-3} \sin [10^7(x + 2y + 3z - \beta t)]$. All numbers are in SI units. (a) What is the direction of propagation? (b) Check that it is a transverse wave. (c) Find β and the angular frequency ω. (d) Find *B*.**

Solution

(a) The general expression for a plane monochromatic wave can be written as

$$\boldsymbol{E} = \boldsymbol{E}_0 \sin [\boldsymbol{k} \cdot \boldsymbol{r} - \omega t + \delta]. \tag{i}$$

Comparing it with the given expression,

$$\boldsymbol{k} \cdot \boldsymbol{r} = 10^7 (x + 2y + 3z)$$

or $$k_x x + k_y y + k_z z = 10^7 (x + 2y + 3z).$$

As this has to be valid for all x, y, z,

$$k_x = 10^7,\ k_y = 2 \times 10^7 \text{ and } k_z = 3 \times 10^7.$$

Thus, $\boldsymbol{k} = 10^7 (\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}})$.

The direction of propagation is along $\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$.

(b) The direction of the electric field is along $(2\hat{\boldsymbol{k}} - 3\hat{\boldsymbol{j}})$. The wave travels in the direction of $\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$.

$$(\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}+3\hat{\boldsymbol{k}})\cdot(2\hat{\boldsymbol{k}}-3\hat{\boldsymbol{j}})=0.$$

So, $\boldsymbol{E}$ is perpendicular to the direction of propagation.

(c) The wave speed is $c=\omega/k$. Comparing the general expression (i) with the given equation,

$\omega = 10^7\beta$ and $k=|\boldsymbol{k}|=|10^7(\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}+3\hat{\boldsymbol{k}})|=\sqrt{14}\times 10^7$.

So, $$3\times 10^8 = \frac{10^7\beta}{\sqrt{14}\times 10^7}$$

or $$\beta = 3\sqrt{14}\times 10^8 \text{ s}^{-1}.$$

So, $$\omega = 10^7\beta = 3\sqrt{14}\times 10^{15}\text{ s}^{-1}.$$

(d) $$\boldsymbol{B}=\frac{\boldsymbol{k}\times\boldsymbol{E}}{\omega}=\frac{10^7(\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}+3\hat{\boldsymbol{k}})\times 10^{-3}(2\hat{\boldsymbol{k}}-3\hat{\boldsymbol{j}})\sin[10^7(x+2y+3z-\beta t)]}{3\sqrt{14}\times 10^{15}}$$

$$=\frac{10^{-11}}{3\sqrt{14}}(13\hat{\boldsymbol{i}}-2\hat{\boldsymbol{j}}-3\hat{\boldsymbol{k}})\sin[10^7(x+2y+3z-3\sqrt{14}\times 10^8 t)].$$

20.4 Energy in EM Waves

An electromagnetic wave consists of electric and magnetic fields. As you know, these fields contain energy. The energy density, that is, the energy per unit volume, corresponding to the electric field $\boldsymbol{E}$ is given by

$$u_E=\frac{1}{2}\varepsilon_0 E^2$$

and that corresponding to the magnetic field $\boldsymbol{B}$ is given by

$$u_B=\frac{1}{2\mu_0}B^2.$$

At any point in the EM wave, the total energy density is

$$u=\frac{1}{2}\varepsilon_0E^2+\frac{1}{2\mu_0}B^2.$$

But $$B=\frac{E}{c}\cdot$$

So $$u=\frac{1}{2}\varepsilon_0E^2+\frac{E^2}{2\mu_0c^2}=\frac{1}{2}\varepsilon_0E^2+\frac{1}{2}\varepsilon_0E^2$$

or $$u=\varepsilon_0E^2.$$

For a wave with $\boldsymbol{E} = \boldsymbol{E}_0 \cos(kx - \omega t)$, this equation shows that

$$u = \varepsilon_0 E_0^2 \cos^2(kx - \omega t).$$

The energy density depends on x and t. At any given location, it changes with time and at any given time, the energy density is different at different places.

Consider a cylindrical volume with its axis along the direction of propagation of the EM wave. Take this axis to be the x-axis (Figure 20.3). Let the area of cross section be A. The ends of the cylinder are at $x = x_1$ and $x = x_2$. How much energy is contained in this volume at a given instant t?

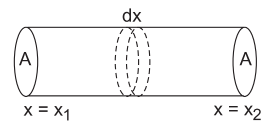


Fig. 20.3

The energy density u at the position x is

$$u = \varepsilon_0 E^2 = \varepsilon_0 E_0^2 \cos^2(kx - \omega t).$$

The energy contained in a small slice of thickness dx at x is

$$dU = uAdx = \varepsilon_0 A E_0^2 \cos^2(kx - \omega t)dx.$$

The energy in the cylindrical volume at time t is

$$U = \varepsilon_0 A E_0^2 \int_{x_1}^{x_2} \cos^2(kx - \omega t)dx.$$

The $\cos^2$ term takes values between 0 and 1 periodically as x is changed. If $x_2 - x_1$ is much larger than the wavelength λ, the average value of the $\cos^2$ term turns out to be 1/2. The integral is then $(x_2 - x_1)/2$. Thus,

$$U = \varepsilon_0 A E_0^2 \frac{x_2 - x_1}{2} = \frac{1}{2}\varepsilon_0 E_0^2 A(x_2 - x_1).$$

Though the energy density u is different at different places in this cylindrical volume, you can talk about an average energy density, which is equal to the total energy in the volume divided by the volume. This average energy density is

$$\langle u \rangle = \frac{U}{A(x_2 - x_1)}$$

or $$\langle u \rangle = \frac{1}{2}\varepsilon_0 E_0^2. \tag{20.15}$$

$\langle u \rangle$ is independent of x or t. So if you average the energy density over a large volume, which contains several wavelengths of the wave, the average will be the same everywhere in the wave and for all time.

The intensity of an EM wave

The intensity of a wave at a point P is defined as follows. Construct a small surface at P with area A perpendicular to the direction of propagation (Figure 20.4). Find how much energy U crosses this area in a time t_0, which is much longer than the time period T of the wave. The intensity at P is then defined as

$$I = \frac{U}{At_0}. \tag{20.16}$$

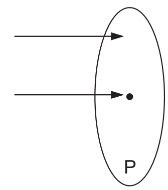


Fig. 20.4

So you can define the intensity of a wave at a point as "the average energy crossing the surface perpendicular to the direction of motion of the wave, per unit area of the surface, per unit time". What is "average energy"? To calculate the energy crossing the surface, you have to wait for a time t_0 and divide the energy that crosses the surface by t_0. It is an average over time. If you take a small t_0 compared to the time period and then divide by t_0, the result will vary with time and will not give you a meaningful quantity but if t_0 is much larger than the time period, the average energy will be the same at any time.

Consider once again the EM wave given by

$$\boldsymbol{E} = \boldsymbol{E}_0 \cos(kx - \omega t).$$

You need the intensity of this wave at a point P. Construct a surface b of area A at P, perpendicular to the direction of propagation (x-direction) of the wave. You have to find the energy crossing this area per unit time. Choose a time interval $t = 0$ to $t = t_0$, large compared to the time period, and construct a cylindrical volume of length ct_0 as shown in Figure 20.5. The wave moves in the x-direction with a speed c. So it covers a distance ct_0 in time t_0. The wave at the rear surface a at time $t = 0$ will reach the front surface b at time $t = t_0$. This means that all the energy contained in the cylinder at $t = 0$ has crossed the surface b in the next time interval t_0. But the energy contained in the cylindrical volume at any instant is

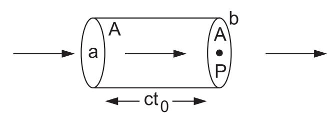


Fig. 20.5

$$U = \langle u \rangle A(ct_0) = \frac{1}{2}\varepsilon_0 E_0^2 Act_0.$$

Thus, the intensity of the wave is

$$I = \frac{U}{At_0} \quad \text{or} \quad I = \frac{1}{2}\varepsilon_0 c E_0^2. \tag{20.17}$$

You can also see that

$$I = c\langle u \rangle. \tag{20.18}$$

The SI unit of energy density is J/m^3 and that of intensity is W/m^2.

20.5 Instantaneous Rate of Energy Transfer

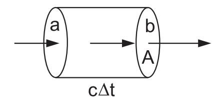


Fig. 20.6

The intensity of an EM wave gives the *average energy* crossing the surface perpendicular to the direction of motion of the wave per unit area *per unit time*. What is the instantaneous rate of energy flow through a given area at a given time t? To obtain it, you have to consider a very short time interval t to $t+\Delta t$, where Δt is much smaller than the time period T. Consider a wave with $\boldsymbol{E} = \boldsymbol{E}_0 \cos(kx - \omega t)$. In Figure 20.6 are drawn two surfaces b and a, perpendicular to the direction of propagation of the wave. The length of the cylinder is $c\Delta t$ and the area of each surface is A. The energy that crosses surface b in Figure 20.6 in this time interval Δt will be the same as that contained in the cylinder at time t. But the energy density at time t and place x is

$$u = \varepsilon_0 E^2 = \varepsilon_0 E_0^2 \cos^2(kx - \omega t).$$

Thus the energy crossing surface b in time Δt is

$$\Delta U = \varepsilon_0 E_0^2 \cos^2(kx - \omega t) A c \Delta t.$$

The instantaneous rate of energy transfer per unit area through surface b is therefore

$$S = c\varepsilon_0 E_0^2 \cos^2(kx - \omega t).$$

The direction of energy flow is the same as that of propagation, i.e., along the x-direction. You can define a vector $\boldsymbol{S}$ giving both the magnitude and the direction of the *instantaneous* rate of energy crossing a unit area perpendicular to the direction of propagation. In the case considered,

$$\boldsymbol{S} = \hat{\boldsymbol{i}}\, c\varepsilon_0 E_0^2 \cos^2(kx - \omega t).$$

The rate at which energy crosses an area da perpendicular to the direction of $\boldsymbol{S}$ is $\boldsymbol{S}da$. Verify that

$$\boldsymbol{S} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0},$$

which is the same as the Poynting vector, as expected.

20.6 Relativistic Transformation of EM Waves

Consider a plane electromagnetic wave $\boldsymbol{E} = \boldsymbol{E}_0 \hat{\boldsymbol{j}} \cos(kx - \omega t)$ travelling along the x-direction in free space with speed $c = \dfrac{\omega}{k}$ as seen from a frame S. The magnetic field is $\boldsymbol{B} = \dfrac{E_0}{c} \hat{\boldsymbol{k}} \cos(kx - \omega t)$. What does this wave look like when viewed from the frame S′, which moves with respect to S with a velocity $v\hat{\boldsymbol{i}}$? In S,

$$E_x = 0,\ E_y = E_0 \cos(kx - \omega t),\ E_z = 0$$

and $$B_x = 0,\ E_y = 0,\ B_z = \frac{E_0}{c}\cos(kx - \omega t)$$

at the point (x, y, z) at time t. At the corresponding point (x', y', z') and corresponding time t', the field components in the S′ frame will be

$$E'_x = E_x = 0,\ E'_y = \frac{E_y - vB_z}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\left(E_0 - \frac{vE_0}{c}\right)\cos(kx-\omega t)}{\sqrt{1-\frac{v^2}{c^2}}},\ E'_z = \frac{E_z + vB_y}{\sqrt{1-\frac{v^2}{c^2}}} = 0$$

and $$B'_x = B_x = 0,\ B'_y = \frac{B_y + \frac{v}{c^2}E_z}{\sqrt{1-\frac{v^2}{c^2}}} = 0,\ B'_z = \frac{B_z - \frac{v}{c^2}E_y}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\left(\frac{E_0}{c} - \frac{v}{c^2}E_0\right)\cos(kx-\omega t)}{\sqrt{1-\frac{v^2}{c^2}}}.$$

Let us discuss the electric field in S′.

$$\boldsymbol{E}' = \hat{\boldsymbol{j}}E_0\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\cos(kx-\omega t).$$

But in this form, we are using the coordinates of the S frame to write the field in S′. We must convert them to S′ coordinates. Using the Lorentz transformations

$$x = \frac{x' + vt'}{\sqrt{1-\frac{v^2}{c^2}}} \quad\text{and}\quad t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}},$$

we get

$$\boldsymbol{E}' = \hat{\boldsymbol{j}}E_0\sqrt{\frac{c-v}{c+v}}\cos\left[\frac{k(x'+vt')}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{\omega\left(t' + \frac{vx'}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}}\right]$$

$$= \hat{\boldsymbol{j}}E'_0\cos\left[x'\frac{\left(k - \frac{\omega v}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}} - t'\left(\frac{\omega - kv}{\sqrt{1-\frac{v^2}{c^2}}}\right)\right] = \hat{\boldsymbol{j}}E'_0\cos(k'x' - \omega' t'), \tag{20.19}$$

where $$k' = \frac{k - \frac{\omega v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \quad\text{and}\quad \omega' = \frac{\omega - kv}{\sqrt{1-\frac{v^2}{c^2}}}.$$

Speed of the wave

Equation (20.19) shows that the wave appears as a plane wave travelling along the x'-axis in S′ too. The speed of the wave in S′ is

$$c' = \frac{\omega'}{k'} = \frac{\omega - kv}{k - \dfrac{\omega v}{c^2}} = \frac{kc - kv}{k - \dfrac{kv}{c}} = \frac{c - v}{1 - \dfrac{v}{c}} = c.$$

The speed of an EM wave in free space is independent of the frame from which you look at the wave.

Doppler effect

The frequency of the wave as it appears in S′

$$\nu' = 2\pi\omega' = \frac{2\pi(\omega - kv)}{\sqrt{1 - \dfrac{v^2}{c^2}}} = \frac{2\pi\left(\omega - \dfrac{\omega}{c}v\right)}{\sqrt{1 - \dfrac{v^2}{c^2}}} = 2\pi\omega\sqrt{\frac{1 - \dfrac{v}{c}}{1 + \dfrac{v}{c}}}$$

or $$\nu' = \nu\sqrt{\frac{c - v}{c + v}}\cdot \tag{20.20}$$

This is the expression for the relativistic Doppler effect. There is no distinction between a moving source and a moving observer. Equation (20.20) is valid when the observer and the wave are going in the same direction as seen from S. If they approach each other, v should be replaced by $-v$, so that

$$\nu' = \nu\sqrt{\frac{c + v}{c - v}}\cdot$$

EXAMPLE 20.4 **The radiation coming from a distant galaxy shows the hydrogen microwave line at 25.0 cm, whereas the laboratory value for this wavelength is 22.5 cm. What is the speed of the galaxy emitting this radiation?**

Solution If the frequency is seen from the galaxy itself, the wavelength must be 22.5 cm. Seen from the earth, the wavelength increases. In other words, the frequency has decreased. Thus, as seen from the galaxy, the observer on the earth (S′ frame) is going in the same direction as the wave. So the frequency will decrease by a factor $\sqrt{\dfrac{c + v}{c - v}}\cdot$ Correspondingly, the wavelength will increase by the same factor. Therefore,

$$\frac{25.0}{22.5} = \sqrt{\frac{c+v}{c-v}}\,.$$

This gives $v = 0.105c$. The galaxy is going away from the earth at a speed $0.105c$.

20.7 Wave Packets

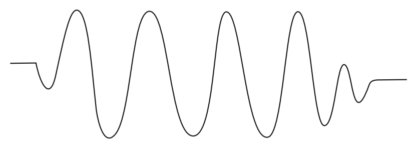


Fig. 20.7

EM waves emitted from actual sources are not really plane waves. The ***E***, ***B*** fields do not extend all over the universe, as demanded by our equations like $\boldsymbol{E} = \boldsymbol{E}_0 \cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)$. Also, no wave from a real source will have a field for all time. You can think of a wave train or wave packet in which the wave extends over a certain length along its direction of propagation and exists at a given point for a certain time. Wave packets can be obtained by superposing a large number of plane waves with wavelengths in a range $\lambda \pm \Delta\lambda$. Thus, such a wave will not be strictly monochromatic. For a tungsten-filament bulb, $\Delta\lambda$ could be several hundreds of nm while for LEDs, it is in the range of 50 nm or so. For good laser sources, $\Delta\lambda$ could be just 1–2 nm.

20.8 Photon Picture

In many phenomena, an electromagnetic wave behaves as if it is made of discrete packets of energy, each packet travelling along the direction of the wave, with speed c. Such packets are called photons. You must have studied photoelectric effect, in which light of appropriate wavelength falls on a metallic surface. The characteristics of photoelectric effect can only be understood if the electromagnetic wave (light) falling on the surface is assumed to be made of photons, each photon having a certain energy and momentum. As in particle–particle collision, a photon hits an electron and gives all of its energy to the electron, which can then come out of the surface. How do we connect the wave parameters to the photon parameters?

A photon is described by its momentum $\boldsymbol{p}$ and energy e. Generally, people use the symbol E for photon energy. But here E is used for electric field so we will use e to denote photon energy. The wave is described by the propagation vector $\boldsymbol{k}$ and angular frequency ω. Write the electric field of the wave as $\boldsymbol{E} = \boldsymbol{E}_0 \cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)$. In the photon picture, a beam of photons, one behind the other, is travelling in the direction of $\boldsymbol{k}$. The relation between these two sets of parameters is

$$\boldsymbol{p} = \hbar\boldsymbol{k},\ e = \hbar\omega.$$

Here $\hbar$ denotes the Planck constant h divided by 2π. The value of the Planck constant is 6.626×10^{-34} J s or 4.136×10^{-15} eV s. It is useful to remember that the value of hc is close to

1240 eV nm and that of $\hbar c$ is close to 197 eV nm. The magnitude of the momentum is

$$p = \frac{h}{2\pi} \cdot \frac{2\pi}{\lambda} = \frac{h}{\lambda} \cdot$$

The energy relation can also be written as

$$e = \frac{h}{2\pi} \times 2\pi\nu = h\nu.$$

How many photons are there in a volume $d\tau$ of this beam? The energy of the wave in this volume is

$$dU = \frac{1}{2}\varepsilon_0 E_0^2 d\tau.$$

If the number of photons per unit volume is n, the number of photons in volume $d\tau$ is $nd\tau$ and the energy in this volume is $(nd\tau)\hbar\omega$. So,

$$\frac{1}{2}\varepsilon_0 E_0^2 = n\hbar\omega$$

or $$n = \frac{\varepsilon_0 E_0^2}{2\hbar\omega} \cdot$$

Each photon carries momentum $\hbar k$ (though it does not have mass). The momentum of all the photons in volume $d\tau$ is

$$dp = (nd\tau)\hbar k = \frac{\varepsilon_0 E_0^2}{2\hbar\omega} \cdot \hbar k d\tau = \frac{1}{2c}\varepsilon_0 E_0^2 d\tau$$

or $$\frac{dp}{d\tau} = \frac{1}{2c}\varepsilon_0 E_0^2 = \frac{u}{c},$$

where u is the average energy density. The momentum per unit volume carried by an electromagnetic wave is its energy per unit volume divided by c. You can write it in terms of the Poynting vector $\boldsymbol{S}$. The direction of the momentum is the same as that of wave propagation, which is also the direction of $\boldsymbol{S}$.

The energy density is given as $u = \frac{S}{c}$ and the momentum density is given by $\frac{d\boldsymbol{p}}{d\tau} = \frac{\boldsymbol{S}}{c^2} \cdot$

20.9 Complex Representation of EM Waves

The $\boldsymbol{E}$-field in a plane monochromatic electromagnetic wave is written as

$$\boldsymbol{E} = \boldsymbol{E}_0 \cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t + \delta).$$

The phase constant δ depends on your choice of $t = 0$ and the origin. You can express the field as

$$\boldsymbol{E} = \mathrm{Re}[\boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t+\delta)}] = \mathrm{Re}[(\boldsymbol{E}_0 e^{i\delta})e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}]$$
$$= \mathrm{Re}[\tilde{\boldsymbol{E}}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}],$$

where Re represents "the real part of". The complex quantity $\tilde{\boldsymbol{E}} = E_0 e^{i\delta}$ is called the complex amplitude of the $\boldsymbol{E}$-field. The quantity in square brackets, that is,

$$\tilde{\boldsymbol{E}} = \tilde{\boldsymbol{E}}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \tag{20.21}$$

is known as the complex electric field of the EM wave. Similarly, the magnetic field can be written as

$$\boldsymbol{B} = \boldsymbol{B}_0 \cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \delta) = \mathrm{Re}[\boldsymbol{B}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t+\delta)}]$$
$$= \mathrm{Re}[\boldsymbol{B}_0 e^{i\delta} e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}] = \mathrm{Re}[\tilde{\boldsymbol{B}}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}].$$

The quantity

$$\tilde{\boldsymbol{B}} = \tilde{\boldsymbol{B}}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \tag{20.22}$$

is known as the complex magnetic field of the EM wave and $\tilde{\boldsymbol{B}} = B_0 e^{i\delta}$, the complex amplitude of the $\boldsymbol{B}$-field. In this form, the phase constant δ is absorbed in $\tilde{\boldsymbol{E}}_0$ and $\tilde{\boldsymbol{B}}_0$. The $\boldsymbol{r}$, t dependence is always written as $e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$.

Check that the amplitude of the real electric field is $E_0 = |\tilde{E}_0| = \sqrt{\tilde{E}_0 \tilde{E}_0^*}$ and the phase constant $\delta = \tan^{-1}\left(\dfrac{\mathrm{Im}\,\tilde{E}_0}{\mathrm{Re}\,\tilde{E}_0}\right)$. Equations 20.21 and 20.22 together represent the plane EM wave. These are complex in value and the actual $\boldsymbol{E}$, $\boldsymbol{B}$ fields are obtained by taking the real parts of these complex functions.

What is the benefit of representing the fields in the EM wave by complex functions $\tilde{\boldsymbol{E}}$, $\tilde{\boldsymbol{B}}$ and then taking the real parts? The benefits show when we have to do several kinds of mathematical operations on the $\boldsymbol{E}$, $\boldsymbol{B}$ functions, as shown in the next section.

20.10 Interference of Two Waves Going in the Same Direction

Suppose two plane electromagnetic waves with the same $\boldsymbol{k}$, and the same ω pass through a point. What is the resultant $\boldsymbol{E}$ field at this point? Let the electric fields of the two waves be given by

$$\boldsymbol{E}_1 = E_{10}\hat{e}\cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t) \tag{i}$$

and $$\boldsymbol{E}_2 = E_{20}\hat{e}\cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \delta). \tag{ii}$$

The corresponding complex ***E, B*** fields are

$$\tilde{\boldsymbol{E}}_1 = \tilde{E}_{10}\,\hat{e}\,e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$$

and $$\tilde{\boldsymbol{E}}_2 = \tilde{E}_{20}\,\hat{e}\,e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)},$$

where $$\tilde{E}_{10} = E_{10} \text{ and } \tilde{E}_{20} = E_{20}e^{i\delta}.$$

The resultant field is $\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2$. So,

$$\begin{aligned}\tilde{\boldsymbol{E}} = \tilde{\boldsymbol{E}}_1 + \tilde{\boldsymbol{E}}_2 &= (\tilde{E}_{10} + \tilde{E}_{20})\hat{e}\,e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \\ &= (E_{10} + E_{20}e^{i\delta})\hat{\boldsymbol{e}}\,e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \\ &= \tilde{E}_0\,\hat{e}\,e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)},\end{aligned}$$

where $$\tilde{E}_0 = E_{10} + E_{20}e^{i\delta}.$$

The real amplitude of the ***E***-field is given by $E_0 = \sqrt{\tilde{E}_0 \cdot \tilde{E}_0^*}$.

$$\begin{aligned}\tilde{E}_0\tilde{E}_0^* &= (E_{10} + E_{20}e^{i\delta})(E_{10} + E_{20}e^{-i\delta}) \\ &= E_{10}^2 + E_{20}^2 + E_{10}E_{20}(e^{i\delta} + e^{-i\delta}) \\ &= E_{10}^2 + E_{20}^2 + 2E_{10}E_{20}\cos\delta\end{aligned}$$

or $$E_0 = \sqrt{E_{10}^2 + E_{20}^2 + 2E_{10}E_{20}\cos\delta}.$$

The phase constant δ_0 is given by

$$\tan\delta_0 = \frac{\text{Im }\tilde{E}_0}{\text{Re }\tilde{E}_0} = \frac{E_{20}\sin\delta}{E_{10} + E_{20}\cos\delta}.$$

The real ***E***-field is $E = E_0\cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \delta_0)$ with these values of E_0 and δ_0. If you add (i) and (ii) directly to get the resultant ***E***-field, you will appreciate the simplification brought about by using complex fields.

Not only in summation, in differentiation and integration too the complex ***E, B*** fields help. Maxwell's equations can also be written in terms of complex fields, differentiation or integration can be done on the complex fields, and real quantities can then be derived from them.

EXAMPLE 20.5 **Let $\boldsymbol{E} = \hat{j}E_0\cos(kx - \omega t + \delta)$. Write the corresponding complex *E*-field, and use the appropriate equation given by Maxwell to find an expression for that complex *B*-field. From this expression, find the actual *B*-field.**

Solution The complex electric field is

$$\tilde{\boldsymbol{E}} = \hat{j}\,\tilde{E}_0 e^{i(kx-\omega t)},$$

where $$\tilde{E}_0 = E_0 e^{i\delta}.$$

Now, $$\nabla \times \tilde{\boldsymbol{E}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & \tilde{E}_0 e^{i(kx-\omega t)} & 0 \end{vmatrix} = \hat{k}\frac{\partial}{\partial x}[\tilde{E}_0 e^{i(kx-\omega t)}] = \hat{k}\,\tilde{E}_0 (ik) e^{i(kx-\omega t)}.$$

Using $$\nabla \times \tilde{\boldsymbol{E}} = -\frac{\partial \tilde{\boldsymbol{B}}}{\partial t},$$

$$\frac{\partial \tilde{\boldsymbol{B}}}{\partial t} = -\hat{k}\,\tilde{E}_0 (ik) e^{i(kx-\omega t)}$$

or $$\tilde{\boldsymbol{B}} = -\hat{k}\,\tilde{E}_0 \frac{(ik)}{(-i\omega)} e^{i(kx-\omega t)} = \hat{k}\,\tilde{E}_0 \frac{k}{\omega} e^{i(kx-\omega t)} = \hat{k}\frac{E_0}{c} e^{i(kx-\omega t+\delta)}.$$

The actual $\boldsymbol{B}$-field is obtained by taking the real part of this complex $\boldsymbol{B}$-field. Thus,

$$\boldsymbol{B} = \hat{k}\frac{E_0}{c}\cos(kx - \omega t + \delta).$$

20.11 Diffraction from a Single Slit

Suppose the complex electric field in a plane monochromatic electromagnetic wave is given by

$$\tilde{\boldsymbol{E}} = \boldsymbol{E}_0 e^{i(kx-\omega t)}. \tag{i}$$

The wave is travelling along the x-direction, where it meets an opaque sheet with a narrow long slit of width d. The situation is represented in Figure 20.8. The width of the slit is along the y-axis, the length is along the z-axis and the centre of the slit is taken as the origin. The complex electric field $\boldsymbol{E}$ at different points of the slit is, from (i),

$$\tilde{\boldsymbol{E}} = \boldsymbol{E}_0 e^{-i\omega t}. \tag{ii}$$

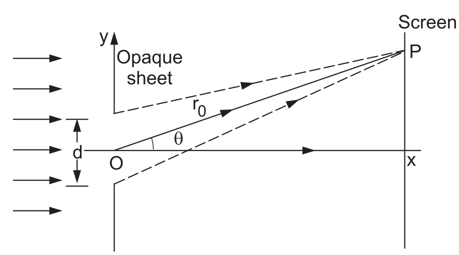


Fig. 20.8

Suppose there is a screen to intercept the wave at a large distance from the slit and parallel to the y–z plane. Take a point P on the screen and join it to the origin O. Assume that the line OP makes an angle θ with the x-axis and its length is r_0. As the screen is far from the slit, all lines joining different parts of the slit to P also make almost the same angle θ with the x-axis. The point P receives EM waves starting from different points of

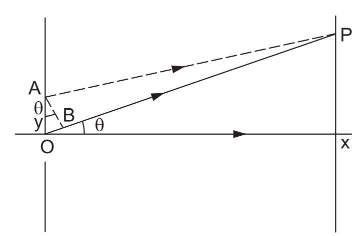


Fig. 20.9

the slits and all these have different phases. The resultant ***E***, ***B*** fields are obtained by adding contributions from these different parts.

We have assumed that ***E***-fields and ***B***-fields from different parts of the slit reaching P have the same directions as those in the incident wave and hence the fields can be added algebraically. Let us calculate the ***E***-field at P at time t.

Take a small element of the slit of width dy at A, a distance y from the origin. The ***E***-field reaching P at time t must have originated from the element at time $t - \dfrac{AP}{c}$.

If you draw a perpendicular AB from A to OP, AB will make an angle θ with the y-direction.

$$AP = \sqrt{AB^2 + BP^2} = BP\left(1 + \frac{AB^2}{BP^2}\right)^{1/2} \approx BP\left(1 + \frac{1}{2}\frac{AB^2}{BP^2}\right).$$

If the screen is far from the slit, you can neglect the second term in parentheses in comparison to 1,

Thus, $\quad AP \approx BP = OP - OB = r_0 - y\sin\theta.$

Thus, the ***E***-field reaching P from the element of the slit at A is $d\tilde{\boldsymbol{E}} = d\boldsymbol{E}_0 e^{-i\omega\left[t - \frac{r_0 - y\sin\theta}{c}\right]}$.

Assume that the amplitude corresponding to the element $d\boldsymbol{E}_0$ is proportional to the width dy. Then,

$$d\tilde{\boldsymbol{E}} = \boldsymbol{A}e^{-i\omega\left[t - \frac{r_0}{c}\right]} \cdot e^{-\frac{i\omega y\sin\theta}{c}}\, dy,$$

where ***A*** is a constant. In fact EM waves spread out in all directions from the slit and the ***E***-field decreases as it moves ahead. But the distances of P from different parts of the slit are more or less equal, and hence any variation in amplitude is neglected. But variation in phase is significant.

The net electric field at P from the whole slit is

$$\tilde{\boldsymbol{E}} = \boldsymbol{A}e^{-i\omega\left(t - \frac{r_0}{c}\right)} \int_{-d/2}^{d/2} e^{-\frac{i\omega y\sin\theta}{c}}\, dy$$

$$= \boldsymbol{A}e^{-i\omega\left(t - \frac{r_0}{c}\right)} \left|\frac{e^{-\frac{i\omega y\sin\theta}{c}}}{\frac{-i\omega\sin\theta}{c}}\right|_{-d/2}^{d/2}$$

$$= \frac{\boldsymbol{A}c}{\omega\sin\theta}\left[\frac{e^{\frac{i\omega d\sin\theta}{2c}} - e^{-\frac{i\omega d\sin\theta}{2c}}}{i}\right] e^{-i\omega\left(t - \frac{r_0}{c}\right)}$$

$$= \frac{2Ac}{\omega \sin\theta} \sin \frac{\omega d \sin\theta}{2c} e^{-i\omega\left(t - \frac{r_0}{c}\right)}.$$

Writing $\frac{\omega}{2c} d \sin\theta = \beta$,

$$\tilde{\boldsymbol{E}} = \frac{2Ac}{(2c\beta/d)} \sin\beta \cdot e^{-i\omega\left(t - \frac{r_0}{c}\right)} = \frac{Ad}{2} \frac{\sin\beta}{\beta} \cdot e^{-i\omega\left(t - \frac{r_0}{c}\right)}.$$

The complex amplitude is $\frac{Ad}{2} \frac{\sin\beta}{\beta}$.

The intensity at P will be proportional to the modulus square of the complex amplitude.

Thus, $$I = I_0 \frac{\sin^2\beta}{\beta^2}, \tag{20.23}$$

where all factors independent of θ are absorbed in I_0.

Check that β can also be written as $\beta = \frac{\pi}{\lambda} d \sin\theta$.

Intensity distribution

At the central point on the screen, just opposite O, $\theta = 0$ and hence $\beta = 0$. Taking limits $\frac{\sin\beta}{\beta} \to 1$. Thus, $I(\theta = 0) = I_0$, the intensity will be zero where

$$\beta = n\pi \qquad (n \neq 0)$$

or $$\frac{\pi}{\lambda} d \sin\theta = n\pi$$

or $$d \sin\theta = n\lambda.$$

The first minimum ($I = 0$) on the two sides of the centre will be at $\sin\theta = \pm\frac{\lambda}{d}$ and then the minimum will occur at $\pm\frac{2\lambda}{d}, \pm\frac{3\lambda}{d}$, etc. Between the consecutive minima, you must have a maximum intensity. The exact positions of the maxima can be obtained by differentiating the expression for intensity with respect to β and equating it to zero. This gives $\beta = \tan\beta$.

How do the intensities of different maxima vary? Let us find out for the first maximum. The first positive value of β, where $\beta = \tan\theta$, is at $\beta = 4.49$.

Then, $$\frac{\sin^2\beta}{\beta^2} = 0.047$$

or $$\frac{I_0}{I} = \frac{1}{0.047} = 21.$$

So, the intensity of the first maximum, next to the central one, on either side, will be reduced by a factor of 21. Similarly, maxima farther away will have gradually decreasing intensities. With this information, you can sketch the intensity as a function of θ. This has been done in Figure 20.10.

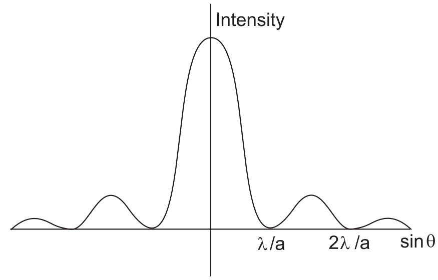


Fig. 20.10

Concepts at a Glance

1. Electromagnetic waves are waves of electric and magnetic fields.
2. A plane electromagnetic wave is represented by $\boldsymbol{E} = \boldsymbol{E}_0 \cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \delta)$, $\boldsymbol{B} = \boldsymbol{B}_0 \cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \delta)$.
3. $|\boldsymbol{k}| = \frac{2\pi}{\lambda}$, $\omega = \frac{2\pi}{T}\cdot$
4. $\boldsymbol{k}$ gives the direction of propagation of a wave.
5. For EM waves in vacuum, $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{k}$ are related as $\boldsymbol{B} = \frac{\boldsymbol{k}\times\boldsymbol{E}}{\omega}\cdot$
6. For EM waves with electric field amplitude E_0, the average energy per unit volume is $\langle u\rangle = \frac{1}{2}\varepsilon_0 E_0^2$.
7. The intensity of an EM wave $= \frac{1}{2}\varepsilon_0 c E_0^2 = c\langle u\rangle$.
8. Relativistic Doppler effect: $\nu' = \nu\sqrt{\frac{c-v}{c+v}}$, where v is the velocity of an observer moving in the same direction as the wave.
9. EM waves can also be described in terms of photons. Each photon has a linear momentum $\boldsymbol{p} = \hbar\boldsymbol{k}$ and an energy $e = \hbar\omega$.
10. Electromagnetic waves have linear momentum. The momentum and energy of the wave in a given volume are related as $p = \frac{U}{c}\cdot$
11. The EM wave is sometimes described by complex electric field and magnetic field waves
    $\tilde{\boldsymbol{E}} = \tilde{\boldsymbol{E}}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}$ and $\tilde{\boldsymbol{B}} = \tilde{\boldsymbol{B}}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}$.
    The actual fields are obtained by taking the real parts of these expressions.
12. The intensity pattern in a single-slit diffraction pattern is given by $I = I_0\frac{\sin^2\beta}{\beta^2}$, where $\beta = \frac{\pi}{\lambda}d\sin\theta$. Here d is the width of the slit and λ is the wavelength of the EM wave being diffracted.

EXERCISES

Based on Concepts

1. What is the value of $\boldsymbol{E} \cdot \boldsymbol{B}$ in an electromagnetic wave moving in free space without boundaries?
2. Does the electric field $\boldsymbol{E} = \dfrac{q}{4\pi\varepsilon_0 r^2}\hat{r}$ satisfy the wave equation?
3. What is the dimension of $\dfrac{\mu_0\varepsilon_0}{u^2}$, where u represents speed?
4. The intensity of EM wave 1 is twice that of EM wave 2. Both the waves are travelling in free space. What is the ratio $\dfrac{E_{01}}{E_{02}}$ of the amplitudes of the electric fields in the two waves?
5. Let $\boldsymbol{S}$ be the Poynting vector and $\boldsymbol{k}$ be the propagation vector of an electromagnetic wave at a point. What is the value of $\boldsymbol{S} \times \boldsymbol{k}$?
6. Electromagnetic wave equations were derived with $\rho = 0$, $j = 0$ everywhere. Can EM waves be produced without any charge anywhere any time?
7. A short wave pulse travels on a rope with a constant speed v without damping. For this wave, can we write (a) $\Phi = \Phi_0 \cos(kx - \omega t)$, (b) $\Phi = f(x - vt)$, (c) $\dfrac{d^2\Phi}{dx^2} = \dfrac{1}{v^2}\dfrac{d^2\Phi}{\partial t}$?

Problems

1. The magnetic field components for an electromagnetic wave in free space are $B_x = B_0 \sin(ky + \omega t)$, $B_y = B_z = 0$. (a) What is the direction of propagation of the wave? (b) Write the electric field components. (c) Is the wave polarized? If so, in what direction?

   [**Ans.** (b) $E_x = E_y = 0$, $E_z = -B_0 C \sin(ky + \omega t)$]

2. Consider a plane electromagnetic wave given by the expression $\boldsymbol{E} = 2\cos\left[2 \times 10^6\left(t - \dfrac{x}{c}\right)\right]\hat{j}$. All numbers represent quantities in SI units.
   (a) Find the values of frequency, wavelength, direction of motion, and amplitude.
   (b) Write an expression for the magnetic field $\boldsymbol{B}$.

   [**Ans.** (a) $\nu = 1$ MHz, $\lambda = 150$ m, $A = 2$ V/m (b) $6.67 \times 10^{-9} \cos\left\{2 \times 10^6\left(t - \dfrac{x}{c}\right)\right\}\hat{k}$]

3. Write expressions for the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields constituting a plane monochromatic wave travelling in the positive z-direction. The wave is linearly polarized in the direction making an angle of 45° with the x-axis. Its amplitude is E_0, wavelength is λ_0 and frequency is ν_0.

   [**Ans.** $\boldsymbol{E} = \dfrac{E_0}{\sqrt{2}}(\hat{i} + \hat{j})\cos 2\pi\left(\dfrac{z}{\lambda_0} - 2\pi\nu_0 t\right)$, $\boldsymbol{B} = \dfrac{E_0}{\sqrt{2}c}(\hat{j} - \hat{i})\cos 2\pi\left(\dfrac{z}{\lambda_0} - 2\pi\nu_0 t\right)$]

4. Write expressions for the electric and magnetic fields for a monochromatic plane wave of amplitude E_0 and frequency ν travelling in free space from the origin to the point (1, 1, 1), with polarization parallel to the x–y plane. [**Ans.** $\boldsymbol{E} = \frac{E_0}{\sqrt{2}}(\hat{\boldsymbol{j}} - \hat{\boldsymbol{i}})\cos\left\{\frac{2\pi\nu}{c\sqrt{3}}(x+y+z) - 2\pi\nu t\right\}$]

5. A linearly polarized monochromatic plane wave with amplitude of electrical field 10^{-3} V/m and wavelength 500 nm is propagating in vacuum in a direction parallel to the x–y plane at 45° to the x-axis. The x–y plane is also its plane of polarization. (a) Write a vector expression describing the wave. (b) Calculate the energy density.

   [**Ans.** $\boldsymbol{E} = \frac{10^{-3}}{\sqrt{2}}(\hat{\boldsymbol{j}} - \hat{\boldsymbol{i}})\cos\{2\sqrt{2}\pi \times 10^6(x+y-3\times 10^8 t)\}$, all numbers with SI units]

6. An electric field in free space is written as $\boldsymbol{E} = A[\sin(\alpha x - \omega t) + \sin(\beta x + \omega t)]\hat{\boldsymbol{j}}$. (a) What should be the relation between α and β for this to be a valid expression for an EM wave in a medium? [**Ans.** $|\alpha| = |\beta|$]

7. Calculate the peak electric field in a 1-mW He–Ne laser having a beam diameter of 1 mm^2. [**Ans.** 980 V/m]

8. The electric field in an electromagnetic wave in free space is given by $\boldsymbol{E} = 10\sin(\omega t + 6\times 10^5 z)\hat{\boldsymbol{i}}$ in SI units. (a) Find the value of ω. (b) Find the value of the Poynting vector at a place, averaged over a long time. (c) How much energy is contained in a cube of length 1 mm? The surfaces of the cube are given by $x = 0$, 1 mm; $y = 0$, 1 mm; $z = 0$, 1 mm. (d) From which of the six surfaces of the cube does the energy enter the cube ? From which of the six surfaces does the energy leave the cube?

   [**Ans.** (a) $1.8\times 10^{-14}\ \text{s}^{-1}$, (b) 0.13 W/m^2, (c) 4.4×10^{-19} J, (d) $z = 1$ mm, $z = 0$]

9. (a) Find the intensity of the electromagnetic radiation emanating from a 100-W lamp at a distance of 1.0 m. (b) Assuming that 20% of the power consumed is radiated as light, what are the peak amplitudes of the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields at this distance? (c) What are the corresponding values of the fields of a 10-W laser beam focussed on a cross-sectional area of $10^{-8}\ \text{m}^2$?

   [**Ans.** (a) 1.6 W/m^2, (b) 34.7 V/m, 1.2×10^{-7} T, (c) 8.7×10^5 V/m, 2.9×10^{-3} T]

10. A plane electromagnetic wave moving in free space with electric field amplitude 0.4 V/m falls perpendicularly on a sheet of area 1 cm^2 and gets absorbed. Find the momentum transferred to the sheet in 15 seconds. [**Ans.** 1.1×10^{-15} kg m/s]

11. A parallel beam of light with wavelength 400 nm has an intensity of 25 mW/mm^2. How many photons cross an area of 1 mm^2 per second placed perpendicular to the beam? [**Ans.** 5×10^{16}]

12. The magnetic field in an electromagnetic wave at $t = 0$ is given by $\tilde{\boldsymbol{B}} = \tilde{\boldsymbol{B}}_0 e^{i(10^7\, m^{-1})x}$. Find the energy of a photon of this wave. [**Ans.** 1.97 eV]

13. The intensity of sunlight in the upper atmosphere is $1.35\times 10^3\ \text{W/m}^2$. Assuming the sunlight to be that of a plane sinusoidal wave, find the electric field amplitude. [**Ans.** 1010 V/m]

14. The electric field in a region is given by $\boldsymbol{E} = 2E_0\cos kz\cos\omega t\,\hat{\boldsymbol{i}}$. (a) Show that this expression satisfies the wave equation $\nabla^2\boldsymbol{E} = \frac{1}{v^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2}$, with $v = \omega/k$. (b) Find the magnetic field corresponding to this E-field. (c) Find the Poynting vector $\boldsymbol{S}$ and discuss the direction of energy flow at a given time t. (d) Find $\langle S\rangle$, the average of S over a time period. [**Ans.** (b) $-\frac{2E_0 k}{\omega}\sin kz\sin\omega t\,\hat{\boldsymbol{j}}$, (c) $\frac{E_0^2 k}{\omega}\sin 2kz\sin 2\omega t\,\hat{\boldsymbol{k}}$, (d) zero]

15. An electromagnetic wave with electric field amplitude E_0 travels in a direction parallel to the x–y plane at an angle θ with the x-axis. The angular frequency of the wave is ω. Consider a frame S′ that moves along the x-direction with velocity $v\hat{i}$ with respect to the lab frame in which the wave was described above. Write an expression for the electric field in this frame S′ and obtain the expression for frequency.

[**Ans.** $\omega' = \omega\left(\dfrac{1-\dfrac{v}{c}\cos\theta}{\sqrt{1-\dfrac{v^2}{c^2}}}\right)$]

16. In the above problem, what will be the direction of the wave as seen from S′?

[**Ans.** $\tan\theta' = \dfrac{\sin\theta\sqrt{1-\dfrac{v^2}{c^2}}}{\cos\theta-\dfrac{v}{c}}$, where θ is the angle with the x'-axis]

17. Consider a plane monochromatic wave getting diffracted by a narrow slit. The wave falls perpendicularly on the slit. Consider the parts of the wave starting at the two edges of the slit and reaching a distant screen at the position of the first minimum on any side of the central maximum. What will be the phase difference between these two parts? [**Ans.** 2π]

❑

21 Magnetism in Materials

Now you know that a magnetic field is produced by electric currents and by changing electric fields. We have discussed at length how currents or changing electric fields are related to the magnetic field they produce. However, we haven't talked much about the most common source of a magnetic field, that is, a permanent magnet. We haven't discussed why an iron nail gets strongly attracted to a magnet and why a copper wire does not. The most familiar sources of magnetism lie all around us—bar magnets in physics laboratories, button magnets in a variety of toys, stickers for decorating refrigerator doors and steel almirahs, and so on. There are no batteries and wires to carry currents and apparently no electric fields that change with time continuously to produce magnetic fields around a magnet. Yet a magnet produces a magnetic field. What is the basic source of this field? It is still electric current but in this case the currents inside the atoms do the job. And they always exist.

21.1 Atomic Magnetic Moment

All materials are made up of atoms and molecules, which have charged particles like protons and electrons. And they are not all at rest. Moving charged particles enclosed in a volume make up a magnetic dipole moment. Thus, each electron and each proton has a magnetic moment due to its motion. We talked about gyromagnetic ratio in a previous chapter. If the charge density in a charge distribution is proportional to its mass density, the magnetic moment μ and the angular momentum $\boldsymbol{I}$ are related as $\mu = \gamma \boldsymbol{l}$, where γ is called gyromagnetic ratio. Its value turns out to be $q/(2m)$, where q is the total charge and m is the total mass. The magnetic moment of an electron due to its motion around the nucleus in an atom is related to its angular momentum as

$$\mu_1 = -\frac{e}{2m} l. \tag{21.1}$$

The gyromagnetic ratio γ is $\left(-\dfrac{e}{2m}\right)$.

This result was derived assuming that the electron behaves like a classical particle moving in a well-defined path around the nucleus. Scientists have now put forward a very different theory, known as quantum mechanics, according to which velocity and path of motion are not meaningful quantities but angular momentum is. Equation 21.1, though derived using the concepts of classical motion, turns out to be correct even in quantum mechanics. In quantum mechanics, the angular momentum resulting from the motion of a particle is called *orbital angular momentum*.

Each electron that has a nonzero orbital angular momentum also has a nonzero magnetic moment. As you know, the s-electrons do not have an orbital angular momentum ($l = 0$) and hence do not have a magnetic moment corresponding to the orbital angular momentum. The p-electrons, d-electrons, f-electrons have orbital angular momenta and corresponding magnetic moments.

An even more interesting quantity related to angular momentum and magnetic moment is the spin of the electron. Electrons have a permanent angular momentum **s**, called its spin angular momentum. They do not have to move or revolve or rotate to have spin angular momentum. It is built in. Interestingly, if you take any z-axis, the z-component of the spin angular momentum of the electron will always be either $\hbar/2$ or $-\hbar/2$, where $\hbar = h/2\pi$ and h is the Planck constant. Together with the spin angular momentum, the electron has a built-in, permanent magnetic moment given by

$$\mu_{\mathbf{s}} = -\frac{e}{m_e}\mathbf{s}. \tag{21.2}$$

The z-component of this quantity is

$$\mu_{sz} = -\frac{e}{m_e}s_z$$

or

$$\mu_{sz} = \frac{e\hbar}{2m_e} \quad \text{or} \quad -\frac{eh}{2m_e} \tag{21.3}$$

as s_z can be either $\hbar/2$ or $-h/2$.

The quantity $\frac{e\hbar}{2m_e}$ is a universal constant and its value is 9.27×10^{-24} A m^2. It is called *Bohr magneton*. So, each electron in the universe has a permanent magnetic moment, the z-component of which has a magnitude of one Bohr magneton. If the electron is part of an atom, it may also have a magnetic moment due to its orbital angular momentum.

The orbital angular momentum too is quantized. Its components can be $n\hbar$, where n is an integer. A p-electron can have $l_z = -\hbar, 0, \hbar$.

Protons too have a built–in magnetic moment. But as the mass of a proton is about 1840 times larger than that of an electron, the magnetic moment is smaller by the same factor and we neglect it.

Now come back to the atom as a whole. In general, it has many electrons, and its magnetic moment is obtained by summing the magnetic momenta of all the electrons vectorially. Because of the Pauli exclusion principle and various interactions, many of the electrons pair up. If one has a spin component of $\hbar/2$, the next will have a spin component of $-\hbar/2$. If the orbital angular momentum component of one is $\hbar$, that of the next will be $-\hbar$, and so on. Not all electrons pair up but many of them do. If all the electrons pair up in an atom, as is definitely the case with closed-shell atoms, the net magnetic moment of the atom is zero. If some of the electrons do not pair up, the atom may have a nonzero magnetic moment. An atom with an odd number of electrons will necessarily have a nonzero magnetic moment.

So, even without being connected to a battery, the atom itself can have a magnetic moment. For the space outside, the atom can be regarded as a tiny source of magnetic field. If a material is placed in an external magnetic field, each atom having a magnetic moment will experience a torque leading to interesting magnetic properties. If atoms join to make molecules, some of the electrons might redistribute. But the situation remains the same. The molecule may or may not have a net magnetic moment depending on how the electrons pair up.

21.2 Magnetization (*M*) in a Material

Any piece of material contains a large number of atoms or molecules, may be of the order of 10^{22} or more. In general, these atoms or molecules are randomly oriented. Even if each of them has a magnetic moment, if you take a volume containing a few thousand molecules, the net magnetic moment in the volume will be zero. A few thousand molecules make up a very small volume, not even a micrometre cube. So, for all practical purposes, you can assume that there is no magnetic moment in the material. But this is not always the case. For example, if the material is kept in an external magnetic field, the molecules may be magnetically aligned to some extent due to the torques acting on them. In this situation, there may be a net magnetic moment in any given volume. To measure the extent of such alignment, one defines a new vector field—the magnetization vector $\boldsymbol{M}$.

Take a small volume $d\tau$ in the given material containing at least a few thousand molecules. Look at the net magnetic moment $d\boldsymbol{\mu}$ of all the molecules in the volume, taken together. The magnetization vector where this small volume is located is defined as

$$\boldsymbol{M} = \frac{d\boldsymbol{\mu}}{d\tau} \cdot \tag{21.4}$$

Thus, magnetization is a vector field defined at each point in space. Take a small volume $d\tau$ at the point where you need to evaluate magnetization and use Equation 21.4 to get $\boldsymbol{M}$ there. If there is no material in $d\tau$, there is no magnetic moment in it, and the magnetization $\boldsymbol{M}$ is zero. The magnetization in empty space is always zero.

The SI unit of magnetization is A/m. Another unit widely used in the research community is emu/cm^3. The abbreviation emu stands for electromagnetic unit. The conversion is 1 A/m $= 10^{-3}$ emu/cm^3. Sometimes, magnetization is mentioned as magnetic moment per unit mass and is measured in emu/g.

Depending on the values of the individual magnetic moments and their interaction with each other in a material, the material can show a variety of interesting properties. Based on these properties, materials are classified into different categories, the most common being diamagnetic, paramagnetic, ferromagnetic and antiferromagnetic.

21.3 Diamagnetic Materials

Materials in which the individual entities (atoms, molecules or other elementary units) have zero magnetic moment are said to be diamagnetic. In general, the magnetization $\boldsymbol{M}$ in such a material is zero. What happens if such a material is placed in an external magnetic field? You may simply think that no change will occur. If the molecules have no magnetic moment, the magnetic field should do nothing to them. But things are not so simple. The zero magnetic moment of the molecule results from the cancellation of the electronic magnetic moments of the paired electrons. Each electron still has a magnetic moment due to its orbital angular momentum and its spin angular momentum. The magnetic moments of the two electrons in a pair are equal in magnitude and opposite in direction and hence the pair as such has zero magnetic moment. When an external magnetic field is applied, the magnetic field affects each of the two electrons in the pair as each has a magnetic moment. The two electrons may get affected in different ways so that their magnetic momenta do not cancel out any more. Thus, an external magnetic field can induce a magnetic moment in a molecule that has a zero magnetic moment otherwise.

Let us now see how a magnetic field changes the magnetic moment corresponding to the orbital motion of an electron going around the nucleus in an atom. We will use a model similar to Bohr model with several simplifying assumptions.

EXAMPLE 21.1 **Consider an electron going around a nucleus containing Z protons. Assume that only the Coulomb force due to the nucleus acts on the electron, which moves along a circle of radius r with speed v. A weak but uniform magnetic field B is switched on, perpendicular to the plane of the motion. Calculate the change in the magnetic moment of this electron corresponding to the orbital motion. Assume that the magnetic field does not change the speed of the electron, but alters the radius of the orbit.**

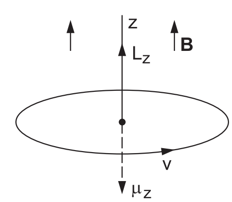


Fig. 21.1

Solution Take the centre of the circle as the origin and the plane of the circle as the x–y plane. Also, assume that the electron moves in the anticlockwise direction as seen from the z-axis. The magnetic field is applied along the z-axis, $\boldsymbol{B} = B\hat{\boldsymbol{k}}$.

Before the magnetic field was applied, the force on the electron towards the centre was $Ze^2/(4\pi\varepsilon_0 r^2)$. Thus,

$$\frac{Ze^2}{4\pi\varepsilon_0 r^2} = m\left(\frac{v^2}{r}\right) \tag{i}$$

or $$r = \frac{Ze^2}{4\pi\varepsilon_0 m v^2}. \tag{ii}$$

Now think of the situation after the magnetic field is applied. It is assumed that the speed remains the same. The magnetic force on the electron is

$$\boldsymbol{F}_{\mathrm{m}} = (-e)(v\hat{\phi}) \times (B\hat{\boldsymbol{k}}) = -evB\hat{\boldsymbol{r}}.$$

The force is towards the centre and will add to the Coulomb force. If the radius changes to $r + \Delta r$, the equation of motion is

$$\frac{Ze^2}{4\pi\varepsilon_0 (r+\Delta r)^2} + evB = \frac{mv^2}{r+\Delta r}$$

or $$\frac{Ze^2}{4\pi\varepsilon_0 r^2}\left(1+\frac{\Delta r}{r}\right)^{-2} + evB = \frac{mv^2}{r}\left(1+\frac{\Delta r}{r}\right)^{-1}$$

or $$\frac{mv^2}{r}\left(1-\frac{2\Delta r}{r}\right) + evB = \frac{mv^2}{r}\left(1-\frac{\Delta r}{r}\right) \qquad (\text{using } \Delta r << r)$$

or $$\frac{mv^2}{r^2}\Delta r = evB$$

or $$\Delta r = \frac{eBr^2}{mv}.$$

The radius is increased.

The magnetic moment of the circulating electron is

$$\mu = -\frac{e}{2m}\boldsymbol{I} = -\frac{evr}{2}\hat{\boldsymbol{k}}.$$

The change in magnetic moment due to the change in the radius is

$$\Delta\mu = -\frac{ev\Delta r}{2}\hat{\boldsymbol{k}} = -\frac{e^2Br^2}{2m}\hat{\boldsymbol{k}}.$$

The change in magnetic moment is opposite to the applied magnetic field.

Is it justified to assume that the speed of the electron remains constant as the magnetic field is applied? It seems logical as the magnetic field does not do any work and hence cannot change speed. However, when you apply the magnetic field, the changing field creates an electric field, which in turn changes the speed of the electron. In fact, in one of the exercises in the chapter on Faraday's law, you were asked to calculate the change in speed in a similar situation. In any case, it is only a crude model to assume that electrons go in closed orbits in an atom and you should use these derivations only to infer that applying a magnetic field induces a magnetic moment in a direction opposite to the applied field.

In the above example, we saw that the change in the magnetic moment was opposite to the applied field. Now an exercise for you. The electron was assumed to move anticlockwise. Take the electron to travel clockwise using the same geometry as in Example 21.1. The original magnetic moment will be $(-\mu)$. Calculate the change in the magnetic moment when the magnetic field is applied. You will find that in this case also the change $\Delta\mu$ is opposite to the applied magnetic field. So, for the first electron, the magnetic moment changes from μ to $\mu + \Delta\mu$, and for the second electron, the magnetic moment changes from $(-\mu)$ to $(-\mu) + \Delta\mu$. Now, look at the two electrons together as a pair. The magnetic moment of the pair was zero before the magnetic field was applied, and becomes $2\Delta\mu$ after the field is applied. So a net magnetic moment is created or induced in the pair and this moment is opposite to the applied field. This result is true for the net magnetic moment of the atom also.

So, in a diamagnetic material, the external magnetic field induces a magnetic moment in each atom/molecule and hence a magnetization $\boldsymbol{M}$ appears in the material. The magnetization has a direction opposite to that of the applied magnetic field.

What happens if the magnetic field is inhomogeneous?

Suppose a diamagnetic material is placed near a cylindrical bar magnet on its axis, as shown in Figure 21.2. The axis of the magnet is taken as the z-axis. A magnetic moment μ is induced in the material opposite to the field, that is, in the negative z-direction. The magnet produces an inhomogeneous magnetic field. The field at any place has a z-component and an s-component in the cylindrical-coordinate system. Thus,

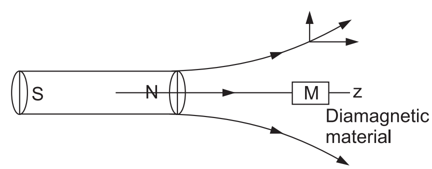


Fig. 21.2

$$\boldsymbol{B} = B_s \hat{\boldsymbol{s}} + B_z \hat{\boldsymbol{k}}.$$

The field gets weaker as you move away from the magnet along its axis. So, $\dfrac{dB_z}{dz}$ is negative. The magnetic moment induced in the diamagnetic material is towards the negative z-direction. The force on the diamagnetic material is

$$\boldsymbol{F} = (\mu \cdot \nabla)\boldsymbol{B}.$$

This force will be along the z-axis. So, only the z-component is to be considered. Thus, the force is

$$F_z = \mu_z \frac{\partial B_z}{\partial z}$$

or $$\boldsymbol{F} = \mu_z \frac{\partial B_z}{\partial z}\hat{\boldsymbol{k}}.$$

As μ_z and $\frac{\partial B_z}{\partial z}$ are both negative, the force is along the positive z-direction.

This means the diamagnetic material is repelled by the magnet. In more concrete terms, a diamagnetic material placed in an inhomogeneous magnetic field will be pushed towards the weaker magnetic field.

You can treat the diamagnetic material in Figure 21.2 itself as a bar magnet. The net magnetic moment of the material is towards the left. Then, the north pole of the magnet corresponding to this material will be towards the left (Figure 21.3). The north pole of the original bar magnet is towards the right. Thus, the diamagnetic material will be repelled by the bar magnet.

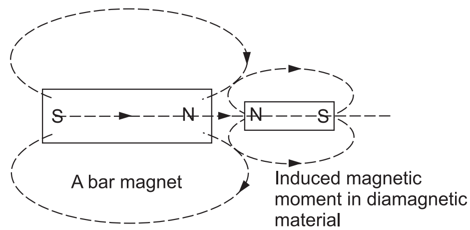


Fig. 21.3

What happens to the net magnetic field at a point inside the diamagnetic material? The magnetic moment of the material itself produces a magnetic field, which is in the direction of the magnetic moment. Some of the field lines due to the external bar magnet and due to the magnetized diamagnetic material are shown in Figure 21.3. The two fields are in opposite directions inside the material (look at the fields at the central line) and hence the net field is reduced inside the material. It is as if the material is trying to expel the magnetic field from inside. The external bar magnet tries to create a field and the material resists it.

Superconducting materials are called perfect diamagnets because if you place them in an external magnetic field, the magnetic field inside the material is exactly zero. Magnetic moments are induced in the material, which create an exactly equal and opposite field to make the field inside zero. However, the mechanism of inducing a magnetic field in a superconducting material is very different from that in usual diamagnetic materials. The phenomenon of complete expulsion of the magnetic field from a superconducting material is called the *Meissner effect*.

Among common solids, copper, gold, silver, mercury and silicon are diamagnetic.

21.4 Paramagnetic Materials

Materials in which the atoms or molecules have permanent dipole moments can be divided into two categories. In one, we place those materials in which the magnetic moments of the neighbouring atoms/molecules do not interact much with each other. In the other category, the neighbouring magnetic moments interact with each other strongly and each moment tries to orient its neighbour's magnetic moment either towards itself or opposite to it. The materials in the first category (neighbouring moments do not interact or interact very weakly) are called paramagnetic. The thermal motion always tries to randomize the direction of the individual magnetic moments. Hence in any given volume $d\tau$ of a paramagnetic material, containing at least a few thousand atoms/molecules, the net magnetic moment is zero. The magnetization vector $\boldsymbol{M}$ is therefore zero everywhere in the material.

What happens if a paramagnetic material is placed in an external magnetic field? The magnetic moment of each atom or molecule comes from the magnetic moments of the electrons in it. The external magnetic field changes each magnetic moment, as in diamagnetic materials. The change in each moment will be in a direction opposite to the applied field. However, there is another effect, which is much more pronounced. The magnetic field will exert a torque on each magnetic moment and will try to align it in its own direction. There will be a competition between the thermal motion, which tries to randomize the directions of the moments, and the external magnetic field, which tries to align the moments along its direction. Generally, the thermal motion is a much stronger player in this competition and the moments are only weakly aligned for common values of applied magnetic fields and temperatures. Any alignment of the atoms/molecules will cause the magnetization $\boldsymbol{M}$ to appear in the direction of the field. If you apply a stronger field, the magnetization will be more. If you raise the temperature, the magnetization will reduce. The magnetization due to this alignment is usually much more than the opposite magnetization induced due to the diamagnetism mechanism. For laboratory fields, say up to a few teslas, and not at very low temperatures, the net magnetization follows the rule

$$M = C\frac{B}{T},$$

where C is a constant for a given material, called the Curie constant.

What happens when a paramagnetic material is placed in a nonuniform magnetic field, for example near the pole of a magnet? The magnetic moment appears in the material and is in the same direction as that of the field of the magnet. Hence, the force will be towards the magnet. So paramagnetic materials are attracted towards the magnet. In general, they tend to move towards regions of stronger magnetic fields.

Copper is diamagnetic and aluminium is paramagnetic. If you bring a magnet towards a copper needle or an aluminium needle, you don't find any attraction or repulsion. Why? Because the attraction and repulsion are very weak.

An Interesting Experiment

Take a test tube and a rectangular piece of thermocol, say $6\ \text{cm} \times 6\ \text{cm} \times 1\ \text{cm}$.

Make a hole in the centre of the thermocol piece and push the test tube in it, say halfway down. Pour water into the tube until it is almost full.

Take a wide tub of water and make the thermocol with the test tube float in it. The tube should remain vertical.

Now bring a strong magnet close to the test tube wall above the thermocol, without making them touch. You should see the test tube moving away from the magnet. The effect is not very strong and it may appear that nothing is happening. Keep trying. The magnet should be strong—rare-earth magnets are preferable. Bring the magnet close to the wall carefully, just avoiding a touch. Sometimes the friction of water creates a late start, so don't give up.

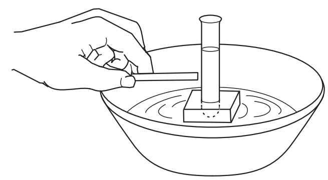


Fig. 21.4

Why is the test tube repelled by the magnet? Because water is diamagnetic.

Repeat the experiment with a test tube filled with $CuSO_4$ powder. You will find that it is attracted towards the magnet. $CuSO_4$ is paramagnetic.

21.5 Ferromagnetic Materials

When a magnet is brought near an iron nail, the nail readily jumps and sticks to the magnet. You can easily lift the nail using the magnet. The force of attraction is more than the weight of the nail. But you cannot lift a brass pin in a similar way. Why is the attraction by the magnet so strong in the case of the iron nail?

Like the atoms of paramagnetic materials, those of iron too have a permanent magnetic dipole moment. But iron has yet another property. The neighbouring atoms interact strongly and tend to have their magnetic moments in the same direction. Even if there is no external magnetic field to orient the magnetic moment, they tend to align with each other. In materials like iron, which we call ferromagnetic, millions of atoms next to each other have magnetic moments in the same direction.

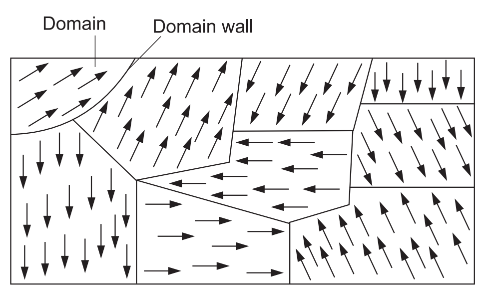


Fig. 21.5

A group of atoms having a large number of aligned atoms is called a magnetic domain. In general, different domains have different orientations of magnetic moment, as indicated in Figure 21.5. The sizes of domains are typically of the order of micrometres. Neighbouring domains, in general, have different orientations. The boundary between the domains is not sharp but has a width. The magnetic moment gradually changes its direction in this boundary. A boundary between the domains is called a domain wall.

If you take a small volume containing thousands of domains of typical size, the magnetic moment of the volume will be almost zero.

Now you can guess what there is in a permanent magnet. A permanent magnet is made of a ferromagnetic material and its domains are largely aligned in the same direction, giving it a net magnetic moment (Figure 21.6). This remains the case even if no external magnetic field is applied. Once the domains of a ferromagnetic material are made to align by some method, they remain aligned to a large extent for long time even without the application of an external magnetic field.

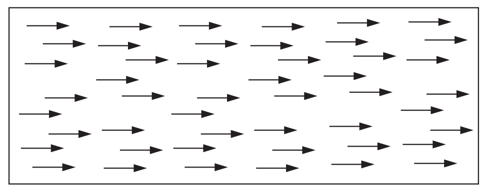


Fig. 21.6

And what happens when an iron nail is put close to a magnet? The magnetic field of the magnet causes two effects.

(i) The domains in the nail for which the magnetic moments are already in the direction of the magnetic field (or close to it) grow in size and the domains for which the magnetic moments are opposite to the direction of the magnetic field (or close to it) diminish in size. How does the size of a domain grow? A schematic presentation is given in Figure 21.7. In Figure 21.7(a) are shown two domains D1 and D2, separated by a domain wall when no magnetic field is applied. The direction of the magnetic moment gradually changes in the domain wall, between D1 and D2. When a magnetic field is applied in the upward direction, the situation is as shown in Figure 21.7(b). This field will try to rotate the magnetic moments to make them point upwards. Some of the magnetic moments in D2, close to the domain wall, have oriented their direction a little bit and become part of the domain wall. Similarly, some of the magnetic moments in the domain wall, close to D1, have changed the orientation to become part of D1. Thus, D1 grows in size and D2 diminishes in size.

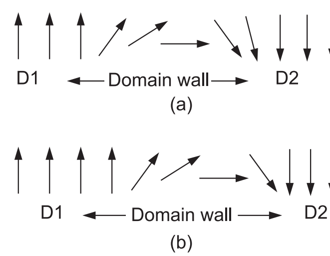


Fig. 21.7

(ii) The domains orient themselves in the magnetic field so that their directions align much more with the magnetic field.

Both these effects create a nonzero magnetization $\boldsymbol{M}$ in the nail in the direction of the magnetic field. Domain-wall movement is easier than domain orientation. So, for small magnetic fields, magnetization is produced mainly due to domain-wall movement and only when larger fields are applied do the domains get oriented.

EXAMPLE 21.2 **The magnetic moment of an iron atom in metallic iron is $2.2\mu_B$. Take the density of iron to be 7.9 g/cm^3 and its atomic mass to be 56 g/mol. Find the saturation magnetization of iron.**

Solution Saturation magnetization will occur when all the domains are perfectly aligned. All atomic magnetic moments will also be in the same direction.

$$\text{Magnetization} = \frac{\text{magnetic moment}}{\text{volume}} = \frac{n(2.2\mu_B)}{V}, \tag{i}$$

where n is the number of iron atoms in volume V. The mass of a sample of iron of volume V is ρV, where $\rho = 7.9\ \text{g/cm}^3$ is the density. The number of iron atoms in 56 g is equal to the Avogadro number 6.023×10^{23}. So the number of atoms in mass m will be

$$n = \frac{m}{56\ \text{g}} \times 6.023 \times 10^{23} = \frac{\rho V}{56\ \text{g}} \times 6.023 \times 10^{23}.$$

So,
$$\frac{n}{V} = \frac{7.9\ \text{g/cm}^3}{56\ \text{g}} \times 6.023 \times 10^{23} = 8.5 \times 10^{28}\ \text{m}^{-3}.$$

From (i),

$$\text{magnetization} = 8.5 \times 10^{28}\ \text{m}^{-3} \times 2.2 \times 9.27 \times 10^{-24}\ \text{A m}^2 = 1.7 \times 10^6\ \text{A/m},$$

which is about 220 emu/g.

Among pure elements, Fe, Co, Ni, Gd and Dy are ferromagnetic. Many alloys too are ferromagnetic. The Fe–Ni–B alloy is used to make very strong permanent magnets.

Why do atoms tend to align their magnetic moments in a ferromagnetic material? This is a purely quantum mechanical effect and the equations described in this book so far cannot explain it. The interaction between neighbouring atoms that tend to align their magnetic moments is called *exchange interaction*. Physicists describe this interaction in terms of potential energy as

$$U_{ij} = -J\boldsymbol{S}_i \cdot \boldsymbol{S}_j, \tag{21.5}$$

where J is a positive constant for the given atoms in the given geometry for a ferromagnetic material and $\boldsymbol{S}_i$, $\boldsymbol{S}_j$ are the spin angular momentum vectors of the two atoms. Many factors have to be looked into in order to understand ferromagnetism. Essentially, it arises from the overlap of the atomic electrons of neighbouring atoms and the Pauli exclusion principle.

If exchange interaction aligns neighbouring atoms to have magnetic moments in the same direction, why don't all the atoms in the given material align? If this happens, every iron nail, sheet and vessel would be a permanent magnet. If the domain size is too big, it becomes unstable and splits into domains with nearly opposite directions of magnetic field.

What happens if you heat a permanent magnet? The thermal energy tries to randomize the directions of the domains, the net magnetic moment of the magnet decreases and so the magnet becomes weaker. At a certain temperature, the domains will have almost random directions and the magnet will lose its magnetization.

What happens if a piece of an ordinary ferromagnetic material (unmagnetized) is heated? The domains are already randomly oriented before heating. Heating cannot randomize it further. But if you heat it to sufficiently high temperatures, the domain structure itself is broken. Domains are formed due to exchange energy $U_i = -J\boldsymbol{S}_i \cdot \boldsymbol{S}_j$. When the material is heated, it gets thermal energy. If the thermal energy becomes larger than the exchange energy, the domains will break into individual randomized magnetic moments. Then, the ferromagnetic material becomes paramagnetic.

Usually, the transition from ferromagnetic to paramagnetic is quite sharp and one can define a temperature T_C of transition. Below T_C, the material is ferromagnetic and above T_C, it is paramagnetic. This transition temperature is called *Curie temperature*.

21.6 Antiferromagnetism and Ferrimagnetism

The atoms of some materials have magnetic moments and the exchange interaction drives neighbouring atoms to align in opposite directions. In the chromium lattice, for example, a chromium atom will try to align in a direction opposite to its neighbouring atom. If you use Equation 21.5, the constant J is negative for these materials. Such materials are called *antiferromagnetic*.

There is another class of materials, in which the magnetic moments of neighbouring atoms are aligned in opposite directions, but in any given volume, the net magnetic moment in one direction is not equal to the net magnetic moment in the opposite direction. Such materials are said to be *ferrimagnetic*. Fe_3O_4 is an example. In this lattice, the oxygen atoms are situated in a face-centred cubic arrangement and iron atoms occupy sites of tetrahedral and octahedral void positions. Two neighbouring iron atoms are far away, their electrons do not overlap and hence they cannot undergo exchange interaction directly. But both these iron atoms are close to a common oxygen atom and the outer electrons of each iron atom overlap with those of the intermediate oxygen atom. Thus, the two iron atoms overlap through the oxygen atom and this causes a kind of exchange interaction, which aligns the two atoms with opposite directions of magnetic moment. An interaction between two magnetic moments via a third atom having no magnetic moment is known as superexchange interaction.

The number of iron atoms located in the octahedral sites is twice that of iron atoms in the tetrahedral sites in Fe_3O_4. The magnetic moments of all the atoms in tetrahedral sites are aligned in one direction and all the atoms in the octahedral sites have an oppositely directed magnetic moment. Because of the difference in the number of iron atoms in the two sites in any given volume, there is a net magnetic moment.

In many respects, the bulk magnetic properties of ferrimagnetic materials are similar to those of ferromagnetic materials. Ferrimagnetic and antiferromagnetic materials too have a domain structure. In each domain, there are millions of atoms with such a magnetic alignment as required by their ferrimagnetic or antiferromagnetic character.

21.7 Superparamagnetism

Superparamagnetism is a phenomenon seen in a class of materials in which the atoms form magnetic domains but still show a magnetic behaviour similar to that of a paramagnetic material. This phenomenon is generally observed when the material is made of very small particles (say of the dimensions of a few nanometres). Each particle is a single domain in itself (no domain wall). Also, the net magnetic moment of the particle is so small that it keeps changing its direction due to the thermal energy of the lattice. At a temperature T, one expects an energy of the order of $\frac{3}{2}kT$ to be available to any particle from thermal motion. If this is sufficient for changing the direction of the magnetic moment frequently, you may not observe the magnetic moment in a given time interval. In other words, the average magnetic moment of the particle over the time of measurement is zero. The particle thus behaves like a paramagnetic one.

Note that the whole particle need not physically rotate to change the direction of the magnetic moment. The particle may retain its orientation, but its magnetic moment may fluctuate, making it show superparamagnetic behaviour.

Thus, superparamagnetic materials are essentially ferrimagnetic, ferromagnetic or antiferromagnetic but due to their small size behave like paramagnetic materials.

Concepts at a Glance

1. Each electron has a magnetic moment due to its motion and its intrinsic spin.
2. The Bohr magneton is given by $\mu_B = \frac{e\hbar}{2m_e}$.
3. Magnetization vector is defined as $\boldsymbol{M} = \frac{d\mu}{d\tau}$, where $d\mu$ is the magnetic moment of the material in volume $d\tau$.

4. Materials are classified as diamagnetic, paramagnetic, ferromagnetic, antiferromagnetic and ferrimagnetic.
5. Ferromagnetism, antiferromagnetism and ferrimagnetism occur due to exchange interaction, which causes some kind of alignment among neighbouring magnetic moments.

EXERCISES

Problems

1. Consider an electron moving round a nucleus having Z protons along a circular path of radius R. Take the origin to be at the centre and the z-axis to be perpendicular to the plane of the circle. The velocity of the electron is $\boldsymbol{v} = v_0\hat{\phi}$. A magnetic field $\boldsymbol{B} = \boldsymbol{B}_0\hat{\boldsymbol{k}}$ is switched on. Assume that the radius of the orbit remains constant. (a) Find the change in the speed of the electron. (b) Work out the change in the magnetic moment of the atom due to the orbital motion of the electron.
2. Consider the situation of the previous problem, the only change being that the velocity of the electron is $\boldsymbol{v} = -v_0\hat{\phi}$. Answer the same questions.
3. Find the magnetic field due to the spin angular momentum of an electron on its spin axis at a distance of 0.1 nm. [**Ans.** 1.85 T]
4. Two electrons are separated by 0.1 nm. Their spin axes are along the line joining the electrons. Find the potential energy corresponding to their magnetic interaction if (a) the spins have the same direction and (b) the spins have opposite directions. [**Ans.** (a) 1.7×10^{-23} J, (b) 1.7×10^{-23} J]
5. A cobalt piece shows a magnetization of 1.5×10^5 A/m. The density of cobalt is 8.9×10^3 kg/m^3 and its atomic mass is 58.9 g/mol. Find the average magnetic moment per atom in Bohr magneton units. [**Ans.** $0.18\mu_B$]
6. Consider a model of the spin angular momentum of an electron in which the electron is a rigid sphere of radius R rotating about one of its diameters with angular speed ω. Also, assume that the maximum linear speed of the rotating parts is equal to the speed of light c. (a) Find an expression for the magnetic moment of the electron in terms of the radius R, charge e and speed of light c. (b) Taking the magnetic moment of the electron to be a Bohr magneton, estimate the radius of the electron. [**Ans.** (a) $\frac{1}{5}ecR$, (b) ≈ 1 pm]

❑

22 Bound Currents and *H*-Vector

In general, when a material is placed in a magnetic field, it attains a magnetization $\boldsymbol{M}$. As you know, the magnetization at a point is defined as the magnetic moment per unit volume evaluated at that point. Magnetization occurs due to the alignment of the magnetic moments of the atoms, ions or molecules. These aligned magnetic moments produce their own magnetic field. For the calculation of the magnetic field due to a magnetized material, one can assume that some kind of an equivalent current flows in the material. This current, which represents the effect of the magnetized material, is called *bound current*.

22.1 Bound Surface Current Density

How do you relate the magnetization to the equivalent current? Where in the material should you assume the current and what magnitude should you take? To start with, take a very simple example. Suppose you are given a disk with thickness t and face area A on either side. Also suppose the disk is magnetized in the direction perpendicular to the faces. Let the magnetization be $\boldsymbol{M}$ everywhere in the disk, as shown in Figure 22.1(a). The total magnetic moment of the disk is

$$\mu = \boldsymbol{M}(At). \tag{i}$$

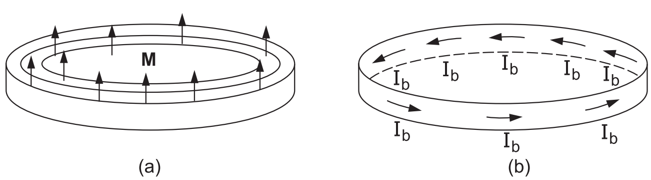


Fig. 22.1

Assume that there is no magnetization anywhere in the material but there is a current I_b along the curved surface (the periphery), as shown in Figure 22.1(b). The magnetic moment of the current loop will be I_bA along the normal to the disk. What if you choose the value of I_b such that the magnetic moment of this current loop is equal to the magnetic moment of the original magnetized disk? Then,

$$I_bA = MAt$$

or $$I_b = Mt \quad \text{or} \quad K_b = \frac{I_b}{t} = M.$$

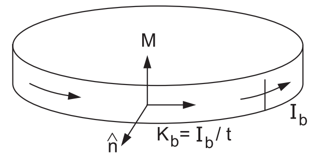


Fig. 22.2

What is K_b? Take any point on the curved surface and draw a line on the surface from this point along the thickness. The length of the line will be t and the current I_b will cross this line perpendicularly (Figure 22.2). Thus, $I_b/t = K_b$ will be the magnitude of the surface current density at this point. The direction of the surface current density is along the current. If you draw a unit vector $\hat{\boldsymbol{n}}$ normal to the curved surface through the chosen point, towards the outside, you can verify that the equation

$$\boldsymbol{K}_b = \boldsymbol{M} \times \hat{\boldsymbol{n}} \tag{22.1}$$

gives the magnitude as well as the direction of the surface current density.

Ok. So, the magnetized disk is equivalent to a surface current across the periphery with the surface current density given by Equation 22.1, as far as the total magnetic moment is concerned. The current density described here gives the same magnetic moment as that of the magnetized disk. But is this sufficient to ensure that the magnetic field everywhere due to the magnetized disk is the same as that of the current drawn? You can draw a number of other current distributions having the same magnetic moment. For example, we can draw a loop of area $A/2$ and current $2I_b$ to get the same magnetic moment. But the two current distributions will give different magnetic fields at the centre of the disk. Then, how will you get a *current distribution* which is *equivalent* to the *uniformly magnetized disk*? Instead of considering the magnetic moment of the whole disk together, divide the disk into small volume elements and then find the equivalent current for each such volume element.

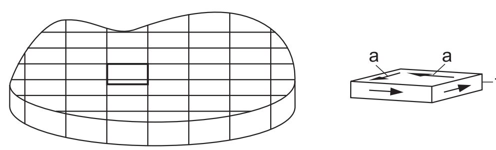


Fig. 22.3

To divide the whole disk into small volume elements, you can draw a rectangular grid on a flat surface of the disk and draw lines perpendicular to the faces from the points of interaction. This will make volume elements of thickness t. If you draw parallel lines at separations a, any volume element not involving the boundary will be rectangular, having a face area a^2 and thickness t (Figure 22.3). The elements involving the boundary will be of different shapes but the same thickness t.

For any element with face area da, the volume is $d\tau = (da)t$ and the magnetic moment is $d\mu = \boldsymbol{M}(da)t$. If you take an equivalent current I_b along the side walls, the magnetic moment will be $I_b(da)$. In order to have the right equivalent current,

$$I_b da = M(da)t$$

or $$I_b = Mt \quad \text{or} \quad K_b = \frac{I_b}{t} = M.$$

Now, consider any volume element not involving the boundary of the disk. All the six surfaces of this element are in the interior. Current I_b moves on the four vertical surfaces, taking the flat surfaces of the disk to be horizontal. Note that each of these four surfaces will coincide with a surface of the adjacent volume element. And this adjacent surface will have current in a direction opposite to that of the first one, the two together making the net current zero. All the currents in the interior thus vanish. No such compensation takes place at the boundary and the periphery again has the same current density K_b. Taking the direction into account,

$$\boldsymbol{K}_b = \boldsymbol{M} \times \hat{\boldsymbol{n}}.$$

This equation remains valid for a material of any shape, not necessarily a disk.

22.2 Bound Volume Current Density

In the above section, the current I_b is taken to be the same for all volume elements because the magnetization $\boldsymbol{M}$ is considered uniform throughout the disk. What if $\boldsymbol{M}$ were not uniform? Take the direction perpendicular to the disk as the z-direction and let the directions of the edges of the volume elements be along the x- and y-axis (Figure 22.4). Suppose the magnetization varies along the y-direction. Consider two adjacent volume elements, one centred on (x, y) and the other on $(x, y+dy)$. Figure 22.4 shows them separated only for clarity, their faces are actually touching.

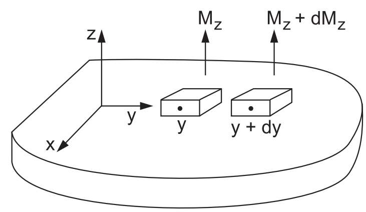


Fig. 22.4

Suppose the magnetization is $\boldsymbol{M} = M_z\hat{\boldsymbol{k}}$ at (x, y) and $(M_z + dM_z)\hat{\boldsymbol{k}}$ at $(x, y+dy)$. The volume element at (x, y) will have a current $M_z t$ along the side surfaces and that at $(x, y+dy)$ will have a current $(M_z + dM_z)t$ along the side surfaces. At $y + \frac{dy}{2}$, where the two elements touch, the currents on the two faces will not exactly cancel each other out. The net current here will be

$$(M_z + dM_z)t - M_z t = (dM_z)t$$

in the x-direction. Look at the figure carefully and think of the array of these currents as you move along the y-axis.

At every interval dy, you get a current $dM_z t$ along the x-axis (Figure 22.5). This current is crossing the area $(dy)t$ perpendicularly. Thus, you have a volume current density in the interior of the disk of value

$$J_b = \frac{(dM_z)t}{(dy)t} = \frac{dM_z}{dy}$$

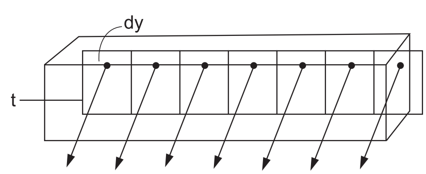


Fig. 22.5

in the x-direction. Verify that for $\boldsymbol{M} = M_z \hat{\boldsymbol{k}}$, and M_z varying only along the y-direction, this current density is the same as

$$\boldsymbol{J}_{\text{b}} = \nabla \times \boldsymbol{M}. \tag{22.2}$$

This expression also gives the bound volume current density for the general case of a magnetized material. In general, a material can be of any arbitrary shape and each component of $\boldsymbol{M} = M_x \hat{\boldsymbol{i}} + M_y \hat{\boldsymbol{j}} + M_z \hat{\boldsymbol{k}}$ can vary in any direction. Take the curl of the magnetization $\boldsymbol{M}$, and this gives the bound volume current density there.

The expression for the bound surface current density remains the same ($\boldsymbol{K}_{\text{b}} = \boldsymbol{M} \times \hat{\boldsymbol{n}}$) even if the magnetization is nonuniform. Equations 22.1 and 22.2 give a current distribution that will produce the same magnetic moment in the volume occupied by the material as actually exists in the material. You can use it to find the magnetic field due to a magnetized material anywhere.

Are bound currents real? Do they represent the flow of charges, which is the usual definition of currents in a conductor? Certainly, no charge moves through long distances. These currents result from the magnetization of each tiny volume. The current for each tiny volume element you took was circulating on the side surfaces of only that volume. The combined effect of these tiny, circulating currents was the current flowing along the periphery of the disk or in the material. As the current at the microscopic level is confined to the tiny volume element it belongs to, it is called bound current. Even here the current need not to be due to the motion of charges. The magnetic moment of the atoms largely results from the spin angular momentum of their electrons, which is an internal property of theirs and has nothing to do with the motion of charges. The current along the side surfaces of a volume element that we assumed was designed to give the total magnetic moment. It is better to think of it as an equivalent current and not a real current.

However, to calculate the magnetic fields due to a magnetized material, you can calculate the bound current densities $\boldsymbol{K}_{\text{b}}$ and $\boldsymbol{J}_{\text{b}}$ and then use the Biot–Savart law or Ampere's law in the same way as you do for real currents.

Don't think that bound current results from the motion of bound charges. Bound charges appear when you have an electrically polarized material. Bound current has no relation with electrical polarization. It results from the alignment of the atomic magnetic moments. In an electrostatic situation, bound charges are static but bound currents may still result if the material is magnetized.

EXAMPLE 22.1 **A solid cylinder has a uniform magnetization *M* along its axis. Find the bound-current density everywhere.**

Solution Take the z-axis to be along the axis of the cylinder so that

$$\boldsymbol{M} = M\hat{\boldsymbol{k}}.$$

As the magnetization is uniform in the interior of the cylinder, its curl is zero at all points within. There is no bound volume current density in the interior. At the curved surface of the cylinder, the normal to the surface is $\hat{\boldsymbol{s}}$ (cylindrical coordinates). So, bound surface current density is

$$\boldsymbol{K}_{\mathrm{b}} = \boldsymbol{M} \times \hat{\boldsymbol{n}} = M\hat{\boldsymbol{k}} \times \hat{\boldsymbol{s}} = M\hat{\phi}.$$

The surface current is along the $\hat{\phi}$-direction. It is equivalent to a solenoid with $M = ni$. At the flat parts of the cylinder, $\boldsymbol{M} \times \boldsymbol{n} = M\hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}}$ or $M\hat{\boldsymbol{k}} \times (-\hat{\boldsymbol{k}})$, which is zero.

These expressions for bound-current densities must have reminded you of the polarization of a dielectric material in an electric field. In that case too, because of the shift in atomic-charge distribution, you get a bound-charge density. If the polarization is $\boldsymbol{P}$, the bound surface charge density and bound volume charge density are given by

$$\sigma_{\mathrm{b}} = \boldsymbol{P} \cdot \hat{\boldsymbol{n}} \text{ and } \rho_{\mathrm{b}} = -\nabla \cdot \boldsymbol{P}.$$

Bound-charge densities result from polarization and bound-current densities result from magnetization.

22.3 The Field *H*

When you place a material in a magnetic field, it gets magnetized. The magnetized material itself produces a magnetic field and the net magnetic field gets modified. From the magnetization $\boldsymbol{M}$, you can get the equivalent bound-current densities and from those the magnetic field due to the magnetized material.

It turns out that one can define a new vector field $\boldsymbol{H}$, which in many cases depends only on the free currents (currents other than bound currents) even in the presence of magnetized materials. This field at any point is defined as

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M},$$

where $\boldsymbol{B}$ is the net magnetic field at the given point, and $\boldsymbol{M}$ is the magnetization at that point. As you can see from this expression, the unit of $\boldsymbol{H}$ is the same as that of $\boldsymbol{M}$, which is the same as the surface-current density, that is, ampere/metre (A/m).

EXAMPLE 22.2 **A long solenoid having n turns per unit length carries a current I. Show that if a cylindrical rod is inserted inside the solenoid to fill it completely, the vector H does not change. Assume that the rod gets magnetized uniformly everywhere in the direction of the field produced by the solenoidal current.**

Solution In the first case, when there is no rod inside the solenoid, $\boldsymbol{M} = 0$ in the space inside the solenoid. The field $\boldsymbol{H}$ in this space is

$$\frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M} = \frac{\boldsymbol{B}_0}{\mu_0},$$

where $\boldsymbol{B}_0 = \mu_0 ni\hat{\boldsymbol{k}}$ is the field due to the solenoid current and $\hat{\boldsymbol{k}}$ is along the axis. So,

$$\boldsymbol{H} = ni\hat{\boldsymbol{k}}.$$

Now, the rod is inserted and gets magnetized in the direction of $\hat{\boldsymbol{k}}$. You can write

$$\boldsymbol{M} = M\hat{\boldsymbol{k}}.$$

The equivalent bound surface current density is

$$\boldsymbol{K}_\text{b} = \boldsymbol{M} \times \hat{\boldsymbol{n}} = M\hat{\boldsymbol{k}} \times \hat{\boldsymbol{s}} = M\hat{\phi}$$

along the curved surface of the rod.

So the surface current is along the $\hat{\phi}$-direction on the rod. This produces a magnetic field

$$\boldsymbol{B}_1 = \mu_0 K_\text{b}\hat{\boldsymbol{k}} = \mu_0 M\hat{\boldsymbol{k}}.$$

inside the rod. The magnetic field due to the solenoid current is $\boldsymbol{B}_0 = \mu_0 ni\hat{\boldsymbol{k}}$. Thus, the net field inside the rod, that is, inside the solenoid, is

$$\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_1 = \mu_0 ni\hat{\boldsymbol{k}} + \mu_0 M\hat{\boldsymbol{k}}.$$

So, $$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M} = (ni\hat{\boldsymbol{k}} + M\hat{\boldsymbol{k}}) - M\hat{\boldsymbol{k}} = ni\hat{\boldsymbol{k}}.$$

Thus, the field $\boldsymbol{H}$ inside the solenoid does not change on inserting the rod. The fields $\boldsymbol{B}$ and $\boldsymbol{M}$ are zero outside the solenoid in both cases. So $\boldsymbol{H}$ also remains zero in this region and does not change when the rod is inserted.

What is the name for the ***H***-field? Unfortunately, there is no standard name for it. The most common name is magnetizing field or magnetizing field intensity. But other names such as magnetic intensity are also used sometimes. However, there is no confusion over the symbol ***H***. So we will continue to call it the ***H***-field.

22.4 Fields *B* and *H* inside a Uniformly Magnetized Sphere

Imagine a sphere with a uniform magnetization $\boldsymbol{M}_0$ throughout its volume. What are the $\boldsymbol{B}$ and $\boldsymbol{H}$ fields inside the sphere due to its magnetization? Taking the centre as the origin and the z-axis along the direction of magnetization, the magnetization can be written as

$$\boldsymbol{M} = M_0 \hat{\boldsymbol{k}}.$$

As the magnetization is uniform, there is no bound current. The surface-current density is

$$\boldsymbol{K} = \boldsymbol{M} \times \hat{\boldsymbol{n}} = M_0 \hat{\boldsymbol{k}} \times \hat{\boldsymbol{r}} = M_0 \sin\theta\hat{\phi}.$$

You already know how to find the magnetic field inside a sphere with a surface-current density $\boldsymbol{K} = C\sin\theta\hat{\phi}$ (see Chapter 15 on vector potential). The vector potential inside the sphere is

$$\boldsymbol{A} = \frac{1}{3}\mu_0 Cr\sin\theta\hat{\phi}.$$

And the magnetic field is

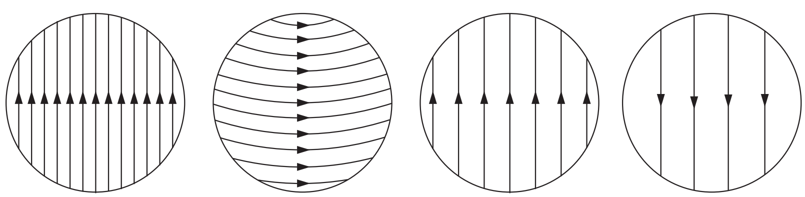
$$\boldsymbol{B} = \frac{2}{3}\mu_0 C\hat{\boldsymbol{k}}.$$

Writing M_0 for C, the magnetic field inside the uniformly magnetized sphere is

$$\boldsymbol{B} = \frac{2}{3}\mu_0 \boldsymbol{M}$$

or $$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0}\boldsymbol{M} - \boldsymbol{M} = -\frac{1}{3}\boldsymbol{M}.$$

The various quantities are schematically shown in Figure 22.6.



(a) **M** (b) $\mathbf{K} = \mathbf{M}_0 \sin\theta\hat{\phi}$ (c) $\mathbf{B} = \frac{2}{3}\mu_0\mathbf{M}$ (d) $\mathbf{H} = -\frac{1}{3}\mathbf{M}$

Fig. 22.6

For points outside, the magnetic field is like that of a magnetic dipole.

$$\boldsymbol{B} = \frac{\mu_0 m}{4\pi\varepsilon_0 r^3}(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\theta}),$$

where $M = M_0 \cdot \frac{4}{3}\pi R^3.$

The field $\boldsymbol{H}$ outside is $\frac{\boldsymbol{B}}{\mu_0}\cdot$

22.5 Relation between *H* and Free-Current Density

Bound current results from magnetization. The current caused by the motion of electrons not bound to specific atoms is called *free current*. When you connect a battery to a wire, electrons flow from the battery, along the loop and enter the battery at the other terminal. This is an example of free current. The volume-current density corresponding to the free current is written as $\boldsymbol{J}_\mathrm{f}$. Inside a magnetized material, one can have a free-current density as well as a bound-current density. So the net current density is

$$\boldsymbol{J} = \boldsymbol{J}_\mathrm{f} + \boldsymbol{J}_\mathrm{b}.$$

We will discuss the case of time-varying fields with materials present in a later chapter. Let us focus here on magnetostatic cases.

In magnetostatics,

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$$

or $$\nabla \times [\mu_0(\boldsymbol{H} + \boldsymbol{M})] = \mu_0(\boldsymbol{J}_\mathrm{f} + \boldsymbol{J}_\mathrm{b})$$

or $$\nabla \times \boldsymbol{H} + \nabla \times \boldsymbol{M} = \boldsymbol{J}_\mathrm{f} + \boldsymbol{J}_\mathrm{b}$$

or $$\nabla \times \boldsymbol{H} + \boldsymbol{J}_\mathrm{b} = \boldsymbol{J}_\mathrm{f} + \boldsymbol{J}_\mathrm{b}$$

or $$\nabla \times \boldsymbol{H} = \boldsymbol{J}_\mathrm{f}. \tag{22.3}$$

This equation looks similar to that for Ampere's law, which is $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$. In place of the total current $\boldsymbol{J}$, you have the free-current density and instead of $\boldsymbol{B}$, you have $\boldsymbol{H}$. Of course, there is the difference of the factor μ_0 but that does not change the mathematical structure. It looks as if $\boldsymbol{H}$ is related to the free-current density $\boldsymbol{J}_\mathrm{f}$ in the same manner as $\boldsymbol{B}$ is related to the total current density $\boldsymbol{J}$. Is $\boldsymbol{H}$ determined solely by $\boldsymbol{J}_\mathrm{f}$? Does the bound current in the materials have no effect on $\boldsymbol{H}$? In Example 22.2, you saw that $H = ni$ was indeed determined only by the free current in the solenoid. There was a bound-surface current on the rod when inserted into the solenoid but that did not change $\boldsymbol{H}$. However, this is not always true, as will be clear from the next example.

EXAMPLE 22.3 **A tightly wound long solenoid of radius R having n turns per unit length carries a current I. A disk of small thickness t and radius only slightly less than that of the solenoid is kept inside the solenoid coaxially with it. The magnetization appearing in the disk along the common axis is m_0. Find the value of H on the axis of the disk at a distance z from its centre. What will be the value of H if the disk is removed?**

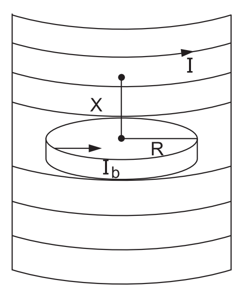


Fig. 22.7

Solution The situation is shown in Figure 22.7. Take the common axis of the solenoid and the disk as the z-axis. The bound surface-current density exists on the curved surface of the disk, and equals $\boldsymbol{K}_\mathrm{b} = m_0 \hat{\boldsymbol{k}} \times \hat{\boldsymbol{s}} = m_0 \hat{\phi}$.

The current is, therefore, $I_\mathrm{b} = K_\mathrm{b} t = m_0 t$ along the $\hat{\phi}$-direction. The magnetic field due to this current at the given point is

$$\boldsymbol{B}_1 = \frac{\mu_0 I_\mathrm{b} R^2}{2(R^2+z^2)^{3/2}}\hat{\boldsymbol{k}} = \frac{\mu_0 m_0 t R^2}{2(R^2+z^2)^{3/2}}\hat{\boldsymbol{k}}.$$

The field due to the solenoid current is

$$\boldsymbol{B}_2 = \mu_0 n I \hat{\boldsymbol{k}}.$$

The net field is $\boldsymbol{B} = \boldsymbol{B}_1 + \boldsymbol{B}_2 = \dfrac{\mu_0 m_0 t R^2}{2(R^2+z^2)^{3/2}}\hat{\boldsymbol{k}} + \mu_0 n I \hat{\boldsymbol{k}}$.

At the given point there is no material, so $\boldsymbol{M} = 0$. Thus,

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M} = \frac{m_0 t R^2}{2(R^2+z^2)^{3/2}}\hat{\boldsymbol{k}} + nI\hat{\boldsymbol{k}}.$$

If the disk is removed,

$$\boldsymbol{H} = nI\hat{\boldsymbol{k}}.$$

Why is the field $\boldsymbol{H}$ dependent on the bound current $I_\mathrm{b} = m_0 t$ in the above example? After all, the equation $\nabla \times \boldsymbol{H} = \boldsymbol{J}_\mathrm{f}$ has no reference to the bound current. The answer is that a vector field is not completely specified by the curl alone. You also have to specify the divergence of it (and also the boundary conditions). Though $\nabla \times \boldsymbol{H} = \boldsymbol{J}_\mathrm{f}$ does not involve bound current, $\nabla \cdot \boldsymbol{H}$ does as shown below.

$$\nabla \cdot \boldsymbol{B} = 0$$

or $$\nabla \cdot [\mu_0(\boldsymbol{H}+\boldsymbol{M})] = 0$$

or $$\nabla \cdot \boldsymbol{H} = -\nabla \cdot \boldsymbol{M}. \tag{22.4}$$

$\boldsymbol{M}$ is related to bound currents and so is $\nabla \cdot \boldsymbol{H}$. In fact, this equation resembles $\nabla \cdot \boldsymbol{E} = \rho/\varepsilon_0$ for electrostatic fields. The divergence $\nabla \cdot \boldsymbol{M}$ acts as the density of point sources for $\boldsymbol{H}$. In Example 22.1, $\boldsymbol{M}$ was uniform everywhere in the cylindrical volume inside the solenoid, giving $\nabla \cdot \boldsymbol{M} = 0$ and so $\boldsymbol{J}_\mathrm{f}$ was the only source of $\boldsymbol{H}$. In Example 22.3, $\boldsymbol{M}$ is confined to the volume of the disk. As you go out, $\boldsymbol{M}$ suddenly changes to zero. So, $\nabla \cdot \boldsymbol{M}$ is not zero on the flat surfaces of the disk. If you think of $-\nabla \cdot \boldsymbol{M}$ in Example 22.3, it is positive on the upper surface and negative on the lower. Like charge density, it produces its own field $\boldsymbol{H}$ and adds to that produced by the free current in the solenoid.

So, the equations governing $\boldsymbol{H}$ are

$$\nabla \cdot \boldsymbol{H} = -\nabla \cdot \boldsymbol{M}$$

and $$\nabla \times \boldsymbol{H} = \boldsymbol{J}_\mathrm{f}.$$

If there is no free-current density anywhere in space, such as in the case of a magnetized object placed in empty space, the equations for $\boldsymbol{H}$ become

$$\nabla \cdot \boldsymbol{H} = -\nabla \cdot \boldsymbol{M}$$

and $$\nabla \times \boldsymbol{H} = 0.$$

These have the same mathematical structure as that for the electrostatic field $\boldsymbol{E}$,

$$\nabla \cdot \boldsymbol{E} = \rho/\varepsilon_0$$

and $$\nabla \times \boldsymbol{E} = 0.$$

Thus, $(-\nabla \cdot \boldsymbol{M})$ plays the same role in getting $\boldsymbol{H}$ as ρ/ε_0 in obtaining $\boldsymbol{E}$. One can write $\rho_H = -\nabla \cdot \boldsymbol{M}$ and treat it as the charge density for the field $\boldsymbol{H}$. Call it $\boldsymbol{H}$-charge density. The quantity $q_H = \int \rho_H d\tau$ in any volume then becomes the total $\boldsymbol{H}$-charge in that volume. The equation for the field $\boldsymbol{H}$ in terms of ρ_m will be

$$\boldsymbol{H} = \frac{1}{4\pi} \int \frac{\rho_H \, |\boldsymbol{r} - \boldsymbol{r}'| \, d\tau}{|\boldsymbol{r} - \boldsymbol{r}'|^3}.$$

Many of the old texts talk about the pole strength of a magnet. The *H*-charge q_H defined here corresponds to this pole strength. The pole strength per unit volume can be called pole density. Remember that we had taken $\boldsymbol{J}_\mathrm{f} = 0$. If there are free currents, you should take their contribution separately. You can use the expression for electric fields for different charge distributions to express the field $\boldsymbol{H}$ due to $\boldsymbol{H}$-charge distributions in the same mathematical form. Substitute *H*-charge for the electric charge and drop ε_0, and the expression shows you $\boldsymbol{H}$.

22.6 The *H*-Charge on a Permanent Magnet

So you can think of $\boldsymbol{H}$-charges or magnetic poles creating the field $\boldsymbol{H}$. Remember that there are no separate particles forming the source of $\boldsymbol{H}$. $\boldsymbol{H}$ is only an equivalent mathematical quantity for a source of the type $\nabla \cdot \boldsymbol{M}$. Now consider a cylindrical bar magnet. Suppose it is uniformly magnetized with $\boldsymbol{M}$ along its length. As $\boldsymbol{M}$ is uniform inside the magnet, $\nabla \cdot \boldsymbol{M} = 0$ and hence there is no *H*-charge or magnetic pole density in the interior of the magnet. But as you cross the end surface, $\boldsymbol{M}$ suddenly becomes zero. So, the value of $\nabla \cdot \boldsymbol{M}$ at the ends becomes infinite. This

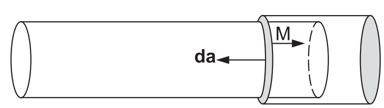


Fig. 22.8

means the ***H***-charge or the magnetic pole density is only at the end surfaces, the surface magnetic pole density being σ_H.

What is the total magnetic pole strength (*H*-charge) on either of the end surfaces? Take a closed cylindrical surface that encloses one of the end surfaces of the bar magnet, as shown in Figure 22.8. The total magnetic pole strength on the surface of the magnet inside this closed surface is

$$q_m = \oint \rho_H d\tau = \int (-\nabla \cdot \boldsymbol{M}) d\tau = -\oint \boldsymbol{M} \cdot \boldsymbol{da},$$

where the surface integration is to be done on the closed surface. The parts of the closed surface outside the magnet do not contribute as ***M*** is zero. The part within the magnet gives

$$-\int \boldsymbol{M} \cdot \boldsymbol{da} = M \cdot A,$$

where ***A*** is the area of cross section of the magnet. Remember that, the direction ***da*** is towards the outward normal to the closed surface and hence is opposite to ***M*** inside the magnet. So an ***H***-charge or magnetic pole strength ***MA*** appears at the end face on the right. Similarly, an *H*-charge –*MA* appears at the end face on the left.

The magnetic dipole moment of the bar magnet is

$$M \times Al = q_H l.$$

This too resembles the expression for the electric dipole moment of a dipole consisting of charges –*q* and +*q*.

The end having positive *H*-charge is called the north pole N of the magnet and the one having negative *H*-charge is called the south pole S. Check that the direction of the dipole moment is from the negative *H*-charge to the positive ***H***-charge.

To remind you once again, the *H*-charge or pole strength is only a mathematical entity that gives correct results but does not have any physical significance.

EXAMPLE 22.4 **A cylindrical bar magnet of length *L* and cross-sectional area *A* is magnetized to have a uniform magnetization M_0 along its length. Find the magnetic field *B* due to the magnet at a point on its axis (a) at a distance *r* from the centre of the magnet with $r >> L$, (b) just outside the ends of the magnet and (c) at the centre of the magnet. Assume the length *L* of the magnet to be large compared to its thickness.**

Solution

(a) The situation outlined in the problem is shown in Figure 22.9. The total *H*-charge at the end N is $q_m = M_0 A$ and that at S is $-q_H = -M_0 A$. For the point *P* far from the magnet ($r >> L$), the pair q_H and $-q_H$ can be treated as a dipole with dipole

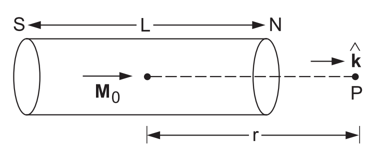


Fig. 22.9

moment $\boldsymbol{m} = q_H L\hat{\boldsymbol{k}} = M_0 AL\hat{\boldsymbol{k}}$. Using spherical coordinates, the field $\boldsymbol{H}$ due to this dipole will equal $\dfrac{\boldsymbol{m}}{4\pi r^3}(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}})$ at points far away from the dipole. This expression is directly taken from the expression for the electric field due to an electric dipole moment. Here, θ is the angle made by the radius vector with the dipole moment.

For points along the axis on the side of N,

$$\theta = 0.$$

So,
$$\boldsymbol{H} = \frac{2m}{4\pi r^3}\hat{\boldsymbol{r}} = \frac{2M_0 AL}{4\pi r^3}\hat{\boldsymbol{k}} = \frac{2\boldsymbol{M}_0 AL}{4\pi r^3}.$$

As $\boldsymbol{M}$ at this point is zero (outside the magnet),

$$\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}) = \frac{2\mu_0 \boldsymbol{M}_0 AL}{4\pi r^3}.$$

(b) For a point just outside the north pole N and close to its centre, the end may be treated as a large surface having a uniform $\boldsymbol{H}$-charge density (Figure 22.10)

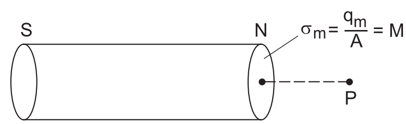


Fig. 22.10

$$\sigma_H = \frac{q_H}{A} = \frac{M_0 A}{A} = M_0.$$

The electric field due to a uniformly charged plane surface with surface density σ just outside it is given by $\dfrac{\sigma}{2\varepsilon_0}\hat{\boldsymbol{n}}$. The field $\boldsymbol{H}$ due to a surface $\boldsymbol{H}$-charge density will have a similar expression:

$$\boldsymbol{H}_1 = \frac{\sigma_H}{2}\hat{\boldsymbol{k}} = \frac{1}{2}\boldsymbol{M}_0.$$

As the length of the magnet is large, the $\boldsymbol{H}$-charge at the south pole can be treated as a point charge to calculate its field $\boldsymbol{H}_2$ at P. Thus,

$$\boldsymbol{H}_2 = \frac{1}{4\pi}\frac{q_H}{L^2}(-\hat{\boldsymbol{k}}) = \frac{1}{4\pi}\frac{M_0 A}{L^2}(-\hat{\boldsymbol{k}}) = -\frac{\boldsymbol{M}_0 A}{4\pi L^2}.$$

Thus, the field $\boldsymbol{H}$ at P is

$$\boldsymbol{H} = \boldsymbol{H}_1 + \boldsymbol{H}_2 = \frac{\boldsymbol{M}_0}{2}\left(1 - \frac{A}{2\pi L^2}\right).$$

This point is also outside the magnet. So,

$$\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}) = \mu_0\boldsymbol{H} = \frac{\mu_0\boldsymbol{M}_0}{2}\left(1 - \frac{A}{2\pi L^2}\right).$$

If you neglect $\dfrac{A}{2\pi L^2}$ in comparison to 1 (because the length L is much larger than the thickness),

$$\boldsymbol{B} = \frac{\mu_0\boldsymbol{M}_0}{2}.$$

(c) At the centre of the magnet, the fields $\boldsymbol{H}$ due to both poles are in the same direction. Treating each pole as a point $\boldsymbol{H}$-charge,

$$\boldsymbol{H} = 2\frac{q_H}{4\pi(L/2)^2}(-\hat{\boldsymbol{k}}) = \frac{8M_0A}{4\pi L^2}(-\hat{\boldsymbol{k}}) = -\frac{2\boldsymbol{M}_0A}{\pi L^2}.$$

Thus,

$$\boldsymbol{B} = \mu_0(\boldsymbol{H}+\boldsymbol{M}) = \mu_0\boldsymbol{M}_0\left[1-\frac{2A}{\pi L^2}\right].$$

Neglecting $\dfrac{2A}{\pi L^2}$,

$$\boldsymbol{B} = \mu_0\boldsymbol{M}_0.$$

Note that the field $\boldsymbol{H}$ inside the magnet and that outside the magnet (on the axis) have opposite directions. But the field $\boldsymbol{B}$ has the same direction at all these points.

You can get $\boldsymbol{B}$ from the bound current itself. The bound current will be on the curved surface, and the bound-current density will equal

$$\boldsymbol{K}_\text{b} = \boldsymbol{M}\times\hat{\boldsymbol{n}} = M_0\hat{\boldsymbol{k}}\times\hat{\boldsymbol{n}} = M_0\hat{\phi}.$$

This expression is like that for a solenoid, and for internal points (long solenoid),

$$\boldsymbol{B} = \mu_0K_\text{b}\hat{\boldsymbol{k}} = \mu_0M_0\hat{\boldsymbol{k}} = \mu_0\boldsymbol{M}_0.$$

So you have two sources of $\boldsymbol{H}$. One is free current, giving the field governed by $\nabla\times\boldsymbol{H} = \boldsymbol{J}_\text{f}$, and the other is $\nabla\cdot\boldsymbol{M}$. If $\nabla\cdot\boldsymbol{M}$ is not zero at certain places, you should use equations corresponding to Coulomb's law to get $\boldsymbol{H}$ from these places.

The field $\boldsymbol{H}$ is important because, in a laboratory, it is easier to control $\boldsymbol{H}$ as it is more directly connected to the free current, which we can play with.

22.7 Boundary Conditions on *H*

Consider a plane surface carrying a free surface current density $\boldsymbol{K}_f$. Let the surface be the x–y plane (Figure 22.11). It could be the surface of a material or the interface between two material media or any other surface. Draw a rectangular loop ABCD, AB being on one side of the plane and CD, on the other side. Let AB and CD be parallel to the y-axis, and BC and DA be parallel to the z-axis. BC and DA are infinitesimally small.

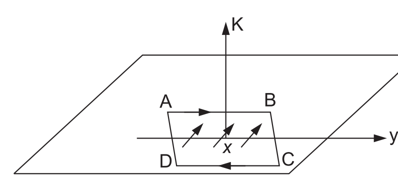


Fig. 22.11

Assume time-independent fields, so that

$$\nabla\times\boldsymbol{H} = \boldsymbol{J}_f$$

or $$\oint \boldsymbol{H}\cdot d\boldsymbol{l} = \int \boldsymbol{J}_f \cdot d\boldsymbol{a} \tag{i}$$

over any closed loop. Using ABCD as the loop,

$$\oint \boldsymbol{H}\cdot d\boldsymbol{l} = H_{1y}l - H_{2y}l, \tag{ii}$$

where H_{1y} is the y-component of $\boldsymbol{H}$ at the location of AB, H_{2y} is that at the location of CD and l is the length of AB or CD. Also, $\int \boldsymbol{J}_f \cdot d\boldsymbol{a}$ is the total current crossing the area $ABCD$. As the width AD or BC is assumed to be infinitesimally small, there will be a current through $ABCD$ only if there is a surface current on the surface and it has an x-component. If K_x be the x-component of the surface current, the net current crossing ABCD will be $(-K_x l)$. Check the direction of the arrow on the curve ABCD and hence the direction of the positive normal to the area $ABCD$. It is along the negative x-direction. Thus, from (i) and (ii),

$$(H_{1y} - H_{2y})l = -K_x l$$

or $$H_{1y} - H_{2y} = -K_x. \tag{iii}$$

Similarly, you can take another loop $A'B'C'D'$ with $A'B'$ and $C'D'$ along the x-axis and on opposite sides of the x–y plane. Doing a similar analysis will give

$$H_{1x} - H_{2x} = K_y. \tag{iv}$$

Writing $\boldsymbol{H}_t = \hat{\boldsymbol{i}}H_x + \hat{\boldsymbol{j}}H_y$ for the component of $\boldsymbol{H}$ parallel to the surface,

$$\boldsymbol{H}_{1t} - \boldsymbol{H}_{2t} = \hat{\boldsymbol{i}}(H_{1x} - H_{2x}) + \hat{\boldsymbol{j}}(H_{1y} - H_{2y}) = \hat{\boldsymbol{i}}K_y - \hat{\boldsymbol{j}}K_x$$

or $$\boldsymbol{H}_{1t} - \boldsymbol{H}_{2t} = \boldsymbol{K}_f \times \hat{\boldsymbol{n}}. \tag{22.5}$$

Remember that $\hat{\boldsymbol{n}} = \hat{\boldsymbol{k}}$ is the unit vector along the normal to the plane drawn from Side 1 of the plane.

This is the boundary condition on the tangential component of the field $\boldsymbol{H}$. To get the boundary condition on the normal component, consider a surface and call one side of it Side 1 and the other side Side 2. Take the unit normal in Side 1 as $\hat{\boldsymbol{n}}$.

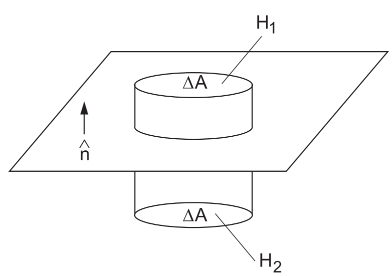


Fig. 22.12

Construct a small area ΔA parallel to the surface in Side 1 (Figure 22.12). Construct a similar area ΔA below it in Side 2. Join corresponding points to make a cylinder. Let the height of the cylinder be infinitesimally small.

$$\nabla \cdot \boldsymbol{H} = -\int \nabla \cdot \boldsymbol{M}.$$

Integrating both sides on the volume of the cylinder,

$$\int \nabla \cdot \boldsymbol{H} d\tau = -\int \nabla \cdot \boldsymbol{M} d\tau$$

or $$\oint \boldsymbol{H} \cdot \boldsymbol{da} = -\oint \boldsymbol{M} \cdot \boldsymbol{da}. \tag{i}$$

The value of $\oint \boldsymbol{H} \cdot \boldsymbol{da}$ on the flat surface in Side 1 is $(\boldsymbol{H}_1 \cdot \hat{\boldsymbol{n}})\Delta A$ and that on the flat surface in Side 2 is $\boldsymbol{H}_2 \cdot (-\hat{\boldsymbol{n}})\Delta A$. Also, $\oint \boldsymbol{H} \cdot \boldsymbol{da}$ on the curved part tends to zero as the height of the cylinder tends to zero.

Thus,

$$\oint \boldsymbol{H} \cdot \boldsymbol{da} = (\boldsymbol{H}_1 \cdot \hat{\boldsymbol{n}})\Delta A - \boldsymbol{H}_2 \cdot (-\hat{\boldsymbol{n}})\Delta A = (H_{1n} - H_{2n})\Delta A.$$

Similarly,

$$\oint \boldsymbol{M} \cdot \boldsymbol{da} = (M_{1n} - M_{2n})\Delta A.$$

From (i),

$$H_{1n} - H_{2n} = M_{2n} - M_{1n}. \tag{22.6}$$

This is the required boundary condition on the normal component of $\boldsymbol{H}$.

You can also write it in terms of the surface $\boldsymbol{H}$-charge density ρ_H.

$$\nabla \cdot \boldsymbol{H} = \rho_H.$$

Compare this with $\nabla \cdot \boldsymbol{E} = \rho/\varepsilon_0$. This leads to the boundary condition on the normal component of $\boldsymbol{E}$ as

$$\boldsymbol{E}_{in} - \boldsymbol{E}_{in} = \frac{\sigma}{\varepsilon_0}\cdot$$

So, $$\nabla \cdot \boldsymbol{H} = \rho_H$$

will give $$\boldsymbol{H}_{in} - \boldsymbol{H}_{in} = \rho_H. \tag{22.7}$$

22.8 Linear Magnetic Material

The behaviour of a material in an applied magnetic field depends on the nature of the material. In many materials, the magnetization $\boldsymbol{M}$ at a point is parallel to the field $\boldsymbol{H}$ at that point, and is proportional to it. Such a material is called a *linear magnetic material*. For such materials,

$$\boldsymbol{M} = \chi \boldsymbol{H}, \tag{22.8}$$

where χ is a constant for the given material at a given temperature and is called *magnetic susceptibility*. Paramagnetic and diamagnetic substances are often linear. But in case of an anisotropic material like a single crystal, the magnetization may be direction-dependent and the material will not be linear.

For diamagnetic materials, the magnetization is produced in the direction opposite to $\boldsymbol{H}$, and so χ is negative. Also, the magnitude of χ is very small, of the order of 10^{-6} or so. For paramagnetic materials, χ is positive and again of the order of 10^{-6}. Ferromagnetic materials are nonlinear and one cannot define a susceptibility that is constant for the material. We will discuss ferromagnetic materials in more detail in the next section. Table 22.1 gives the values of magnetic susceptibility for some common diamagnetic and paramagnetic materials.

Table 22.1

Diamagnetic material	Susceptibility	Paramagnetic material	Susceptibility
Bismuth	-1.6×10^{-4}	Oxygen	1.9×10^{-6}
Gold	-3.4×10^{-5}	Sodium	8.5×10^{-6}
Silver	-2.4×10^{-5}	Aluminium	2.1×10^{-5}
Copper	-9.7×10^{-6}	Tungsten	2.8×10^{-4}
Water	-9.0×10^{-6}	Platinum	7.8×10^{-5}
Carbon dioxide	-1.2×10^{-8}	Liquid oxygen (–200° C)	3.9×10^{-3}
Hydrogen	-2.2×10^{-9}	Gadolinium	4.8×10^{-1}

Another useful constant for linear magnetic materials is relative permeability, written as μ_r. The magnetic field inside a magnetic material is

$$\begin{aligned} \boldsymbol{B} &= \mu_0(\boldsymbol{H}+\boldsymbol{M}) = \mu_0(\boldsymbol{H}+\chi\boldsymbol{H}) \\ &= \mu_0(1+\chi)\boldsymbol{H}. \end{aligned} \tag{22.9}$$

The quantity $1+\chi$ is called *relative permeability* and is written as μ_r. Thus,

$$\boldsymbol{B} = \mu_0\mu_r\boldsymbol{H},$$

where $$\mu_r = 1+\chi. \tag{22.10}$$

The quantity $\mu_0\mu_r$ is denoted by μ and is called the *permeability* of the material. Thus, for linear materials,

$$\boldsymbol{B} = \mu\boldsymbol{H}. \tag{22.11}$$

22.9 The *M–H* Relation for Ferromagnetic Materials

Ferromagnetic and ferrimagnetic materials have magnetic domains aligned along certain

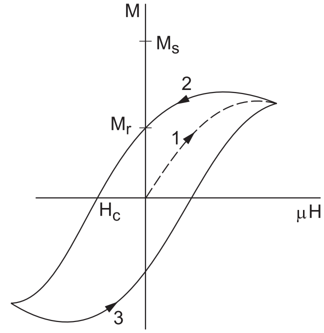


Fig. 22.13

directions. In the unmagnetized state, the domains are randomly oriented. When the materials are placed in an external magnetic field, the domain walls move, and the domains are oriented. With a moderately strong field, say $\mu_0 \boldsymbol{H} = 1$ tesla or so, many ferromagnetic materials saturate, meaning that all the domains get perfectly aligned and the further application of a field does not increase magnetization.

The magnetization of a ferromagnetic material depends not only on the magnetic field applied at that instant but also on the history of the applied magnetic field. In general, the field is applied by passing a current *i* through a coil and is changed by changing *i*. The current produces the field ***H***. This field is taken as the independent variable and magnetization is studied against it.

Suppose you start with an unmagnetized piece of a ferromagnetic material, apply a field ***H*** and increase its value in steps. The magnetization of the piece will increase with ***H*** initially, and will finally saturate. For common ferromagnetic materials, saturation comes about around 10^6 A/m. If you talk in terms of $\mu_0 H$, it will be ~ 1T. The part from zero magnetization to the maximum value of magnetization (or *saturation magnetization*) is labelled 1 in Figure 22.13. Saturation magnetization is denoted by the symbol M_s. If you now reduce ***H*** in slow steps, the magnetization decreases, but not along the same path. For any value of ***H*** in the decreasing steps, the magnetization is more than the value for the same ***H*** when it was increasing (Path 2 in Figure 22.13). When ***H*** reduces to zero, there is still some magnetization which is denoted by M_r. This quantity is called *remanent magnetization*. If you apply ***H*** in opposite direction at this stage and increase the magnitude in slow steps, the magnetization decreases from the value of M_r, and becomes zero at a particular value of ***H***. This value of ***H*** is called the *coercive field*. Increasing the magnitude of ***H*** raises the magnitude *M* till it finally saturates. You can now decrease the magnitude of ***H*** and the magnetization will follow Path 3.

The phenomenon of magnetization lagging behind in the decreasing cycle of ***H*** compared to the increasing cycle is called *hysteresis* and the *M–H* curve for the full cycle of variation of ***H*** is called the *hysteresis loop*. Energy is converted into thermal energy when a ferromagnetic material goes through a hysteresis loop.

As the value of *M* depends on the history of the applied field, one cannot define the magnetic susceptibility of ferromagnetic materials. However, when you apply a magnetic field on an unmagnetized material ($M = 0$), it has a definite $\frac{dM}{dH}$. This quantity $\frac{dM}{dH}$ at $M = 0$ is called the initial magnetic susceptibility of the material, and is an important characteristic of ferromagnetic materials. Its value can be in tens of thousands.

Concepts at a Glance

1. Bound currents appear in a magnetized material.
2. $\boldsymbol{J}_b = \nabla \times \boldsymbol{M}$, $\boldsymbol{K}_b = \boldsymbol{M} \times \hat{\boldsymbol{n}}$.
3. The field $\boldsymbol{H}$ is defined as $\dfrac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}$.
4. $\nabla \times \boldsymbol{H} = \boldsymbol{J}_f$ (free-current density).
5. $\nabla \cdot \boldsymbol{H} = -\nabla \cdot \boldsymbol{M}$.
6. For linear materials,

   $\boldsymbol{M} = \chi \boldsymbol{H}$, $\boldsymbol{B} = \mu_0(1+\chi)\boldsymbol{H}$, $\mu_r = 1+\chi$.

   χ is called magnetic susceptibility and μ_r is called relative permeability.
7. Ferromagnetic materials show hysteresis where magnetization depends not only on the value of $\boldsymbol{H}$ or $\boldsymbol{B}$ but also on the history of the applied field.
8. The tangential component of $\boldsymbol{H}$ is continuous across a surface if there is no free current on the surface.
9. If there is a free current on a surface, the field $\boldsymbol{H}$ is discontinuous across it.

   $\boldsymbol{H}_{1t} - \boldsymbol{H}_{2t} = \boldsymbol{K} \cdot \hat{\boldsymbol{n}}$.
10. The normal component of $\boldsymbol{H}$ across a surface is discontinuous if that of the magnetization $\boldsymbol{M}$ is discontinuous across the surface. $\boldsymbol{H}_{1n} - \boldsymbol{H}_{2n} = \boldsymbol{M}_{2n} - \boldsymbol{M}_{1m}$.
11. One can define a mathematical quantity *H*-charge density as $\rho_H = -\nabla \cdot \boldsymbol{M}$. The quantity $\int \rho_H d\tau$ can then be termed the total *H*-charge in the volume considered. In early texts, this quantity was called pole strength.

EXERCISES

Based on Concepts

1. Give one example in which there is no free current anywhere but the field $\boldsymbol{H}$ is still nonzero.
2. Give one example where there is a free-current density but $\boldsymbol{H}$ is zero (or very close to zero).
3. A bar magnet is neatly cut in two parts. Will the magnetization at a point in the magnet decrease, increase or remain the same?
4. $\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M})$. Does this mean that $\boldsymbol{B} > \mu_0 \boldsymbol{M}$? Does the answer depend on whether the material is diamagnetic or paramagnetic?
5. Wherever there is a point-like source of $\boldsymbol{H}$, there is a source of $\boldsymbol{M}$. Comment on this statement.
6. Is it necessary that the magnetic poles (*H*-charge) be situated only at the ends of a bar magnet?
7. A circular cylinder of radius a and length L carries a uniform magnetization $\boldsymbol{M}$ parallel to its axis. Sketch the field $\boldsymbol{H}$ due to the cylinder if (a) $L >> a$, (b) $l << a$, and (c) $L \approx a$.

Problems

1. A long solenoid with 40 turns/cm has an aluminium core and carries a current of 2.0 A. Calculate the magnetization developed in the core and the field $\boldsymbol{B}$ at the centre. The magnetic susceptibility of aluminium is 2.3×10^{-5}. [**Ans.** 0.10 A/m]

2. A long, tightly wound solenoid with 50 turns/cm carries a current of 4.00 A. (a) Find the value of the fields $\boldsymbol{H}$ and $\boldsymbol{B}$ at the centre. (b) What will be the values of these quantities if an iron core is inserted in the solenoid and the magnetization in the core is 2.00×10^{6} A/m?

   [**Ans.** (a) 2.00×10^{4} A/m, 25.1 mT; (b) 2.00×10^{4} A/m, 2.54 T]

3. An infinitely long cylinder of radius R has a uniform magnetization $\boldsymbol{M}$ parallel to its axis. Find the field $\boldsymbol{H}$ due to this cylinder everywhere. [**Ans.** Zero]

4. An infinitely long cylinder of radius R has a magnetization $\boldsymbol{M} = ks\hat{\boldsymbol{k}}$, parallel to the axis, where k is a constant and s is the distance from the axis. There is no free current anywhere. (a) Find $\boldsymbol{H}$ inside and outside the cylinder by using Ampere's law. (b) From this, get $\boldsymbol{B}$.

   [**Ans.** (a) Zero everywhere, (b) $\mu_0 ks\hat{\boldsymbol{k}}$ inside and zero outside]

5. A long, circular cylinder of radius R has a magnetization $\boldsymbol{M} = ks^2\hat{\phi}$, where k is a constant, s is the distance from the axis and $\hat{\phi}$ is the usual azimuthal unit vector. (a) Find the bound volume current density. (b) Using Ampere's law for $\boldsymbol{B}$, find the magnetic field inside the cylinder. (c) From these results, find the value of $\boldsymbol{H}$ inside the cylinder. [**Ans.** (a) $\boldsymbol{J}_\text{b} = 3ks\hat{\boldsymbol{k}}$, $\boldsymbol{K}_\text{b} = -kR^2\hat{\boldsymbol{k}}$, (c) Zero]

6. The length as well as the radius of a cylindrical material are equal to L. The sample is uniformly magnetized along its axis and the magnetization is $\boldsymbol{M}$. Find the magnetic field at the centre. [**Ans.** $\dfrac{\mu_0\boldsymbol{M}}{\sqrt{5}}$]

7. An infinitely long, hollow cylinder of inner radius a and outer radius b has a frozen-in magnetization $\boldsymbol{M} = \dfrac{C}{s}\hat{\varphi}$, in cylindrical (s, ϕ, z) coordinates with C as a constant and the z-axis along the axis of the cylinder. (a) Find the surface and volume bound-current densities. (b) Find $\boldsymbol{H}$ everywhere.

   [**Ans.** (a) $\boldsymbol{J}_\text{b} = 0$, $\boldsymbol{K}_\text{b} = -\dfrac{C}{b}\hat{\boldsymbol{k}}$ on the outer surface and $\dfrac{C}{a}\hat{\boldsymbol{k}}$ on the inner surface]

8. In a large, uniformly magnetized piece of material, the fields $\boldsymbol{H}$ and $\boldsymbol{B}$ are given by $H_0\hat{\boldsymbol{k}}$ and $B_0\hat{\boldsymbol{k}}$ respectively. The material has a small cavity. Find $\boldsymbol{B}$ and $\boldsymbol{H}$ at the centre of the cavity if the cavity has the shape of (a) a very thin but long cylinder with its axis parallel to $\hat{\boldsymbol{k}}$, (b) a sphere of small radius R and (c) a thin but wide disk with its axis parallel to $\hat{\boldsymbol{k}}$.

   [**Ans.** (a) $\mu_0 H_0\hat{\boldsymbol{k}}$, $H_0\hat{\boldsymbol{k}}$; (b) $\dfrac{(B_0+2\mu_0 H_0)}{3}\hat{\boldsymbol{k}}$, $\dfrac{(B_0+2\mu_0 H_0)}{(3\mu_0)}\hat{\boldsymbol{k}}$; (c) $B_0\hat{\boldsymbol{k}}$, $\dfrac{B_0}{\mu_0}\hat{\boldsymbol{k}}$]

9. A long, cylindrical shell of inner and outer radii a and b rotates about its axis with a uniform angular speed ω. The magnetization in the shell is $\boldsymbol{M} = M_0\hat{\boldsymbol{k}}$, and the magnetic susceptibility is 5000. (a) Find $\boldsymbol{H}$ and $\boldsymbol{B}$ everywhere. (b) Find the voltage generated between the inner and outer surfaces.

   [**Ans.** (a) $H = 0$ everywhere, $\boldsymbol{B} = \mu_0\boldsymbol{M}$ inside the magnet; (b) $\dfrac{\mu_0 M\omega(b^2-a^2)}{2}$]

10. A coaxial cable consists of two very long, cylindrical tubes, separated by a linear insulating material of magnetic susceptibility χ_m. A current I flows along the inner conductor and returns along the outer one. In each case, the current distributes itself uniformly over the surface of the tube. Find the magnetic field in the insulating material. [**Ans.** $\dfrac{\mu_0(1+\chi)I}{2\pi s}\hat{\phi}$]

11. A current I flows down a long, straight copper wire of radius a and magnetic susceptibility χ_m, distributed uniformly over its cross section. (a) What is the magnetic field inside the wire at a distance s from the axis? (b) Find the bound surface current density in the wire.[**Ans.** (a) $\frac{\mu_0(1+\chi_m)Is}{2\pi a^2}\hat{\phi}$, (b) $\frac{\chi I}{2\pi a}\hat{\boldsymbol{k}}$]

12. A large conducting plate extends from $z=-a$ to $z=a$. It carries a free-current density $\boldsymbol{J}_f = J_0\frac{z}{a}\hat{\boldsymbol{i}}$. The material of the plate has a magnetic susceptibility χ_m. Find the (a) field $\boldsymbol{H}$ everywhere and (b) bound-current densities in the material. [**Ans.** (a) $\frac{J_0}{2a}(a^2-z^2)\hat{\boldsymbol{j}}$, (b) $\boldsymbol{K}_b=0$, $\boldsymbol{J}_b=\chi_m J_0\frac{z}{1}\hat{\boldsymbol{i}}$]

13. A sphere of magnetic susceptibility χ_m is placed in an otherwise uniform magnetic field $\boldsymbol{B}_0$. Find (a) the magnetization developed in the sphere, assuming it to be uniform, (b) the magnetic field $\boldsymbol{B}$ inside the sphere. [**Ans.** (a) $\frac{3\chi_m}{3+\chi_m}\frac{\boldsymbol{B}_0}{\mu_0}$, (b) $\frac{3(1+\chi_m)}{3+\chi_m}\boldsymbol{B}_0$]

14. A large slab of a material with $\mu_r=2.5$ fills up the region $0<z<2$ m. The field $\boldsymbol{H}$ in this region is $10y\hat{\boldsymbol{k}}-5x\hat{\boldsymbol{j}}$ in SI units. Find (a) the free-current density, (b) the bound-current density and (c) the magnetization in the slab. [**Ans.** (a) $10\hat{\boldsymbol{i}}-5\hat{\boldsymbol{k}}$, (b) $15\hat{\boldsymbol{i}}-7.5\hat{\boldsymbol{k}}$, (c) $15y\hat{\boldsymbol{k}}-7.5x\hat{\boldsymbol{j}}$]

15. The region $y\le x+2$ is filled with a medium of relative permeability $\mu_r=1.5$. The field $\boldsymbol{H}$ here is given by $= H_0(2\hat{\boldsymbol{i}}+6\hat{\boldsymbol{j}}+4\hat{\boldsymbol{k}})$. All quantities are in SI units. Calculate the field $\boldsymbol{B}$ in the region $y\ge x+2$. There is no free current anywhere and the material in the region $y>x+2$ is air. [**Ans.** $\mu_0 H_0(\hat{\boldsymbol{i}}+7\hat{\boldsymbol{j}}+4\hat{\boldsymbol{k}})$]

16. There is a free surface current density $\boldsymbol{K}=\frac{10^4}{4\pi}\hat{\boldsymbol{j}}$ on the x–y plane. The medium in $z<0$ has relative permeability 6 and that in $z>0$ has relative permeability 4. The magnetic field at $z=\varepsilon$ is $\boldsymbol{B}=(5\hat{\boldsymbol{i}}+8\hat{\boldsymbol{k}})B_0$, where ε is a small positive quantity. All quantities are in SI units. Find the fields $\boldsymbol{H}$ and $\boldsymbol{B}$ at $z=-\varepsilon$. [**Ans.** $\frac{1}{\mu_0}\left(\frac{5}{4}\hat{\boldsymbol{i}}-\frac{4}{3}\hat{\boldsymbol{k}}\right), \left(\frac{15}{2}\hat{\boldsymbol{i}}-8\hat{\boldsymbol{k}}\right)$]

17. Two magnetically linear media having relative permeability μ_{r1} and μ_{r2} are separated by a plane surface. There is no free current on this surface but there may be other currents in space. The magnetic field $\boldsymbol{B}$ in the first medium near the boundary makes an angle α_1 with the normal and the field in the second medium makes an angle α_2 with the normal there. Find the relation between α_1 and α_2. What will be the relation between the angles made by $\boldsymbol{H}$ in the two media with the normal? [**Ans.** $\mu_{r2}\tan\alpha_1=\mu_{r1}\tan\alpha_2$ in both cases]

18. An iron rod having a square cross section is bent in the shape of a circle with a narrow gap at one place (Figure 22.E1). The gap has a width w and the circle has a mean radius R. The cross section has an area a^2. Find the magnetic field in the gap. Assume $w \ll a \ll R$. [**Ans.** $\mu_0 M\left(1-\frac{2\sqrt{2}w}{\pi a}\right)$]

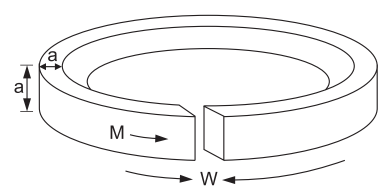


Fig. 22.E1

19. A toroidal coil of 2000 turns with a square cross section is wound over a ring with inner radius 10 mm, outer radius 15 mm and magnetic susceptibility 500. A very long, straight wire is placed along the axis of the toroid. Find the mutual inductance between the wire and the toroid. [**Ans.** 4.0 mH]

❑

23 Maxwell's Equations in a Material Medium

23.1 Maxwell's Equations

The four equations that govern electric field, magnetic field, charge and current density are

$$\nabla \cdot \boldsymbol{E} = \rho/\varepsilon_0, \tag{23.1}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{23.2}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{23.3}$$

and

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}. \tag{23.4}$$

As you know, these equations are called Maxwell's equations. They are valid in all situations (within classical limits). For fields inside a material, these equations can be rewritten in a slightly different manner. We have already discussed the case of electrically polarized materials. If you have a material with polarization field $\boldsymbol{P}$, bound charges appear on the surface and possibly inside it. The bound-charge density can be written as

$$\rho_b = -\nabla \cdot \boldsymbol{P} \text{ and } \sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{n}}.$$

Thus the charge density ρ can be written in two parts, $\rho = \rho_b + \rho_f$, where ρ_b is the bound-charge density and ρ_f is the free-charge density. You also know that Gauss's law (Equation 23.1) can be written in term of free-charge density as

$$\nabla \cdot \boldsymbol{D} = \rho_f, \tag{23.5}$$

where $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$ is the displacement vector. Why would one be interested in writing Gauss's law in terms of ρ_f when it can be expressed in terms of the total charge density ρ? The reason is that in a laboratory situation, you have better control on free charges than on bound charges. We had discussed all this in the context of electrostatics but the law is universal, and is valid even if the fields are time-varying. Thus, Equation 23.5 is also valid in all cases.

Next, consider Equation 23.4

$$\nabla \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{J} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right).$$

The current density $\boldsymbol{J}$ in a material can come from various sources. You already know about the free-current density $\boldsymbol{J}_\mathrm{f}$ and the bound-current density $\boldsymbol{J}_\mathrm{b}$. The free currents arise from the motion of electrons or positive charges that are not bound to the atoms. The bound current appears due to the alignment of atomic magnetic moments. When the electric field is time-dependent, there is one more source of current density. As the electric field changes, so does the polarization $\boldsymbol{P}$ and hence the bound-charge density ρ_b. But ρ_b can change only due to the movement of bound charges. This causes a current that is different from free and bound currents. The current density due to the motion of bound charges is called *polarization current density* and is denoted by $\boldsymbol{J}_\mathrm{p}$.

How is polarization current density related to rate of change of polarization? An obvious guess is

$$\boldsymbol{J}_\mathrm{p} = k \frac{\partial \boldsymbol{P}}{\partial t}, \tag{i}$$

where k is a constant. Do a dimensional analysis to find the dimensions of k. As $\boldsymbol{P} \cdot \hat{\boldsymbol{n}} = \sigma_\mathrm{b}$,

$$\mathrm{Dim}\left[\frac{\partial \boldsymbol{P}}{\partial t}\right] = \mathrm{Dim}\left[\frac{\sigma}{t}\right] = \mathrm{Dim}\left[\frac{\text{Charge}}{\text{Area} \times \text{Time}}\right] = \mathrm{Dim}\left[\frac{\text{Current}}{\text{Area}}\right].$$

Thus, $\frac{\partial \boldsymbol{P}}{\partial t}$ has the same dimensions as $\boldsymbol{J}_\mathrm{p}$, meaning that k must be dimensionless. You can get the value of k by using the law of charge conservation. Since polarization current results from the motion of bound charges, it must satisfy the same equation of continuity that free current and free charges do. Thus,

$$\nabla \cdot \boldsymbol{J}_\mathrm{p} = -\frac{\partial \rho_\mathrm{b}}{\partial t}. \tag{ii}$$

Remember, this is the mathematical expression for the fact that the total charge leaving a closed surface is equal to the decrease in the total charge contained in the volume enclosed by this surface.

From (i) and (ii),

$$\nabla \cdot \left(k \frac{\partial \boldsymbol{P}}{\partial t} \right) = -\frac{\partial \rho_\mathrm{b}}{\partial t}$$

or

$$k \frac{\partial}{\partial t} (\nabla \cdot \boldsymbol{P}) = -\frac{\partial \rho_\mathrm{b}}{\partial t}$$

or $$k\frac{\partial}{\partial t}(-\rho_\mathrm{b}) = -\frac{\partial \rho_\mathrm{b}}{\partial t}$$

or $$k = 1.$$

Thus,

$$\boldsymbol{J}_\mathrm{p} = \frac{\partial \boldsymbol{P}}{\partial t}\cdot$$

The net current density in a material can therefore be written as

$$\boldsymbol{J} = \boldsymbol{J}_\mathrm{f} + \boldsymbol{J}_\mathrm{b} + \boldsymbol{J}_\mathrm{p} = \boldsymbol{J}_\mathrm{f} + \nabla\times\boldsymbol{M} + \frac{\partial \boldsymbol{P}}{\partial t}\cdot$$

The Maxwell–Ampere law given by Equation 23.4 becomes

$$\nabla\times\boldsymbol{B} = \mu_0\left[\boldsymbol{J}_\mathrm{f} + \nabla\times\boldsymbol{M} + \frac{\partial \boldsymbol{P}}{\partial t} + \varepsilon_0\frac{\partial \boldsymbol{E}}{\partial t}\right]$$

or $$\nabla\times\left(\frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}\right) = \boldsymbol{J}_\mathrm{f} + \frac{\partial}{\partial t}(\varepsilon_0\boldsymbol{E} + \boldsymbol{P})$$

or $$\nabla\times\boldsymbol{H} = \boldsymbol{J}_\mathrm{f} + \frac{\partial \boldsymbol{D}}{\partial t},$$

where $\boldsymbol{D} = \varepsilon_0\boldsymbol{E} + \boldsymbol{P}$ is the displacement vector. This equation contains only the free-current density on which you have much better control. Bound current and polarization current are things you do not measure directly. Based on the above discussion, Maxwell's equations can be written as

$$\nabla\cdot\boldsymbol{D} = \rho_\mathrm{f}, \tag{23.6}$$

$$\nabla\cdot\boldsymbol{B} = 0, \tag{23.7}$$

$$\nabla\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{23.8}$$

and $$\nabla\times\boldsymbol{H} = \boldsymbol{J}_\mathrm{f} + \frac{\partial \boldsymbol{D}}{\partial t}\cdot \tag{23.9}$$

The $\nabla\times\boldsymbol{B}$ and $\nabla\times\boldsymbol{E}$ equations are left in their original form. This is because these equations do not contain charge or current density, and hence it is not difficult to handle bound charge, bound current, etc. In this form of Maxwell's laws, we have used four fields ***E, B, D*** and ***H***. Though it may look odd, it is quite useful. If you try to write all the equations in terms of ***E*** and ***B***, you retain ρ and ***J***, which are difficult to know because of bound charges, bound current and polarization current. If you try to write all the equations in terms of ***D*** and ***H*** then also you don't get simple equations. The $\nabla\cdot\boldsymbol{B}$ equation will be

$$\nabla\cdot\boldsymbol{B} = 0$$

or $$\nabla\cdot(\mu_0\boldsymbol{H}+\mu_0\boldsymbol{M})=0$$

or $$\nabla\cdot\boldsymbol{H}=-\nabla\cdot\boldsymbol{M}.$$

So, a new field $\boldsymbol{M}$ appears. Similarly, the $\nabla\times\boldsymbol{E}$ equation gives

$$\nabla\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t}$$

or $$\frac{1}{\varepsilon_0}\nabla\times(\boldsymbol{D}-\boldsymbol{P})=-\mu_0\frac{\partial}{\partial t}(\boldsymbol{H}+\boldsymbol{M}).$$

Thus, a new vector $\boldsymbol{P}$ appears. Again, you end up with four fields ***D, M, H*** and ***P***.

23.2 Linear Electric and Magnetic Materials

In linear electric materials, $\boldsymbol{D}=\varepsilon\boldsymbol{E}$, and in linear magnetic materials $\boldsymbol{B}=\mu\boldsymbol{H}$. Here $\varepsilon=\varepsilon_0\varepsilon_r$ is the permittivity and $\mu=\mu_0\mu_r$ is the permeability of the material. For a material which is both electrically and magnetically linear, the $\nabla\cdot\boldsymbol{B}$ equation gives

$$\nabla\cdot\boldsymbol{H}=0$$

and the $\nabla\times\boldsymbol{E}$ equation gives

$$\nabla\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t}$$

or $$\nabla\times\left(\frac{1}{\varepsilon}\boldsymbol{D}\right)=-\frac{\partial}{\partial t}(\mu\boldsymbol{H})$$

or $$\nabla\times\boldsymbol{D}=\mu\varepsilon\frac{\partial\boldsymbol{H}}{\partial t}\cdot$$

Thus, for linear materials, the four equations can be written in terms of just two fields $\boldsymbol{D}$ and $\boldsymbol{H}$.

$$\nabla\cdot\boldsymbol{D}=\rho_f, \tag{23.10}$$

$$\nabla\cdot\boldsymbol{H}=0, \tag{23.11}$$

$$\nabla\times\boldsymbol{D}=-\mu\varepsilon\frac{\partial\boldsymbol{H}}{\partial t} \tag{23.12}$$

and $$\nabla\times\boldsymbol{H}=\boldsymbol{J}_f+\frac{\partial\boldsymbol{D}}{\partial t}\cdot \tag{23.13}$$

23.3 Displacement Current

We have talked about displacement current earlier in the context of time-varying electric fields in free space. The general expression for displacement current density is $\boldsymbol{J}_{\mathrm{d}} = \frac{\partial \boldsymbol{D}}{\partial t}$. The quantity $\int \boldsymbol{J}_{\mathrm{d}} \cdot d\boldsymbol{a}$ over an area is called displacement current through that area. It is not strictly a current in the sense of charge flowing through an area. As $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$, the displacement current gets contributions from $\varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$ and $\frac{\partial \boldsymbol{P}}{\partial t}$. The latter comes from the movement of physical charges (bound charges) but the former, $\varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$, does not correspond to any charge transfer. However, looking at Equation 23.13, you can see that $\frac{\partial \boldsymbol{D}}{\partial t}$ is related to magnetic field in the same way as the current density $\boldsymbol{J}$ or $\boldsymbol{J}_{\mathrm{f}}$ is related to it. Thus, $\frac{\partial \boldsymbol{D}}{\partial t}$ is equivalent to a current density as far as calculations of magnetic field are concerned.

For free space, the polarization $\boldsymbol{P}$ is zero. So, the displacement current density is $\varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$.

23.4 Boundary Conditions on Fields Across a Surface

The boundary conditions on ***E, D, B*** and ***H*** for electrostatic and magnetostatic situations have been derived. These were done in different chapters as and when a field was defined. These boundary conditions together with the equations from which these are derived, are given in the table below. These are valid in the presence of materials too.

Equation	Boundary condition
$\nabla \cdot \boldsymbol{D} = \rho_{\mathrm{f}}$	$D_{1n} - D_{2n} = \sigma_{\mathrm{f}}$
$\nabla \times \boldsymbol{E} = 0$	$\boldsymbol{E}_{1t} = \boldsymbol{E}_{2t}$
$\nabla \cdot \boldsymbol{B} = 0$	$B_{1n} = B_{2n}$
$\nabla \times \boldsymbol{H} = \boldsymbol{J}_{\mathrm{f}}$	$\boldsymbol{H}_{1t} - \boldsymbol{H}_{2t} = \boldsymbol{K}_{\mathrm{f}} \times \hat{\boldsymbol{n}}$

The subscripts 1 and 2 are used for Medium 1 and Medium 2 separated by the surface (Figure 23.1). The vector $\hat{\boldsymbol{n}}$ denotes the unit vector perpendicular to the surface drawn from a point on the surface towards Medium 1. The subscript n denotes the normal component towards $\hat{\boldsymbol{n}}$ and the subscript t denotes the tangential component.

Fig. 23.1

When you allow time-varying fields, the two divergence equations do not change. Thus, the boundary conditions on the normal components of $\boldsymbol{D}$ and $\boldsymbol{B}$ remain valid in the same form. What happens to the other two boundary conditions, where the curl equations get modified? First, check the $\nabla \times \boldsymbol{E}$ equation. For time-varying fields,

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}\cdot$$

In integral form,

$$\oint \boldsymbol{E}\cdot \boldsymbol{dl} = -\int \frac{\partial \boldsymbol{B}}{\partial t}\cdot \boldsymbol{da}. \tag{i}$$

Figure 23.2 shows two materials, 1 and 2, separated by the x–y plane. Take a rectangular path abcd as shown in the figure. The sides ab and cd are parallel to the surface, along the y-direction. The sides bc and da are perpendicular to the surface. These sides are small as you need a relation between the fields close to the surface on the two sides. Thus, $\oint \boldsymbol{E}\cdot \boldsymbol{dl}$ will have no contribution from bc and da as these sides become infinitesimally small. Then,

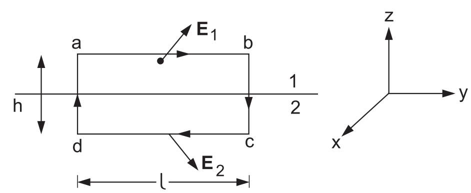


Fig. 23.2

$$\oint \boldsymbol{E}\cdot \boldsymbol{dl} = \int \boldsymbol{E}_1 \cdot (ab)\,\hat{\boldsymbol{j}} + \int \boldsymbol{E}_2 \cdot (cd)\cdot(-\hat{\boldsymbol{j}})$$
$$= (E_{1y} - E_{2y})l, \tag{ii}$$

where l is the length ab or cd.

Now, look at $-\int \frac{\partial \boldsymbol{B}}{\partial t}\cdot \boldsymbol{da}$, that is the RHS of equation (i). As the height of the path abcd tends to zero, the area itself tends to zero. Thus $\oint \frac{d\boldsymbol{B}}{\partial t}\cdot \boldsymbol{da}$ will also tend to zero. Using this in (i) and (ii),

$$E_{1y} - E_{2y} = 0$$

or $$E_{1y} = E_{2y}.$$

Similarly, by taking a loop with length parallel to the x-axis,

$$E_{1x} = E_{1y}.$$

Combining these two,

$$\boldsymbol{E}_{1t} = \boldsymbol{E}_{2t}.$$

This is the same as for the electrostatic case. Thus, the boundary condition on $\boldsymbol{E}_t$ is valid even for time-varying fields.

You can give a simple argument to get this result. As the curl equation differs from that of the static-field situation only by the term $-\frac{\partial \boldsymbol{B}}{\partial t}$, and this term makes no contribution to the integration in (i), the boundary condition arising from this curl equation will be the same.

Finally, consider the equation $\nabla \times \boldsymbol{H} = \boldsymbol{J}_\mathrm{f} + \dfrac{\partial \boldsymbol{D}}{\partial t}\cdot$ In integral form,

$$\oint \boldsymbol{H}\cdot d\boldsymbol{l} = I_{\mathrm{f,\,encl}} + \int \frac{\partial}{\partial t}\boldsymbol{D}\cdot d\boldsymbol{a}. \tag{iii}$$

This equation differs from that in magnetostatics by the term $\int \frac{\partial}{\partial t}\boldsymbol{D}\cdot d\boldsymbol{a}$. The boundary condition is obtained by applying equation (iii) to a path like abcd shown in Figure 23.3 with height bc and da tending to zero. Whatever be the values of $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$, the value of $\int \frac{\partial}{\partial t}\boldsymbol{D}\cdot d\boldsymbol{a}$ tends to zero as the area itself tends to zero. Thus there is no effect of this term on the boundary condition, and hence the boundary condition on $\boldsymbol{H}_t$ as given in the table is valid even if the fields are time-varying.

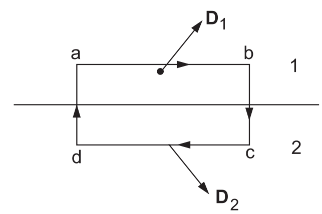


Fig. 23.3

So, all the boundary conditions across a surface separating two materials are the same, regardless of whether the fields are steady or time-varying.

Concepts at a Glance

1. Maxwell's equations for $\nabla\cdot\boldsymbol{E}$ and $\nabla\times\boldsymbol{B}$ can also be written as $\nabla\cdot\boldsymbol{D} = \rho_\mathrm{f}$ and $\nabla\times\boldsymbol{H} = \boldsymbol{J}_\mathrm{f} + \dfrac{\partial \boldsymbol{D}}{\partial t}\cdot$
2. The total current density is $\boldsymbol{J} = \boldsymbol{J}_\mathrm{f} + \boldsymbol{J}_\mathrm{b} + \boldsymbol{J}_\mathrm{p}$, where $\boldsymbol{J}_\mathrm{f}$ is free current, $\boldsymbol{J}_\mathrm{b}$ is bound current and $\boldsymbol{J}_\mathrm{p} = \dfrac{\partial \boldsymbol{P}}{\partial t} =$ polarization current.
3. For linear materials, all the four equations may be written in terms of $\boldsymbol{D} = \varepsilon\boldsymbol{E}$ and $\boldsymbol{H} = \dfrac{\boldsymbol{B}}{\mu}\cdot$
4. The boundary conditions on ***E, B, D*** and ***H*** remain the same even if the fields vary with time.

EXERCISES

Based on Concepts

1. The curl equation for a magnetic field can be written as $\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{J} + \mu_0\varepsilon_0\dfrac{\partial \boldsymbol{E}}{\partial t}$ or $\nabla\times\boldsymbol{H} = \boldsymbol{J}_\mathrm{f} + \dfrac{\partial \boldsymbol{D}}{\partial t}\cdot$ Which of the two is valid for free space and which is valid in a material medium? If you say that the first is valid for free space and the second for a medium, your answer may not reflect the situation appropriately.

2. The equation of continuity can be used for free currents and polarization currents. Is it also valid for bound currents?
3. Is the equation $\nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}$ valid for electrostatic fields?

Problems

1. Consider the model of a dielectric material in which each atom initially has positive and negative charges distributed symmetrically so that the negative charge centre coincides with the positive charge centre (nucleus). The material is polarized when the negative charges shift by a displacement $\boldsymbol{dr}$, the same for all atoms. Using this model, show that if the polarization varies with time, the current due to the motion of the negative charges is given by $\boldsymbol{J}_{\text{p}} = \dfrac{d\boldsymbol{P}}{dt}$.
2. Show that the fields $\boldsymbol{H}$ and $\boldsymbol{D}$ are zero inside a perfect conductor, even for time-varying external fields.
3. A parallel-plate capacitor with large plate areas is filled with a dielectric material with electrical susceptibility $\chi = 1.25$. The plates are being charged so that the surface charge density changes as $\sigma = At$. Find the polarization current density $\boldsymbol{J}_{\text{p}}$. [**Ans.** $\dfrac{\chi A}{1+\chi}$]
4. The surface $z = 0$ separates two linear media. The medium in $z > 0$ has a dielectric constant K and that in $z < 0$ has a dielectric constant $K/2$. The electric field in the region $z > 0$ is $\boldsymbol{E} = [(10x+5)\hat{\boldsymbol{i}} + (5y+10)\hat{\boldsymbol{j}} + (z+1)\hat{\boldsymbol{k}}]e^{-t/\tau}$. All numbers represent quantities in SI units. Find the electric field at a point (2 m, 2 m, $-\varepsilon$), where ε is a small positive number. [**Ans.** $(25\hat{\boldsymbol{i}} + 20\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})e^{-t/\tau}$]
5. An electromagnetic wave travels in free space. As $\nabla \cdot \boldsymbol{E} = 0$, you can define a vector potential $\boldsymbol{A}'$ so that $\boldsymbol{E} = \nabla \times \boldsymbol{A}'$. (a) Write the differential wave equation in terms of $\boldsymbol{A}'$. (b) Express $\boldsymbol{B}$ in terms of $\boldsymbol{A}'$.

   [**Ans.** (a) $\nabla^2 \boldsymbol{A}' = \varepsilon_0\mu_0 \dfrac{\partial^2}{\partial t^2}\boldsymbol{A}'$, (b) $\boldsymbol{B} = \varepsilon_0\mu_0 \dfrac{\partial \boldsymbol{A}'}{\partial t}$]

❑

24 Electromagnetic Waves in a Material Medium

24.1 Energy Dissipation in EM Waves

We have already discussed how electric and magnetic fields can travel as electromagnetic waves, and the various properties of such waves. You have also learnt that electromagnetic waves travel in free space with a particular speed independent of wavelength or angular frequency. Energy is propagated but without any loss as the wave travels.

Electromagnetic waves can also travel in a material medium. The most common evidence of such wave motion is the propagation of light in water, glass, clear plastic and other transparent materials. Several wave properties that correspond to the propagation of EM waves in free space are modified when the waves move in a medium, and this modification depends on the nature of the material. The material may be linear where $\boldsymbol{P} \propto \boldsymbol{E}$ and $\boldsymbol{M} \propto \boldsymbol{H}$ or nonlinear. It may be a very good conductor like silver or a poor conductor like glass or diamond. For insulators, the band gap (energy difference between the top of the valence band and the bottom of the conduction band) may be small or large. We will consider only linear materials, which are characterized by relative permittivity ε_r, relative permeability μ_r and electrical conductivity σ.

A major effect of the electrical conductivity of a medium is that the energy of a wave is dissipated as it travels through the medium. This happens because the electric field drives a current $\boldsymbol{J}_f = \sigma\boldsymbol{E}$ in the medium, resulting in Ohmic heating. There are several other causes of energy dissipation too. The electric field also causes polarization in the medium by orienting the atomic electron clouds along itself. But the field changes rapidly and the electron clouds try to follow this change. There might be some loss of energy if the material itself resists this kind of electronic motion. There may be scattering centres in the medium, from which part of the EM wave is scattered away from the original direction, so that the intensity of the wave keeps decreasing along the direction of propagation. The molecules in the material have some natural vibrational frequencies. If the frequency of the electromagnetic wave matches one of these natural frequencies, a large amount of energy may be absorbed from the EM waves to cause and maintain these vibrations.

We will consider only those materials in which any possible energy dissipation takes place only due to electrical conductivity. We will first discuss materials with zero conductivity and hence no loss of energy as the EM waves go through them. Such materials may be termed loss-free or perfect linear dielectrics. Perfect here means $\sigma = 0$. We will discuss materials with nonzero conductivity later in the chapter.

As EM waves can go through a loss-free dielectric medium without loss of energy, the medium can be called transparent. A loss-free dielectric material is transparent to EM waves.

24.2 EM Waves in a Loss-free Linear Dielectric

A linear dielectric is characterized by relative permittivity ε_r and relative permeability μ_r. Often $\mu_r \approx 1$ for dielectrics. Only for ferromagnetic materials is μ_r much larger but such materials are magnetically nonlinear and mostly conducting (mostly, but not always). For linear dielectrics, $\boldsymbol{D} = \varepsilon \boldsymbol{E}$ and $\boldsymbol{H} = \boldsymbol{B}/\mu$, where $\varepsilon = \varepsilon_0 \varepsilon_r$ and $\mu = \mu_0 \mu_r$. As already discussed, it is convenient to write Maxwell's equations in the form

$$\nabla \cdot \boldsymbol{D} = \rho_f,$$

$$\nabla \cdot \boldsymbol{B} = 0,$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

and $$\nabla \times \boldsymbol{H} = \boldsymbol{J}_f + \frac{\partial \boldsymbol{D}}{\partial t}$$

while discussing fields in a material medium. In a perfect dielectric, there are no free charges. That is why the conductivity σ is zero. As there are no free charges, there is no free current. So, $\rho_f = 0$ and $\boldsymbol{J}_f = 0$. Using $\boldsymbol{D} = \varepsilon \boldsymbol{E}$ and $\boldsymbol{H} = \boldsymbol{B}/\mu$, Maxwell's equations in this case become

$$\nabla \cdot \boldsymbol{E} = 0,$$

$$\nabla \cdot \boldsymbol{B} = 0,$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

and $$\nabla \times \boldsymbol{B} = \varepsilon\mu \frac{\partial \boldsymbol{E}}{\partial t}.$$

Look at Maxwell's equations given for free space in an earlier chapter. You can see that the only difference between the two sets of equations is that the constant $\varepsilon_0 \mu_0$ appearing in the $\nabla \times \boldsymbol{B}$ equation for free space is replaced by $\varepsilon\mu$ in the case of a dielectric. Thus, all the results derived for EM waves in free space can be used for a loss-free dielectric with ε replacing ε_0 and μ replacing μ_0. So the EM wave in such a medium is also transverse, the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are perpendicular to each other and in the same phase, and so on.

Refractive index

The wave equations for $\boldsymbol{E}$ and $\boldsymbol{B}$ are

$$\nabla^2 \boldsymbol{E} = \frac{1}{\varepsilon\mu}\frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

and $$\nabla^2 \boldsymbol{B} = \frac{1}{\varepsilon\mu}\frac{\partial^2 \boldsymbol{B}}{\partial t^2}.$$

Thus, the speed of the wave is

$$\begin{aligned} v &= \frac{1}{\sqrt{\varepsilon\mu}} = \frac{1}{\sqrt{\varepsilon_0\varepsilon_r\mu_0\mu_r}} \\ &= \sqrt{\frac{1}{\varepsilon_0\mu_0}} \times \frac{1}{\sqrt{\varepsilon_r\mu_r}} \\ &= \frac{c}{\sqrt{\varepsilon_r\mu_r}} = \frac{c}{n}, \end{aligned} \tag{24.1}$$

where $n = \sqrt{\varepsilon_r\mu_r}$.

So the speed of an EM wave in a dielectric medium is less than that in free space by a factor $n = \sqrt{\varepsilon_r\mu_r}$. As you know, this factor is called the *refractive index* of the medium.

For most dielectric mediums, $\mu_r \approx 1$.

So the refractive index is $n = \sqrt{\varepsilon_r}$. (24.2)

The relative permittivity of a material depends on the frequency of the wave. For example, the relative permittivity of water is 81 for steady electric fields or low-frequency waves, but becomes around 3–4 for GHz waves. Because of the variation of relative permittivity with frequency of the EM waves, the refractive index also changes with frequency. This is the reason why white light is dispersed in glass prisms.

Relation between *E* and *B*

For a plane monochromatic wave in a loss-free dielectric medium,

$$\boldsymbol{E} = \boldsymbol{E}_0 \cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)$$

and $$\boldsymbol{B} = \boldsymbol{B}_0 \cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t).$$

The wave speed v is related to the angular frequency ω and the magnitude of the propagation vector $\boldsymbol{k}$ as

$$v = \frac{\omega}{|\boldsymbol{k}|}.$$

The waves are transverse and the ***E, B*** fields are in phase, as in the case of free space. The amplitudes of the ***E, B*** fields are related as

$$B_0 = \frac{E_0}{v}.$$

In fact,

$$\boldsymbol{B} = \frac{\boldsymbol{k} \times \boldsymbol{E}}{\omega}$$

holds good for waves in a material medium also.

Intensity

We had derived the expression for the intensity of an EM wave in free space as $I = \frac{1}{2}c\varepsilon_0 E_0^2$, where E_0 is the amplitude of the electric field. The intensity of the wave in a dielectric medium is

$$I = \frac{1}{2}v\varepsilon E_0^2.$$

The Poynting vector ***S*** representing the flow of energy per unit area per unit time is

$$\boldsymbol{S} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu}.$$

The role of the medium

Why is the speed of the EM wave in a dielectric different from that in free space? The fact that the speeds are different shows that the mechanisms for EM wave propagation in free space and in a material medium are different. The electric field and the magnetic field of the wave interact with atoms and molecules when the wave moves in a material. The material gets polarized and magnetized, and the polarization and magnetization change as a function of time. Thus, time-varying bound charges and bound currents appear and generate new electric and magnetic fields. It is amazing that these waves interfere to give rise to a single wave travelling in a well-defined direction with well-defined amplitudes and a well-defined speed.

24.3 EM Waves in a Conducting (Lossy) Medium

The free charge density P_f

Now consider a medium with nonzero conductivity σ. It is still assumed to be linear, with relative permittivity ε_r and relative permeability μ_r. It differs in a major way from a dielectric medium—there are free-charge carriers that respond to any applied electric field and result in currents and Ohmic loss of energy. Such a medium is also called a lossy medium.

Though you have free charges in a conducting medium, in general there is no free charge density ρ_f. The positive ions in the conductor together with the free electrons make ρ_f zero. Even if some free charge density is created at a certain time, it readily moves to the surface and ρ_f becomes zero in the interior. When you connect a wire to a battery, there is an electric field inside the wire and a free current $\boldsymbol{J}_f = \sigma\boldsymbol{E}$ results. This free current exists there because of the motion of free electrons. However, the free charge density ρ_f in the wire is still zero, because you have positive ions in the wire making the charge density zero (do not consider relativistic effects as the drift speeds are very small). The number of free electrons entering a volume in any time internal is the same as that leaving the volume. You can easily work out the time scales for which a significant value of ρ_f exists in a conducting medium once it is as given below.

The free current density $\boldsymbol{J}_f$ results from the motion of free charges. The continuity equation expressing the law of conservation of charge gives

$$\nabla \cdot \boldsymbol{J}_f = -\frac{\partial \rho_f}{\partial t}$$

or $$\nabla \cdot \sigma\boldsymbol{E} = -\frac{\partial \rho_f}{\partial t}$$

or $$\frac{\sigma}{\varepsilon}\nabla \cdot \boldsymbol{D} = -\frac{\partial \rho_f}{\partial t}$$

or $$\frac{\sigma}{\varepsilon}\rho_f = -\frac{\partial \rho_f}{\partial t}$$

or $$\rho_f = \rho_{f_0} e^{-(\sigma/\varepsilon)t}. \tag{24.3}$$

So, once there is a free charge density in a conducting medium, its value will exponentially decay with a time constant of ε/σ. Let us see how much this time constant is in common situations. For metals, $\sigma \approx 10^7$ S/m and $\varepsilon \approx 10^{-12}$ F/m—so $\varepsilon/\sigma \approx 10^{-19}$ s. For sea water, $\sigma \approx 4$ S/m and $\varepsilon \approx 10^{-11}$ F/m—so $\varepsilon/\sigma \approx 10^{-11}$ s. In fact, these estimates are not very accurate because Ohm's law $\boldsymbol{J}_f = \sigma\boldsymbol{E}$ itself describes an average phenomenon resulting from the large number of collisions or scattering of free electrons taking place in the conducting medium. The collision time is typically 10^{-14} s and you should not give much importance to the numerical values of 10^{-19} s, etc. The important fact is that the charge density decays very quickly with time and finally one gets $\rho_f \approx 0$.

Maxwell's equations and their wave solutions

Using $\boldsymbol{J}_f = \sigma\boldsymbol{E}$, $\boldsymbol{D} = \varepsilon\boldsymbol{E}$ and $\boldsymbol{H} = \boldsymbol{B}/\mu$, you can write Maxwell's equations (with $\rho_f = 0$ and $\boldsymbol{J}_f = \sigma\boldsymbol{E}$) as

$$\nabla \cdot \boldsymbol{E} = 0,$$

$$\nabla \cdot \boldsymbol{B} = 0,$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

and $$\nabla \times \boldsymbol{B} = \mu(\nabla \times \boldsymbol{H}) = \mu\left(J_\mathrm{f} + \frac{\partial D}{\partial t}\right) = \mu\sigma\boldsymbol{E} + \mu\varepsilon\frac{\partial \boldsymbol{E}}{\partial t}\cdot$$

To get the wave equation in $\boldsymbol{E}$, take the curl of the $\nabla \times \boldsymbol{E}$ equation and eliminate $\boldsymbol{B}$ from the other curl equation. Then,

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\frac{\partial}{\partial t}(\nabla \times \boldsymbol{B})$$

or $$\nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} = -\frac{\partial}{\partial t}\left(\mu\sigma\boldsymbol{E} + \mu\varepsilon\frac{\partial \boldsymbol{E}}{\partial t}\right)$$

or $$\nabla^2 \boldsymbol{E} = \mu\varepsilon\frac{\partial^2 \boldsymbol{E}}{\partial t^2} + \mu\sigma\frac{\partial \boldsymbol{E}}{\partial t}\cdot \tag{24.4}$$

Similarly, by taking the curl of the $\nabla \times \boldsymbol{B}$ equation,

$$\nabla^2 \boldsymbol{B} = \mu\varepsilon\frac{\partial^2 \boldsymbol{B}}{\partial t^2} + \mu\sigma\frac{\partial \boldsymbol{B}}{\partial t}\cdot \tag{24.5}$$

These equations also allow wave-like solutions. Assume that the wave moves in the x-direction and define the complex electric field as

$$\tilde{\boldsymbol{E}} = \tilde{\boldsymbol{E}}_0 e^{i(kx-\omega t)}. \tag{24.6}$$

Putting this in Equation 24.4,

$$\frac{\partial^2}{\partial x^2}[\tilde{\boldsymbol{E}}_0 e^{i(kx-\omega t)}] = \mu\varepsilon\frac{\partial^2}{\partial t^2}[\tilde{\boldsymbol{E}}_0 e^{i(kx-\omega t)}] + \mu\sigma\frac{\partial}{\partial t}[\tilde{\boldsymbol{E}}_0 e^{i(kx-\omega t)}]$$

or $$-k^2[\tilde{\boldsymbol{E}}_0 e^{i(kx-\omega t)}] = \mu\varepsilon(-\omega^2)[\tilde{\boldsymbol{E}}_0 e^{i(kx-\omega t)}] + \mu\sigma(-i\omega)\tilde{\boldsymbol{E}}_0 e^{i(kx-\omega t)}$$

or $$k^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega. \tag{24.7}$$

Thus the expression in 24.6 satisfies Equation 24.4 provided k is a complex quantity satisfying Equation 24.7. So, k in Equation 24.6 is not the propagation constant in the usual sense. It is called complex propagation constant. Write $k = k' + ik''$, where k' and k'' are the real and imaginary parts of k. From Equation 24.7, you get

$$(k' + ik'')^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega,$$

giving $$k'^2 - k''^2 = \mu\varepsilon\omega^2 \tag{i}$$

and $$2k'k'' = \mu\sigma\omega.$$

From these,

$$k'^2 + k''^2 = [(\mu\varepsilon\omega^2)^2 + (\mu\sigma\omega)^2]^{1/2}. \tag{ii}$$

From (i) and (ii),

$$\left.\begin{aligned} k' &= \sqrt{\frac{\mu\varepsilon\omega^2}{2}}\left[\left\{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2\right\}^{1/2}+1\right]^{1/2} \\ \text{and}\qquad k'' &= \sqrt{\frac{\mu\varepsilon\omega^2}{2}}\left[\left\{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2\right\}^{1/2}-1\right]^{1/2}. \end{aligned}\right\} \tag{24.8}$$

Write $k = k' + ik''$ in Equation 24.6 and you get

$$\tilde{\boldsymbol{E}} = \tilde{\boldsymbol{E}}_0 e^{i(k'x+ik''x-\omega t)} = \tilde{\boldsymbol{E}}_0 e^{-k''x} e^{i(k'x-\omega t)}.$$

You can write the actual electric field as

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{-k''x} \cos(k'x - \omega t + \delta), \tag{24.9}$$

where $\tilde{\boldsymbol{E}}_0 = \boldsymbol{E}_0 e^{i\delta}$.

Since $\tilde{\boldsymbol{E}}$ satisfies the wave equation, so does $\boldsymbol{E}$. Hence, Equation 24.9 gives the electric field.

The wave equation (Equation 24.5) for the magnetic field has the same mathematical structure as that for $\boldsymbol{E}$. Thus you can directly write the equation for $\boldsymbol{B}$ as

$$\tilde{\boldsymbol{B}} = \tilde{\boldsymbol{B}}_0 e^{-k''x} e^{i(k'x-\omega t)}$$

or

$$\boldsymbol{B} = \boldsymbol{B}_0 e^{-k''x} \cos(k'x - \omega t + \delta'). \tag{24.10}$$

The characteristics of EM waves in a conducting medium

The amplitude is reduced as the wave moves The magnitude of the electric field is $E = E_0 e^{-k''x} \cos(k'x - \omega t + \delta)$. Plot E as a function of x for a given time t. It will be as shown in Figure 24.1. The cos function oscillates between +1 and –1 but has to be multiplied by $E_0 e^{-k''x}$, which decays exponentially.

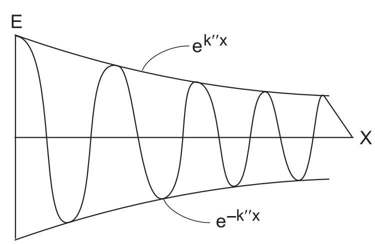


Fig. 24.1

Thus the EM wave in a conducting medium is attenuated as it moves ahead. This is what you should expect on physical grounds, as discussed in the beginning of the chapter.

Attenuation length, or skin depth The amplitude of the electric field (and also of the magnetic field) decreases by a factor of $1/e$ as the wave moves a distance $dx = 1/k''$ in the medium. This distance is called *attenuation length*. As you will see shortly, the attenuation length for good conductors is very small and in such cases it is also called *skin depth*.

The imaginary part k'' of the complex propagation constant k describes the attenuation.

EXAMPLE 24.1 **A plane EM wave travels in a medium that absorbs energy. If the intensity at a point P is I_0, what would be the intensity at a point ahead of P by (a) one attenuation length and (b) six attenuation lengths?**

Solution (a) The amplitude decreases by a factor of $1/e$ in one attenuation length. All other things remaining the same, the intensity is proportional to the square of the amplitude. So the intensity will decrease by the factor $1/e^2$.

Thus, $I = I_0/e^2 = 0.13 I_0$.

(b) In six attenuation lengths, the amplitude will decrease by the factor $1/e^6$. So, the intensity becomes

$$I_0 = \frac{I_0}{e^{12}} = 5.7 \times 10^{-6} I_0.$$

Wavelength The wave has some kind of periodicity because of the $\cos(k'x - \omega t + \delta)$ term. For a given t, this function is periodic in x with a periodicity $\lambda = 2\pi/k'$. The wavelength of the wave given by Equations 24.9 and 24.10 is therefore $2\pi/k'$. Thus the real part k' of the complex propagation constant k describes the wavelength.

Why the term "some kind of"? Because the amplitude is decreasing, and $\boldsymbol{E}$ and $\boldsymbol{B}$ do not repeat their values as x increases.

Wave speed The terms $\boldsymbol{E}_0 e^{-k''x}$ and $\boldsymbol{B}_0 e^{-k''x}$ in Equations 24.9 and 24.10 give the amplitudes, and $\cos(k'x - \omega t + \delta)$ determines the phase of the wave. If E is maximum at some position x and at some time t, it will again be maximum at $x + \Delta x$ and time $t + \Delta t$, where $k'\Delta x = \omega \Delta t$. The wave speed is $v = \dfrac{\Delta x}{\Delta t} = \dfrac{\omega}{k'}$.

Magnetic field *B* The electric field in an EM wave travelling along the x-direction is given by $\boldsymbol{E} = \boldsymbol{E}_0 e^{-k''x} \cos(k'x - \omega t + \delta)$. Taking the direction of the electric field as the y-axis and writing $k = k' + ik''$, you can express the complex electric field as

$$\tilde{\boldsymbol{E}} = \hat{\boldsymbol{j}}\,\tilde{E}_0 e^{i(kx - \omega t)}, \text{ where } \tilde{E}_0 = E_0 e^{i\delta}.$$

You can also write it directly from Equation 24.6.

$$\nabla \times \tilde{\boldsymbol{E}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & \tilde{E}_0 e^{i(kx-\omega t)} & 0 \end{vmatrix} = \hat{k}\frac{\partial}{\partial x}[\tilde{E}_0 e^{i(kx-\omega t)}] = \hat{k}\,\tilde{E}_0 (ik) e^{i(kx-\omega t)}.$$

Thus $$\frac{\partial \tilde{\boldsymbol{B}}}{\partial t} = -\hat{k}\,\tilde{E}_0(ik)e^{i(kx-\omega t)}$$

or $$\tilde{\boldsymbol{B}} = -\hat{k}\,\tilde{E}_0\left(\frac{ik}{-i\omega}\right)e^{i(kx-\omega t)}$$

or $$\tilde{\boldsymbol{B}} = \hat{k}\,\tilde{E}_0\,\frac{k}{\omega}\,e^{i(kx-\omega t)}. \tag{i}$$

You don't need an integration constant, because any time-invariant component does not form part of the wave. Notice that k appears in the exponential factor as well as outside. You have to take the real part to write the actual magnetic field in the wave. Take care that k itself is complex. Write

$$k = k' + ik'' = Ke^{i\phi},$$

where $$K = \sqrt{k'^2 + k''^2},\ \tan\phi = \frac{k''}{k'}\cdot$$

Then,

$$\tilde{\boldsymbol{B}} = \hat{k}\,E_0 e^{i\delta}\,\frac{K}{\omega}\,e^{i\phi}e^{i[(k'+ik'')x-\omega t]} = \hat{\boldsymbol{k}}\,\frac{KE_0 e^{-k''x}}{\omega}\,e^{i(k'x-\omega t+\delta+\phi)}$$

or $$\boldsymbol{B} = \hat{\boldsymbol{k}}\,\frac{KE_0}{\omega}\,e^{-k''x}\cos(k'x-\omega t+\delta+\phi) = \hat{k}\,B_0 e^{-k''x}\cos(k'x-\omega t+\delta+\phi), \tag{24.11}$$

where $B_0 = \dfrac{KE_0}{\omega} = \sqrt{k'^2+k''^2}\,\dfrac{E_0}{\omega}\cdot$

Please note that B_0 is not E_0/v as is the case with a loss-free dielectric medium. The wave speed is ω/k', whereas

$$B_0 = \frac{E_0}{\omega\big/\sqrt{k'^2+k''^2}}\cdot$$

Another major difference in the relation between the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields of an EM wave in a lossy medium is that they are not in phase. Compare Equations 24.9 and 24.11, and you will find that they differ in phase by $\phi = \tan^{-1}(k''/k')$. The magnetic field lags behind the electric field.

Complex refractive index The propagation constant k has been assumed to be complex, taking care of absorption in a lossy medium. The relation between the propagation constant k and the refractive index n for a non-lossy material is $n = \dfrac{c}{v} = \dfrac{ck}{\omega}\cdot$ You can retain this relation and define a complex refractive index $n = n' + in''$, where $n' = \dfrac{c}{\omega}k' = \dfrac{c}{v}$ and $n'' = \dfrac{c}{\omega}k''$. The wave equation for the electric field can then be written as

$$\boldsymbol{E} = \tilde{\boldsymbol{E}}_0 e^{i\left(\frac{\omega}{c}nx - \omega t\right)} = \tilde{\boldsymbol{E}}_0 e^{i\omega\left(\frac{n}{c}x - t\right)} = \tilde{\boldsymbol{E}}_0 e^{-\frac{\omega}{c}n''x} e^{i\omega\left(\frac{n'}{c}x - t\right)}.$$

Similarly,

$$\boldsymbol{B} = \tilde{\boldsymbol{B}}_0 e^{-\frac{\omega}{c}n''x} e^{i\omega\left(\frac{n'}{c}x - t\right)}.$$

So the real part of n gives the refractive index ($n' = c/v$) and its imaginary part gives attenuation.

Good conductors and poor conductors

The characteristics of EM waves in a conducting medium (a medium having nonzero conductivity) are dependent on k' and k''. If you put $\sigma = 0$ in the expression for k' and k'' (Equation 24.8), you will find that $k'' = 0$ and $k' = \sqrt{\mu\varepsilon\omega}$. These are the same values as those for a loss-free dielectric medium. From these expressions, you can also see that the conductivity σ appears in the dimensionless combination $\sigma/(\varepsilon\omega)$. Thus σ has to be compared with $\varepsilon\omega$. If $\sigma << \varepsilon\omega$, you call the medium a poor conductor and if $\sigma >> \varepsilon\omega$, you call it a good conductor. So, the response of the conductivity of the material to the EM wave propagation depends on the frequency. The same material that is a good conductor at low frequencies may be a poor conductor at high frequencies.

The quantity $\dfrac{\sigma}{\varepsilon\omega}$ is called the loss tangent of the material.

EXAMPLE 24.2 **A dielectric material characterized by $\varepsilon_r = 4$ also has a conductivity $\sigma = 10^{-2}$ S/m. Find the loss tangent for an EM wave of frequency (a) 1 kHz, (b) 1 GHz and (c) 10^{15} Hz. In each case, would you call it a good conductor or a poor conductor?**

Solution

(a) Loss tangent $= \dfrac{\sigma}{\varepsilon\omega} = \dfrac{10^{-2}}{4 \times 8.85 \times 10^{-12} \times 2\pi \times 10^3} = 4.5 \times 10^4.$

This is much larger than 1. So, for a 1-kHz EM wave, this material is a good conductor.

(b) Loss tangent $= \dfrac{\sigma}{\varepsilon\omega} = \dfrac{10^{-2}}{4 \times 8.85 \times 10^{-12} \times 2\pi \times 10^9} \approx 4.5 \times 10^{-2}.$

For an EM wave of this frequency, the material may be termed a somewhat poor conductor.

(c) Loss tangent $= \dfrac{\sigma}{\varepsilon\omega} = \dfrac{10^{-2}}{4 \times 8.85 \times 10^{-12} \times 2\pi \times 10^{15}} = 4.5 \times 10^{-8}.$

As this is much smaller than 1, the material is a very poor conductor for a 10^{15}-Hz EM wave.

24.4 EM Waves in Good Conductors

For good conductors, $\frac{\sigma}{\varepsilon\omega} >> 1$. Thus, in Equation 24.8, you can neglect 1 in comparison to $\frac{\sigma}{\varepsilon\omega}\cdot$

Then, $k' \approx k'' = \sqrt{\frac{\mu\varepsilon\omega^2}{2}} \times \sqrt{\frac{\sigma}{\varepsilon\omega}} = \sqrt{\frac{\mu\sigma\omega}{2}}\cdot$

So, the skin depth or the attenuation length is $d = \frac{1}{k''} = \sqrt{\frac{2}{\mu\sigma\omega}}\cdot$

<u>EXAMPLE 24.3</u> **Find the skin depth for an EM wave moving through copper if the frequency of the wave is (a) 50 Hz, (b) 1 MHz and (c) 10^{14} Hz (optical frequency). Take the conductivity of copper to be 5.76×10^7 S/m and its relative permeability ≈ 1.**

Solution $d = \sqrt{\frac{2}{\mu\sigma\omega}}\cdot$

(a) $d = \sqrt{\frac{2}{4\pi\times10^{-7}\times5.76\times10^{7}\times2\pi\times50}} = 9.38\text{ mm}.$

(b) $d = \sqrt{\frac{2}{4\pi\times10^{-7}\times5.76\times10^{7}\times2\pi\times10^{6}}} = 6.6\times10^{-5}\text{ m} = 66\ \mu\text{m}.$

(c) $d = \sqrt{\frac{2}{4\pi\times10^{-7}\times5.76\times10^{7}\times2\pi\times10^{14}}} = 6.6\times10^{-9}\text{ m} = 6.6\text{ nm}.$

The skin depth at optical frequencies is only a few nanometres. Now you know why metals are opaque to light. Also, there is a large difference in skin depth as you change frequency.

The wavelength of an EM wave in a good conductor is $\lambda = \frac{2\pi}{k'}\cdot$ The skin depth is $\frac{1}{k''} \approx \frac{1}{k'} = \frac{\lambda}{2\pi}\cdot$ You can see that the wavelength is more than six times the skin depth. [The amplitude of the EM wave therefore decreases by a factor of e (one skin depth) even before it can go one wavelength.]

The wave speed is $v = \frac{\omega}{k'} = \sqrt{\frac{2\omega}{\mu\sigma}}\cdot$ It is proportional to $\sqrt{\omega}$. Let us look at the speed of an EM wave in copper.

EXAMPLE 24.4 **Calculate the speed of an EM wave in copper at a frequency of (a) 1 MHz and (b) 10^{14} Hz.**

Solution (a) $v = \sqrt{\dfrac{2\omega}{\mu\sigma}} = \sqrt{\dfrac{2 \times 2\pi \times 10^6}{4\pi \times 10^{-7} \times 5.76 \times 10^7}} = \dfrac{10^3}{2.4} = 417 \text{ m/s}.$

(b) $v = \sqrt{\dfrac{2\omega}{\mu\sigma}} = \sqrt{\dfrac{2 \times 2\pi \times 10^{14}}{4\pi \times 10^{-7} \times 5.76 \times 10^7}} = 4.17 \times 10^6 \text{ m/s}.$

The phase difference between $\boldsymbol{B}$ and $\boldsymbol{E}$ is $\tan^{-1}\phi = \dfrac{k''}{k'}\cdot$ For good conductors, $k' \approx k''$. So $\tan^{-1}\phi = 1$ or $\phi = \pi/4$. The magnetic field lags behind the electric field by $\pi/4$.

24.5 EM Waves in Poor Conductors

If $\sigma/\varepsilon\omega << 1$, we term the material a poor conductor for electromagnetic waves of those frequencies. Dielectrics having small conductivity come under this category. You may call them lossy or leaky dielectrics. Look at the expression for k'.

$$k' = \sqrt{\frac{\mu\varepsilon\omega^2}{2}}\left[\left\{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2\right\}^{1/2} + 1\right]^{1/2}.$$

As $\left(\dfrac{\sigma}{\varepsilon\omega}\right)^2$ is much smaller than 1, you can neglect it in comparison to 1 to get

$$k' = \sqrt{\frac{\mu\varepsilon\omega^2}{2}} \times \sqrt{2} = \sqrt{\mu\varepsilon}\,\omega. \tag{i}$$

The wave speed is $v = \dfrac{\omega}{k'} = \sqrt{\dfrac{1}{\mu\varepsilon}}\cdot$

This expression is the same as that for loss-free dielectrics. For $\sigma << \varepsilon\omega$, the wave speed is almost the same as that obtained by neglecting the conductivity altogether.

Similarly, $\lambda = 2\pi/k' = \dfrac{2\pi}{\sqrt{\mu\varepsilon}\omega} = \dfrac{v}{\nu}$ as usual.

Now look at k''.

$$k'' = \sqrt{\frac{\mu\varepsilon\omega^2}{2}}\left[\left\{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2\right\}^{1/2} - 1\right]^{1/2}.$$

Can you neglect $\left(\frac{\sigma}{\varepsilon\omega}\right)^2$ in comparison with 1? If you do so, $k'' = 0$. So what you neglect is more than what you retain. So, you should not neglect it altogether. You can make a binomial expansion and retain the first term after 1. Then,

$$k'' \approx \sqrt{\frac{\mu\varepsilon\omega^2}{2}}\left[\left\{1+\frac{1}{2}\left(\frac{\sigma}{\varepsilon\omega}\right)^2+\ldots\right\}-1\right]^{1/2} = \sqrt{\frac{\mu\varepsilon\omega^2}{2}}\frac{1}{\sqrt{2}}\frac{\sigma}{\varepsilon\omega} = \frac{1}{2}\sqrt{\frac{\mu}{\varepsilon}}\,\sigma. \qquad \text{(ii)}$$

Attenuation length

$$d = \frac{1}{k''} = \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}.$$

It is independent of ω and inversely proportional to σ. For small σ, d is large.

....

Concepts at a Glance

1. Electromagnetic waves have the same characteristics in a loss-free dielectric as in vacuum, except that ε_0 is to be replaced with ε and μ_0 with μ.
2. There is a loss in the intensity of an electromagnetic wave travelling through a dielectric medium with some conductivity.
3. The wave equation for a lossy medium gets modified and an extra term $\mu\sigma\frac{\partial \boldsymbol{E}}{\partial t}$ or $\mu\sigma\frac{\partial \boldsymbol{B}}{\partial t}$ appears in it.
4. In the complex electric field written as $\tilde{\boldsymbol{E}} = \tilde{\boldsymbol{E}}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$, the wave vector $\boldsymbol{k}$ becomes complex. The real part of $\boldsymbol{k}$ gives the wavelength and the imaginary part gives the attenuation length.
5. The attenuation in an EM wave can also be expressed in terms of the complex refractive index.
6. If $\sigma >> \varepsilon\omega$, the medium can be treated as a good conductor. If $\sigma << \sigma\omega$, it is a poor conductor for EM wave propagation.

EXERCISES

Based on Concepts

1. What is the relative permittivity of water at optical frequencies?
2. How does the resistance of a wire change for high-frequency transmission?
3. Why should the increased conductivity of a material result in the absorption of more energy from an electromagnetic wave?

4. The electric and magnetic fields in an EM wave have a phase difference. Can this phase difference be 60°?
5. When white light goes through a prism, it splits into different colours. Which property of the dielectric constant of glass is responsible for this?
6. Think of a dielectric that brings down the speed of light to a few metres per second. What is the difficulty in having such a dielectric? Think in terms of the polarization of the material.

Problems

1. The electric field in an electromagnetic wave passing through a dielectric medium is given by $\boldsymbol{E} = 500\hat{\boldsymbol{j}}\cos(1.2\times10^7 x - 2.04\times10^{15} t)$, where all quantities are in SI units. Find the relative permittivity of the medium. [**Ans.** 1.33]
2. A plane electromagnetic wave with angular frequency $\omega = 5\times10^{14}$ rad/s travels along the direction $\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ in a medium with relative permittivity 2.25 and relative permeability 1. The electric field has an amplitude of 10 V/m. Write an expression for a possible electric field as a function of x, y, z, t.

   [**Ans.** $(10\ \text{V/m})\cos\left\{\left(\frac{2.5\times10^6}{\sqrt{6}}\ \text{m}^{-1}\right)(x+2y+z) - 5\times10^4\ \text{s}^{-1}t\right\}$]
3. A plane, monochromatic, linearly polarized, light wave has an electric field given by $\boldsymbol{E} = \hat{\boldsymbol{k}}E_0\cos\left[3.14\times10^{15}\left(t - \frac{x}{0.65c}\right)\right]$ while travelling through a piece of glass. Find (a) the frequency of light, (b) the wavelength and (c) the index of refraction of glass for this wave.

   [**Ans.** (a) 5.0×10^{14} Hz, (b) 390 nm, (c) 1.54]
4. An EM wave is travelling in a dielectric medium of refractive index n. Using the relevant Maxwell's equation, show that $|\boldsymbol{B}| = \frac{n|\boldsymbol{E}|}{c}$.
5. Imagine you have embedded some free charge in a piece of glass (refractive index 1.5). How long would it take for the charge to flow to the surface? Take the conductivity of glass to be about 10^{-12} S/m.

   [**Ans.** ~70,000 years]
6. If the amplitude of the electric field of light travelling in glass (n = 1.5) is 100 V/m, find the average magnitude of the Poynting vector. [**Ans.** 25 μW/m^2]
7. The complex electric field of a plane electromagnetic wave propagating in a lossy medium is given by $\boldsymbol{E} = \tilde{\boldsymbol{E}}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$, where $\boldsymbol{k} = k_0(1+2i)\hat{\boldsymbol{i}}$. (a) What is the wavelength? (b) Find the distance at which the intensity is attenuated by a factor $1/e$. [**Ans.** (a) $\frac{2\pi}{k_0}$, (b) $\frac{1}{2k_0}$]
8. A 3-GHz EM wave travels in a conducting medium with dielectric constant 2.5 and conductivity 7.1×10^7 S/m. The relative permeability is close to 1.
   (a) Determine the distance over which the amplitude of the wave is reduced to half its value.
   (b) Determine the wavelength and the phase velocity.

(c) Assuming that $\tilde{\boldsymbol{E}} = 50\,\hat{\boldsymbol{j}}\,e^{-(3\pi\times10^5 x)}\cos(80x - 6\pi\times10^9 t + \pi/3$ in SI units, write the expression of the magnetic field for general (x, t).

[**Ans.** (a) 0.76 μm; (b) 6.8 μm, 2.0×10^4 m/s; (c) $4.09\times10^{-15}\,\hat{\boldsymbol{k}}\cos(-3\pi\times10^{-5}x)\cos\left(80 - 6\pi\times10^{-9}t + \dfrac{7\pi}{12}\right)$]

9. Suppose you wish to design a microwave experiment to operate at a frequency of 10 MHz. How thick would you like to make the silver coating so that the microwaves do not penetrate the coating (intensity falling by a factor $1/e^{10}$) and also the cost of silver remains low. The conductivity of silver is 6.2×10^7 S/m. [**Ans.** Around 0.1 mm, which is 5 times the skin depth]

10. An electromagnetic wave travels through salty water, which has a conductivity of 4 S/m. The frequency of the wave is 5 MHz and it moves along the z-direction. At this frequency, the relative permittivity of water is 72. Find the (a) skin depth, (b) wave speed and (c) wavelength.

    [**Ans.** (a) 11.2 cm, (b) 3.53×10^6 m/s, (c) 70.7 cm]

11. Consider a monochromatic electromagnetic wave incident on a thick slab of conducting material. What percentage of the total energy delivered to the slab is deposited within the skin depth.

    [**Ans.** 86.5%]

12. Show that for complex sinusoidally varying electric and magnetic fields, $\nabla\times\tilde{\boldsymbol{E}} = -i\omega\mu\,\tilde{\boldsymbol{H}}$.

13. A plane electromagnetic wave is travelling in a medium with permeability μ, permittivity ε and conductivity σ. Find the ratio u_B/u_E, where u_B and u_E are energy densities in the wave corresponding to the magnetic and electric fields. Which contribution is larger? [**Ans.** $\sqrt{1+\left(\dfrac{\sigma}{\varepsilon\omega}\right)^2}$]

14. An electromagnetic wave of frequency 1 GHz is travelling in a medium. The loss tangent of the medium at this frequency is 0.001. At a certain place, the amplitude of the electric field is 125 V/m. Find the average power dissipated per unit volume at that place. [**Ans.** 1.1 W/m^3]

❑

25 Reflection and Refraction of EM Waves

When you get ready for your college, you look at yourself in the mirror. You do this many more times during the day. Which phenomenon helps you see the image? That of reflection of light. And which phenomenon helps you see through the glass windows of your house? The phenomenon of refraction. You have used these phenomena from early childhood but have studied them in detail only in middle and high school. You must have studied the so-called laws of reflection and those of refraction.

Laws of reflection

(a) The incident ray, the reflected ray and the normal to the reflecting surface are in the same plane.

(b) The angle of incidence θ_i and the angle of reflection θ_r are equal.

Laws of refraction

(a) The incident ray, the refracted ray and the normal to the refracting surface are in the same plane.

(b) The angle of incidence θ_i and the angle of refraction θ_t are related as

$$\frac{\sin\theta_i}{\sin\theta_t} = \text{constant} \qquad \text{(Snell's law)}$$

for a given pair of media. This constant is written as $\frac{n_2}{n_1}$, where n_1 and n_2 are the refractive indices of the media containing the incident ray and the refracted ray respectively.

You also know that the refractive index of a medium is given by

$$n = \frac{c}{v},$$

where c and v are the speeds of light in vacuum and in the medium respectively. Then,

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{v_1}{v_2}$$

where v_1 and v_2 are the speeds of light in the media containing the incident ray and refracted ray respectively.

Are the laws of reflection and refraction fundamental laws of nature which we have discovered from experiments or they can be derived from more fundamental laws by doing some mathematics? Light is an electromagnetic wave and the laws of reflection and refraction follow from the properties of electric and magnetic fields. This means that one can derive these laws from Maxwell's equations and the material properties of the media concerned. In this chapter, we will do these derivations which will also allow us to deduce some more results relating the intensities of incident, reflected and refracted waves.

25.1 Derivation of Laws of Reflection and Refraction

Suppose two linear, loss-free media called 1 and 2 are separated by the surface $z = 0$ (Figure 25.1). x- and z-axis have been shown in the plane of the paper. The y-axis is perpendicular to this plane and going into it. The region $z > 0$ is filled with medium 1 having permittivity ε_1 and permeability μ_1. The region $z < 0$ is filled with medium 2 with permittivity ε_2 and permeability μ_2. An electromagnetic wave given by

$$\tilde{\boldsymbol{E}}_I = \tilde{\boldsymbol{E}}_{0I}\, e^{i(\boldsymbol{k}_I\cdot \boldsymbol{r} - \omega t)}$$

$$\tilde{\boldsymbol{B}}_I = \tilde{\boldsymbol{B}}_{0I}\, e^{i(\boldsymbol{k}_I\cdot \boldsymbol{r} - \omega t)}$$

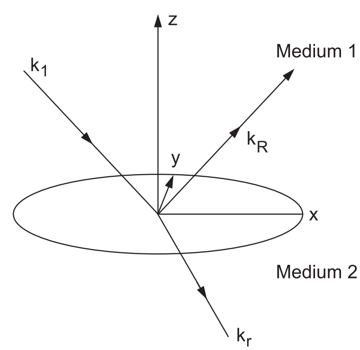


Fig. 25.1

travels in medium 1 in the direction of the propagation vector $\boldsymbol{k}_I$ and falls on the surface $z = 0$. Suppose the wave is reflected along $\boldsymbol{k}_R$ and transmitted along $\boldsymbol{k}_T$. Does the frequency ω remain the same? It does remain the same, because the frequency of a wave is decided by the source which generates the wave. But let us pretend that we do not know this. The boundary conditions will give the answer.

The reflected wave is given by

$$\boldsymbol{E}_R = \tilde{\boldsymbol{E}}_{0R}\, e^{i(\boldsymbol{k}_R\cdot \boldsymbol{r} - \omega' t)}$$

$$\boldsymbol{B}_R = \tilde{\boldsymbol{B}}_{0R}\, e^{i(\boldsymbol{k}_R\cdot \boldsymbol{r} - \omega' t)}$$

and the refracted wave is given by

$$\boldsymbol{E}_T = \tilde{\boldsymbol{E}}_{0T}\, e^{i(\boldsymbol{k}_T\cdot \boldsymbol{r} - \omega'' t)}$$

$$\boldsymbol{B}_T = \tilde{\boldsymbol{B}}_{0T}\, e^{i(\boldsymbol{k}_T \cdot \boldsymbol{r} - \omega'' t)}.$$

Please note that any change in phase is taken care of by the possible complex values of the amplitudes. The electric field at a point $\boldsymbol{r}$ in medium 1 is $\tilde{\boldsymbol{E}}_1 = \tilde{\boldsymbol{E}}_I + \tilde{\boldsymbol{E}}_R$ and that at a point $\boldsymbol{r}$ in medium 2 is $\tilde{\boldsymbol{E}}_2 = \tilde{\boldsymbol{E}}_T$. Similar expressions may be written for the magnetic fields. These fields have to satisfy the boundary conditions

$$D_{1n} = D_{2n} \tag{i}$$

$$E_{1t} = E_{2t} \tag{ii}$$

$$B_{1n} = B_{2n} \tag{iii}$$

$$H_{1t} = H_{2t} \tag{iv}$$

as there are no free charges or free currents. Now consider two points $(x, y, 0^+)$ and $(x, y, 0^-)$, on the two sides of the surface $z = 0$. Start with any one of the four boundary conditions for these points. For example, take the first one. It gives

$$\varepsilon_1(\tilde{\boldsymbol{E}}_1)_z = \varepsilon_2(\tilde{\boldsymbol{E}}_2)_z$$

or $$\varepsilon_1(\tilde{\boldsymbol{E}}_I + \tilde{\boldsymbol{E}}_R)_z = \varepsilon_2(\tilde{\boldsymbol{E}}_T)_z$$

or $$\varepsilon_1(\tilde{\boldsymbol{E}}_{0I})_z e^{i(\boldsymbol{k}_I \cdot \boldsymbol{r} - \omega t)} + \varepsilon_1(\tilde{\boldsymbol{E}}_{0R})_z e^{i(\boldsymbol{k}_R \cdot \boldsymbol{r} - \omega' t)} = \varepsilon_2(\tilde{\boldsymbol{E}}_{0T})_z e^{i(\boldsymbol{k}_T \cdot \boldsymbol{r} - \omega'' t)}$$

or $$Ae^{i(\boldsymbol{k}_I \cdot \boldsymbol{r} - \omega t)} + Be^{i(\boldsymbol{k}_R \cdot \boldsymbol{r} - \omega' t)} = Ce^{i(\boldsymbol{k}_T \cdot \boldsymbol{r} - \omega'' t)}, \tag{i}$$

where A, B, C are constants not depending on time. Here, $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$ as the fields are evaluated infinitesimally close to the surface $z = 0$.

If you start with any other boundary condition, you will get a similar equation. This equation is valid for all values of x, y and t. Take $x = y = 0$. Equation (i) becomes

$$Ae^{-i\omega t} + Be^{-i\omega' t} = Ce^{i\omega'' t}, \tag{ii}$$

valid for all t. By taking different values of t, you can write any number of linear equations in A, B and C. This can only be possible if either

$$A = B = C = 0 \text{ or } \omega = \omega' = \omega''$$

and $A + B = C$. Obviously, all three (A, B, C) cannot be zero and hence the frequency of the reflected and transmitted waves is the same as that of the incident wave.

Remember that we have confined ourselves to classical physics. There also exist quantum phenomena, which will not emerge from the theory we are discussing. For example, you do have Raman effect—light of frequency ω is passed through a liquid and the transmitted wave has additional frequencies $\omega \pm \Delta\omega$ apart from the original frequency. Dr C V Raman got the Nobel Prize for this discovery. But for the time being, let us assume only the classical rules.

Next, take $y = 0$, $t = 0$ in equation (i). You get

$$Ae^{i(\boldsymbol{k}_I)_x x} + Be^{i(\boldsymbol{k}_R)_x x} = C^{i(\boldsymbol{k}_T)_x x}.$$

As this is true for all values of x,

$$(\boldsymbol{k}_I)_x = (\boldsymbol{k}_R)_x = (\boldsymbol{k}_T)_x. \tag{iii}$$

Similarly, by taking $x = 0$, $t = 0$ in equation (i), you get

$$(\boldsymbol{k}_I)_y = (\boldsymbol{k}_R)_y = (\boldsymbol{k}_T)_y. \tag{iv}$$

Look at Figure 25.1. The incident direction given by $\boldsymbol{k}_I$ is in the x–z plane. Thus,

$$(\boldsymbol{k}_I)_y = 0.$$

From equation (iv), $(\boldsymbol{k}_R)_y = 0$ and $(\boldsymbol{k}_T)_y = 0$. This means the vectors $\boldsymbol{k}_R$ and $\boldsymbol{k}_T$ are also in the x–z plane. The normal to the interface between the two media, that is, the z-axis, is any way in the x–z plane. If you draw $\boldsymbol{k}_I, \boldsymbol{k}_R, \boldsymbol{k}_T$ from the same point on the surface, say from the origin (Figure 25.2), they will be coplanar (in the x–z plane) and the z-axis also will be in the same plane. If you identify the direction of $\boldsymbol{k}$, drawn from a point as a ray from that point, you get the results described as the first law of reflection and the first law of refraction.

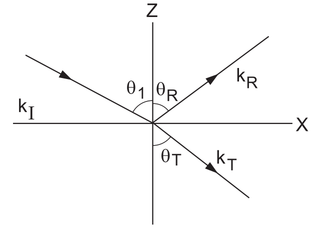


Fig. 25.2

Now look at equation (iii). From Figure 25.2, you can see that

$$(\boldsymbol{k}_I)_x = k_I \sin \theta_I$$

$$(\boldsymbol{k}_R)_x = k_R \sin \theta_R$$

and $$(\boldsymbol{k}_T)_x = k_T \sin \theta_T.$$

Now, $k = \dfrac{\omega}{v}$, so, $k_I = \omega/v_1$, $k_R = \omega/v_1$ and $k_T = \omega/v_2$, where v_1 and v_2 are the speeds of the electromagnetic wave in medium 1 and medium 2 respectively. Using (iii),

$$(\boldsymbol{k}_I)_x = (\boldsymbol{k}_R)_x$$

or $$k_I \sin \theta_1 = k_R \sin \theta_R$$

or $$\frac{\omega}{v_1} \sin \theta_I = \frac{\omega}{v_1} \sin \theta_R$$

or $$\theta_I = \theta_R.$$

This is the second law of reflection. The angle of incidence is equal to the angle of reflection.

Also, $$(\boldsymbol{k}_I)_x = (\boldsymbol{k}_T)_x$$

or $$k_I \sin \theta_I = k_T \sin \theta_T$$

or $$\frac{\omega}{v_1} \sin \theta_1 = \frac{\omega}{v_2} \sin \theta_T$$

or $$\frac{\sin \theta_I}{\sin \theta_T} = \frac{v_1}{v_2} = \text{constant},$$

which is Snell's law.

So, the laws of reflection and refraction arise from the behaviour of ***E, B*** fields in materials. Light is an electromagnetic wave, and ***E, B*** fields have to follow Maxwell's equations and the boundary conditions across the material boundary.

25.2 Reflection and Transmission at Normal Incidence

Suppose a train is stationed at a platform. It is night time and the light conditions at the platform are poor. Say you are standing on the platform in front of an AC coach which is well lighted. You can see all the passengers and objects inside the coach clearly. This means light from the inside goes through the window glass and comes out on the platform. Now go inside the coach and look at the window. You will see your own image and the images of other objects in the coach. This means light from inside gets reflected by the window.

So reflection and refraction both take place simultaneously when light falls on the window glass. A part of the light gets reflected, because of which the persons inside see the images of the objects inside. Another part gets through the glass, because of which people on the platform see the objects inside the coach.

When any electromagnetic wave travelling in one medium falls on the surface of another transparent medium, it gets partially reflected and partially transmitted (refracted).

We will now derive the relation between the intensities of the incident, reflected and transmitted waves when an EM wave travelling in a loss-free medium falls normally on the surface of another loss-free medium. Figure 25.3 shows the situation schematically. Two loss-free dielectric media are separated by the surface at $x = 0$ ($y-z$ plane). The permittivity and permeability for medium 1 are ε_1 and μ_1, and for medium 2 these are ε_2 and μ_2. As said earlier, for most dielectrics, $\mu \approx \mu_0$, whereas ε can be several times ε_0.

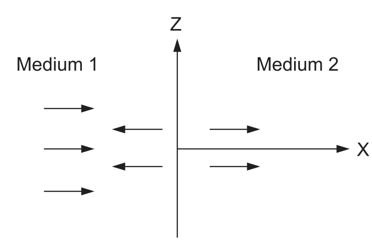


Fig. 25.3

Suppose an electromagnetic wave travelling in medium 1 along the x-direction reaches the plane $x = 0$ where there is a change of medium. Suppose the electric field $\boldsymbol{E}_I$ in the incident wave is in the y-direction. Then,

$$\tilde{\boldsymbol{E}}_I = \tilde{E}_{0I}\,\hat{\boldsymbol{j}}\,e^{i(k_1 \cdot x - \omega t)}$$

$$\tilde{\boldsymbol{B}}_I = \tilde{B}_{0I}\,\hat{k}\,e^{i(k_1 \cdot x - \omega t)} = \frac{\tilde{E}_{0I}}{v_1}\,\hat{k}\,e^{i(k_1 x - \omega t)},$$

where $k_1 = \dfrac{\omega}{v_1}$ and $\tilde{E}_{0I}$ and $\tilde{B}_{0I}$ are the complex amplitudes of the electric and magnetic fields. The reflected wave moves in medium 1 along the negative x-direction. What is the direction of the electric field in the reflected wave? It has to be in the y–z plane as the wave is propagating in the (negative) x-direction. The incident wave has $\boldsymbol{E}$ in the $\hat{\boldsymbol{j}}$-direction. Can the reflected wave have a different direction of $\boldsymbol{E}$-field? Allowing for this possibility, write the $\boldsymbol{E}$-field in the reflected wave as

$$\tilde{\boldsymbol{E}}_R = (\tilde{E}'_{0R}\,\hat{j} + \tilde{E}''_{0R}\,\hat{k})e^{i(-k_1 x - \omega t)}.$$

Then, the magnetic field in the reflected wave is

$$\tilde{\boldsymbol{B}}_R = \frac{\boldsymbol{k}_R \times \tilde{\boldsymbol{E}}_R}{\omega} = \frac{(-\hat{i}k_1)\times \tilde{\boldsymbol{E}}_R}{\omega}$$

$$= \frac{(-\hat{i}k_1)\times(\tilde{E}'_{0R}\,\hat{j} + \tilde{E}''_{0R}\,\hat{k})e^{i(-k_1 \cdot x - \omega t)}}{\omega}$$

$$= \frac{-\tilde{E}'_{0R}\,\hat{k} + \tilde{E}''_{0R}\,\hat{j}}{v_1}\,e^{i(-k_1 \cdot x - \omega t)}.$$

Similarly, you can write the field in medium 2 as

$$\tilde{\boldsymbol{E}}_T = [\tilde{E}'_{0T}\,\hat{j} + \tilde{E}''_{0T}\,\hat{k}]e^{i(k_2 \cdot x - \omega t)}.$$

And $$\tilde{\boldsymbol{B}}_T = \frac{E'_{0T}\,\hat{k} - \tilde{E}''_{0T}\,\hat{j}}{v_2}\,e^{i(k_2 \cdot x - \omega t)},$$

where $k_2 = \omega/v_2$.

Now let us apply the boundary conditions at $x = 0$. The normal components (x-components) of $\boldsymbol{E}$ and $\boldsymbol{B}$ are zero everywhere, and so the conditions $\tilde{E}_{1n} = \tilde{E}_{2n}$ and $\tilde{H}_{1n} = \tilde{H}_{2n}$ are automatically satisfied. Look at the tangential components. The electric field in medium 1 is

$$\tilde{\boldsymbol{E}}_1 = \tilde{\boldsymbol{E}}_I + \tilde{\boldsymbol{E}}_R = \tilde{E}_{0I}\,\hat{j}\,e^{i(k_1 \cdot x - \omega t)} + (\tilde{E}'_{0R}\,\hat{j} + \tilde{E}''_{0R}\,\hat{k})e^{i(-k_1 x - \omega t)}$$

and that in medium 2 is

$$\tilde{\boldsymbol{E}}_2 = \tilde{\boldsymbol{E}}_T = (\tilde{E}'_{0T}\,\hat{j} + \tilde{E}''_{0T}\,\hat{k})e^{i(k_2 x - \omega t)}.$$

These fields are parallel to the surface $x = 0$. At the boundary $x = 0$, the tangential components should be equal. Matching the values at $t = 0$,

$$\tilde{E}_{0I}\hat{j} + \tilde{E}_{0R}\hat{j} + E''_{0R}\hat{k} = \tilde{E}'_{0T}\hat{j} + \tilde{E}''_{0T}\hat{k}.$$

Thus,

$$\tilde{E}_{0I} + \tilde{E}'_{0R} = \tilde{E}'_{0T} \tag{i}$$

and $$\tilde{E}''_{0R} = \tilde{E}''_{0T}. \tag{ii}$$

The magnetic field in medium 1 is

$$\tilde{\boldsymbol{B}}_1 = \tilde{\boldsymbol{B}}_I + \tilde{\boldsymbol{B}}_R = \left[\frac{\tilde{E}_{0I}}{v_1}\hat{k}\,e^{i(k_1x-\omega t)} + \frac{-\tilde{E}'_{0R}\hat{k} + \tilde{E}''_{0R}\hat{j}}{v_1}e^{i(-k_1x-\omega t)}\right]$$

and that in medium 2 is

$$\tilde{\boldsymbol{B}}_2 = \tilde{\boldsymbol{B}}_T = \frac{E'_{0T}\hat{k} - E''_{0T}\hat{j}}{v_2}e^{i(k_2x-\omega t)}.$$

The boundary condition at $x = 0$, $t = 0$ gives

$$\tilde{\boldsymbol{H}}_{1t} = \tilde{\boldsymbol{H}}_{2t}$$

or $$\frac{\tilde{\boldsymbol{B}}_{1t}}{\mu_1} = \frac{\tilde{\boldsymbol{B}}_{2t}}{\mu_2}.$$

Taking $\mu_1 \approx \mu_2 (\approx \mu_0)$, it becomes

$$\frac{\tilde{E}_{0I}}{v_1}\hat{k} + \frac{-\tilde{E}'_{0R}\hat{k} + \tilde{E}''_{0R}\hat{j}}{v_1} = \frac{\tilde{E}'_{0T}\hat{k} - E''_{0T}\hat{j}}{v_2}$$

Equating the z-components,

$$\frac{\tilde{E}_{0I} - \tilde{E}'_{0R}}{v_1} = \frac{\tilde{E}'_{0T}}{v_2}$$

or $$\tilde{E}_{0I} - \tilde{E}'_{0R} = \frac{v_1}{v_2}\tilde{E}'_{0T} \tag{iii}$$

and equating the y-components,

$$\frac{E''_{0R}}{v_1} = \frac{E''_{0T}}{v_2}. \tag{iv}$$

What do you conclude from (ii) and (iv)? As $v_1 \neq v_2$, the only way these two equations can be satisfied is by having $E''_{0R} = E''_{0T} = 0$. So the direction of polarization is not changed because of reflection or refraction.

Thus, the equations for the electric field in the reflected and the transmitted waves are

$$\boldsymbol{E}_R = \tilde{E}'_{0R}\,\hat{j}e^{i(-k_1x-\omega t)}$$

$$\boldsymbol{E}_T = E'_{0T}\,\hat{j}e^{i(k_2x-\omega t)}.$$

You can easily get the expression for $\tilde{E}'_{0R}$ and $\tilde{E}'_{0T}$ in terms of $\tilde{E}_{0I}$. Adding equations (i) and (iii),

$$2\tilde{E}_{0I} = \left(1+\frac{v_1}{v_2}\right)E'_{0T}$$

or
$$\tilde{E}'_{0T} = \frac{2v_2}{v_1+v_2} = \tilde{E}_{0I}$$

or
$$\tilde{E}'_{0T} = \frac{2n_1}{n_1+n_2} = \tilde{E}_{0I}. \tag{25.1}$$

From (i),
$$\tilde{E}'_{0R} = \tilde{E}'_{0T} - \tilde{E}_{0I} = \frac{2n_1}{n_1+n_2}\tilde{E}_{0I} - \tilde{E}_{0I}$$

or
$$\tilde{E}'_{0R} = \frac{n_1-n_2}{n_1+n_2}\tilde{E}_{0I}. \tag{25.2}$$

The ratio $t = \frac{E_{0T}}{E_{0I}}$ is called transmission coefficient and $r = \frac{E_{0R}}{E_{0I}}$ is called reflection coefficient. Check that the ratio of the complex amplitudes is the same as that of the real amplitudes.

Thus, for normal incidence,

$$\frac{\tilde{E}_{0R}}{\tilde{E}_{0I}} = \frac{E_{0R}}{E_{0I}} \text{ and } \frac{\tilde{E}_{0T}}{\tilde{E}_{0I}} = \frac{E_{0T}}{E_{0I}}.$$

Reflectance and transmittance

The intensity of the incident wave is

$$I_i = \frac{1}{2}\varepsilon_1 v_1 |\tilde{E}_{0I}|^2,$$

that of the reflected wave is

$$I_r = \frac{1}{2}\varepsilon_1 v_1 |\tilde{E}'_{0R}|^2$$

and that of the transmitted wave is

$$I_t = \frac{1}{2}\varepsilon_2 v_2 |\tilde{E}'_{0T}|^2 .$$

The ratio I_r/I_i is called *reflectance* and I_t/I_i is called *transmittance*. These quantities give the fraction of the incident energy that gets reflected and that which gets transmitted. The reflectance is

$$R = \frac{I_r}{I_i} = \left|\frac{\tilde{E}'_{0R}}{\tilde{E}_{0I}}\right|^2.$$

Using Equation 25.2,

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2. \tag{25.3}$$

$$T = \frac{I_t}{I_i} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\frac{|\tilde{E}'_{0T}|^2}{|\tilde{E}_{0I}|^2} = \frac{n_2^2}{n_1^2}\times\frac{n_1}{n_2}\left|\frac{\tilde{E}'_{0T}}{\tilde{E}_{0I}}\right|^2 = \frac{n_2}{n_1}\left|\frac{\tilde{E}'_{0T}}{\tilde{E}_{0I}}\right|^2.$$

Using Equation 25.1, $T = \dfrac{n_2}{n_1}\left(\dfrac{2n_1}{n_1 + n_2}\right)^2$

or
$$T = \frac{4n_1 n_2}{(n_1 + n_2)^2}. \tag{25.4}$$

If $n_1 < n_2$, $(n_1 - n_2)$ is negative and so from Equation 25.2, $\tilde{E}_{0R}$ and $\tilde{E}_{0I}$ are in opposite phase. This means that there is a sudden phase change of π when a light wave is reflected from a medium of higher refractive index. No such phase change occurs if $n_1 > n_2$. The transmitted wave is always in phase with the incident wave at the boundary as shown by Equation 25.1. Check that $R + T = 1$.

EXAMPLE 25.1 **Calculate the reflectance and transmittance when light travelling in air falls normally on a glass surface ($n = 1.5$).**

Solution Here $n_1 = 1$, $n_2 = 1.5$.

So, $$R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 = \left(\frac{0.5}{2.5}\right)^2 = \frac{1}{25}$$

and $$T = 1 - R = 1 - \frac{1}{25} = \frac{24}{25}.$$

Thus only 4% of the incident light energy gets reflected and 96% is transmitted. Now you know why you can see through the window glass so clearly.

25.3 Reflectance from a Conducting Surface

Suppose an EM wave travelling in a loss-free dielectric medium falls normally on the surface of a lossy medium. As you know, the equations for wave propagation in a lossy medium have the same form as in the case of a loss-free medium provided you use the complex propagation constant or complex refractive index, defined in the previous chapter. In terms of the complex ***E*** and ***B*** fields, the boundary conditions also remain the same and so the expressions for reflectance and transmittance will also be the same. If you write the incident, reflected and transmitted waves as

$$\tilde{\boldsymbol{E}}_I = \tilde{E}_{0I}\,\hat{j}e^{i(k_1x-\omega t)}$$

$$\tilde{\boldsymbol{E}}_R = \tilde{E}_{0R}\,\hat{j}e^{i(-k_1x-\omega t)}$$

$$\tilde{\boldsymbol{E}}_T = \tilde{E}_{0T}\,\hat{j}e^{i(k_2x-\omega t)}$$

$$\frac{\tilde{E}_{0R}}{\tilde{E}_{0I}} = \frac{n_1-n_2}{n_2-n_1}.$$

The reflectance is $R = \left|\dfrac{\tilde{E}_{0R}}{\tilde{E}_{0I}}\right|^2 = \left|\dfrac{n_1-n_2}{n_1+n_2}\right|^2$ · Here n_1 is real and n_2 is complex.

$$n_1 = \frac{ck_1}{\omega} \text{ and } n_2 = \frac{c}{\omega}k_2 = \frac{c}{\omega}(k_2' + ik_2'').$$

Thus $$R = \left|\frac{k_1 - k_2' - ik_2''}{k_1 + k_2' + ik_2''}\right|^2 = \frac{(k_1-k_2')^2 + k_2''^2}{(k_1+k_2')^2 + k_2''^2}$$ and the transmittance is $T = 1 - R$.

EXAMPLE 25.2 **A light wave travelling in air with angular frequency $\omega = 3\times 10^{15}$ rad/s falls normally on a flat silver surface. What fraction of the incident light (energy) will reflect back? Take the conductivity $\sigma = 5.8\times 10^7$ S/m, $\mu = \mu_0$.**

Solution For silver, $\dfrac{\sigma}{\varepsilon\mu} >> 1.$ So we use the good-conductor limit

$$k_2' = k_2'' = \sqrt{\frac{\mu\omega\sigma}{2}}$$

$$= \sqrt{\frac{4\pi\times 10^{-7}\times 3\times 10^{15}\times 5.8\times 10^{7}}{2}}$$

$$= 4.6\times 10^8 \text{ m}^{-1}.$$

Also, $$k_1 = \frac{\omega}{c} = \frac{3\times 10^{15}}{3\times 10^8} = 10^7 \text{ m}^{-1}.$$

Thus $$R = \frac{(k_1 - k_2')^2 + k_2''^2}{(k_1 + k_2')^2 + k_2''^2} = \frac{(1-46)^2 + 46^2}{(1+46)^2 + 46^2} \approx 0.96.$$

96% of the light will be reflected back. In general, good conductors are good reflectors.

Role of silvering in plane mirror

Mirrors are made by coating a thin silver layer on one side of a glass plate. What is the role of this silver layer? In fact, it plays the most important role in reflecting light. When light travelling in air falls on the glass surface, only about 4% of it is reflected and the rest goes in. This transmitted light falls on the silver surface and most of it is reflected back into the glass. This reflected light falls on the air–glass surface and most of it comes out for us to see the image.

The glass plate essentially supports the silver film and prevents its wear and tear.

25.4 Reflection and Transmission at Oblique Incidence

Consider two loss-free dielectric media separated at the plane $z = 0$. The permittivity and permeability for the first medium are ε_1, μ_1 and those for the second medium are ε_2, μ_2. Also assume that $\mu_1 \approx \mu_2 \approx \mu_0$. An electromagnetic wave travelling in medium 1 is incident obliquely on the interface. The situation is the same as that described in Section 25.1.

We have already discussed how laws of reflection and refraction are obtained by matching the field components at the interface. In this section, we will derive relations for reflectance and transmittance. What fractions of the incident EM wave intensity are the reflected and transmitted intensities?

The expressions for reflectance and transmittance depend on the direction of polarization of the incident wave. Let us consider two cases, one in which the ***E***-field of the incident wave is parallel to the plane of incidence and the other in which it is perpendicular to the plane of incidence.

E_1 parallel to the plane of incidence

The directions of propagation for all three waves, meeting at *O*, are coplanar and in the *x–z* plane. The ***E*** vector for the incident wave is taken to be in this plane (Figure 25.4). Will the ***E*** fields for the reflected and transmitted wave also be in the same plane? Yes, but as of now, we don't have an argument to assume that. So we will allow a component in the $\hat{j}$-direction also.

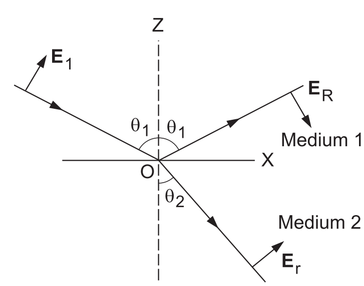


Fig. 25.4

We will now write equations for the boundary conditions. You know from Section 25.1 that all the exponential terms like $e^{i(k_i \cdot \boldsymbol{r} - \omega t)}$ are the same for the same point $\boldsymbol{r}$ (on the surface) and the same time t. So, in the boundary-condition equations, these terms will cancel out. So we will ignore these terms and write

$$\tilde{\boldsymbol{E}}_I = \tilde{E}_{0I}(\cos\theta_1 \hat{i} + \sin\theta_1 \hat{k}),$$

$$\tilde{\boldsymbol{E}}_R = \tilde{E}_{0R}(\cos\theta_1 \hat{i} - \sin\theta_1 \hat{k}) + \tilde{E}' \hat{j}$$

and
$$\tilde{\boldsymbol{E}}_T = \tilde{E}_{0T}(\cos\theta_2 \hat{i} + \sin\theta_2 \hat{k}) + \tilde{E}'' \hat{j}.$$

Write the corresponding $\boldsymbol{B}$ fields using $\boldsymbol{B} = \dfrac{\boldsymbol{k} \times \boldsymbol{E}}{\omega}$ with

$$\boldsymbol{k}_I = (\sin\theta_1 \hat{i} - \cos\theta_1 \hat{k})k_1,$$

$$\boldsymbol{k}_R = (\sin\theta_1 \hat{i} + \cos\theta_1 \hat{k})k_1$$

and
$$\boldsymbol{k}_T = (\sin\theta_2 \hat{i} - \cos\theta_2 \hat{k})k_2.$$

You get

$$\tilde{\boldsymbol{B}}_I = -\frac{\tilde{E}_{0I}}{u_1}\hat{j},$$

$$\tilde{\boldsymbol{B}}_R = \frac{\tilde{E}_{0R}}{v_1}\hat{j} + \frac{\tilde{E}'}{v_1}(\sin\theta_1 \hat{k} - \cos\theta_1 \hat{i})$$

and
$$\tilde{\boldsymbol{B}}_T = -\frac{\tilde{E}_{0T}}{v_2}\hat{j} + \frac{\tilde{E}''}{v_2}(\sin\theta_2 \hat{k} + \cos\theta_2 \hat{i}).$$

Now apply the boundary conditions. First, $D_{1n} = D_{2n}$.

$$\varepsilon_1[(\tilde{\boldsymbol{E}}_I)_z + (\tilde{\boldsymbol{E}}_R)_z] = \varepsilon_2(\tilde{\boldsymbol{E}}_T)_z$$

or
$$\varepsilon_1(\tilde{E}_{0I}\sin\theta_1 - \tilde{E}_{0R}\sin\theta_1) = \varepsilon_2 \tilde{E}_{0T}\sin\theta_2$$

or
$$\tilde{E}_{0I} - \tilde{E}_{0R} = \frac{\varepsilon_2}{\varepsilon_1}\frac{\sin\theta_2}{\sin\theta_1}\tilde{E}_{0T}.$$

We assume that $\mu_1 \approx \mu_2 \approx \mu_0$, $\varepsilon_2 = n_2^2$ and $\varepsilon_1 = n_1^2$. Also $\dfrac{\sin\theta_2}{\sin\theta_1} = \dfrac{n_1}{n_2}\cdot$ Thus,

$$\tilde{E}_{0I} - \tilde{E}_{0R} = \frac{n_2}{n_1}\tilde{E}_{0T}. \tag{i}$$

Now apply $\tilde{\boldsymbol{E}}_{1t} = \tilde{\boldsymbol{E}}_{2t}$.

$$(\tilde{E}_{0I}\cos\theta_1 + \tilde{E}_{0R}\cos\theta_1)\hat{i} + \tilde{E}'\hat{j} = \tilde{E}_{0T}\cos\theta_2 \hat{i} + \tilde{E}''\hat{j}$$

or $$\tilde{E}_{0I} + \tilde{E}_{0R} = \frac{\cos\theta_2}{\cos\theta_1}\tilde{E}_{0T} \tag{ii}$$

and $$\tilde{E}' = \tilde{E}''. \tag{iii}$$

Next apply $B_{1n} = B_{2n}$.

$$0 + \frac{\tilde{E}'}{v_1}\sin\theta_1 = \frac{\tilde{E}''}{v_2}\sin\theta_2$$

or $$\tilde{E}' = \tilde{E}''\frac{v_1}{v_2}\frac{\sin\theta_2}{\sin\theta_1} = \tilde{E}''\frac{v_1 v_2}{v_2 v_1},$$

which again gives $\tilde{E}' = \tilde{E}''$.

Finally, apply $H_{1t} = H_{2t}$.

$$-\frac{\tilde{E}_{0I}}{\mu_1 v_1}\hat{j} + \frac{\tilde{E}_{0R}}{\mu_1 v_1}\hat{j} - \frac{\tilde{E}'}{\mu_1 v_1}\cos\theta_1\hat{i} = -\frac{\tilde{E}_{0T}}{\mu_2 v_2}\hat{j} + \frac{\tilde{E}''}{\mu_2 v_2}\cos\theta_2\hat{i}. \tag{iv}$$

Comparing the x-components and using $\tilde{E}' = \tilde{E}''$, $\mu_1 \approx \mu_2$,

$$\tilde{E}'\left(\frac{\cos\theta_1}{v_1} + \frac{\cos\theta_2}{v_2}\right) = 0.$$

As the quantity in parentheses is not zero,

$$\tilde{E}' = \tilde{E}'' = 0.$$

The $\boldsymbol{E}$ fields of reflected and transmitted waves are also parallel to the plane of incidence.

Comparing the y-components on either side in equation (iv),

$$\frac{\tilde{E}_{0I}}{v_1} - \frac{\tilde{E}_{0R}}{v_1} = \frac{\tilde{E}_{0T}}{v_2}$$

or $$\tilde{E}_{0I} - \tilde{E}_{0R} = \frac{v_1}{v_2}\tilde{E}_{0T} = \frac{n_2}{n_1}\tilde{E}_{0T},$$

which is the same as (i).

Adding equations (i) and (ii),

$$2\tilde{E}_{0I} = \left(\frac{n_2}{n_1} + \frac{\cos\theta_2}{\cos\theta_1}\right)\tilde{E}_{0T}$$

or $$\tilde{E}_{0T} = \frac{2n_1\cos\theta_1}{n_2\cos\theta_1 + n_1\cos\theta_2}\tilde{E}_{0I}. \tag{25.5}$$

You can write the expression for $\tilde{E}_{0R}$ in terms of $\tilde{E}_{0I}$ using equations (i) and 25.5.

$$\tilde{E}_{0R} = \tilde{E}_{0I} - \frac{n_2}{n_1}\frac{2n_1\cos\theta_1}{n_2\cos\theta_1 + n_1\cos\theta_2}\tilde{E}_{0I}$$

or $$\tilde{E}_{0R} = \frac{n_1\cos\theta_2 - n_2\cos\theta_1}{n_2\cos\theta_1 + n_1\cos\theta_2}\tilde{E}_{0I}. \qquad (25.6)$$

Equations 25.5 and 25.6 are called *Fresnel's equations for parallel polarization*. You can now get the reflectance R and transmittance T as

$$R = \left|\frac{\tilde{E}_{0R}}{\tilde{E}_{0I}}\right|^2 \text{ and } T = 1 - R.$$

Figure 25.5 shows the variation of R and T with the angle of incidence for the case of electric fields in the plane of incidence.

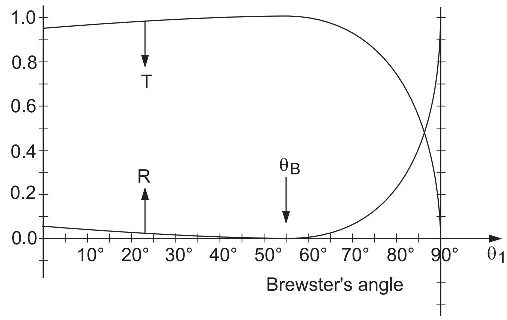


Fig. 25.5

Brewster's angle

From Equation 25.6 you can see that for a certain angle of incidence θ_1, the amplitude of the reflected wave may become zero. For this to happen,

$$n_1\cos\theta_2 = n_2\cos\theta_1$$

or $$n_1^2(1-\sin^2\theta_2) = n_2^2(1-\sin^2\theta_1)$$

or $$n_1^2\left[1-\frac{n_1^2}{n_2^2}\sin^2\theta_1\right] = n_2^2(1-\sin^2\theta_1)$$

or $$n_1^2 - n_2^2 = \left(\frac{n_1^4}{n_2^2} - n_2^2\right)\sin^2\theta_1$$

or $$\sin\theta_1 = \frac{n_2}{\sqrt{n_1^2+n_2^2}}$$

or $$\tan\theta_1 = \frac{n_2}{n_1}. \qquad (25.7)$$

If the EM wave falls on the surface at an angle of incidence $\theta_1 = \tan^{-1}\left(\frac{n_2}{n_1}\right)$, with the electric field parallel to the plane of incidence, there will be no reflected wave. Reflectance in this situation will be zero and transmittance will be equal to 1. This phenomenon is widely used to get polarized light (***E*** field in a specific direction).

The angle $\theta_p = \tan^{-1}(n_2/n_1)$ is called *Brewster's angle or polarization angle*. Unpolarized light reflected from the surface of a dielectric at Brewster's angle will become linearly polarized as the E-field component parallel to the plane of incidence will not be reflected.

Phase inversion on reflection

Now look at the ratio of the complex amplitudes of the reflected and incident waves. Call it reflection coefficient.

$$r = \frac{\tilde{E}_{0R}}{\tilde{E}_{0I}} = \frac{n_1 \cos\theta_2 - n_2 \cos\theta_1}{n_2 \cos\theta_1 + n_1 \cos\theta_2}. \tag{25.8}$$

How does this change with the angle of incidence θ_1?

At Brewster's angle θ_p, the ratio $r = 0$. For $\theta_1 < \theta_p$, the ratio will be negative and for $\theta_1 > \theta_p$, it will be positive. What is the significance of negative r? The expression in Equation 25.7 was derived on the basis of the geometry of Figure 25.4. A negative value of r means that the electric field drawn in the figure is to be reversed.

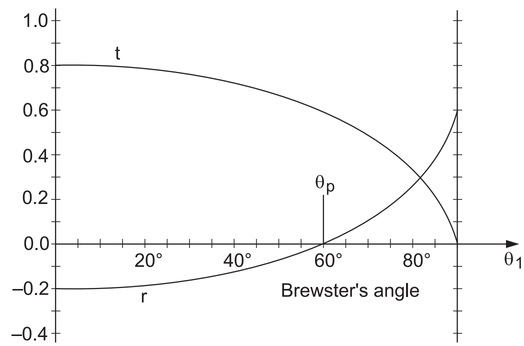


Fig. 25.6

The ratio of transmitted to incident electric field amplitude is positive for all angles of incidence. The variations of r and t as a function of angle of incidence are shown in Figure 25.6 for the specific case of light falling from air on a glass surface ($n = 1.5$) with electric field parallel to the plane of incidence. This shows that there is no sudden phase change in the electric field of the transmitted wave.

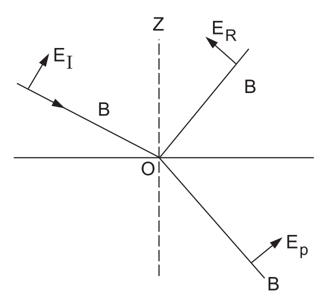


Fig. 25.7

The electric field vectors for angle of incidence less than Brewster's angle are shown in Figure 25.7. For angles of incidence larger than Brewster's angle, the electric field vector will be as shown in Figure 25.4.

E_I Perpendicular to the plane of incidence

If the direction of the electric field in the incident wave is perpendicular to the plane of incidence, the directions of the $\boldsymbol{E}$ fields in the reflected and transmitted waves will also be perpendicular to this plane. You can start with tangential components in reflected and transmitted waves, as done in the previous derivation, but the boundary conditions will make these components zero. So you can write

$$\tilde{\boldsymbol{E}}_I = \tilde{E}_{0I}\,\hat{\boldsymbol{j}},$$

$$\tilde{\boldsymbol{E}}_R = \tilde{E}_{0R}\,\hat{\boldsymbol{j}}$$

and $$\tilde{\boldsymbol{E}}_T = \tilde{E}_{0T}\,\hat{\boldsymbol{j}}.$$

The corresponding $\boldsymbol{B}$ fields will be

$$\tilde{\boldsymbol{B}}_I = \frac{\boldsymbol{k}_I \times \tilde{\boldsymbol{E}}_I}{\omega} = \frac{(\sin\theta_1\hat{k} + \cos\theta_1\hat{i})}{v_1}\tilde{E}_{0\hat{\jmath}}$$

$$\tilde{\boldsymbol{B}}_R = \frac{\boldsymbol{k}_R \times \tilde{\boldsymbol{E}}_R}{\omega} = \frac{(\sin\theta_1\hat{k} - \cos\theta_1\hat{i})}{v_1}\tilde{E}_{0R}$$

and $$\tilde{\boldsymbol{B}}_T = \frac{\boldsymbol{k}_T \times \tilde{\boldsymbol{E}}_T}{\omega} = \frac{\sin\theta_2\hat{k} + \cos\theta_2\hat{i}}{v_2}\tilde{E}_{0T}.$$

Apply the boundary condition $E_{1t} = E_{2t}$. It gives

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}. \tag{v}$$

Apply the boundary condition $H_{1t} = H_{2t}$. It gives

$$(\tilde{E}_{0I} - \tilde{E}_{0R})\frac{\cos\theta_1}{v_1} = \frac{\cos\theta_2}{v_2}\tilde{E}_{0T}$$

or $$\tilde{E}_{0I} - \tilde{E}_{0R} = \frac{v_1}{v_2}\frac{\cos\theta_2}{\cos\theta_1}\tilde{E}_{0T}$$

or $$\tilde{E}_{0I} - \tilde{E}_{0R} = \frac{n_2\cos\theta_2}{n_1\cos\theta_1}\tilde{E}_{0T}. \tag{vi}$$

From (v) and (vi),

$$2\tilde{E}_{0I} = \left(1 + \frac{n_2\cos\theta_2}{n_1\cos\theta_1}\right)\tilde{E}_{0T}$$

or $$\tilde{E}_{0T} = \frac{2n_1\cos\theta_1}{n_1\cos\theta_1 + n_2\cos\theta_2}\tilde{E}_{0I}. \tag{25.9}$$

From (v),

$$\tilde{E}_{0R} = \left[\frac{2n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 \cos\theta_2} - 1\right]\tilde{E}_{0I}$$

or
$$\tilde{E}_{0R} = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2}\tilde{E}_{0I}. \tag{25.10}$$

Taking the specific case of light falling from air to glass ($n = 1.5$), the reflectance R and transmittance T can be calculated as in the case of electric fields in the plane of incidence. The plots of R and T as functions of the angle of incidence are shown in Figure 25.8.

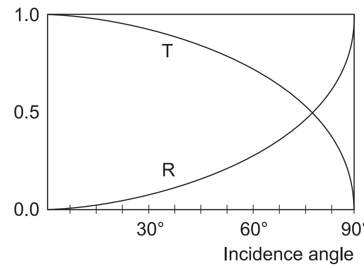


Fig. 25.8

Equations 25.8 and 25.9 are *Fresnel's equations for perpendicular polarization*.

Phase inversion

Check the expressions for $r = \dfrac{\tilde{E}_{0R}}{\tilde{E}_{0I}}$ and $t = \dfrac{\tilde{E}_{0T}}{\tilde{E}_{0I}}$ for the case of electric field perpendicular to the plane of incidence. You should work out that the sign of r is negative if $n_1 < n_2$ and positive if $n_1 > n_2$. This is true for any angle of incidence. There is a sudden phase change of π at reflection in this case. The ratio t is always positive.

General of direction *E*

What if the incident wave has components parallel to the plane of incidence as well as perpendicular to it? The two components are reflected according to the equations derived above. At Brewster's angle, $\theta = \theta_p$, the E-fields are as shown in Figure 25.9.

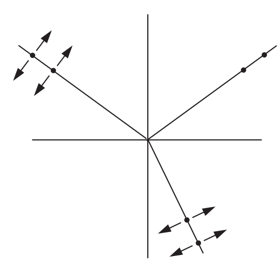


Fig. 25.9

25.5 Total Internal Reflection

According to Snell's law, when an electromagnetic wave falls on the interface between two dielectric media, the angle of incidence and the angle of refraction are related as

$$\frac{\sin\theta_I}{\sin\theta_T} = \frac{v_1}{v_2} = \frac{n_2}{n_1}$$

$$\sin\theta_T = \frac{n_1}{n_2}\sin\theta_I,$$

where v_1 and v_2 are speeds of the wave in the first and second medium respectively. Consider the situation shown in Figure 25.10. The complex electric field of the transmitted wave will be

$$\tilde{\boldsymbol{E}}_T = \tilde{E}_{0T}e^{i\boldsymbol{k}_T\cdot\boldsymbol{r}-\omega t},$$

where $$\boldsymbol{k}_T = |k_T|(\sin\theta_T\hat{i}+\cos\theta_T\hat{j}).$$

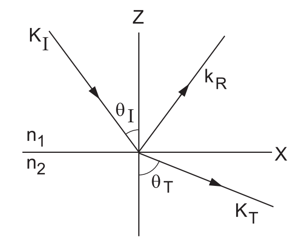


Fig. 25.10

Suppose the speed of the EM wave in medium 1 is smaller than that in medium 2. In everyday language one says that the wave is going from an optically denser medium to an optically rarer medium.

For $$\theta_I = \theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right),$$

$$\sin\theta_T = \frac{n_1}{n_2}\sin\theta_I = 1 \text{ and } \cos\theta_T = 0. \text{ Thus } \boldsymbol{k}_T = k_T\hat{i}.$$

The transmitted wave goes along the surface. The angle θ_c is called the critical angle for this combination of dielectrics.

What if $\theta_1 > \sin^{-1}\left(\frac{n_2}{n_1}\right)$?

In this case, $\sin\theta_T > 1$ and you do not get any angle θ_T at which the wave can be transmitted. Does it mean there is no transmitted wave? Can one have reflection without refraction? Suppose the electric field of the incident wave is in the plane of incidence. Refer to Figure 25.4 and apply the boundary condition with $E_T = 0$. First,

$$D_{1n} = D_{2n}$$

or $$\varepsilon_1[(\tilde{\boldsymbol{E}}_I)_z + (\tilde{\boldsymbol{E}}_R)_z] = 0$$

or $$\tilde{E}_{0I}\sin\theta_I - \tilde{E}_{0R}\sin\theta_I = 0$$

or $$\tilde{E}_{0I} = \tilde{E}_{0R}. \tag{i}$$

Next, apply $\tilde{\boldsymbol{E}}_{1t} = \tilde{\boldsymbol{E}}_{2t}$ along the x-direction.

$$\tilde{E}_{0I}\cos\theta_I + \tilde{E}_{0R}\cos\theta_I = 0$$

or $$\tilde{E}_{0I} + \tilde{E}_{0R} = 0. \tag{ii}$$

From (i) and (ii), $\tilde{E}_{0I} = \tilde{E}_{0R} = 0$. But you do have an incident wave and a reflected wave. So, you must have a transmitted wave even if the angle of incidence is larger than the critical angle!

But then, in which direction should the wave go in the second medium? Snell's law does not give us any real angle θ_T. Let us still write the propagation vector $\boldsymbol{k}_T$ for the second medium, as

$$\boldsymbol{k}_T = |\boldsymbol{k}_T| (\sin\theta_T \hat{i} - \cos\theta_T \hat{k}). \tag{iii}$$

$|\boldsymbol{k}_T| = \dfrac{\omega}{v_2}$ is a well-defined quality, $\sin\theta_T = (\sin\theta_I)\dfrac{n_1}{n_2}$ is a well-defined real quantity having a value larger than 1.

$$\cos\theta_T = \sqrt{1-\sin^2\theta_T} = \sqrt{1-\frac{(\sin^2\theta_I)n_1^2}{n_2^2}} = \pm i\sqrt{\frac{n_1^2\sin^2\theta_I}{n_2^2}-1},$$

where the quantity under the square root sign is positive. The sign + or – will be chosen from physical arguments. From (iii),

$$\boldsymbol{k}_T = |\boldsymbol{k}_T|\left(\frac{n_1\sin\theta_1}{n_2}\hat{i} \pm i\sqrt{\frac{n_1^2\sin^2\theta_I}{n_2^2}-1}\,\hat{k}\right) = k'\,\hat{i} \pm i\beta\hat{k}, \text{ where } k' = |\tilde{\boldsymbol{k}}_T|\frac{n_1\sin\theta_I}{n_2} \text{ and } \beta = \sqrt{\frac{n_1^2\sin^2\theta_I}{n_2^2}-1}.$$

The transmitted wave is

$$\begin{aligned}\tilde{\boldsymbol{E}}_T &= \tilde{\boldsymbol{E}}_{0T}e^{i(\boldsymbol{k}_T\cdot\boldsymbol{r}-\omega t)} = \tilde{E}_{0T}e^{i[(k'\hat{i}+i\beta\hat{k})\cdot\boldsymbol{r}-\omega t]}\\ &= \tilde{\boldsymbol{E}}_{0T}e^{-\beta z}e^{i(k'x-\omega t)}.\end{aligned}$$

You can see that there is a transmitted wave going in medium 2 in the *x*-direction, that is, parallel to the surface of separation of the two media. The amplitude of the wave decreases exponentially as one goes away from the surface.

We have chosen a proper sign for the complex part of $\boldsymbol{k}_1$ to get the $e^{-\beta z}$ factor. If you choose the opposite sign, this factor will be $e^{\beta z}$ which is unphysical as it would say that intensity of the transmitted wave keeps increasing as you go away from the surface.

Such a wave is called an *evanescent wave*.

We have seen the existence of the evanescent wave according to the requirements of electrodynamics. Such a wave can also be observed in an experimental situation. If you place another dielectric close to the surface (at a distance $\lesssim 1/\beta$), you can see the wave being transmitted into this dielectric in medium 2 (Figure 25.11).

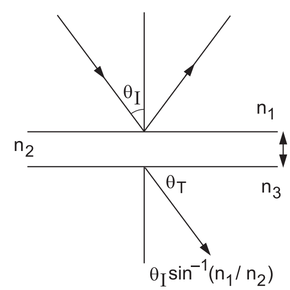


Fig. 25.11

The electric field of the incident wave was taken to be in the plane of the incidence. It is your homework to do it for the electric field perpendicular to the plane of incidence.

EXERCISES

Based on Concepts

1. Which of the following does not change as an electromagnetic wave is refracted at a surface?
   (a) The magnitude of the propagation vector ***k***
   (b) Phase of the electric field
   (c) The ratio of amplitude of the electric field to that of the magnetic field
2. When light falls on a glass surface normally, about 4% of the intensity falling on the surface reflects and 96% transmits. Think in terms of photons. Is each photon divided into two parts, with 4% energy and 96% energy, the first one being reflected and the second one being transmitted? Or are 4% of the total number of photons falling in a given time reflected and are 96% transmitted? If this be the case, how do the photons keep track of the photons previously reflected or transmitted to maintain 4% reflection?
3. A pair of polarizing goggles cuts out nearly 50% of natural light, by allowing either horizontally or vertically polarized light to pass through. In order to block the glare of the twilight sun (i.e., when the sun's rays are reflected at a grazing angle), a specific polarization direction is blocked out. Analyze and determine which polarization state is blocked out (horizontal or vertical).
4. Light moving in air falls on a glass surface at Brewster's angle. The electric field is linearly polarized in a direction that has components parallel to the plane of incidence as well as perpendicular to it. Will there be a reflected wave? If so, will it be linearly polarized? If so, will the direction of polarization be the same as that of the incident light?

Problems

1. Show that Fresnel's equations for ***E***–field in the plane of incidence and for ***E***-field perpendicular to the plane of incidence reduce to the same equations if $\theta_1 = 0$, that is, for normal incidence.
2. Find the critical angle and Brewster's angle for diamond. The speed of light in diamond is 1.25×10^8 m/s. [**Ans.** 24.6°, 67.4°]
3. A light wave is incident on crown glass ($n = 1.52$) at angle $\pi/6$. Determine the amplitude reflection coefficient and transmission coefficient when the beam is linearly polarized (i) in the plane of incidence and (ii) normal to the plane of incidence. [**Ans.** (i) 2.7%, (ii) 6.2%]
4. In the above question, determine the angle at which the reflected wave would be completely polarized. Does such an angle exist for both polarizations?
   [**Ans.** 56.7° for ***E*** parallel to the plane of incidence]
5. Consider light going from water $\left(n = \dfrac{4}{3}\right)$ to air. (a) Calculate Brewster's angle θ_p. (b) If the angle of incidence is greater than the critical angle θ_c, the transmitted intensity becomes zero. Find θ_c. (c) In this case you will find $\theta_p < \theta_c$. Is this relation valid for all values of n_1 and n_2 with $n_1 > n_2$ (light going from denser medium to rarer medium)? [**Ans.** (a) 36.9°, (b) 48.7°]

6. A film of a linear dielectric having refractive index 1.2 is deposited on a thick glass substrate of refractive index 1.5. A plane monochromatic light wave of wavelength 600 nm travelling in air falls perpendicularly on the film. Consider the wave reflected from the top surface (wave 1) and the wave which emerges in air after one reflection at the film-glass interface (wave 2). (a) What is the minimum thickness of the film for these two waves to have a phase difference of π? (b) Find the ratio $\frac{E_{02}}{E_{01}}$, where E_{01} and E_{02} represent the amplitudes of electric field in wave 1 and wave 2.

[**Ans.** (a) 500 nm, (b) 120/99]

7. Light travelling through air falls on a glass plate ($n = 1.5$). The electric field is in the plane of incidence. What should be the angle of incidence so that 25% of the photons get reflected and 75% get transmitted. [**Ans.** 6.2°]

8. A light wave travelling in a medium of refractive index n falls on its surface open to air. The angle of incidence is more than the critical angle. What is the reflectance I_R/I_I for electric field in the plane of incidence and also for electric field perpendicular to the plane of incidence? [**Ans.** 1 in each case]

9. Light going in water ($\mu = 4/3$) falls on the water–air interface. Calculate the fraction of the intensity that is reflected if the angle of incidence is $\theta_I = 40°$ and the electric field is (a) parallel to the plane of incidence and (b) perpendicular to the plane of incidence. [**Ans.** (a) 4.94%, (b) 22.4%]

10. Consider the situation of the previous problem. What would be the fraction of the intensity reflected if the angle of incidence is 60° and the electric field is (a) parallel to the plane of incidence and (b) perpendicular to the plane of incidence? [**Ans.** 1]

❑

Appendix 1

Line, Surface and Volume Integration

A1.1 Line Integral

You are familiar with the integration of a function $f(x)$ over a given range of x. You can write it as

$$\int_a^b f(x)\,dx.$$

Suppose you have a scalar quantity $\phi(\boldsymbol{r})$, which is a function of points in space. This means that at any given point with position vector $\boldsymbol{r}$, you have a definite value of ϕ. To start with, suppose ϕ is a scalar quantity. We call it a scalar field.

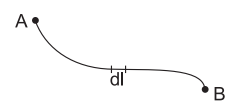


Fig. A1.1

Suppose you are given a curve AB (Figure A1.1). You can define a line integral $\int_{\text{curve}} \phi\, dl$ as follows. Take an element dl on the curve. At the location of this element, find the value of ϕ. Multiply this ϕ by the length dl. Taking the element successively forward, starting from A and ending at B, add all these values of $\phi\, dl$. Take the limit $dl \to 0$ and you get $\int_{\text{curve}} \phi\, dl$. This is called the line integral of $\phi(\boldsymbol{r})$ over the given curve.

Another useful line integral is that of a vector field $\boldsymbol{A}(\boldsymbol{r})$ over a curve, written as $\int_{\text{curve}} \boldsymbol{A}\cdot d\boldsymbol{l}$. Here you have a vector quantity $\boldsymbol{A}(\boldsymbol{r})$ which is a function of space points. Such a quantity is called a vector field.

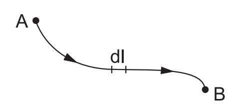


Fig. A1.2

A curve AB is given to you with an arrow on it (Figure A1.2). Take an element $d\boldsymbol{l}$ on this curve. The arrow on the curve indicates the direction of $d\boldsymbol{l}$. At this location find the vector $\boldsymbol{A}(\boldsymbol{r})$. Take the dot product of $\boldsymbol{A}$ and $d\boldsymbol{l}$. Add $\boldsymbol{A}\cdot d\boldsymbol{l}$ for all length elements $d\boldsymbol{l}$ starting from one end and reaching the other end. Taking the limit where $d\boldsymbol{l} \to 0$, you get $\int_{\text{curve}} \boldsymbol{A}\cdot d\boldsymbol{l}$.

You can also have a closed curve with an arrow on it. In that case you start taking $\boldsymbol{dl}$ from any point on the curve you choose and then reach the same point evaluating the integral.

When the integration is over a closed curve, we put a small circle on the integration sign and write it as $\oint \boldsymbol{A} \cdot \boldsymbol{dl}$.

In practice, one uses a certain coordinate system to describe the points. Using Cartesian coordinates, the point $\boldsymbol{r}$ is given by the three coordinates x, y, z.

$$\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}.$$

The vector $\boldsymbol{A}$ is $\boldsymbol{A} = A_x\hat{\boldsymbol{i}} + A_y\hat{\boldsymbol{j}} + A_z\hat{\boldsymbol{k}}$.

Each of the three components can be a function of the space point, that is, a function of x, y, z. Also,

$$\boldsymbol{dl} = dx\hat{\boldsymbol{i}} + dy\hat{\boldsymbol{j}} + dz\hat{\boldsymbol{k}}$$

so that

$$\boldsymbol{A} \cdot \boldsymbol{dl} = A_x dx + A_y dy + A_z dz.$$

You can use other coordinate systems too to describe the space points. A description of two of these coordinate systems is given in Appendix 2.

Depending on the problem at hand, the line integral can be converted into integrations over x, y, z with proper limits.

A1.2 The Surface Integral of a Vector Field

A surface can be a plane surface or it can be curved. It can be open or closed. A closed surface encloses a volume. The whole surface of a sphere, a cube, a cylinder with both ends closed with flat surfaces, etc., are examples of closed surfaces. For such surfaces, you have an "inside" and an "outside". If you are inside and want to go outside, you have to cross the surface somewhere.

Open surfaces do not enclose a volume. A circular disk is an example. You still have two sides of the surface but not an inside and an outside. An open surface has a boundary curve. Closed surfaces do not have boundary curves.

Take a small part on a given surface. Let the area of this element be da. You can define an area vector $\boldsymbol{da}$ corresponding to this element. If the given surface is a closed surface, the normal to the chosen area element, towards the outside, is taken as the direction of this vector $\boldsymbol{da}$. The geometric area da is of course the magnitude of $\boldsymbol{da}$.

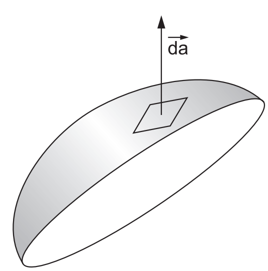


Fig. A1.3

If the surface is not closed, there is no inside or outside. In that case, the choice of the direction of $\boldsymbol{da}$ is yours. Define one side of the surface as the positive side. Draw a normal to the chosen area element towards the positive side. The area vector $\boldsymbol{da}$ is to be taken in this direction.

Now if you are given a vector field $\boldsymbol{A}(\boldsymbol{r})$, you can take the dot product of $\boldsymbol{A}(\boldsymbol{r})$ with $\boldsymbol{da}$ at any location on the given surface. Divide the surface into such elements, evaluate $\boldsymbol{A}\cdot\boldsymbol{da}$ at each element and add. Take the limit $\boldsymbol{da}\to 0$. This gives the surface integral $\int_{\text{surface}} \boldsymbol{A}\cdot\boldsymbol{da}$ of $\boldsymbol{A}$ over the given surface.

For the actual evaluation of a surface integral, both $\boldsymbol{A}$ and $\boldsymbol{da}$ are expressed in terms of the coordinates and integration is done on these coordinates under proper limits.

If the surface is closed, the integration is written as $\oint \boldsymbol{A}\cdot\boldsymbol{da}$.

You can also have surface integration of a scalar field $\phi(\boldsymbol{r})$. Here the quantity ϕ is a scalar quantity and is a function of the space points $\boldsymbol{r}$. Just take the area da, without making a vector out of it, and do the integration as $\int_{\text{surface}} \phi\, da$.

A1.3 Volume Integral

Volume integration is generally done for a scalar field $\phi(\boldsymbol{r})$. Take a small element $d\tau$ in a given volume and find $\phi(\boldsymbol{r})$ at the location of this $d\tau$. Multiply to get $\phi(\boldsymbol{r})d\tau$. Divide the given volume into such volume elements, calculate $\phi d\tau$ for all such elements and add. As $d\tau\to 0$, this becomes the volume integral $\int_{\text{volume}} \phi\, d\tau$.

❑

Appendix 2

Coordinate Systems

A2.1 Cartesian Coordinates

The Cartesian coordinate system is the first coordinate system you must have been taught. Three perpendicular lines from a point are chosen as x, y, z axes. The point itself is called the origin. The three coordinates describing a point in space are x, y, z, which are perpendicular distances of the point from the y–z, x–z, and x–y planes respectively. The unit vectors along the x, y and z axes are generally written as $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$. The axes are to be labelled in such a way that $\hat{\boldsymbol{i}} \times \hat{\boldsymbol{j}} = \hat{\boldsymbol{k}}$.

Suppose you keep x = constant and allow y and z to vary in the full range $-\infty$ to $+\infty$. What kind of surface do you get? You get a plane parallel to the y–z plane. Similarly, the surface y = constant is a plane parallel to the x–z plane and z = constant is a plane parallel to the x–y plane. You may call these *coordinate surfaces*.

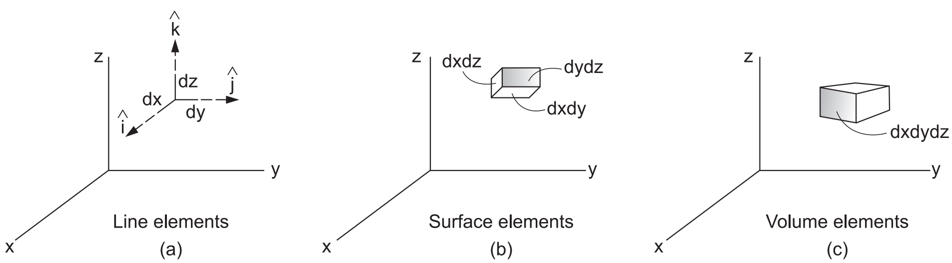


Fig. A2.1

From a given point (x, y, z), you can construct three line elements dx, dy, dz by varying one coordinate at a time. If you only vary x and take its value from x to $x + dx$, you get a line element $dx\hat{\boldsymbol{i}}$. Similarly, by varying y from y to $y + dy$, you get a line element $dy\hat{\boldsymbol{j}}$ and by varying z from z to $z + dz$, you get a line element $dz\hat{\boldsymbol{k}}$. By varying two coordinates at a time you make a surface element. If you change x and y and keep z fixed, you get a small surface area $dx\,dy$. This area is perpendicular to $\hat{\boldsymbol{k}}$. Similarly, you can construct the surface elements $dy\,dz$ perpendicular to $\hat{\boldsymbol{i}}$ and $dx\,dz$ perpendicular to $\hat{\boldsymbol{j}}$. If you vary all three coordinates, you get a small volume element $dx\ dy\ dz$ at the location (x, y, z).

A2.2 Spherical Polar Coordinates

In the spherical polar coordinate system also, we have three perpendicular axes from the origin, and we designate them as x, y and z axes. The three coordinates in this system are r, θ and ϕ. The distance of the given point P from the origin O is r and the angle between the z-axis and OP is θ (Figure A2.2). To get ϕ, you drop a perpendicular PA from the given point P on the x–y plane. The angle OA makes with the x-axis is ϕ. This angle has to be measured along the direction from the x-axis to the y-axis. Thus the range of ϕ is 0 to 2π while θ goes from 0 to π. The coordinate r can go from 0 to ∞. Spherical polar coordinates are also called spherical coordinates.

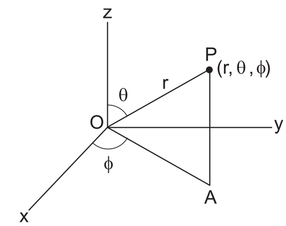


Fig. A2.2

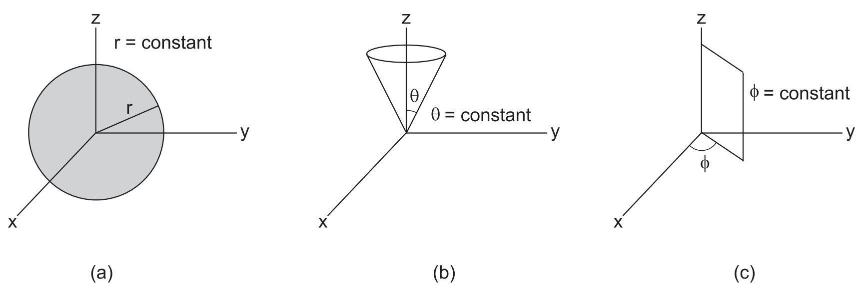


Fig. A2.3

Keep r constant, say $r = R$, and allow θ and ϕ to vary in their full range, that is, θ from 0 to π and ϕ from 0 to 2π. You get a spherical surface of radius R centred on the origin (Figure A2.3). Now think of the surface $\theta =$ constant, r going from 0 to ∞ and ϕ going from 0 to 2π. Convince yourself that it forms the surface of a cone with the vertex at the origin and semi-vertical angle θ. Finally, look at the surface $\phi =$ constant. It is a plane containing the z-axis. If you rotate the x–z plane by an angle ϕ about the z-axis you get this surface $\phi =$ constant. In fact $\phi =$ constant represents only half the plane. For example, if you take $\phi = 0$ and allow r to go from 0 to ∞ and θ from 0 to π, you get half of the x–z plane, with $x \geq 0$. If you take $\phi = \pi$, you get the other half of the x–z plane, that is, with $x \leq 0$. Similarly, $\phi = \dfrac{\pi}{2}$ represents half of the y–z plane, with $y \geq 0$, and $\phi = 3\pi/2$ represents the other half of the y–z plane, with $y \leq 0$.

$\hat{r}, \hat{\theta}, \hat{\phi}$ vectors

Focus your attention on the point P having coordinates r, θ, ϕ. If you keep θ and ϕ fixed and vary

r from r to $r + dr$, you move a distance dr in the direction of OP. The unit vector along this direction is written as $\hat{\boldsymbol{r}}$ (Figure A2.4).

Once again, start from (r, θ, ϕ), keep r and ϕ fixed, and vary θ from θ to $\theta + d\theta$. In which direction do you move from P? As ϕ is constant, you have to remain in the plane OZPA. As r is also fixed, you will move in the direction perpendicular to OP, in fact in a small circular arc. The unit vector in this direction is written as $\hat{\theta}$.

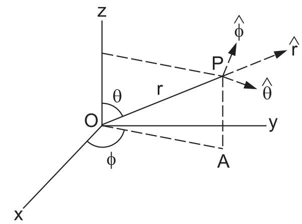


Fig. A2.4

Now suppose you are at P and asked to change ϕ to $\phi + d\phi$, keeping r and θ fixed. As θ is fixed you have to remain on the surface of the cone with semi-vertical angle θ. r is also fixed. So you will move in a direction parallel to the x–y plane. This direction is perpendicular to the plane OZPA. The unit vector in this direction is written as the $\hat{\phi}$-direction.

If you are not familiar with spherical polar coordinates from your mathematics course and are learning them for the first time here, you may have to go through the above description two or three times, each time visualizing the geometry well. It will help if you sit near a corner of the room, set up x, y axes along the edges of the floor and the z-axis along the vertical edge of the walls there. Take a stick, put one end at the origin and keep it in a slanting position. The tip of the stick represents (r, θ, ϕ). Now move the stick in different ways, taking a cue from the statements made in the previous paragraphs. See that $\hat{\boldsymbol{r}}$ has different directions for different points P. Similar is the case with $\hat{\theta}$ and $\hat{\phi}$. In contrast, $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$ do not change direction when you draw them from different points.

Line, surface and volume elements

Focus your attention on the point P(r, θ, ϕ). Fix up θ, ϕ and change r from r to $r + dr$. Make a small line element of length dr in the $\hat{\boldsymbol{r}}$-direction (Figure A2.5a). If you fix r and ϕ, and change θ from θ to $\theta + d\theta$, you get a line element of length $rd\theta$ in the $\hat{\theta}$-direction [Figure A2.5(b)]. And if you fix r and θ, and change ϕ from ϕ to $\phi + d\phi$, you get a line element of length $r \sin\theta\, d\phi$ along the $\hat{\phi}$-direction [Figure A.5(c)]. Thus the three basic line elements obtained by varying one coordinate at a time are dr, $rd\theta$ and $r \sin\theta\, d\phi$, as shown in Figure A2.5.

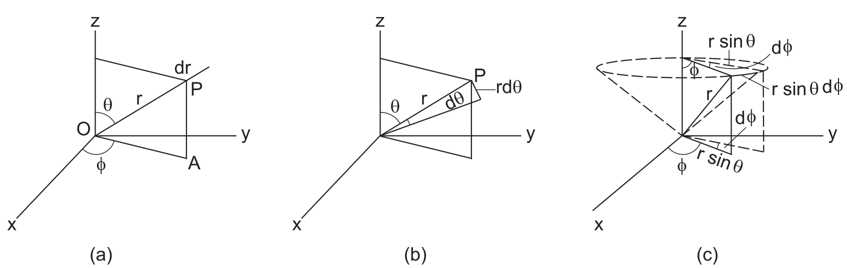


Fig. A2.5

If you allow two of the coordinates to vary a little and keep the third fixed, you get a surface element. Suppose r is kept constant, and θ and ϕ are changed by $d\theta$ and $d\phi$. You move on the surface of the sphere r = constant, and make two perpendicular line elements $rd\theta$ and $r\sin\theta\, d\phi$. With these two as adjacent sides, you make a small, nearly rectangular surface of area $(rd\theta)(r\sin\theta\, d\phi) = r^2 \sin\theta\, d\theta\, d\phi$ [Figure A2.6(a)]. This surface area will be perpendicular to $\hat{\boldsymbol{r}}$.

If you keep ϕ constant and vary r and θ by dr and $d\theta$, you make line elements dr and $rd\theta$. With these two as adjacent sides, make a rectangular area $(dr)(rd\theta) = r\, dr\, d\theta$ on the plane ϕ = constant [Figure A2.6(b)]. This area element will be perpendicular to $\hat{\phi}$.

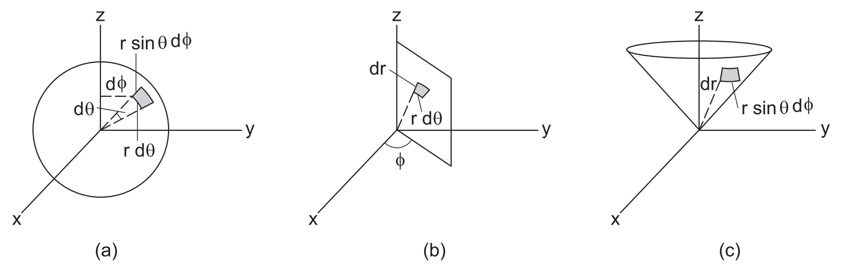


Fig. A2.6

Finally, consider varying r and ϕ while keeping θ constant. You can only move on the surface of the cone θ = constant. If r and ϕ are changed by dr and $d\phi$ from the point (r, θ, ϕ), two line elements dr and $r\sin\theta\, d\phi$ will be formed. The surface element formed by these two variations will have area $(dr)(r\sin\theta\, d\phi) = r\sin\theta\, dr\, d\theta$ [Figure A2.6(c)]. This area will be perpendicular to $\hat{\theta}$. If you vary all three coordinates r, θ, ϕ by $dr, d\theta, d\phi$, respectively, you get a volume element

$$d\tau = (dr)(rd\theta)(r\sin\theta\, d\phi) = r^2 \sin\theta dr\, d\theta\, d\phi.$$

A2.3 Relation between Cartesian and Spherical Polar Coordinates

Let P be a point having spherical polar coordinates (r, θ, ϕ). From P drop a perpendicular PA on the x–y plane and a perpendicular PB on the z-axis. OAPB makes a plane and OA and OB are projections of OP along two perpendicular lines in this plane. Thus $OB = r\cos\theta$ and $OA = r\sin\theta$. From A, now drop perpendiculars AC and AD on the x- and y-axis. Then

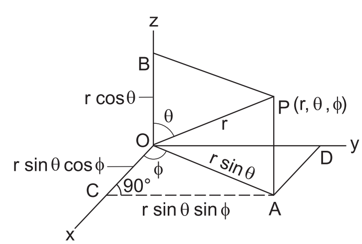


Fig. A2.7

$$OC = r\sin\theta\cos\phi \ \text{ and } \ OD = r\sin\theta\sin\phi.$$

Thus $\boldsymbol{OP} = \boldsymbol{OB} + \boldsymbol{OA} = \boldsymbol{OB} + \boldsymbol{OC} + \boldsymbol{OD}$

$$= (r\cos\theta)\hat{k} + (r\sin\theta\cos\phi)\hat{i} + (r\sin\theta\sin\phi)\hat{j}.$$

If (x, y, z) be the Cartesian coordinates of P,

$$\boldsymbol{OP} = x\hat{i} + y\hat{j} + z\hat{k}.$$

So,

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta.$$

A2.4 Cylindrical Polar Coordinates

In the cylindrical polar coordinate system, the three coordinates used to represent the location of a point will be shown by s, ϕ and z. Generally, the first coordinate is written as ρ, but as in electrodynamics ρ is frequently used for charge density, we will represent it by s. The famous book *Introduction to Electrodynamics* by D J Griffiths uses the same symbol. From the point P, drop a perpendicular PB on the z-axis. The length of this perpendicular is the s-coordinate of P. So, s is the perpendicular distance of P from the z-axis. The coordinate ϕ is defined in the same manner as in the spherical polar coordinate system. Drop a perpendicular PA on the x–y plane (Figure A2.8). The angle made by OA with the x-axis (going from the x- to the y-direction) is ϕ. The z-coordinate of P in the cylindrical polar coordinate system is the same as the z-coordinate in the Cartesian system. Thus, in Figure A2.8, $z = OB$.

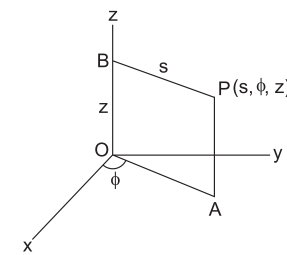


Fig. A2.8

Cylindrical polar coordinates are also called cylindrical coordinates.

Now that we have discussed Cartesian and spherical polar coordinate systems in such detail, it will be easy to see the surfaces in cylindrical coordinates when one of the coordinates is fixed. The surface s = constant is a cylindrical surface with its axis along the z-axis and radius s. It extends from $-\infty$ to $+\infty$ along the z-direction. The coordinate ϕ takes values from 0 to 2π. The surface ϕ = constant is a plane containing the z-axis and making an angle ϕ with the x–z plane, the same as in spherical polar coordinates. The surface z = constant is a plane parallel to the x–y plane. These surfaces are shown in Figure A2.9.

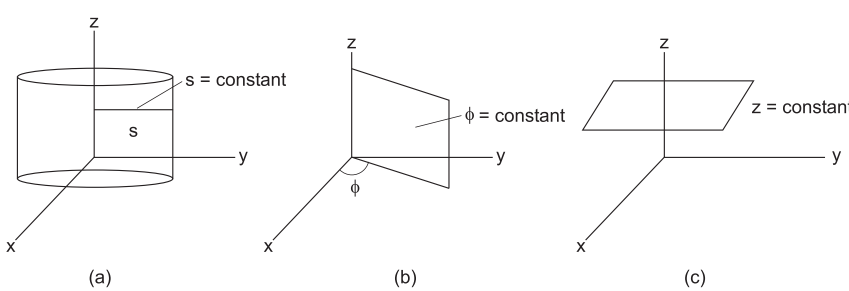


Fig. A2.9

$\hat{s}$, $\hat{\phi}$ and $\hat{z}$ vectors

If you keep ϕ and z fixed and vary s a little, in which direction do you move? You go away from the z-axis. The unit vector in this direction is written as $\hat{s}$. Similarly, keep s and z fixed and vary ϕ. You move perpendicular to the plane *OBPA,* that is, parallel to the x–y plane. The unit vector in this direction is $\hat{\phi}$. And if you keep s and ϕ fixed and vary z, you move parallel to the z-direction. The unit vector in this direction is $\hat{z}$. It is the same as the $\hat{k}$ of the Cartesian system. These vectors are shown in Figure A2.10.

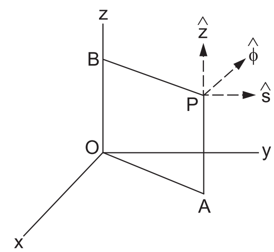


Fig. A2.10

Line, surface and volume elements

As usual, we draw three basic line elements by varying one coordinate at a time. If you vary s, the line element is ds along the $\hat{s}$-direction; if you vary ϕ, you get the line element $sd\phi$ in the ϕ-direction and if you vary z, you get the line element dz in the $\hat{z}$-direction (Figure A2.11).

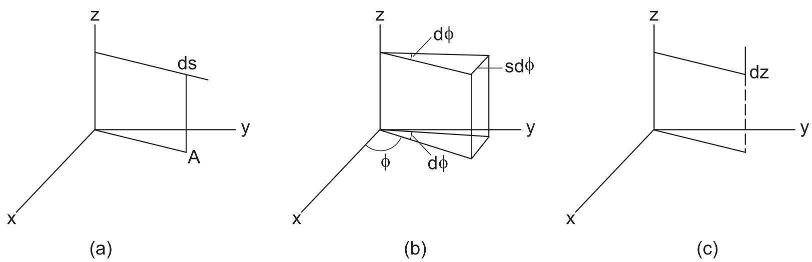


Fig. A2.11

The elementary surface elements are constructed by varying two coordinates at a time. If you keep s fixed and vary ϕ and z, you get a surface element $sd\phi dz$ on the cylindrical surface of radius s [Figure A2.12(a)]. If you vary s and z, you get a surface element $ds dz$ on the plane ϕ = constant [Figure A2.12(b)]. If you vary s and ϕ, keeping z fixed, you get a surface element $sds d\phi$ in

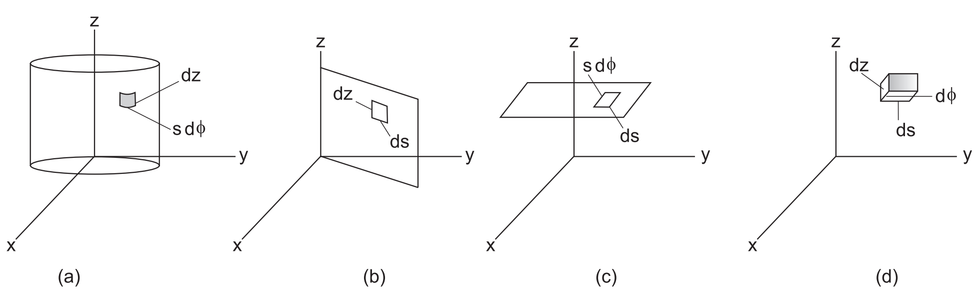


Fig. A2.12

the plane z = constant [Figure A2.12(c)]. When you vary all the three coordinates you obtain the volume element

$$d\tau = s\,ds\,d\phi\,dz,$$

as shown in Figure A2.12(d).

Best is to sit with a cylinder, think of the origin somewhere on the axis and visualize different line and surface elements.

❑

Appendix 3

The Dirac Delta Function

A3.1 The Dirac Delta Function in One Dimension

The one-dimensional Dirac delta function $\delta(x - x_0)$ is a function of x, defined by the following three statements:

(a) $\delta(x - x_0) = 0 \qquad$ for all $x \neq x_0$

(b) $\delta(x - x_0) = \infty \qquad$ for $x = x_0$

(c) $\int_{-\infty}^{\infty} \delta(x - x_0)\,dx = 1.$

So it is zero everywhere except at $x = x_0$, where it goes to infinity in such a way that its integration in the whole range is 1. In fact, the contribution to the integration comes only from $x = x_0$ because everywhere else, the function is zero. Hence you may write

$$\begin{aligned} \int_a^b \delta(x - x_0) &= 1 \qquad \text{if } a < x_0 < b. \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

if x_0 falls in the range a to b, the integration is 1. If it does not, the integration is zero. $\delta(x - x_0)$ is called the Dirac delta function centred at $x = x_0$. The Dirac delta function is also called delta function in short.

A very important result involving the δ-function is the following.

$$\int_{x_1}^{x_2} f(x)\delta(x - x_0)\,dx = f(x_0)$$

if the integration range x_1 to x_2 includes x_0. This is because at all points where $x \neq x_0$, the delta function is zero and hence no contribution to the integration comes from there. At $x = x_0$, $f(x) = f(x_0)$. Hence

$$\int_{x_1}^{x_2} f(x)\delta(x - x_0)\,dx = \int_{x_1}^{x_2} f(x_0)\delta(x - x_0)\,dx = f(x_0)\int_{x_1}^{x_2} \delta(x - x_0)\,dx = f(x_0).$$

A3.2 The Dirac Delta Function in Three Dimensions

The three-dimensional Dirac delta function is a function of space points. A space point can be described by its position vector $\boldsymbol{r}$. The 3-D delta function can be written as $\delta^3(\boldsymbol{r}-\boldsymbol{r}_0)$. Here $\boldsymbol{r}_0$ is the position vector of a fixed point. The function $\delta^3(\boldsymbol{r}-\boldsymbol{r}_0)$ is zero at all points $\boldsymbol{r}$ except at $\boldsymbol{r}_0$, where it goes to infinity. It is defined by the following three statements.

(a) $\delta^3(\boldsymbol{r}-\boldsymbol{r}_0)=0$ for all points except $\boldsymbol{r}=\boldsymbol{r}_0$

(b) $\delta^3(\boldsymbol{r}-\boldsymbol{r}_0)=\infty$ for $\boldsymbol{r}=\boldsymbol{r}_0$

(c) $\displaystyle\int_{\text{all space}} \delta^3(\boldsymbol{r}-\boldsymbol{r}_0)\,d\tau = 1.$

Here $d\tau$ is a small volume element at the position $\boldsymbol{r}$ and integration is to be done over entire space. Once again, the contribution to the integration comes only from the point $\boldsymbol{r}=\boldsymbol{r}_0$. Hence the third condition can be written as

$$\int_{\text{volume}} \delta^3(\boldsymbol{r}-\boldsymbol{r}_0)\,d\tau = 1 \quad \text{if the volume contains, } \boldsymbol{r}_0$$
$$= 0 \quad \text{otherwise.}$$

The integration is 1 if the volume of integration contains the point $\boldsymbol{r}=\boldsymbol{r}_0$. If the volume of integration does not contain $\boldsymbol{r}_0$, the integration is zero. The function $\delta^3(\boldsymbol{r}-\boldsymbol{r}_0)$ may also be written as

$$\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$$

where (x_0, y_0, z_0) are the Cartesian coordinates corresponding to the point $\boldsymbol{r}_0$.

How can you write $\delta^3(\boldsymbol{r}-\boldsymbol{r}_0)$ in terms of the spherical polar coordinates (r, θ, ϕ)? Suppose the point $\boldsymbol{r}_0$ has coordinates (r_0, θ_0, ϕ_0). If you are thinking that the delta function can be written as $\delta(r-r_0)\delta(\theta-\theta_0)\delta(\phi-\phi_0)$, you are wrong. It is $\delta^3(r-r_0)=\dfrac{1}{r^2\sin\theta}\delta(r-r_0)\delta(\theta-\theta_0)\delta(\phi-\phi_0)$.

To verify this, evaluate the integral $\int \delta^3(\boldsymbol{r}-\boldsymbol{r}_0)\,d\tau$ over all space,

$$\int_{\text{all space}} \delta^3(\boldsymbol{r}-\boldsymbol{r}_0)\,d\tau = \int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{1}{r^2\sin\theta}\delta(r-r_0)\delta(\theta-\theta_0)\delta(\phi-\phi_0)\cdot(r^2\sin\theta\,dr\,d\theta\,d\phi)$$

$$= \int_0^{\infty}\delta(r-r_0)\,dr\int_0^{\pi}\delta(\theta-\theta_0)\,d\theta\int_0^{2\pi}\delta(\phi-\phi_0)\,d\phi.$$

Each of these integrations is 1 and so $\displaystyle\int_{\text{all space}} \delta^3(\boldsymbol{r}-\boldsymbol{r}_0)\,d\tau = 1.$

A3.3 Some Charge Distributions

A point charge at $\boldsymbol{r} = \boldsymbol{r}_0$

Suppose a point charge q is placed at the point $\boldsymbol{r} = \boldsymbol{r}_0$, and no other charge exists at any other point. What is the volume charge density $\rho(\boldsymbol{r})$? As there is no charge anywhere other than $\boldsymbol{r} = \boldsymbol{r}_0$, the volume charge density $\rho(\boldsymbol{r})$ should be zero at all points other than $\boldsymbol{r}_0$. At $\boldsymbol{r} = \boldsymbol{r}_0$, you have a finite charge q. Thus you have a finite charge in zero volume (talking like a mathematician and not like a physicist) and so here the charge density $\rho(\boldsymbol{r})$ should be infinity. Now look at $\int_{\text{volume}} \rho(\boldsymbol{r})d\tau$ integrated over a volume. Take the volume as all space. The integration then gives the total charge in space, which is q.

So $\dfrac{1}{q}\displaystyle\int_{\text{all space}} \rho(\boldsymbol{r})d\tau = 1.$

Now consider the function $\dfrac{1}{q}\displaystyle\int_{\text{all space}} \rho(\boldsymbol{r})d\tau = 1.$

The volume integration of this function is 1 if the volume of integration contains the point $\boldsymbol{r} = \boldsymbol{r}_0$ and is zero if it does not. Also $\dfrac{1}{q}\rho(\boldsymbol{r})$ is zero everywhere except at $\boldsymbol{r} = \boldsymbol{r}_0$, where it goes to infinity. These are precisely the conditions for the function $\delta^3(\boldsymbol{r} - \boldsymbol{r}_0)$. Thus

$$\frac{1}{q}\rho(\boldsymbol{r}) = \delta^3(\boldsymbol{r} - \boldsymbol{r}_0)$$

or $$\rho(\boldsymbol{r}) = q\delta^3(\boldsymbol{r} - \boldsymbol{r}_0).$$

In Cartesian coordinates,

$$\rho(\boldsymbol{r}) = q(x - x_0)(y - y_0)(z - z_0).$$

A point charge can therefore be written as a volume charge distribution with volume charge density in terms of the delta function.

A uniform charge distribution on a spherical surface

Suppose you have a spherical surface of radius R with its centre at the origin. It contains a total charge Q uniformly distributed. How do you write the charge density ρ? Use spherical coordinates.

The charge density is zero at all (r, θ, ϕ) for which $r \neq R$. If $r = R$, the charge density ρ is infinity if you assume no thickness of the surface. Remember that charge density ρ is charge/volume and for a surface of zero thickness, the volume is zero. So the charge density should be of the form $k\delta(r-R)$.

Now
$$\int_{\text{all space}} \rho(\boldsymbol{r})d\tau = Q$$

or
$$\int_{r=0}^{R}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \rho(\boldsymbol{r})r^2 \sin\theta \, dr\, d\theta\, d\phi = Q$$

or
$$\int_0^R k\delta(r-R)r^2 dr \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi = Q$$

or
$$4\pi k \int r^2\delta(r-R)dr = Q$$

or
$$4\pi k R^2 = Q$$

or
$$k = \frac{Q}{4\pi R^2}\cdot$$

Thus,
$$\rho(\boldsymbol{r}) = \frac{Q}{4\pi R^2}\delta(r-R)$$

or
$$\rho(\boldsymbol{r}) = \sigma\delta(r-R),$$

where σ is the surface charge density on the spherical surface.

Some more charge distributions

Suppose you have a linear charge distribution along the z-axis. Let the linear charge density be $\lambda(z)$. This distribution can be written as $\rho(\boldsymbol{r}) = \lambda(z)\delta(x)\delta(y)$.

Assume you have a surface charge distribution on the x–y plane. Let the surface charge density at a point $(x, y, 0)$ be $\sigma(x, y)$. The volume charge density corresponding to this distribution may be written as $\rho(\boldsymbol{r}) = \sigma(x, y)\delta(z)$.

❑

Appendix 4

Divergence, Gradient and Curl

A4.1 Divergence of a Vector Field

If you have a vector quantity that has a unique value at each point in space, it is called a vector field. Suppose you are given a vector field $\boldsymbol{A}(\boldsymbol{r})$. The field has a definite direction and a definite magnitude at a given point. You can show the field at any point by an arrow of proper length. Take any general point P with position vector $\boldsymbol{r}$ and construct a small closed surface surrounding this point (Figure A4.1). It encloses a volume $\Delta\tau$. From each point of this surface you can draw a vector to show the magnitude and direction of $\boldsymbol{A}$ at that point. Let da be an elementary surface area on the closed surface. Draw a vector of magnitude da and direction along the outward normal to the chosen surface area. The vector is the area vector $\boldsymbol{da}$. Find $\boldsymbol{A}\cdot\boldsymbol{da}$ and integrate over the whole surface to find the surface integral $\oint \boldsymbol{A}\cdot\boldsymbol{da}$. This quantity is called the flux of the vector field through the closed surface. Divide this surface integral by the volume enclosed by the closed surface and take the limit as this volume Δt goes to zero. That means we have to take the closed surface to be very small. The scalar quantity

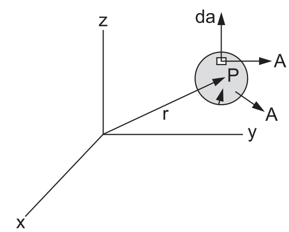


Fig. A4.1

$$\lim_{\Delta\tau\to 0} \oint \frac{\boldsymbol{A}\cdot\boldsymbol{da}}{\Delta\tau}$$

is called the divergence of $\boldsymbol{A}$ at P and is written as $\nabla\cdot\boldsymbol{A}$. The symbol ∇ is also read as del.

Thus,

$$\nabla\cdot\boldsymbol{A} = \lim_{\nabla\tau\to 0} \oint \frac{\boldsymbol{A}\cdot\boldsymbol{da}}{\Delta\tau}. \tag{A4.1}$$

What does this signify? Consider a point P and a tiny volume around it. In general, the direction of the vector $\boldsymbol{A}$ at the closed surface of this volume will go out of the surface at certain points of the closed surface and will go into it at certain other points. If it is going out, $\boldsymbol{A}\cdot\boldsymbol{da}$ will be positive and if it is going into the surface, $\boldsymbol{A}\cdot\boldsymbol{da}$ will be negative. Suppose the vector $\boldsymbol{A}$ goes out of the surface everywhere on the closed surface as shown in Figure A4.2(a). Then $\oint \boldsymbol{A}\cdot\boldsymbol{da}$ will

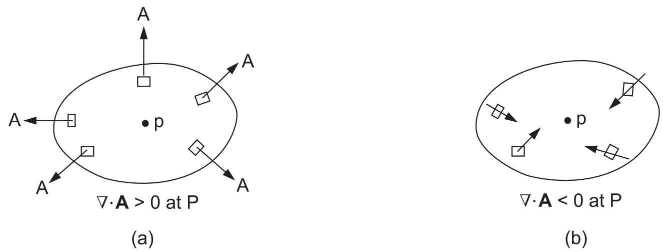


Fig. A4.2

definitely be positive and hence $\nabla \cdot \boldsymbol{A}$ at P will be positive. On the other hand, if $\boldsymbol{A}$ is going into the closed surface at all points of the closed surface (Figure A4.2b), $\nabla \cdot \boldsymbol{A}$ at P will be negative. If $\boldsymbol{A}$ goes into the surface at certain points and comes out at certain other points such that the net $\oint \boldsymbol{A} \cdot \boldsymbol{da} = 0$, then $\nabla \cdot \boldsymbol{A} = 0$ at P.

Imagine the vector $\boldsymbol{A}$ as a quantity flowing in a direction. Then $\nabla \cdot \boldsymbol{A}$ at P tells you if there is a source of $\boldsymbol{A}$ at P that is sending out $\boldsymbol{A}$ in all directions. If the field $\boldsymbol{A}$ is generated at P and diverges in all directions, $\nabla \cdot \boldsymbol{A}$ will have a positive value there. Similarly, if the field $\boldsymbol{A}$ converges from all directions towards the point P, $\nabla \cdot \boldsymbol{A}$ at P will be negative. We say that there is a sink at P that absorbs the field $\boldsymbol{A}$. If there is no such source or sink for $\boldsymbol{A}$ at P, $\nabla \cdot \boldsymbol{A}$ there will be zero.

It turns out that if the vector field $\boldsymbol{A}$ is given to you in terms of the coordinates of the points where it is defined, you can get $\nabla \cdot \boldsymbol{A}$ by doing simple differentiations without going to equation A4.1. The definition A4.1 is equivalent to the following expressions.

Cartesian coordinates:
$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\cdot \tag{A4.2}$$

Spherical polar coordinates:
$$\nabla \cdot \boldsymbol{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(A_\phi). \tag{A4.3}$$

Cylindrical coordinates:
$$\nabla \cdot \boldsymbol{A} = \frac{1}{s}\frac{\partial}{\partial s}(sA_s) + \frac{1}{s}\frac{\partial}{\partial \phi}(A_\phi) + \frac{\partial A_z}{\partial z}\cdot \tag{A4.4}$$

If the vector field $\boldsymbol{A}$ is given in terms of Cartesian coordinates, i.e., $\boldsymbol{A}(x, y, z) = A_x(x, y, z)\hat{i} + A_y(x, y, z)\hat{j} + A_z(x, y, z)\hat{k}$, use equation A4.2 to get $\nabla \cdot \boldsymbol{A}$. Similarly if $\boldsymbol{A}$ is given in spherical polar coordinates $\boldsymbol{A}(r, \theta, \phi) = A_r(r, \theta, \phi)\hat{r} + A_\theta(r, \theta, \phi)\hat{\theta} + A_\phi(r, \theta, \phi)\hat{\phi}$, use equation A4.3. Use equation A4.4 if the field is given in cylindrical coordinates $\boldsymbol{A}(s, \phi, z) = A_s(s, \phi, z)\hat{s} + A_\phi(s, \phi, z)\hat{\phi} + A_z(s, \phi, z)\hat{z}$.

A4.2 The Gauss Divergence Theorem

From the definition of divergence one can show that for any closed surface S and the enclosed volume τ, the following relation holds for any vector field $\boldsymbol{A}$.

$$\oint_S \boldsymbol{A}\cdot d\boldsymbol{a} = \int_\tau \nabla\cdot \boldsymbol{A}\,d\tau. \tag{A4.5}$$

This result is known as the *Gauss divergence theorem*. You take the vector field $\boldsymbol{A}$ at points on the given closed surface S and evaluate the surface integration $\oint \boldsymbol{A}\cdot d\boldsymbol{a}$. As usual, the direction of $d\boldsymbol{a}$ is towards the outward normal. In other words, get the flux of $\boldsymbol{A}$ through the closed surface. That is the LHS. To get the RHS, obtain $\nabla\cdot\boldsymbol{A}$ at all points in the volume enclosed by S and get the volume integral of $\nabla\cdot\boldsymbol{A}$ on this volume. To evaluate the LHS, you only need to know the values of $\boldsymbol{A}$ at the points on the surface, you need not know the values at any point inside or outside the volume. On the other hand, to evaluate the RHS, you need $\nabla\cdot\boldsymbol{A}$ at all points inside the volume. Still the two are equal. This is truly amazing!

A4.3 Gradient of a Scalar Field

Consider a scalar field $V(\boldsymbol{r})$, which is defined as a function of space points. It has a certain value V at a given point P. If you now look at a nearby point Q at a distance dl, the value of V may be different, say $V + dV$. So, as you move a distance dl from P in the direction PQ, the value of V changes from V to $V + dV$. We say that the rate of change of V in the direction of PQ is dV/dl. If you shift from P by the same distance dl but in different directions, the change in V, in general, will be different and so the rate dV/dl will also be different. Look at the direction in which this rate dV/dl is the highest and note down this highest rate. You have obtained the "*gradient of V at* P".

The gradient of a scalar field $V(\boldsymbol{r})$ is itself a vector field having a magnitude and a direction at each space point. It is written as ∇V and is read as grad V. At a given point, its direction is the one in which V increases most rapidly and its magnitude is the same as the maximum rate of increase in V at that point.

EXAMPLE A4.1 **Consider a scalar function $V(\boldsymbol{r}) = kr^2$ where r gives the distance from the origin and k is a positive constant. Find the value of ∇V at a point with spherical polar coordinates $r = R, \theta = \frac{\pi}{3}, \phi = \frac{\pi}{2}$.**

Solution The value of the function f at the given point is $f(P) = kR^2$. Suppose the position vector of P is $\boldsymbol{r}$ and you move through a small distance dl from P and get to a nearby point Q (FigureA4.3). Let PQ make an angle α with the radial direction OP.

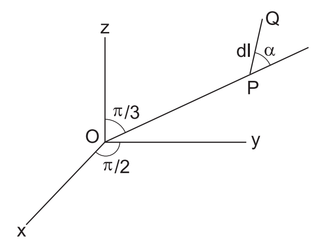


Fig. A4.3

$$\boldsymbol{OQ} = \boldsymbol{OP} + \boldsymbol{PQ} = \boldsymbol{r} + \boldsymbol{dl}$$

$$|OQ|^2 = (\boldsymbol{r} + \boldsymbol{dl})\cdot(\boldsymbol{r} + \boldsymbol{dl}) = R^2 + 2\boldsymbol{r}\cdot\boldsymbol{dl} + dl^2$$

$$\approx R^2 + 2Rdl\cos\alpha, \text{ neglecting } dl^2.$$

Thus, $V(Q) = k\,|OQ|^2 \approx k(R^2 + 2Rdl\cos\alpha)$

and $dV = V(Q) - V(P) = 2kRdl\cos\alpha.$

Therefore,

$$\frac{dV}{dl} = 2kR\cos\alpha.$$

In which direction should we take PQ so that $\frac{dV}{dl}$ is maximum? Surely in the direction of OP. Then $\alpha = 0$, $\cos\alpha = 1$ and $\frac{dV}{dl}$ will be maximum. So ∇V at P has the same direction as OP, i.e., the $\hat{r}$-direction. And in this direction,

$$\frac{dV}{dl} = 2kR.$$

So ∇V at P has magnitude $2kR$. Thus

$$\nabla V = 2kR\hat{r}.$$

There are simple rules to get the gradient of a scalar function V by doing differentiations with respect to the coordinates. The expressions of ∇V in the different coordinate systems are as follows.

Cartesian coordinates:
$$\nabla V(x, y, z) = \frac{\partial}{\partial x}V(x, y, z)\hat{i} + \frac{\partial}{\partial y}V(x, y, z)\hat{j} + \frac{\partial}{\partial z}V(x, y, z)\hat{k}. \quad \text{(A4.6)}$$

Spherical coordinates:

$$\nabla V(r, \theta, \phi) = \frac{\partial}{\partial r}V(r, \theta, \phi)\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}V(r, \theta, \phi)\hat{\theta} + \frac{1}{r\sin\phi}\frac{\partial}{\partial \phi}V(r, \theta, \phi)\hat{\phi}. \quad \text{(A4.7)}$$

Cylindrical coordinates:
$$\nabla V(s, \phi, z) = \frac{\partial}{\partial s}V(s, \phi, z)\hat{s} + \frac{1}{s}\frac{\partial}{\partial \phi}V(s, \phi, z)\hat{\phi} + \frac{\partial}{\partial z}V(s, \phi, z)\hat{z}. \quad \text{(A4.8)}$$

Suppose the value of ∇V is known at a point P and you move to a nearby point Q where $\boldsymbol{PQ} = \boldsymbol{dl}$. This $\boldsymbol{dl}$ need not be in the direction of ∇V. How much is the change in V?

The easiest way to find the answer is to use the expression of ∇V in Cartesian coordinates. If P and Q have coordinates (x, y, z) and $(x + dx, y + dy, z + dz)$ then $\boldsymbol{dl} = dx\hat{i} + dy\hat{j} + dz\hat{k}$.

$$\begin{aligned} V(Q) - V(P) = dV &= \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz \\ &= \left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right)\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k}) \\ &= \nabla V \cdot \boldsymbol{dl}. \end{aligned}$$

Thus, $dV = \nabla V \cdot \boldsymbol{dl}$. (A4.9)

This equation is quite useful. If the angle between ∇V and $\boldsymbol{dl}$ is α,

$$dV = |\nabla V|\, dl \cos\alpha.$$

If you take $\boldsymbol{dl}$ in the direction of ∇V, $\alpha = 0$ and

$$dV = |\nabla V|\, dl$$

or

$$|\nabla V| = \frac{dV}{dl},$$

as defined earlier.

A4.4 The Curl of a Vector Function

Curl is an operation to be done on a vector field $\boldsymbol{A}(\boldsymbol{r})$. The curl of a vector field $\boldsymbol{A}$ is itself a vector field and is written as $\nabla \times \boldsymbol{A}$. It is read as curl A or Del cross A. It essentially tells us how large is the line integral of the field over a closed loop around the given point. If the field itself turns with the loop, you will get a large value of this integration and hence a large value of curl of A at least at some of the points on the loop. Let us define curl in more detail. Be patient—it will take long to arrive at the definition.

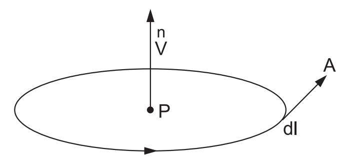


Fig. A4.4

Suppose you have a vector field $\boldsymbol{A}$. Consider a point P and let $\hat{\boldsymbol{n}}$ be the unit vector from P in a particular direction. Draw a small closed loop around P in the plane perpendicular to $\hat{\boldsymbol{n}}$. Put an arrow on the loop. If $\hat{\boldsymbol{n}}$ is towards you, the arrow should be such that the loop looks anticlockwise and if $\hat{\boldsymbol{n}}$ is away from you, the arrow should be such that the loop looks clockwise to you. Take a length element $\boldsymbol{dl}$ on the loop. The vector field $\boldsymbol{A}$ has some value at the location of the line element. Evaluate $\boldsymbol{A} \cdot \boldsymbol{dl}$ and integrate over the whole loop to get $\oint \boldsymbol{A} \cdot \boldsymbol{dl}$. Now divide it by the area Δa enclosed by the loop (this area contains P) and take the limit when the loop shrinks towards P and the area Δa goes to zero. This gives you the component of curl $\boldsymbol{A}$ at P, along $\hat{\boldsymbol{n}}$. The component of curl $\boldsymbol{A}$ along $\hat{\boldsymbol{n}}$ can be written as $(\nabla \times \boldsymbol{A}) \cdot \hat{\boldsymbol{n}}$. Thus,

$$(\nabla \times \boldsymbol{A}) \cdot \hat{\boldsymbol{n}} = \lim_{\Delta a \to 0} \frac{\oint \boldsymbol{A} \cdot \boldsymbol{dl}}{\Delta a}. \tag{A4.10}$$

By taking loops in different orientations, you can find the direction of $\nabla \times \boldsymbol{A}$. It is the direction $\hat{\boldsymbol{n}}$ for which $\oint \boldsymbol{A} \cdot \boldsymbol{dl}$ is largest for the same infinitesimal area.

The above definition leads to Stokes' theorem

$$\oint_{\text{loop}} \boldsymbol{A} \cdot \boldsymbol{dl} = \int_{\text{surface}} (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{da}, \tag{A4.11}$$

valid for any closed loop. To evaluate the LHS, you only need values of $\boldsymbol{A}$ at the boundary points of the loop, whereas to evaluate the RHS, you need values of $\nabla \times \boldsymbol{A}$ at all the points in the surface bounded by the loop.

If the vector field $\boldsymbol{A}$ is given to you in terms of the coordinates of the space point, you can write $\nabla \times \boldsymbol{A}$ using simple differentiation. The expressions of $\nabla \times \boldsymbol{A}$ in the three coordinate systems commonly used are as follows.

Cartesian coordinates: $$\nabla \times \boldsymbol{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\boldsymbol{i}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\boldsymbol{j}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\boldsymbol{k}}.$$

Spherical coordinates:

$$\nabla \times \boldsymbol{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{\boldsymbol{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta}A_r\right]\hat{\phi}.$$

Cylindrical coordinates: $$\nabla \times \boldsymbol{A} = \left[\frac{1}{s}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{\boldsymbol{s}} + \left[\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(sA_\phi) - \frac{\partial A_s}{\partial\phi}\right]\hat{\boldsymbol{z}}.$$

Conservative field

If the curl of a vector field $\boldsymbol{A}(\boldsymbol{r})$ is zero at all points, the field is said to be conservative. Thus for conservative fields, $\nabla \times \boldsymbol{A} = 0$ at all points. Suppose you have a scalar field $V(\boldsymbol{r})$. The gradient of this V makes a vector field $\boldsymbol{A}(\boldsymbol{r}) = \nabla \boldsymbol{V}(\boldsymbol{r})$. Then,

$$\nabla \times \boldsymbol{A} = \nabla \times (\nabla V).$$

Now the curl of any gradient function is zero. This can be most easily seen if we use Cartesian coordinates.

Let $\quad V(\boldsymbol{r}) = V(x, y, z).$

Then $$\boldsymbol{A} = \nabla V = \hat{\boldsymbol{i}}\frac{\partial V}{\partial x} + \hat{\boldsymbol{j}}\frac{\partial V}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial V}{\partial z}\cdot$$

So, $$A_x = \frac{\partial V}{\partial x}, A_y = \frac{\partial V}{\partial y}, A_z = \frac{\partial V}{\partial z}\cdot$$

Now $$[\nabla \times A]_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial y}\right) = 0.$$

Similarly $\quad [\nabla \times A]_y = 0$ and $[\nabla \times A]_z = 0.$

So $\quad \nabla \times \boldsymbol{A} = 0.$

For any vector field that can be written as a gradient of a scalar field, the curl must be zero. In other words, any vector field that can be written as gradient of a scalar field is a conservative field. This can also be taken as the definition of a conservative field.

The line integral of a conservative vector field

Suppose you have a conservative vector field A. Then, $\nabla \times \boldsymbol{A} = 0$ at all points of space. We also call such a field curl-free. Then,

$$\oint_{\text{loop}} \boldsymbol{A} \cdot \boldsymbol{dl} = \int_{\text{surface}} (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{da} = 0$$

for any closed loop. Thus the line integral of a conservative vector field over a closed loop is always zero.

Now suppose ACBDA is a closed loop and $\boldsymbol{A}$ is a curl-free vector (Figure A4.5).

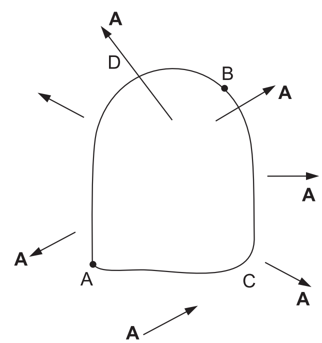


Fig. A4.5

Then,

$$\oint_{\text{ACBDA}} \boldsymbol{A} \cdot \boldsymbol{dl} = 0$$

or $$\int_{\text{ACB}} \boldsymbol{A} \cdot \boldsymbol{dl} + \int_{\text{BDA}} \boldsymbol{A} \cdot \boldsymbol{dl} = 0$$

or $$\int_{\text{ACB}} \boldsymbol{A} \cdot \boldsymbol{dl} - \int_{\text{ADB}} \boldsymbol{A} \cdot \boldsymbol{dl} = 0$$

or $$\int_{\text{ACB}} \boldsymbol{A} \cdot \boldsymbol{dl} = \int_{\text{ADB}} \boldsymbol{A} \cdot \boldsymbol{dl}.$$

This equation tells you that if you connect two given points A and B through any path, ACB or ADB or any other path, the line integral of A on all these paths will be the same. The line integral of a conservative vector field depends only on the end points and not on the path taken.

A4.5 Laplacian Operator

Suppose you have a scalar field $V(\boldsymbol{r})$. You can take the gradient of this to get a vector field $\nabla V(\boldsymbol{r})$. Next, take the divergence of this vector field ∇V. This is called the Laplacian of V and is written as $\nabla^2 V$. It is often read as Del square of V.

So $\nabla^2 V(\boldsymbol{r}) = \nabla \cdot [\nabla V(\boldsymbol{r})]$.

Its expressions in different coordinate systems are as follows.

Cartesian coordinates: $\nabla^2 V = \dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} \cdot$

Spherical polar coordinates: $\nabla^2 V = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial^2 V}{\partial r^2}\right) + \dfrac{1}{r^2 \sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta \dfrac{\partial V}{\partial\theta}\right) + \dfrac{1}{r^2 \sin^2\theta}\dfrac{\partial^2 V}{\partial\phi} \cdot$

Cylindrical polar coordinates: $\nabla^2 V = \dfrac{1}{\rho_r}\dfrac{\partial}{\partial\rho_r}\left(\rho_r \dfrac{\partial V}{\partial\rho_r}\right) + \dfrac{1}{\rho r^2}\dfrac{\partial^2 V}{\partial\phi^2} + \dfrac{\partial^2 V}{\partial z^2} \cdot$

❑

Appendix 5

Derivation of Ampere's Law from the Biot–Savart Law

The Biot–Savart law for the magnetic field due to a current distribution is given by $\boldsymbol{B} = \frac{\mu_0}{4\pi}\int \boldsymbol{J}(\boldsymbol{r}')\times\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\,d\tau'$. Taking the curl on both sides,

$$\nabla\times\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\nabla\times\int \boldsymbol{J}'(\boldsymbol{r}')\times\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\,d\tau'.$$

Now the integration is over a volume containing the current distribution and $d\tau'$ is the volume element at $\boldsymbol{r}'$. On the other hand, the curl operation on the RHS involves differentiation with respect to the coordinates of the field point $\boldsymbol{r}$. So the order of differentiation and integration can be reversed and we can write

$$\nabla\times\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int\left[\nabla\times\left\{\boldsymbol{J}(\boldsymbol{r}')\times\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\right\}\right]d\tau'.$$

Denote $\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$ by $\boldsymbol{A}$. The integrand is then $\nabla\times(\boldsymbol{J}\times\boldsymbol{A})$.

Use Cartesian coordinates and expand this expression.

$$\begin{aligned}\nabla\times(\boldsymbol{J}\times\boldsymbol{A}) &= \nabla\times[\hat{\boldsymbol{i}}(J_yA_z - J_zA_y) + \hat{\boldsymbol{j}}(J_zA_x - J_xA_z) + \hat{\boldsymbol{k}}(J_xA_y - J_yA_x)]\\ &= \hat{\boldsymbol{i}}\left[\frac{\partial}{\partial y}(J_xA_y - J_yA_x) - \frac{\partial}{\partial z}(J_zA_x - J_xA_z)\right]\\ &\quad + \hat{\boldsymbol{j}}\left[\frac{\partial}{\partial z}(J_yA_z - J_zA_y) - \frac{\partial}{\partial x}(J_xA_y - J_yA_x)\right]\\ &\quad + \hat{\boldsymbol{k}}\left[\frac{\partial}{\partial x}(J_zA_x - J_xA_z) - \frac{\partial}{\partial y}(J_yA_z - J_zA_y)\right].\end{aligned}$$

Now $\boldsymbol{J}$ is a function of x', y', z' and hence does not depend on x, y, z. Thus the above expression is

$$\hat{\boldsymbol{i}}\left[\left(J_x\frac{\partial A_y}{\partial y} - J_y\frac{\partial A_x}{\partial y}\right) - \left(J_z\frac{\partial A_x}{\partial z} - J_x\frac{\partial A_z}{\partial z}\right)\right]$$

$$+\hat{\boldsymbol{j}}\left[\left(J_y\frac{\partial A_z}{\partial z}-J_z\frac{\partial A_y}{\partial z}\right)-\left(J_x\frac{\partial A_y}{\partial x}-J_y\frac{\partial A_x}{\partial x}\right)\right]$$

$$+\hat{\boldsymbol{k}}\left[\left(J_z\frac{\partial A_z}{\partial x}-J_x\frac{\partial A_z}{\partial x}\right)-\left(J_y\frac{\partial A_z}{\partial y}-J_z\frac{\partial A_y}{\partial y}\right)\right].$$

Rearranging the terms,

$$\frac{\partial A_x}{\partial x}(\hat{\boldsymbol{j}}J_y+\hat{\boldsymbol{k}}J_z)+\frac{\partial A_y}{\partial y}(\hat{\boldsymbol{k}}J_z+\hat{\boldsymbol{i}}J_x)+\frac{\partial A_z}{\partial z}(\hat{\boldsymbol{i}}J_x+\hat{\boldsymbol{j}}J_y)$$

$$-\hat{\boldsymbol{i}}\left(J_y\frac{\partial A_x}{\partial y}+J_z\frac{\partial A_x}{\partial z}\right)-\hat{\boldsymbol{j}}\left(J_z\frac{\partial A_y}{\partial z}+J_x\frac{\partial A_y}{\partial x}\right)-\hat{\boldsymbol{k}}\left(J_x\frac{\partial A_z}{\partial x}+J_y\frac{\partial A_z}{\partial y}\right)$$

$$=\left(\frac{\partial A_x}{\partial x}\boldsymbol{J}+\frac{\partial A_y}{\partial y}\boldsymbol{J}+\frac{\partial A_z}{\partial z}\boldsymbol{J}\right)-\hat{\boldsymbol{i}}\left(J_x\frac{\partial A_x}{\partial x}+J_y\frac{\partial A_x}{\partial y}+J_z\frac{\partial A_x}{\partial z}\right)$$

$$-\hat{\boldsymbol{j}}\left(J_x\frac{\partial A_y}{\partial x}+J_y\frac{\partial A_y}{\partial y}+J_z\frac{\partial A_y}{\partial z}\right)-\hat{\boldsymbol{k}}\left(J_x\frac{\partial A_z}{\partial x}+J_y\frac{\partial A_z}{\partial y}+J_z\frac{\partial A_z}{\partial z}\right). \qquad \text{(i)}$$

Let us look at the first expression in parentheses. It is

$$\boldsymbol{J}(\nabla\cdot\boldsymbol{A})=\boldsymbol{J}\nabla\cdot\left(\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}|^3}\right).$$

Do you remember this divergence. For a point charge q at $\boldsymbol{r}'$, the electric field at $\boldsymbol{r}$ is $\boldsymbol{E}=\dfrac{q}{4\pi\varepsilon_0}\left(\dfrac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}|^3}\right)$.

Using Gauss's law,

$$\nabla\cdot\boldsymbol{E}=\frac{\rho}{\varepsilon_0}$$

or $$\nabla\cdot\left[\frac{q}{4\pi\varepsilon_0}\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\right]=\frac{1}{\varepsilon_0}q\delta^3(\boldsymbol{r}-\boldsymbol{r}')$$

or $$\nabla\cdot\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}=4\pi\delta^3(\boldsymbol{r}-\boldsymbol{r}').$$

Thus, the term in the first parentheses in (i) is equal to

$$A_1=4\pi\boldsymbol{J}\delta^3(\boldsymbol{r}-\boldsymbol{r}'). \qquad \text{(ii)}$$

Now look at the term in the second parentheses. It is

$$J_x \frac{\partial A_x}{\partial x} + J_y \frac{\partial A_x}{\partial y} + J_z \frac{\partial A_x}{\partial z}.$$

What is A_x? It is

$$A_x = \frac{x - x'}{\sqrt{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{3/2}}}.$$

You can easily check that

$$\frac{\partial A_x}{\partial x} = -\frac{\partial A_x}{\partial x'};\ \frac{\partial A_x}{\partial y} = -\frac{\partial A_x}{\partial y'};\ \frac{\partial A_x}{\partial z} = -\frac{\partial A_x}{\partial z'}.$$

So,

$$J_x \frac{\partial A_x}{\partial x} + J_y \frac{\partial A_x}{\partial y} + J_z \frac{\partial A_x}{\partial z}$$

$$= -\left[J_x \frac{\partial A_x}{\partial x'} + J_y \frac{\partial A_x}{\partial y'} + J_z \frac{\partial A_x}{\partial z'}\right]$$

$$= -\left[\nabla' \cdot (\boldsymbol{J} A_x) - \left(A_x \frac{\partial J_x}{\partial x'} + A_x \frac{\partial J_y}{\partial y'} + A_x \frac{\partial J_z}{\partial z'}\right)\right] = -[\nabla' \cdot (\boldsymbol{J} A_x) - A_x \nabla' \cdot \boldsymbol{J}].$$

Here the divergence is taken with respect to the primed coordinates (x', y', z'). As $\boldsymbol{J}$ is a function of primed coordinates, when we take the divergence of $\boldsymbol{J} A_x$, the components of $\boldsymbol{J}$ will also the differentiated.

But $\nabla' \cdot \boldsymbol{J} = 0$ for steady current distributions. Hence

$$J_x \frac{\partial A_x}{\partial x} + J_y \frac{\partial A_x}{\partial y} + J_z \frac{\partial A_x}{\partial z} = -\nabla' \cdot (\boldsymbol{J} A_x). \tag{iii}$$

Similarly,

$$J_x \frac{\partial A_y}{\partial x} + J_y \frac{\partial A_y}{\partial y} + J_z \frac{\partial A_y}{\partial z} = -\nabla' \cdot (\boldsymbol{J} A_y) \tag{iv}$$

And

$$J_x \frac{\partial A_z}{\partial x} + J_y \frac{\partial A_z}{\partial y} + J_z \frac{\partial A_z}{\partial z} = -\nabla' \cdot (\boldsymbol{J} A_z). \tag{v}$$

Putting (ii) to (v) in (i),

$$\nabla \cdot \left[\boldsymbol{J}(\boldsymbol{r}) \times \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3}\right] = 4\pi \boldsymbol{J} \delta^3(\boldsymbol{r} - \boldsymbol{r}') + \nabla' \cdot (\boldsymbol{J} A_x)\hat{\boldsymbol{i}} + \nabla' \cdot (\boldsymbol{J} A_y)\hat{\boldsymbol{j}} + \nabla' \cdot (\boldsymbol{J} A_z)\hat{\boldsymbol{k}}.$$

Thus,

$$\nabla \times \boldsymbol{B} = \frac{\mu_0}{4\pi}\left[\int 4\pi \boldsymbol{J}(\boldsymbol{r}')\delta^3(\boldsymbol{r}-\boldsymbol{r}')d\tau' + \int \nabla' \cdot (\boldsymbol{J}A_x)\hat{\boldsymbol{i}}\,d\tau' + \int \nabla' \cdot (\boldsymbol{J}A_y)\hat{\boldsymbol{j}}\,d\tau' + \int \nabla' \cdot (\boldsymbol{J}A_z)\hat{\boldsymbol{k}}\,d\tau'\right]. \qquad \text{(vi)}$$

As stated earlier, the integration has to be performed over a volume that contains the current distribution. But you can choose many such volumes. In fact you can choose the whole of space as the volume for integration. Whenever there is no current, $\boldsymbol{J} = 0$ and the contribution to the integral will be zero. Taking the whole of space as the volume for integration, the first integral becomes

$$\int_{\substack{\text{whole}\\ \text{space}}} \boldsymbol{J}(\boldsymbol{r}')\delta^3(\boldsymbol{r}-\boldsymbol{r}')d\tau' = \boldsymbol{J}(\boldsymbol{r}).$$

Now look at the second integration in (vi)

$$\int_{\substack{\text{whole}\\ \text{space}}} \nabla' \cdot (\boldsymbol{J}A_x)d\tau = \oint_{\substack{\text{bounding}\\ \text{surface}}} (\boldsymbol{J}A_x)\cdot \boldsymbol{da}'$$

using the Gauss divergence theorem. What is the bounding surface for all space? Surely the whole of this bounding surface is at infinity. Taking the current distribution to be finite, $\boldsymbol{J}$ at the bounding surface is then zero and the integral

$$\oint (\boldsymbol{J}A_x)\cdot \boldsymbol{da}' = 0.$$

This means that $\quad \int \nabla' \cdot (\boldsymbol{J}A_x)d\tau = 0.$

Similarly, $\quad \int \nabla' \cdot (\boldsymbol{J}A_y)d\tau = \int \nabla' \cdot (\boldsymbol{J}A_z)d\tau = 0.$

Thus, from (vi), $\quad \nabla \times \boldsymbol{B}(\boldsymbol{r}) = \mu_0 \boldsymbol{J}(\boldsymbol{r}).$

❑