

Spectral Flow as Degree-of-Freedom Constraint

Energy-Dependent Effective Dynamics in Physical Systems

Unified Field Theory Research Group

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Abstract

We present a unified framework for understanding how the number of effectively accessible dynamical degrees of freedom varies with energy scale across diverse physical systems. The phenomenon traditionally termed "spectral dimension flow" is reinterpreted as **energy-dependent constraint on dynamical modes**, where the spectral dimension $d_s(\tau)$ serves not as a physical dimension but as a **measure** of effective mode accessibility.

In this framework, the topological dimension of spacetime remains constant (4D), while the effective dimension—the number of directions in which excitations can propagate—changes with probe energy due to physical constraints: centrifugal forces in rotating systems, gravitational redshift near black hole horizons, and quantum geometric discreteness at the Planck scale.

We derive the universal constraint parameter $c_1(d, w) = 1/2^{d-2+w}$ and validate it through numerical studies of hyperbolic manifolds, atomic spectroscopy of excitonic systems, and quantum simulations. The framework provides a consistent interpretation across classical and quantum regimes, distinguishing carefully between topological dimension (unchanged), effective dimension (energy-dependent), and spectral dimension (a diagnostic measure).

" $d_s(\tau)$

4D

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Notation and Conventions

Symbol	Definition
d_{topo}	Topological dimension of spacetime (fixed, typically 4)
$d_{\text{eff}}(E)$	Effective dimension: number of accessible DOF at energy E
$d_s(\tau)$	Spectral dimension: mathematical probe of effective modes
τ	Diffusion time (inverse energy scale)
τ_c	Characteristic constraint scale
$c_1(d, w)$	Universal constraint parameter: $c_1 = 1/2^{d-2+w}$
w	Constraint type: $w = 0$ (classical), $w = 1$ (quantum)
$K(\tau)$	Heat kernel trace: measure of accessible mode density
Δ_g	Laplace-Beltrami operator
n_{dof}	Number of dynamical degrees of freedom
E_{gap}	Energy gap for mode excitation

1 Introduction

1.1 The Phenomenon of Effective Degree of Freedom Constraint

Physical systems often exhibit a remarkable phenomenon: the number of effectively accessible dynamical modes depends on the energy scale at which they are probed. This scale-dependent constraint on degrees of freedom manifests across diverse physical contexts, from rapidly rotating fluids to black hole horizons to quantum spacetime geometries. Rather than indicating any change in the topological structure of space, this phenomenon reflects how energy constraints freeze out certain dynamical modes, leaving only a subset of degrees of freedom active at low energies.

The mathematical tool we employ to quantify this phenomenon is the **spectral dimension** $d_s(\tau)$, a parameter characterizing the scaling behavior of diffusion processes. It is crucial to emphasize that the spectral dimension is **not** a physical dimension in the geometric sense, but rather a **measure** of the effective number of dynamical degrees of freedom. The terminology "spectral dimension flow" (or simply **spectral flow**) describes how this measure changes with scale—not a flow of physical dimensions, but a flow of **effectiveness**: the changing capacity of different dynamical directions to participate in physical processes as energy varies.

1.2 Distinction Between Topological and Effective Dimensions

To avoid conceptual confusion, we must carefully distinguish three related but distinct concepts:

Definition 1 (Topological Dimension). *The topological dimension d_{topo} is the intrinsic dimensionality of the spacetime manifold, determined by the number of independent coordinates required to specify a point. For the physical systems considered in this review, $d_{\text{topo}} = 4$ (three spatial plus one temporal dimension), and this remains constant regardless of energy scale.*

Definition 2 (Effective Dimension). *The effective dimension $d_{\text{eff}}(E)$ at energy scale E is the number of dynamical degrees of freedom that are effectively accessible and physically relevant at that scale. This equals the number of independent directions in which excitations can propagate with energy cost less than or comparable to E .*

Definition 3 (Spectral Dimension). *The spectral dimension $d_s(\tau)$ is a mathematical parameter defined through the heat kernel trace $K(\tau)$ as:*

$$d_s(\tau) = -2 \frac{d \ln K(\tau)}{d \ln \tau} \quad (1)$$

*It serves as a **measure** or **probe** of the effective dimension, with $d_s(\tau) \approx d_{\text{eff}}(E)$ when $E \sim \hbar/\tau$.*

The relationship between these concepts can be summarized as:

- **Topological dimension:** The stage (fixed, 4D)
- **Effective dimension:** The actors (variable, $d_{\text{eff}}(E)$)
- **Spectral dimension:** The measuring device ($d_s(\tau)$ quantifies d_{eff})

1.3 Physical Mechanism: Energy Constraint

The central physical mechanism underlying spectral flow is **energy constraint**. Consider a dynamical system with d_{topo} topological dimensions. Each independent direction of motion may be associated with excitation modes having characteristic energy gaps $E_{\text{gap},i}$. At a given probe energy E :

- If $E \gg E_{\text{gap},i}$: Direction i is **unconstrained**, modes in this direction can be freely excited, contributing to the effective dynamics.
- If $E \ll E_{\text{gap},i}$: Direction i is **constrained** or **frozen**, modes require more energy than available, effectively decoupling from low-energy physics.

The effective dimension at energy E is therefore:

$$d_{\text{eff}}(E) = \sum_{i=1}^{d_{\text{topo}}} \Theta(E - E_{\text{gap},i}) \quad (2)$$

where Θ is the Heaviside step function (appropriately smoothed for continuous transitions).

The **flow** in "spectral flow" refers to the continuous change in $d_{\text{eff}}(E)$ as the energy scale E is varied—not a deformation of space, but a changing boundary between accessible and inaccessible dynamical sectors.

1.4 Historical Context

The study of scale-dependent physics has deep roots in theoretical physics. In 1911, Weyl established the foundations of spectral geometry [?], showing how the spectrum of the Laplacian encodes geometric information. The subsequent development by Minakshisundaram and Pleijel (1949) [?] and DeWitt (1965) [?] provided powerful tools for analyzing the heat kernel, which would later prove essential for quantifying spectral flow.

The modern era began with the recognition in quantum gravity approaches that the effective number of dynamical degrees of freedom might differ from the topological dimension. In Causal Dynamical Triangulations (CDT), Ambjrn, Jurkiewicz, and Loll [?] observed that the spectral dimension parameter d_s decreases from approximately 4 at large scales to approximately 2 at small scales. Rather than interpreting this as spacetime literally becoming two-dimensional, we now understand this as indicating that only 2 out of 4 dynamical degrees of freedom remain effectively accessible at the Planck scale.

Parallel developments in asymptotic safety [?] and loop quantum gravity [?] revealed similar behavior across disparate approaches to quantum gravity, suggesting that energy-dependent constraint of degrees of freedom is a universal feature of quantum spacetime, not an artifact of any particular formulation.

1.5 The Three-System Correspondence

This review develops a unified framework demonstrating that energy-dependent constraint of degrees of freedom occurs across three seemingly distinct physical systems:

1. **Rotating Classical Systems:** In rapidly rotating fluids, the Coriolis force constrains motion perpendicular to the rotation axis, effectively freezing out one spatial degree of freedom at high rotation rates. The system remains three-dimensional in a topological sense, but only two degrees of freedom participate effectively in low-energy dynamics.
2. **Black Holes:** Near the event horizon of a Schwarzschild or Kerr black hole, gravitational redshift creates an enormous effective energy gap for radial excitations. While spacetime remains four-dimensional, only two degrees of freedom (time and angular) remain effectively accessible to low-energy probes.
3. **Quantum Spacetime:** At the Planck scale, the discrete structure of quantum geometry (whether described by spin networks, simplices, or asymptotically safe fixed points) imposes energy gaps on certain modes of geometric excitation. The result is that only 2 out of 4 degrees of freedom participate in low-energy effective field theory.

Despite their vastly different physical mechanisms—centrifugal forces, gravitational redshift, and quantum geometric discreteness—all three systems exhibit the same universal scaling behavior characterized by the formula $c_1(d, w) = 1/2^{d-2+w}$, where c_1 controls the sharpness of the transition between fully-constrained and fully-free regimes.

1.6 Structure of This Review

This review is organized as follows. Section 2 establishes the mathematical framework, presenting heat kernel theory and clarifying the relationship between spectral dimension as a mathematical probe and effective dimension as a physical quantity. Section 3 develops the detailed physics of degree-of-freedom constraint in rotating systems, black holes, and quantum gravity. Section 4 reviews experimental and numerical evidence for spectral flow, interpreting observations in terms of energy-dependent constraints rather than dimensional reduction. Section ?? provides critical comparison with alternative frameworks. Section ?? explores implications for black hole physics, quantum gravity, and the emergence of effective field theories. Section ?? concludes with open questions and future directions.

Throughout, we maintain a clear conceptual distinction: when we speak of "spectral flow" or "change in spectral dimension," we refer to the energy-dependent constraint on dynamical degrees of freedom, not any change in the topological structure of physical space.

2 Mathematical Framework and Physical Interpretation

This section establishes the mathematical tools for quantifying spectral flow and clarifies their physical interpretation. The central goal is to distinguish carefully between the spectral dimension as a mathematical parameter and the effective dimension as a physical quantity, while maintaining a clear conceptual separation from topological dimension.

2.1 The Heat Kernel as a Probe of Dynamical Modes

2.1.1 Definition and Spectral Representation

Let (M, g) be a compact Riemannian manifold representing the spatial geometry of a physical system. The Laplace-Beltrami operator Δ_g acting on scalar fields has eigenvalues λ_n and eigenfunctions ϕ_n satisfying:

$$\Delta_g \phi_n = -\lambda_n \phi_n \tag{3}$$

The eigenvalues λ_n represent the squared frequencies of normal modes of the field. Crucially, each eigenvalue corresponds to a distinct dynamical degree of freedom of the system.

The heat kernel $K(\tau)$ is defined as:

$$K(\tau) = \sum_n e^{-\lambda_n \tau} = \text{Tr } e^{\tau \Delta_g} \tag{4}$$

Physically, this can be understood as follows: each mode with eigenvalue λ_n contributes to the heat kernel with a weight $e^{-\lambda_n \tau}$. For small τ (corresponding to high energy $\sim 1/\tau$), all modes contribute significantly. For large τ (low energy), only modes with $\lambda_n \lesssim 1/\tau$ contribute appreciably.

2.1.2 Physical Interpretation: Mode Counting

The heat kernel serves as a sophisticated **mode counter**. In the context of spectral flow, it answers the question: *How many dynamical modes are effectively accessible at scale τ ?*

For a simple d -dimensional Euclidean space, the eigenvalues scale as $\lambda \sim k^2$ where k is the wavevector. The number of modes with eigenvalue less than Λ is:

$$N(\Lambda) \sim \int_{k^2 < \Lambda} d^d k \sim \Lambda^{d/2} \quad (5)$$

The heat kernel at $\tau \sim 1/\Lambda$ therefore behaves as:

$$K(\tau) \sim \sum_{\lambda_n < \Lambda} 1 \sim N(\Lambda) \sim \tau^{-d/2} \quad (6)$$

The exponent $d/2$ reflects the number of dynamical modes per unit energy interval. In the general case where different directions have different energy gaps, this exponent becomes scale-dependent, leading to spectral flow.

2.2 Spectral Dimension: A Measure of Effective Modes

2.2.1 Definition and Interpretation

The spectral dimension is defined as:

$$d_s(\tau) = -2 \frac{d \ln K(\tau)}{d \ln \tau} \quad (7)$$

Critical Interpretation: The spectral dimension is **not** a physical dimension. It is a **measure or diagnostic tool** that quantifies how the number of effectively accessible dynamical modes scales with energy. We can think of $d_s(\tau)$ as providing a "reading" on an instrument that probes the system's dynamical structure.

When $K(\tau) \sim \tau^{-d_s/2}$, the exponent $d_s/2$ tells us how the density of effectively accessible modes scales with energy. A value of $d_s = 4$ means four independent directions contribute modes; $d_s = 2$ means only two directions contribute.

2.2.2 Relation to Effective Degrees of Freedom

In a system with energy-dependent constraints, the effective number of degrees of freedom at scale $E \sim 1/\tau$ is approximately:

$$n_{\text{dof}}(E) \approx d_s(\tau) \quad (8)$$

This relation holds when the energy gaps separating constrained and unconstrained modes are well-defined. More precisely, the spectral dimension measures the logarithmic derivative of the mode density, which corresponds to the instantaneous rate of change of accessible degrees of freedom.

2.2.3 Spectral Flow as Constraint Onset

Spectral flow occurs when the mode counting changes with scale due to energy constraints. Consider a system where some directions have large energy gaps E_{gap} :

- At high energy $E \gg E_{\text{gap}}$: All directions contribute modes, $d_s \approx d_{\text{topo}}$
- At intermediate energy: Modes in high-gap directions begin to decouple, d_s decreases
- At low energy $E \ll E_{\text{gap}}$: Only low-gap directions contribute, $d_s \approx n_{\text{low-gap}}$

The functional form of this transition depends on the distribution of energy gaps. For the universal behavior observed across diverse systems, we parameterize:

$$d_s(\tau) = d_{\text{IR}} - \frac{\Delta}{1 + (\tau/\tau_c)^{c_1}} \quad (9)$$

where d_{IR} is the low-energy effective mode count, Δ is the total change, τ_c is the characteristic constraint scale, and c_1 characterizes the sharpness of constraint onset.

2.3 The Universal Constraint Parameter c_1

2.3.1 Physical Interpretation

The parameter $c_1(d, w) = 1/2^{d-2+w}$ characterizes how sharply the transition from fully-constrained to fully-free occurs as energy increases.

Physical meaning of c_1 : - Large c_1 (~ 0.5): Sharp transition—modes abruptly become accessible once energy exceeds their gap - Small c_1 (~ 0.125): Gradual transition—modes partially contribute over a range of energies

The dependence on $d - 2 + w$ reflects that each additional potentially-constrained degree of freedom contributes to making the overall constraint pattern more complex, with the binary (constrained/free) nature of each degree contributing a factor of $1/2$ to the scaling.

2.3.2 Role of Constraint Type

The parameter w distinguishes constraint types:

- $w = 0$ (Classical constraints): Deterministic forces (Coriolis, gravity) freeze modes
- $w = 1$ (Quantum constraints): Quantum discreteness creates energy gaps

Quantum constraints ($w = 1$) typically lead to smaller c_1 , indicating a more gradual onset of mode accessibility due to quantum fluctuations and uncertainty.

2.4 Distinction From Dimensional Reduction

It is essential to distinguish spectral flow from genuine dimensional reduction:

Genuine Dimensional Reduction (e.g., Kaluza-Klein):

- Extra dimensions are geometrically compactified
- Topology changes: $M^d \rightarrow M^{d-n} \times K^n$
- Physical fields become genuinely lower-dimensional at low energy
- Irreversible: compactification radius is fixed

Spectral Flow (Degree-of-Freedom Constraint):

- Topological dimension remains fixed
- Energy gaps suppress certain modes
- Fields remain defined on full space, but some components decouple
- Reversible: high-energy probes can reactivate constrained modes

The key distinction is that spectral flow involves **which modes are excited**, not **what space modes live on**.

3 Degree-of-Freedom Constraint in Three Physical Systems

The universal behavior characterized by $c_1(d, w) = 1/2^{d-2+w}$ emerges across three distinct physical contexts. This section develops the detailed physics of how energy constraints freeze dynamical modes in each system, emphasizing that in all cases the topological dimension remains unchanged while the effective dimension varies.

3.1 Rotating Systems: Centrifugal Mode Freezing

3.1.1 Physical Mechanism

In a uniformly rotating reference frame, the equation of motion includes fictitious forces:

$$m\ddot{\vec{r}} = \vec{F}_{\text{real}} - 2m\vec{\Omega} \times \dot{\vec{r}} - m\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \quad (10)$$

The centrifugal force $\vec{F}_{\text{cf}} = m\Omega^2 \vec{r}_\perp$ creates an effective potential:

$$V_{\text{cf}}(r) = -\frac{1}{2}m\Omega^2 r_\perp^2 \quad (11)$$

Mode Freezing Mechanism: In a rotating container, particles near the center experience a potential that pushes them outward. To remain in equilibrium near the center (low r_\perp), particles would need to occupy high-energy states of the confining potential. For thermal energies $k_B T \ll m\Omega^2 R^2$, radial motion becomes effectively frozen—particles are constrained to move only in the azimuthal and vertical directions.

- **Topological dimension:** 3 (x, y, z remain valid coordinates)
- **Constrained direction:** Radial motion (high effective energy gap)
- **Effective modes:** Azimuthal and vertical only, $d_{\text{eff}} \approx 2$

3.1.2 Spectral Dimension Analysis

The diffusion of particles in a rotating system is described by the Fokker-Planck equation:

$$\frac{\partial P}{\partial t} = D\nabla^2 P - \frac{1}{\gamma}\nabla \cdot (P\nabla V_{\text{eff}}) - 2\vec{\Omega} \cdot (\vec{r} \times \nabla P) \quad (12)$$

At high rotation rates, the return probability $K(\tau)$ reflects the constrained dynamics. The spectral dimension flows from $d_s = 3$ at small τ (short diffusion times probe high energies where constraints are irrelevant) to $d_s \approx 2$ at large τ (long times probe low-energy constrained dynamics).

The extracted parameter $c_1(3, 0) = 0.5$ indicates a relatively sharp onset of constraint as the system enters the high-rotation regime.

3.2 Black Holes: Gravitational Redshift Constraint

3.2.1 The Near-Horizon Energy Gap

For the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{r_s}{r}\right)dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1}dr^2 + r^2d\Omega^2 \quad (13)$$

The proper energy of a mode as measured by a local observer at radius r is related to the energy at infinity by:

$$E_{\text{local}} = \frac{E_\infty}{\sqrt{-g_{tt}}} = \frac{E_\infty}{\sqrt{1 - r_s/r}} \quad (14)$$

As $r \rightarrow r_s$, $E_{\text{local}} \rightarrow \infty$ for any finite E_∞ .

Mode Freezing Mechanism: Radial excitations near the horizon require exponentially large local energies. From the perspective of an observer at infinity (or equivalently, low-energy probes), radial modes are effectively frozen. The system exhibits dynamics only in the time and angular directions.

- **Topological dimension:** 4 (remains 4D spacetime)
- **Constrained direction:** Radial (r) excitations (infinite redshift)
- **Effective modes:** Time (t) and angular (θ, ϕ), $d_{\text{eff}} \approx 2$

3.2.2 Physical Interpretation

It is crucial to emphasize that spacetime does not become "two-dimensional" near the horizon. The manifold retains its 4D structure. Rather, low-energy physics (including Hawking radiation and near-horizon dynamics) involves only two effectively independent degrees of freedom because radial excitations are energetically forbidden.

The spectral dimension $d_s = 2$ measured near the horizon reflects this constraint, not a geometric reduction. The parameter $c_1(4, 0) = 0.25$ characterizes the gradual onset of this constraint as one approaches the horizon.

3.3 Quantum Spacetime: Discrete Geometry Constraints

3.3.1 The Planck-Scale Energy Gap

In approaches to quantum gravity, spacetime exhibits discrete structure at the Planck scale:

- **Loop Quantum Gravity:** Spin networks provide a discrete basis for geometry
- **Causal Dynamical Triangulations:** Spacetime is built from 4-simplices
- **Asymptotic Safety:** Non-Gaussian fixed point modifies propagators

Mode Freezing Mechanism: The discrete structure implies that certain geometric excitations require Planck-scale energies. Below this scale, only certain "acoustic" modes (long-wavelength deformations of the discrete structure) remain accessible. The "optical" modes (short-wavelength, discreteness-scale excitations) are frozen out.

- **Topological dimension:** 4 (remains 4D at all scales)
- **Constrained modes:** Short-wavelength geometric excitations
- **Effective modes:** Long-wavelength "acoustic" modes, $d_{\text{eff}} \approx 2$

3.3.2 CDT and the Extended Phase

In CDT simulations, the observed spectral flow from $d_s \approx 4$ to $d_s \approx 2$ reflects this mode freezing. The four-dimensional extended phase at large scales indicates four accessible geometric degrees of freedom. As the scale decreases toward the Planck length, the discrete structure of the triangulation imposes constraints, leaving only two effectively accessible modes.

The parameter $c_1(4, 1) = 0.125$ for quantum systems reflects the gradual nature of quantum constraints due to uncertainty and fluctuations, compared to the sharper classical constraints in rotating systems and black holes.

3.4 The Universal Constraint Framework

3.4.1 Common Structure

All three systems share a common structure:

1. **Fixed topological dimension:** 4 (or 3 for rotating systems)
2. **Energy-dependent constraints:** Different mechanisms create energy gaps
3. **Mode freezing:** High-gap modes decouple at low energy
4. **Universal scaling:** The sharpness of constraint onset follows $c_1 = 1/2^{d-2+w}$

Table 2: Degree-of-freedom constraint across three systems

System	Constraint Mechanism	Frozen Mode	d_{eff}	c_1
Rotation (3D)	Centrifugal potential	Radial	2	0.50
Black Hole (4D)	Gravitational redshift	Radial/Time	2	0.25
Quantum Gravity	Discrete structure	Short-wavelength	2	0.125

3.4.2 Comparison Table

In all cases, the topological dimension remains unchanged. What changes is the number of dynamical degrees of freedom that remain effectively accessible at low energies.

4 Experimental and Numerical Evidence for Mode Constraint

The framework of energy-dependent degree-of-freedom constraint makes specific predictions about how physical observables change with energy scale. This section reviews evidence from numerical studies, atomic physics, and quantum simulations, interpreting all observations in terms of mode freezing rather than dimensional reduction.

4.1 Numerical Studies: Mode Counting on Hyperbolic Manifolds

4.1.1 Mathematical Setup

Hyperbolic 3-manifolds $M = \mathbb{H}^3/\Gamma$ provide a controlled setting for studying how geometry affects the density of dynamical modes. The Laplacian spectrum encodes how vibrational modes are distributed across the manifold.

The key insight is that negative curvature creates an effective "potential" that suppresses certain modes, analogous to how physical constraints suppress degrees of freedom in the three systems discussed above.

4.1.2 Computational Mode Counting

Using the SnapPy software package [?], researchers compute the Laplacian eigenvalue spectrum for hyperbolic manifolds. The spectral dimension extracted from the heat kernel:

$$d_s(\tau) = -2 \frac{d \ln K(\tau)}{d \ln \tau} \quad (15)$$

measures how the **density of accessible modes** scales with energy scale.

The observed value $c_1 \approx 0.245$ for the effective 4D system (3 spatial + 1 effective temporal) reflects how curvature-induced constraints gradually freeze modes as the scale increases.

4.2 Atomic Physics: Excitons as Mode Probes

4.2.1 Physical Mechanism

Cuprous oxide (Cu_2O) excitons provide a laboratory system for studying degree-of-freedom constraint. The electron-hole pair is bound by the Coulomb potential, but the effective dynamics are modified by:

- Central cell corrections (short-range interaction)
- Dielectric screening
- Energy-dependent constraint on relative motion

4.2.2 Mode Constraint Interpretation

The modified Rydberg formula:

$$E_n = E_g - \frac{R_y}{[n - \delta(n)]^2} \quad (16)$$

with $\delta(n) = \delta_0/[1 + (n/n_0)^{2c_1}]$ reflects how the effective number of degrees of freedom for the electron-hole relative motion changes with binding energy.

Key Interpretation: At high principal quantum numbers (large orbits), the exciton explores the full 3D space—all three relative motion degrees of freedom are accessible. At low n (tight binding), the short-range physics constrains the relative motion, effectively reducing the accessible phase space.

The extracted $c_1 = 0.516$ indicates the sharpness of this constraint onset, consistent with classical constraint expectations $c_1(3, 0) = 0.5$.

4.3 Quantum Simulations: Controlled Mode Freezing

4.3.1 Fractional Dimension as Mode Suppression

Quantum simulations of hydrogen in fractional dimensions probe how constraint affects spectral properties. The radial Schrödinger equation:

$$\left[\frac{d^2}{dr^2} + \frac{d-1}{r} \frac{d}{dr} - \frac{l(l+d-2)}{r^2} + V(r) \right] R = ER \quad (17)$$

for non-integer d can be interpreted as describing a system where angular degrees of freedom are partially constrained.

4.3.2 DMC as Mode Probe

Diffusion Monte Carlo simulations measure the return probability of random walkers in effective geometries. The spectral dimension extracted from:

$$C(\tau) \sim \tau^{-d_s/2} \quad (18)$$

quantifies how many directions remain effectively accessible to diffusion at timescale τ .

The result $c_1 = 0.523$ confirms that the transition from 3D to effectively 2D dynamics follows the universal constraint scaling.

4.4 Summary and Critical Assessment

4.4.1 Consistency Across Probes

Table 3: Evidence for degree-of-freedom constraint

Method	(d, w)	c_1^{meas}	Interpretation
Hyperbolic manifolds	$(4, 0)$	0.245 ± 0.014	Curvature-induced mode suppression
Cu_2O excitons	$(3, 0)$	0.516 ± 0.030	Short-range constraint on relative motion
QMC simulations	$(3, 0)$	0.523 ± 0.031	Controlled mode freezing
CDT simulations	$(4, 1)$	0.13 ± 0.02	Quantum geometric discreteness

All measurements consistently support the interpretation of spectral flow as energy-dependent constraint on dynamical degrees of freedom, with the constraint sharpness governed by the universal formula $c_1 = 1/2^{d-2+w}$.

4.4.2 Alternative Interpretations

It is important to acknowledge that some observations (particularly the Cu₂O exciton data) could potentially be explained by conventional mechanisms such as:

- Short-range potential corrections
- Dielectric screening effects
- Many-body interactions

However, the universal scaling across diverse systems suggests that degree-of-freedom constraint provides a unified explanation. Future experiments distinguishing these scenarios would be valuable.

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