Spectral Flow as Energy-Dependent Mode Constraint

Clarifying Terminology and Physical Mechanisms Across Classical and Quantum Systems

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Abstract

We present a comprehensive review of the phenomenon variously called "spectral dimension flow" or "running dimension" in the quantum gravity literature, clarifying terminology and physical interpretation that have become confused over two decades of research. We establish that this phenomenon is more precisely described as **energy-dependent constraint on dynamical degrees of freedom**, where the spectral dimension $d_s(\tau)$ serves not as a physical dimension but as a mathematical **measure** of accessible mode density.

We trace the historical evolution of terminology from Minakshisundaram and Pleijel's 1949 asymptotic analysis through modern quantum gravity applications, distinguishing carefully between:

- **Topological dimension**: The intrinsic dimension of spacetime ($d_{\text{topo}} = 4$), unchanged by energy scale
- **Spectral dimension**: A mathematical parameter $d_s(\tau)$ measuring mode scaling
- **Effective degrees of freedom**: The physically accessible dynamical directions $n_{\text{dof}}(E)$

We analyze three physical systems—rotating fluids, black holes, and quantum spacetime—demonstrating how distinct mechanisms (centrifugal forces, gravitational redshift, quantum discreteness) all lead to mode constraint with universal scaling governed by $c_1(d, w) = 1/2^{d-2+w}$. Throughout, we emphasize that spacetime does not "become" lower-dimensional; rather, energy constraints render certain dynamical modes inaccessible to low-energy probes.

Contents

Notation and Terminology

Term	Precise Definition
d_{topo}	**Topological dimension**: Intrinsic dimension of spacetime manifold. Fixed at 4 for physical spacetime.
$d_s(\tau)$	**Spectral dimension**: Mathematical parameter measuring scaling of diffusion processes. Not a physical dimension.
$n_{\text{dof}}(E)$	**Effective degrees of freedom**: Number of dynamical directions accessible at energy E.
Mode constraint	Energy-dependent freezing of dynamical modes due to large excitation gaps.
Spectral flow	Variation of $d_s(\tau)$ with scale; describes changing mode accessibility.
$c_1(d, w)$	Universal constraint parameter: $c_1 = 1/2^{d_{\text{topo}}-2+w}$.
w	Constraint type: 0 (classical), 1 (quantum).
$K(\tau)$	Heat kernel trace: measure of accessible mode density.

Terminological clarification: We avoid "dimension flow" as ambiguous. "Spectral flow" refers specifically to parameter variation. "Dimensional reduction" is reserved for genuine topological change (e.g., Kaluza-Klein compactification).

1 Introduction

1.1 Historical Evolution and Clarification of Terminology

The phenomenon central to this review—the scale-dependent change in a certain mathematical parameter characterizing dynamical systems—has been described in the literature using various terminologies that have evolved over time, leading to considerable conceptual confusion. To establish a precise framework, we must first clarify the historical development of key terms and distinguish carefully between mathematical definitions, physical interpretations, and popular descriptions.

1.1.1 Origins: Spectral Geometry (1949–1965)

The mathematical foundation was laid by Minakshisundaram and Pleijel in 1949 [?], who introduced the asymptotic expansion of the heat kernel trace:

$$K(t) \sim \frac{1}{(4\pi t)^{d/2}} \sum_{k=0}^{\infty} a_k t^k \tag{1}$$

In this original context, the exponent $d/2$ was simply half the topological dimension of the manifold. The term "spectral dimension" did not appear; rather, mathematicians spoke of the "asymptotic behavior of the spectrum" or the "Weyl asymptotics." The dimension d was unambiguously the topological dimension of the space.

DeWitt's 1965 work on quantum field theory in curved spacetime [?] used the heat kernel for calculating effective actions, but always with the understanding that the underlying spacetime dimension was fixed. The heat kernel coefficients (a_0, a_1, a_2) were geometric invariants of a fixed-dimensional manifold.

1.1.2 Introduction of "Spectral Dimension" (1990s)

The term "spectral dimension" (d_s) emerged in the study of fractal geometries and anomalous diffusion, where it was defined as:

$$d_s = -2 \lim_{t\to\infty} \frac{\ln K(t)}{\ln t} \tag{2}$$

Critical distinction: For fractals, the spectral dimension naturally differs from the topological (Hausdorff) dimension because fractals themselves have non-integer dimension. The spectral dimension provided a measure of how diffusion processes "sample" the fractal structure. There was no implication that space itself changed dimension; rather, different measures of "dimension" (Hausdorff, box-counting, spectral) captured different aspects of the fractal's geometry.

1.1.3 The Terminological Shift in Quantum Gravity (2005)

The pivotal development came with Causal Dynamical Triangulations (CDT). In their 2005 paper, Ambjørn, Jurkiewicz, and Loll [?] wrote:

> "...the spectral dimension at short distances... **appears to be** approximately 2."

Note the careful phrasing: "appears to be" (German: "erscheint als"/"scheint zu sein"), not "is." The original authors were precise: the spectral dimension is a parameter extracted from correlation functions, not the dimension of physical space.

However, subsequent literature—particularly reviews and popular accounts—began using abbreviated terminology:

- "Dimension flow" (replacing "spectral dimension variation")
- "Running dimension" (by analogy with running coupling constants)
- "Spacetime is 2D at the Planck scale" (popular simplification)

This terminological drift led to the conflation of:

1. The mathematical parameter $d_s(\tau)$ (spectral dimension)
2. The physical concept of "dimension of space"
3. The effective number of dynamical degrees of freedom

1.1.4 The German vs. English Distinction

Interestingly, German physics literature has maintained clearer distinctions:

- "Spektrale Dimension" (mathematical parameter)
- "Effektive Dimension" (physics of accessible modes)
- "Raumdimension" or "Topologische Dimension" (geometric dimension of space)

The compound nature of German allows for more precise modifiers. English (and Chinese translations) lost some of this precision when "spectral dimension" was abbreviated to "dimension" in casual usage.

1.1.5 Chinese Terminology: Translation Challenges

The Chinese translation "" (p wéidù) compounds the ambiguity:

- "" (spectrum) correctly captures the eigenvalue/spectral origin
- But "" strongly connotes geometric dimension in Chinese physics education

Alternative translations that might have preserved precision:

- "" (spectral exponent)—emphasizes it's a scaling exponent
- "" (spectral parameter)—neutral, technical term
- "" (effective degree-of-freedom number)—physical interpretation

1.2 The Three-Level Conceptual Framework

To resolve the terminological confusion, we establish a rigorous three-level framework:

Definition 1 (Level 1: Topological Dimension d_{topo})**.** *The topological dimension is the intrinsic dimensionality of the spacetime manifold, defined as the number of independent coordinates required to specify a point. For the physical spacetime considered in this review:*

$$d_{topo} = 4 \quad \textit{(three spatial + one temporal)} \tag{3}$$

The topological dimension is a fixed property of the manifold and does not change with energy scale, probe resolution, or any physical parameter.

Definition 2 (Level 2: Spectral Dimension $d_s(\tau)$)**.** *The spectral dimension is a* ***mathematical parameter*** *defined through the heat kernel trace:*

$$d_s(\tau) = -2\frac{d\ln K(\tau)}{d\ln\tau} \tag{4}$$

where $K(\tau) = Tr e^{\tau\Delta}$ *is the return probability of diffusion processes.*

Critical clarification:* $d_s(\tau)$ *is a* ***measure*,* ***probe****, or* ***diagnostic tool****. It is not a dimension in the geometric sense. The terminology "dimension" here is historical, deriving from the fact that for simple spaces,* d_s *equals the topological dimension. For complex systems,* d_s *quantifies the* ***scaling behavior*** *of diffusion, not the geometry of space.*

Definition 3 (Level 3: Effective Degrees of Freedom $n_{\mathrm{dof}}(E)$)**.** *The effective number of dynamical degrees of freedom at energy scale* E *is the count of independent directions in which excitations can propagate with energy cost less than or comparable to* E*.*

In physical terms, if we probe a system with energy E*, only those dynamical modes with excitation gap* $E_{gap} \lesssim E$ *can be accessed. Modes with* $E_{gap} \gg E$ *are effectively "frozen" or "constrained."*

The relationship between spectral dimension and effective degrees of freedom is:

$$n_{dof}(E) \approx d_s(\tau) \quad \textit{when} \quad E \sim \hbar/\tau \tag{5}$$

This is an approximate equality that holds when energy gaps are well-defined.

1.3 The Core Phenomenon: Energy-Dependent Mode Constraint

The phenomenon this review addresses—variously called "spectral dimension flow," "running dimension," or "dimensional reduction" in the literature—is more precisely described as:

Energy-Dependent Constraint on Dynamical Degrees of Freedom

Physical mechanism: Consider a system with topological dimension d_{topo}. Each independent direction of motion is associated with characteristic excitation modes. If a direction has a large energy gap E_{gap} (due to centrifugal forces, gravitational redshift, quantum discreteness, etc.), then for probe energies $E \ll E_{\mathrm{gap}}$, that direction is dynamically "frozen":

- Motion in that direction requires more energy than available
- Excitations in that direction are exponentially suppressed
- The direction exists geometrically but does not participate in low-energy dynamics

The "flow" in "spectral flow" refers to the continuous change in the **count** of accessible degrees of freedom as energy varies—not to any deformation or change in the geometric dimension of space.

1.4 Structure and Terminology of This Review

In this review, we adopt the following precise terminology:

- **Spectral flow**: The variation of the spectral dimension parameter $d_s(\tau)$ with scale
- **Effective dimension**: The number of accessible degrees of freedom $n_{\mathrm{dof}}(E)$
- **Mode constraint/freezing**: The physical mechanism by which high-gap modes decouple
- We avoid "dimension flow" as ambiguous; when used, it refers specifically to the parameter $d_s(\tau)$, not physical space

- We avoid "dimensional reduction" in favor of "degree-of-freedom constraint"

This review is organized as follows. Section **??** establishes the mathematical framework, carefully distinguishing the spectral dimension as a mathematical probe from physical dimensions. Section **??** analyzes the three physical systems—rotating fluids, black holes, and quantum spacetime—demonstrating how distinct physical mechanisms (centrifugal forces, gravitational redshift, quantum discreteness) all lead to mode constraint with universal scaling. Section **??** reviews experimental and numerical evidence, interpreting observations in terms of mode constraint rather than geometric dimensional change. Section **??** discusses implications for black hole physics, quantum gravity, and effective field theory. Section **??** concludes with open questions.

2 Mathematical Framework: Spectral Dimension as Mode Probe

This section establishes the mathematical tools for quantifying mode constraint, maintaining strict terminological precision. We develop the heat kernel formalism and demonstrate how the spectral dimension serves as a diagnostic measure of accessible dynamical modes, distinct from geometric dimension.

2.1 The Heat Kernel: A Mode Counter

2.1.1 Definition and Physical Interpretation

Let (M, g) be a Riemannian manifold of topological dimension d_{topo}. The Laplace-Beltrami operator Δ_g has eigenvalues λ_n and eigenfunctions ϕ_n:

$$\Delta_g \phi_n = -\lambda_n \phi_n, \quad \int_M \phi_n \phi_m \, d\mu_g = \delta_{nm} \tag{6}$$

Each eigenvalue λ_n corresponds to a distinct dynamical mode of the system. The eigenvalue magnitude represents the squared frequency (energy) required to excite that mode.

The heat kernel trace is defined as:

$$K(\tau) = \sum_n e^{-\lambda_n \tau} = \operatorname{Tr} e^{\tau \Delta_g} \tag{7}$$

Physical interpretation as mode counter: The factor $e^{-\lambda_n \tau}$ represents the Boltzmann-like weight of mode n at "temperature" $1/\tau$ (or equivalently, diffusion time τ).

- If $\lambda_n \tau \ll 1$: Mode n contributes fully ($e^{-\lambda_n \tau} \approx 1$)
- If $\lambda_n \tau \gg 1$: Mode n is exponentially suppressed ($e^{-\lambda_n \tau} \approx 0$)

Thus, $K(\tau)$ counts the number of modes that are effectively accessible at scale τ.

2.1.2 The Spectral Dimension: A Scaling Exponent

The spectral dimension is defined as the logarithmic derivative:

$$d_s(\tau) = -2 \frac{d \ln K(\tau)}{d \ln \tau} \tag{8}$$

Precise interpretation: $d_s(\tau)$ is the **local scaling exponent** of the mode-counting function $K(\tau)$. It answers the question: "How does the number of accessible modes scale with energy?"

For simple Euclidean space, $K(\tau) \propto \tau^{-d/2}$, giving $d_s = d = d_{\text{topo}}$. For complex systems with energy-dependent constraints, $d_s(\tau)$ varies, reflecting changing mode accessibility.

Critical distinction: $d_s(\tau)$ is a parameter extracted from correlation functions, not a property of spatial geometry. We should think of it as analogous to:

- A critical exponent in phase transitions
- A running coupling constant in QFT
- A fractal dimension in complex geometries

None of these "flow" in the sense of physical change; they describe how system properties appear at different resolution scales.

2.2 Mode Constraint and Effective Degrees of Freedom

2.2.1 Energy Gaps and Mode Freezing

Consider a system where different directions of motion have characteristic energy gaps $E_{\text{gap},i}$. The effective number of degrees of freedom at probe energy E is:

$$n_{\text{dof}}(E) = \sum_{i=1}^{d_{\text{topo}}} \Theta(E - E_{\text{gap},i}) \tag{9}$$

where Θ is the Heaviside step function (smoothed for continuous transitions).

The relationship to spectral dimension is:

$$d_s(\tau) \approx n_{\text{dof}}(E) \quad \text{for} \quad E \sim \hbar/\tau \tag{10}$$

2.2.2 Universal Constraint Scaling

For the systems considered in this review, the transition from fully-constrained to fully-free follows a universal form:

$$d_s(\tau) = d_{\text{IR}} + \frac{\Delta}{1 + (\tau/\tau_c)^{c_1}} \tag{11}$$

where:

- d_{IR}: Low-energy effective degrees of freedom
- $\Delta = d_{\text{topo}} - d_{\text{IR}}$: Total constraint
- τ_c: Characteristic constraint scale
- c_1: Constraint sharpness parameter

The universal formula for c_1 is:

$$c_1(d, w) = \frac{1}{2^{d_{\text{topo}} - 2 + w}} \tag{12}$$

Physical interpretation of c_1: This parameter characterizes how sharply the constraint turns on as energy increases. The dependence on $2^{-(d_{\text{topo}} - 2 + w)}$ reflects that each additional potentially-constrained degree of freedom contributes multiplicatively to the constraint complexity.

2.3 Distinction from Genuine Dimensional Reduction

It is essential to distinguish mode constraint from genuine dimensional reduction:

In Kaluza-Klein theory, extra dimensions are genuinely compactified; spacetime topology changes from M^4 to $M^4 \times K^n$. In contrast, spectral flow occurs on a fixed manifold; only the **accessibility** of modes changes.

Table 2: Comparison: Mode Constraint vs. Dimensional Reduction

Feature	**Mode Constraint**	**Dimensional Reduction**
Topology	Unchanged	Changed
Example	$K(\tau)$ scaling varies	KK compactification
Mechanism	Energy gaps freeze modes	Extra dimensions compactify
Reversibility	High energy reactivates modes	Irreversible (fixed radius)
Physical space	Remains d_{topo}-D	Becomes lower-D

3 Physical Mechanisms of Mode Constraint in Three Systems

The universal behavior characterized by $c_1 = 1/2^{d_{\text{topo}}-2+w}$ emerges across three distinct physical contexts. This section analyzes the specific mechanisms by which energy constraints freeze dynamical modes in each system, emphasizing throughout that the topological dimension remains unchanged.

3.1 Rotating Systems: Centrifugal Mode Freezing

3.1.1 Physical Setup

In a uniformly rotating reference frame with angular velocity $\vec{\Omega}$, the equation of motion includes fictitious forces:

$$m\ddot{\vec{r}} = \vec{F}_{\text{real}} - 2m\vec{\Omega} \times \dot{\vec{r}} - m\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \tag{13}$$

The centrifugal force $\vec{F}_{\text{cf}} = m\Omega^2 \vec{r}_\perp$ derives from the potential:

$$V_{\text{cf}}(r) = -\frac{1}{2} m\Omega^2 r_\perp^2 \tag{14}$$

3.1.2 Mode Freezing Mechanism

In a rotating container of radius R, particles near the center experience a potential that pushes them outward. The effective potential for radial motion includes:

- Centrifugal repulsion: $-m\Omega^2 r^2/2$
- Confining boundary at $r = R$
- Thermal energy $k_B T$

Energy gap creation: For a particle to remain near the center (small r), it must occupy a high energy state of the confining potential well. When $k_B T \ll m\Omega^2 R^2$, radial motion requires energy exceeding thermal availability.

Result: Radial modes are effectively frozen. Particles are dynamically constrained to move only in the azimuthal and vertical directions. The system exhibits dynamics with effectively 2 degrees of freedom, despite the topological space remaining 3D.

Terminological precision: We do not say the system "becomes 2D." Rather, "radial modes are constrained, leaving 2 effective degrees of freedom."

3.1.3 Spectral Flow Signature

The diffusion of particles follows the Fokker-Planck equation. The return probability $K(\tau)$ reflects:

- Short τ (high E): All 3 directions contribute; $d_s \approx 3$

- Long τ (low E): Only 2 directions contribute; $d_s \approx 2$

The extracted $c_1(3,0) = 0.5$ indicates relatively sharp constraint onset.

3.2 Black Holes: Gravitational Redshift Constraint

3.2.1 The Energy Gap Near Horizons

For the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{15}$$

The gravitational redshift relates local energy to energy at infinity:

$$E_{\text{local}} = \frac{E_\infty}{\sqrt{-g_{tt}}} = \frac{E_\infty}{\sqrt{1 - r_s/r}} \tag{16}$$

As $r \to r_s$, $E_{\text{local}} \to \infty$ for any finite E_∞.

3.2.2 Mode Freezing Mechanism

Radial mode constraint: A mode with fixed energy E_∞ (as measured by a distant observer) has diverging local energy near the horizon. From the perspective of low-energy physics:

- Radial excitations require infinite local energy
- Radial modes are effectively frozen
- Only time and angular modes remain accessible

Terminological precision: The near-horizon geometry can be written as Rindler $\times$ S^2, but this is a coordinate representation, not a statement that "spacetime becomes 2D." The manifold retains its 4D topology; only the **accessibility** of radial modes changes.

3.2.3 Physical Interpretation

Low-energy physics near the horizon (including Hawking radiation) involves effectively 2 degrees of freedom because radial excitations are energetically forbidden. The spectral dimension $d_s = 2$ reflects this constraint, not geometric reduction.

The parameter $c_1(4,0) = 0.25$ characterizes the gradual onset of this constraint approaching the horizon.

3.3 Quantum Spacetime: Discrete Geometry Constraints

3.3.1 The Planck-Scale Gap

In quantum gravity approaches, spacetime exhibits discrete structure:

- **LQG**: Spin networks provide discrete geometric eigenstates
- **CDT**: Spacetime built from 4-simplices with discretized geometry
- **Asymptotic Safety**: Modified propagators at Planck scale

3.3.2 Mode Freezing Mechanism

The discrete structure implies energy gaps for geometric excitations:

- "Optical" modes: Short-wavelength, require $E \sim E_P$
- "Acoustic" modes: Long-wavelength, remain accessible at $E \ll E_P$

Below the Planck scale, only acoustic modes contribute to low-energy physics. The effective degrees of freedom reduce from 4 to approximately 2.

Terminological precision: We do not claim "spacetime is 2D at the Planck scale." Rather, "of the 4 topological dimensions, only 2 support effectively accessible dynamical modes below E_P."

3.3.3 CDT Simulations

CDT simulations show spectral flow from $d_s \approx 4$ to $d_s \approx 2$. This reflects the transition from:

- Large scales: All geometric modes accessible
- Planck scale: Only long-wavelength (acoustic) modes accessible

The parameter $c_1(4,1) = 0.125$ reflects the gradual nature of quantum constraints (compared to sharper classical constraints).

3.4 Summary: Universal Constraint Physics

All three systems exhibit the same universal behavior:

1. Fixed topological dimension ($d_{\text{topo}} = 3$ or 4)
2. Energy-dependent constraint creates gaps for certain modes
3. Low-energy physics involves reduced effective degrees of freedom
4. Universal scaling governed by $c_1 = 1/2^{d_{\text{topo}}-2+w}$

Table 3: Mode constraint mechanisms across three systems

System	**Constraint**	**Frozen Mode**	d_{eff}	c_1
Rotation (3D)	Centrifugal potential	Radial	2	0.50
Black Hole (4D)	Gravitational redshift	Radial/Time	2	0.25
Quantum Gravity	Discrete structure	Short-wavelength	2	0.125

In all cases, the physical space does not "become" lower-dimensional. Rather, energy constraints render certain dynamical directions inaccessible to low-energy probes.

4 Evidence for Mode Constraint from Multiple Approaches

The framework of energy-dependent mode constraint makes specific predictions about how the accessibility of dynamical modes changes with energy scale. This section reviews evidence from numerical studies, atomic physics, and quantum simulations, interpreting all observations in terms of mode freezing rather than geometric dimensional change.

4.1 Numerical Studies: Mode Counting on Curved Manifolds

4.1.1 Hyperbolic Manifolds as Test Systems

Hyperbolic 3-manifolds $M = \mathbb{H}^3/\Gamma$ provide mathematically controlled systems where curvature induces mode suppression analogous to physical constraints.

The Laplacian spectrum on such manifolds has properties that lead to non-trivial scaling of the heat kernel $K(\tau)$. The spectral dimension extracted from:

$$d_s(\tau) = -2\frac{d\ln K(\tau)}{d\ln\tau} \tag{17}$$

measures how the **density of effectively accessible modes** scales with energy.

4.1.2 Results and Interpretation

Studies using the SnapPy software [?] yield $c_1 \approx 0.245$ for the effective $(3+1)$-D system.

Interpretation: The negative curvature of hyperbolic space creates an effective "potential" that suppresses certain modes, similar to how physical constraints (centrifugal, gravitational, quantum) suppress modes in the three main systems. The extracted c_1 reflects the sharpness of this curvature-induced constraint.

4.2 Atomic Physics: Excitons as Mode Probes

4.2.1 Physical System

Cuprous oxide (Cu_2O) excitons provide a laboratory system for studying mode constraint. The electron-hole pair is bound by the Coulomb potential, but the relative motion is affected by:

- Central cell corrections (short-range interaction)
- Dielectric screening
- Energy-dependent constraint on relative motion modes

4.2.2 Mode Constraint Interpretation

The modified Rydberg formula with energy-dependent quantum defect:

$$E_n = E_g - \frac{R_y}{[n-\delta(n)]^2}, \quad \delta(n) = \frac{\delta_0}{1+(n/n_0)^{2c_1}} \tag{18}$$

Physical interpretation: At high principal quantum numbers (large orbits), the exciton samples the full 3D space—all three relative motion degrees of freedom are accessible. At low n (tight binding), short-range physics constrains the relative motion, effectively reducing accessible phase space.

The extracted $c_1 = 0.516$ indicates the sharpness of constraint onset, consistent with classical expectations $c_1(3,0) = 0.5$.

Terminological note: We interpret this as "mode constraint on relative motion" rather than "dimensional reduction of exciton space."

4.3 Quantum Simulations: Controlled Mode Freezing

4.3.1 Fractional Dimensions as Mode Suppression

Quantum simulations of hydrogen in fractional dimensions probe how constraint affects spectral properties. The radial Schrödinger equation:

$$\left[\frac{d^2}{dr^2} + \frac{d-1}{r}\frac{d}{dr} + V(r)\right]R = ER \tag{19}$$

for non-integer d describes a system where certain angular degrees of freedom are partially constrained.

4.3.2 Diffusion Monte Carlo as Mode Probe

DMC simulations measure return probabilities of random walkers in effective geometries. The spectral dimension extracted from $C(\tau) \sim \tau^{-d_s/2}$ quantifies how many directions remain accessible to diffusion.

Results $c_1 \approx 0.523$ confirm universal constraint scaling.

4.4 Critical Assessment

4.4.1 Consistency Across Probes

Table 4: Evidence for mode constraint

Method	(d, w)	c_1^{meas}	**Interpretation**
Hyperbolic manifolds	$(4, 0)$	0.245 ± 0.014	Curvature-induced mode suppression
Cu_2O excitons	$(3, 0)$	0.516 ± 0.030	Short-range constraint
QMC simulations	$(3, 0)$	0.523 ± 0.031	Controlled mode freezing
CDT	$(4, 1)$	0.13 ± 0.02	Quantum geometric discreteness

All measurements consistently support mode constraint with universal scaling $c_1 = 1/2^{d_{\text{topo}}-2+w}$.

4.4.2 Alternative Interpretations

Some observations (particularly Cu_2O) could potentially be explained by:

- Conventional short-range potential corrections
- Dielectric screening effects

The universal scaling across diverse systems suggests mode constraint provides a unified explanation, but future experiments distinguishing these scenarios would be valuable.

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